# Recognizing Certain Rational Homogeneous Manifolds of Picard Number 1 from their Varieties of Minimal Rational Tangents

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Let $X$ be a uniruled projective manifold, i.e., a projective manifold that can be filled up by rational curves. Fixing a polarization and minimizing degrees among free rational curves we have the notion of minimal rational curves. Collecting the set of all tangents to minimal rational curves passing through a given general point we obtain a variety of minimal rational tangents. In a series of articles with Jun-Muk Hwang we have been studying uniruled projective manifolds, especially Fano manifolds of Picard number 1, in terms of the geometry of varieties of minimal rational tangents. Here Fano manifolds admit rational curves by the pioneering work of Mori [18], and they are uniruled by Miyaoka-Mori [16].

In our geometric study of uniruled projective manifolds the general philosophy is that one should be able to recover the complex structure and algebro-geometric properties of a projective uniruled manifold, especially in the case of Fano manifolds of Picard number 1, from the geometry of its varieties of minimal rational tangents. In this direction, when varieties of minimal rational tangents of a Fano manifold $X$ of Picard number 1 are positive-dimensional, under very mild conditions we have shown in Hwang-Mok [8] that $X$ is determined up to biholomorphism by the germ of the fiber space of varieties of minimal rational tangents at a general base point, where such germs are identified whenever there exists a local biholomorphism which transforms the varieties of minimal rational tangents to one another. In the case of an irreducible Hermitian symmetric space $S$ of the compact type and of rank $\geq 2$, there is a theory of geometric structures, called $S$-structures, arising from isotropy representations. In this case the result of Ochiai [19] says that a local biholomorphism $S$ preserving $S$-structures must necessarily extend to an automorphism. To relate to rational curves we note that a local biholomorphism on $S$ preserves the $S$-structure if and only if it preserves varieties of minimal rational tangents. Thus, the result of [8] cited above can be interpreted as uniqueness results for "variable" geometric structures.

In the case of a Hermitian symmetric space $S$ of the compact type and of rank $\geq 2$, there are other results on the characterization of $S$ in terms of the geometry of varieties of minimal rational tangents. Ochiai's result [19] says that a simply connected compact complex manifold $X$ admitting a flat $S$-structure is biholomorphic to $S$. When $X$ is assumed to be a uniruled projective manifold, Hwang-Mok [5] shows that flatness is automatic. As a step in the proof Hwang-Mok [5] shows that a uniruled projective manifold $X$ is biholomorphic to $S$ whenever varieties of minimal rational curves at *every* point agrees with those of the model space $S$. When $X$ is of Picard number 1, as suggested by [8,9], ideally geometric characterizations should depend on varieties of minimal rational tangents on a nonempty open subset. In this article we prove a result in this direction, showing that a uniruled

*Supported by a CERG grant of the Research Grants Council of Hong Kong.

projective manifold $X$ of Picard number 1 is biholomorphic to $S$ if and only if the variety of minimal rational tangents of $X$ at a *general* point is projectively equivalent to those of the model space $S$. We also generalize the latter result to the case of Fano contact homogeneous manifolds of Picard number 1 other than odd-dimensional projective spaces. In the proof we make use of Hong's characterization [3] of such manifolds among uniruled projective manifolds in terms of geometric structures, which in turn relies on the theory of geometric structures and differential systems of Tanaka [20] and Yamaguchi [21]

On the uniruled projective manifold $X$ equipped with a minimal rational component $\mathcal{K}$, there is *a priori* a subvariety $E \subsetneq X$ outside which varieties of minimal rational tangents agree with those of the model space. The principal difficulty in the proof of our main results is that $E$ may contain a divisor $H$. To overcome this we prove what one may call a 'removable singularity theorem' for a general point of $H$, by showing that the variety of minimal rational tangents at such a point must still agree with those of the model space. For the proof we make use of a notion of parallel transport of the second fundamental form along the tautological lifting of a standard minimal rational curve, and in the case of contact Fano manifolds, parallel transport for the third fundamental form. In an implicit form these notions were first introduced in Mok [17] in connection with a problem of reconstruction of the 3-dimensional hyperquadric and the 5-dimensional Fano contact homogeneous manifold pertaining to $G_2$, in which cases it is sufficient to ascertain that the second and third fundamental forms remain nondegenerate over a general point of the divisor $H$. In the current article, application of the notion of parallel transport in the cases at hand are enough to show that the varieties of minimal rational tangents on the uniruled projective manifold $X$ agree with those of the model symmetric or contact homogeneous manifold outside a subvariety of codimension $\geq 2$. In the proof, of special importance is the consideration of varieties of minimal rational tangents $\mathcal{C}_x$ themselves as uniruled projective manifolds. For instance, for $S$ symmetric in most cases $\mathcal{C}_x$ is itself an irreducible Hermitian symmetric space of rank 2 whose complex structure is completely determined by its own varieties of minimal rational tangents, which are in turn determined by the second fundamental form which we show to be invariant under parallel transport along tautological liftings of standard minimal rational curves.

Acknowledgement At the International Congress of Chinese Mathematicians (ICCM), December 2004, held at the Chinese University of Hong Kong, the author gave a lecture entitled "Geometry of Fano Manifolds", in which he explained a general program of research on uniruled projective manifolds basing on the study of varieties of minimal rational tangents. The current article is an outgrowth of that program. The author would like to thank the organizers of the ICCM for their invitation. He also wishes to thank the referee for corrections and for pointing out that some of the extension results on varieties of minimal rational tangents in the present article (parts of Propositions 2.3 and 3.2) follow from existing results on the characterization of certain Hermitian symmetric spaces in terms of local differential projective-geometric invariants.

## §1 Background about minimal rational curves on uniruled projective manifolds

(1.1) To put the results of the current article in perspective and in line with the style of presentation of the the original lecture addressed to non-specialists, we will provide a broader overview of the background on rational curves than is absolutely necessary. A standard reference for the study of rational curves on projective manifolds is Kollár [12]. For the set-up pertinent to the discussion in the current article the reader may also consult Hwang-Mok [6,7].

Let $X$ be a projective manifold. By a parametrized rational curve on $X$ we mean a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$. $f$ is said to be free if and only if $f^*T_X$ is semipositive. A projective manifold $X$ is uniruled if and only if there exists a free parametrized rational curve $f : \mathbb{P}^1 \to X$ on $X$. In this case deformations of $f$ sweep out a Zariski-open subset of $X$.

Let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the set of holomorphic mappings of $\mathbb{P}^1$ into the projective manifold $X$. $\mathrm{Hom}(\mathbb{P}^1, X)$ is naturally a subscheme of the Hilbert scheme of $\mathbb{P}^1 \times X$ (Kollár [12, (1.9), p.16ff.]) and as such it decomposes into irreducible components. We will say that an irreducible component $\mathcal{H}$ of $\mathrm{Hom}(\mathbb{P}^1, X)$ is dominant whenever the rational curves $C := f(\mathbb{P}^1)$, $f \in \mathcal{H}$, sweep out a Zariski open subset of $X$. This is the case if and only if $\mathcal{H}$ contains a free parametrized rational curve.

Given a dominant irreducible component $\mathcal{H}$ of $\mathrm{Hom}(\mathbb{P}^1, X)$ whose general member $f : \mathbb{P}^1 \to X$ is generically injective, consider the subset $\mathcal{H}' \subset \mathcal{H}$ consisting of those $f \in \mathcal{H}$ which are free and generically injective. The deformation of any $f \in \mathcal{H}'$ is unobstructed, and $\mathcal{H}'$ can be given the structure of a quasi-projective manifold on which $\mathrm{Aut}(\mathbb{P}^1) \cong \mathbb{P}GL(2;\mathbb{C})$ acts freely and algebraically, realizing the canonical map $\lambda : \mathcal{H}' \to \mathcal{H}'/\mathrm{Aut}(\mathbb{P}^1) := \mathcal{K}$ as a principal bundle. Here $\mathrm{Aut}(\mathbb{P}^1)$ acts without fixed points because $f : \mathbb{P}^1 \to X$ is generically injective for every $f$ belonging to $\mathcal{H}'$, and the base space $\mathcal{K}$ admits the structure of a quasi-projective manifold. $\mathcal{K}$ is an irreducible component of $\mathrm{RatCurves}^n(X)$, in the notations of Kollár [12, (2.11), p.108ff.], and will be called a rational component in what follows. A member of $\mathcal{K}$ defined by $f \in \mathcal{H}'$ corresponds to its image $C = f(\mathbb{P}^1)$, and will be referred to as a $\mathcal{K}$-rational curve. To avoid confusion we will sometimes write $[C]$ for a $\mathcal{K}$-rational curve $C$, when the latter is regarded as a member of $\mathcal{K}$. Associated to $\mathcal{K}$ we have the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ of $\mathcal{K}$-rational curves. Thus, $\rho : \mathcal{U} \to \mathcal{K}$ is a holomorphic $\mathbb{P}^1$-bundle, and $\mu(\rho^{-1}([C]))$ projects onto the rational curve $C \subset X$.

Let $X$ be a uniruled projective manifold and $\{\mathcal{H}_\alpha\}_{\alpha \in A}$ be the nonempty set of all dominant irreducible components of $\mathrm{Hom}(\mathbb{P}^1, X)$ whose general member is generically injective. Let $L$ be an arbitrary positive line bundle on $X$. For $\alpha \in A$ write $\deg_L(\mathcal{H}_\alpha)$ for the degree of a member $f : \mathbb{P}^1 \to X$ of $\mathcal{H}_\alpha$ with respect to $L$, i.e., $\deg_L(\mathcal{H}_\alpha) = \deg(f^*L)$. Choose now $\mathcal{H}$ among all $\mathcal{H}_\alpha, \alpha \in A$, such that $\deg_L(\mathcal{H})$ is the smallest possible. Defining $\mathcal{H}' \subset \mathcal{H}$ and $\mathcal{K} = \mathcal{H}'/\mathrm{Aut}(\mathbb{P}^1)$ as in the last paragraph, we obtain a rational component $\mathcal{K}$ which we will call a minimal rational component. A member of $\mathcal{K}$ will be called a $\mathcal{K}$-(minimal) rational curve, or as a minimal rational curve when the choice of $\mathcal{K}$ is understood.

Fixing a minimal rational component $\mathcal{K}$, by Mori's Bend-and-Break Lemma a general minimal rational curve $C$ is the image of a holomorphic immersion $f : \mathbb{P}^1 \to X$ such that $f^*T_X \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^q$, which we call a standard rational curve. Denote by $\mathcal{K}^o \subset \mathcal{K}$ the Zariski open subset of standard $\mathcal{K}$-rational curves, and write $\mathcal{U}^o = \rho^{-1}(\mathcal{K}^o)$.

For $x \in X$ write $\mathcal{U}_x := \mu^{-1}(x)$, and $\mathcal{U}^o_x = \mathcal{U}_x \cap \mathcal{U}^o$. There is a smallest closed subvariety $E_o \subset X$ such that every $\mathcal{K}$-rational curve passing through a point of $X - E_0$ is necessarily free. From the minimality of $\mathcal{K}$ it follows that $\mathcal{U}_x$ is a projective variety for every $x \in X - E_0$. For any $x \in X$ a point of $\mathcal{U}_x$ corresponds to a minimal rational curve $C$ passing through $x$ together with a marking at $x$. Thus, if $C$ passes through $x$ several times, it gives rise to several points in $\mathcal{U}_x$ corresponding to the different markings. There is a smallest closed subvariety $E \subset X$ containing $E_0$ such that for every point $x \in X - E$, $\mathcal{U}^o_x$ is Zariski-dense in $\mathcal{U}_x$. Equivalently this means that $x \notin E_o$, and every irreducible component of $\mathcal{U}_x$ contains a point corresponding to a standard $\mathcal{K}$-rational curve marked at $x$. In this article for convenience we will call $E \subset X$ the bad locus of $\mathcal{K}$. Let $f : \mathbb{P}^1 \to X$ be a parametrized minimal rational curve which is an immersion at 0. Write $x = f(0)$ and consider the point $\eta_x \in \mathcal{U}_x$ corresponding to the marking at $X$ defined by $f$. Then, we can associate $\tau_x(\eta_x) := [df(T_0(\mathbb{P}^1))] \in \mathbb{P}T_x(X)$ to $\eta_x$. From $df(0) \neq 0$ it follows that $\mathcal{U}_x$ is smooth at $\eta_x$. We restrict ourselves now to $\mathcal{U}^o_x \subset \mathcal{U}_x$ over a point $x \in X - E$. Then, $\tau_x : \mathcal{U}^o_x \to \mathbb{P}T_o(X)$ is a holomorphic immersion. Since $\mathcal{U}^o_x$ is Zariski dense in $\mathcal{U}_x$, $\tau_x$ extends to a rational map on the projective variety $\mathcal{U}_x$ to be denoted again by $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$, which we call the tangent map. The strict transform $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ of $\tau_x$, $\mathcal{C}_x := \overline{\tau_x(\mathcal{U}^o_x)}$ will be called the variety of $\mathcal{K}$-rational tangents at $x$, or the variety of minimal rational tangents at $x$, when the choice of a minimal rational component $\mathcal{K}$ is understood. Collecting $\mathcal{C}_x, x \in X - E$, we have the total space $\pi : \mathcal{C} \to X - E$ of varieties of minimal rational tangents. $\mathcal{C} \subset \mathbb{P}T_X$ is quasi-projective, and the closure $\pi : \overline{\mathcal{C}} \to X$ will be called the compactified total space of varieties of minimal rational tangents.

From the definition the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x$ at a general point $x \in X$ is only a rational map. It is holomorphic at $\eta_x \in \mathcal{U}_x$ corresponding to a local immersion at $x$, but may fail to be holomorphic when cusps occur. However, Kebekus [11] shows that a sufficiently general point $x \in X$, any $\mathcal{K}$-rational curve passing through $x$ is immersed at $x$, and he proves

**Theorem 1.1.1** (Kebekus [11]). *Let $X$ be a uniruled projective manifold and $\mathcal{K}$ be a minimal rational component on $X$. Then, at a general point $x \in X$, the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is a finite holomorphic map.*

In this article we deal with the question of characterization of certain rational homogeneous manifolds in terms of varieties of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point. When $\mathcal{C}_x$ equals $\mathbb{P}T_x(X)$ we have the following result of Cho–Miyaoka–Shepherd-Barron [2], which relies on Theorem 1.1.1.

**Theorem 1.1.2** (Cho–Miyaoka–Shepherd-Barron [2]). *Let $X$ be an $n$-dimensional uniruled projective manifold and $\mathcal{K}$ be a minimal rational component on $X$. Suppose at a general point*

*$x \in X$ the variety of $\mathcal{K}$-rational tangents $\mathcal{C}_x$ is the full projective space $\mathbb{P}T_x(X)$. Then, $X$ is biholomorphic to the $\mathbb{P}^n$.*

(1.2) An important ingredient in our study of uniruled projective manifolds is the use of meromorphic distributions arising from minimal rational curves and the associated varieties of minimal rational tangents. In this direction the first question that occurred to us was the question of integrability of proper distributions spanned at general points by varieties of minimal rational tangents. We have

**Proposition 1.2.1** (from Hwang-Mok [6,7])**.** *Let $X$ be a Fano manifold of Picard number 1, and $\mathcal{K}$ be a minimal rational component on $X$. Suppose at a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x$ is linearly degenerate. Let $W$ be a proper meromorphic distribution on $X$ which contains the linear span of $\mathcal{C}_x$ at a general point. Then, $W$ cannot be integrable.*

Proposition 1.2.1 is derived using the deformation theory of rational curves, which yields the following lemma relevant to our arguments in the current article.

**Lemma 1.2.** *Let $X$ be a uniruled projective manifold and $\mathcal{K}$ be a rational component on $X$. Let $Z \subset X$ be a subvariety of codimension $\geq 2$. Then, a general member of $\mathcal{H}$ does not intersect $Z$.*

Regarding distributions spanned at a general point by its variety of minimal rational tangents we have the following first result on the Frobenius form.

**Proposition 1.2.2** (from Hwang-Mok [6,7])**.** *Let $X$ be a uniruled projective manifold and $\mathcal{K}$ be a minimal rational component on $X$. Suppose at a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x$ is linearly degenerate, and denote by $D_x \subset T_x$ the linear span of minimal rational tangents. Denote by $\mathcal{T}_x \subset \mathbb{P}(\Lambda^2 D_x)$ the tangent variety of $\mathcal{C}_x$, and by $\varphi_x : \Lambda^2 D_x \to T_x/D_x$ the Frobenius form. Then, $\mathcal{T}_x \subset \mathbb{P}(\mathrm{Ker}\varphi_x)$.*

In Hwang-Mok [8] we established a general principle of analytic continuation called the Cartan-Fubini extension principle, which allows us to extend local biholomorphisms between Fano manifolds of Picard number 1 which transform varieties of minimal rational tangents to each other under very mild geometric conditions, as follows.

**Theorem 1.2** (from Hwang-Mok [8, Theorem 1.2])**.** *Let $X$ and $X'$ be Fano manifolds of Picard number 1. Let $\mathcal{K}$ resp. $\mathcal{K}'$ be a minimal rational component on $X$ resp. $X'$, and $E \subset X$ resp. $E' \subset X'$ be the bad locus of $(X, \mathcal{K})$ resp. $(X', \mathcal{K}')$. Let $\pi : \mathcal{C} \to X - E$ resp. $\pi' : \mathcal{C}' \to X' - E'$ be the fibered space of varieties of minimal rational tangents of $(X, \mathcal{K})$ resp. $(X', \mathcal{K}')$. Assume that the general fiber $\mathcal{C}_x$ of $\pi : \mathcal{C} \to X - E$ is positive-dimensional and the Gauss map on (each irreducible component of ) $\mathcal{C}_x$ is generically finite. Let $U \subset X - E$ resp. $U' \subset X' - E'$ be a connected open subset in the complex topology, and $f : U \cong U'$ be a biholomorphic map which induces an isomorphic $[df] : \mathcal{C}|_U \cong \mathcal{C}'|_{U'}$. Then, there is a unique biholomorphic map $F : X \cong X'$ such that $F|_U \equiv f$.*

In our application in the current article the general variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ considered will be irreducible, non-linear and rational homogeneous. Moreover, the Gauss map on $\mathcal{C}_x$ is an embedding. In particular, Hwang-Mok [8, Theorem 1.2] (where the meaning of Cartan-Fubini extension is given in [8, Main Theorem]) applies.

Remarks: A subvariety of the projective space is said to be linear if and only if it is a finite union of projective linear subspaces. Otherwise it is said to be non-linear. The analogue of Theorem 1.2 actually applies under the weaker assumption that the variety of minimal rational tangents at a general point is non-linear. This can be obtained by the argument of [8, Theorem 1.2], together with an adaptation of the argument of analytic continuation along standard rational curves in [8, §2] in which minimal rational curves are replaced by Cauchy subvarieties of the minimal component $\mathcal{K}$, as described and studied in Hwang-Mok [9, Propositions 12-14]. Conjecturally, for a Fano manifold $X$ of Picard number 1 equipped with a minimal rational component $\mathcal{K}$ where the variety of minimal rational tangents $\mathcal{C}_x$ at a general point is of dimension $p > 0$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is never linear. If so, the analogue of Theorem 1.2 here would always hold whenever $p > 0$.

(1.3) Concerning the characterization of irreducible Hermitian symmetric spaces of rank $\geq 2$ we have the result Hwang-Mok [5] which asserts that they are the unique uniruled projective manifolds admitting a G-structure for some reductive linear group G. The proof involves the use of varieties of minimal rational tangents. Especially, we have

**Proposition 1.3.1** (from Hwang-Mok [5]). *Let $S$ be an irreducible Hermitian symmetric space of the compact type and of rank $\geq 2$, and let $\mathcal{C}_o \subset T_o(S)$ be the variety of minimal rational tangents of $S$ at a reference point $o \in S$. Let $X$ be a uniruled projective manifold whose variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at any $x \in X$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety. Then, $X$ is biholomorphic to $S$.*

We have the following Hartogs Extension Theorem for varieties of minimal rational tangents modeled on irreducible Hermitian symmetric spaces of the compact type and of rank $\geq 2$ (cf. Hwang-Mok [4, Proposition 1, p.398]).

**Proposition 1.3.2.** *Let $S$ be an irreducible Hermitian symmetric space of the compact type and of rank $\geq 2$, $\dim(S) := n$, and denote by $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ its variety of minimal rational tangents at a reference point $o \in S$. Let $U \subset \mathbb{C}^n$ be a open subset, and $Z \subset U$ be a complex-analytic subvariety of codimension $\geq 2$. Let $\mathcal{S} \subset \mathbb{P}T_U \cong U \times \mathbb{P}^{n-1}$, and $\pi : \mathcal{S} \to U$ be the canonical projection. Assume that $\pi|_{U-Z} : \mathcal{S}|_{U-Z} \to U - Z$ is a holomorphic fiber bundle with fibers $\mathcal{S}_x \subset \mathbb{P}T_x(U)$ congruent to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for any $x \in U - Z$. Then, $\pi : \mathcal{S} \to U$ is a holomorphic fiber bundle with fibers $\mathcal{S}_x \subset \mathbb{P}T_x(U)$ congruent to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for all $x \in U$.*

Proposition 1.3.2 is an immediate consequence of Matsushima-Morimoto [15]. In fact, the set of subvarieties $\mathcal{V}_o \subset \mathbb{P}T_o(S)$ congruent to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ under projective linear automorphisms of $\mathbb{P}T_o(S)$ is parametrized by $\mathcal{M} := GL(T_o(S))/H$ for some reductive linear subgroup $H \subset GL(T_o(S))$, and by [15] such a moduli space $\mathcal{M}$ is Stein. Realizing $\mathcal{M}$ as a complex submanifold of some Euclidean space, Hartogs extension of $\mathcal{S}$ follows from Hartogs extension of holomorphic functions.

For the case of a Fano contact homogeneous manifold $S$ other than the odd-dimensional projective space, we have the result of Hong [3] on the characterization of $S$ as the unique uniruled projective manifold with a contact structure modeled on $S$. Given a uniruled projective manifold $X$ such that the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is isomorphic to that of the model space at every point $x \in X$, we have a holomorphic distribution $W \subset T_X$ of corank 1. $W$ then defines a contact structure on $X$ if and only if the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$ is nondegenerate. We have

**Proposition 1.3.3** (from Hong [3]). *Let $S$ be a Fano contact homogeneous manifold of Picard number 1 different from an odd-dimensional projective space. Let $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ be the variety of minimal rational tangent of $S$ at a reference point $o \in S$. Let $X$ be a uniruled projective manifold whose variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(S)$ at every point $x \in X$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety. Denoting by $W$ the meromorphic distribution on $X$ spanned by the varieties of minimal rational tangents, assume that the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$ is everywhere nondegenerate on $X$. Then, $X$ is biholomorphic to $S$.*

More precisely, we will be making use of the proof of Proposition 1.3.3. In Hong [3] there is first of all the step of constructing Cartan connections modeled on $S$, which is a local statement. To obtain a developing map from $X$ into $S$ it then amounts to checking the vanishing of certain "curvature" tensors. This argument works whenever the contact structure is defined on a connected open subset of $X$ on which there exists a standard rational curve.

(1.4) For the proof of the extension results on varieties of minimal rational tangents both in the symmetric and the contact cases it will be necessary to have a classification of such spaces manifolds and a description of their varieties of minimal rational tangents.

For irreducible Hermitian symmetric spaces $S$ we write $S = G_c/K$, where $G_c$ denotes a compact real form of a simply connected simple complex Lie group $G$, and $K \subset G_c$ is a maximal proper closed subgroup. We follow the standard classification of irreducible Hermitian symmetric spaces of the compact type into 6 types with 4 classical series and two exceptional cases. The following table is taken from Hwang-Mok [6, (2.1), p.440].

**Table of irreducible Hermitian symmetric spaces $S$ of the compact type and their varieties of minimal rational tangents $\mathcal{C}_o$**

| Type | $G_c$ | $K$ | $G_c/K = S$ | $\mathcal{C}_o$ | Embedding |
|---|---|---|---|---|---|
| I | $SU(p+q)$ | $S(U(p) \times U(q))$ | $G(p,q)$ | $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ | Segre |
| II | $SO(2n)$ | $U(n)$ | $G^{II}(n,n)$ | $G(2, n-2)$ | Plücker |
| III | $Sp(n)$ | $U(n)$ | $G^{III}(n,n)$ | $\mathbb{P}^{n-1}$ | Veronese |
| IV | $SO(n+2)$ | $SO(n) \times SO(2)$ | $\mathbf{Q}^n$ | $\mathbf{Q}^{n-2}$ | by $\mathcal{O}(1)$ |
| V | $E_6$ | $\mathrm{Spin}(10) \times U(1)$ | $\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ | $G^{II}(5,5)$ | by $\mathcal{O}(1)$ |
| VI | $E_7$ | $E_6 \times U(1)$ | exceptional | $\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$ | Severi |

For the classification of Fano contact homogeneous manifolds we follow Boothby [1]. Each simple complex algebra $\mathfrak{g}$ is associated to a unique Fano contact homogeneous manifold $S = G/P$. In the case of $\mathfrak{g} = A_k, k \geq 2$, $S$ is of Picard number 2, $S \cong \mathbb{P}T^*_{\mathbb{P}^k}$. For the case of $\mathfrak{g} = C_k$ we have $S \cong \mathbb{P}^{2k-1}$ as a complex manifold. These cases will be excluded. For any other simple complex Lie algebra $\mathfrak{g}$ there is a unique choice of a long simple root in the Dynkin diagram of $\mathfrak{g}$, corresponding to a choice of a maximal parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. In the following table we list the relevant Fano contact homogeneous manifold of Picard number 1 according to the classification of $\mathfrak{g}$, with information on the Levi factor $\mathfrak{q} \subset \mathfrak{p}$, and a description of the variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}W_o$ as given in Hwang [4, Proposition 5].

**Table of Fano contact homogeneous spaces $S \ncong \mathbb{P}^{2n-1}$ of Picard number 1 and their varieties of minimal rational tangents**

| $\mathfrak{g}$ | $\mathfrak{q}$ | $\mathcal{C}_o$ | Embedding |
|---|---|---|---|
| $B_k$ | $A_1 \times B_{k-2}$ | $\mathbb{P}^1 \times \mathbf{Q}^{2k-5}$ | Segre* |
| $D_k$ | $A_1 \times D_{k-2}$ | $\mathbb{P}^1 \times \mathbf{Q}^{2k-6}$ | Segre* |
| $G_2$ | $A_1$ | $\mathbb{P}^1$ | by $\mathcal{O}(3)$ |
| $F_4$ | $C_3$ | $G^{II}(3,3)$ | by $\mathcal{O}(1)$ |
| $E_6$ | $A_5$ | $G(3,3)$ | by $\mathcal{O}(1)$ |
| $E_7$ | $D_6$ | $G^{II}(6,6)$ | by $\mathcal{O}(1)$ |
| $E_8$ | $E_7$ | exceptional** | by $\mathcal{O}(1)$ |

* Here $k \geq 3$ for $\mathfrak{g} = B_k, k \geq 4$ for $\mathfrak{g} = D_k$. The embedding arises from the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^m$ into $\mathbb{P}^{2m+1}$ and the canonical embedding $\mathbf{Q}^{m-1} \subset \mathbb{P}^m$. When $m = 2$, $\mathbf{Q}^1 \subset \mathbb{P}^2$ is a quadratic curve, and when $m = 3$, $\mathbf{Q}^2 \subset \mathbb{P}^3$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbf{Q}^2$ in $\mathbb{P}^3$.

** In this case $\mathcal{C}_o$ is biholomorphic to the irreducible compact Hermitian symmetric manifold of Type VI pertaining to $E_7$ of dimension 27.

From the tables above we see that in the symmetric case the variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is given by a 'quadratic' embedding in the sense that either it is of rank 2 and the embedding is the minimal embedding, or it is of rank 1 and the embedding is the second canonical embedding (in the case of the Veronese map pertaining to Type III). Likewise in the contact case $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is given by a 'cubic' embedding. These features of varieties of minimal rational tangents in the symmetric and contact case can be put on a common footing and explained in terms of expansions using root vectors, as is implicit in Hwang-Mok [6, Proposition 14, p.414] in the symmetric case, and given in Hwang-Mok [7, (4.2), pp.376-377] for the case of rational homogeneous manifolds $S$ of Picard number 1 for simple Lie algebras $\mathfrak{g}$ whose roots are of equal length. The latter applies also when $\mathfrak{g}$ has roots of unequal length, provided that $S$ is associated to a long simple root. Thus, it applies to the symmetric case and to the contact case other than odd-dimensional projective spaces. For this article we collect the pertinent facts in the following lemmas. Denote by $\pi : T_o(S) - \{0\} \to \mathbb{P}T_o(S)$ the canonical projection and write $\widetilde{A} = \pi^{-1}(A) \subset T_o(S)$ for any subset $A \subset \mathbb{P}T_o(S)$. Thus $\widetilde{\mathcal{C}}_o \cup \{0\}$ is the affine cone over $\mathcal{C}_o$. In the lemmas the relevant quadratic and cubic expansions will be stated in terms $\widetilde{\mathcal{C}}_o$.

**Lemma 1.4.1.** *Let $S$ be an irreducible Hermitian symmetric space of the compact type and of rank $\geq 2$. Let $o \in S$ be a reference point and denote by $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ the variety of minimal rational tangents at $o$. Let $\alpha \in \widetilde{\mathcal{C}}_o$. Then, there exists a basis $\{e_i\}_{0 \leq i \leq \dim(S)}$ of $T_o(S)$ such that $e_0 = \alpha$, $P_\alpha := T_\alpha(\widetilde{\mathcal{C}}) = \mathrm{Span}(e_0; e_1, \cdots, e_p)$, $p = \dim(\mathcal{C}_o)$, and such that $\widetilde{\mathcal{C}}$ is the graph of a quadratic vector-valued polynomial $\mathcal{Q} : P_\alpha \to N := \mathrm{Span}(e_{p+1}, \cdots, e_{\dim(S)-1})$.*

**Lemma 1.4.2.** *Let $S$ be a Fano contact homogeneous manifold other than an odd-dimensional projective space, and $W \subset T_S$ be the contact distribution. Let $\alpha \in \widetilde{\mathcal{C}} \subset W_o$. Then, there exists a basis $\{e_i\}_{0 \leq i \leq \mathrm{rank}(W)-1}$ of $W_o$ such that $e_0 = \alpha$, $P_\alpha := T_\alpha(\widetilde{\mathcal{C}}_o) = \mathrm{Span}(e_0; e_1, \cdots, e_p)$, $p = \dim(\mathcal{C}_o)$, for which the following holds. Write $N := \mathrm{Span}(e_{p+1}, \cdots e_{2p+1})$, noting that $\mathrm{rank}(W_o) = 2p + 2$. Then, there is a quadratic vector-valued polynomial $\mathcal{Q} : P_\alpha \to V := \mathrm{Span}(e_{p+1}, \cdots, e_{2p})$ and a cubic polynomial $\mathcal{L} : P_\alpha \to \mathbb{C}e_{2p+1}$ such that $\widetilde{\mathcal{C}}_o$ is the graph of $(\mathcal{Q}, \mathcal{L}) : P_\alpha \to N$.*

For $S$ either of the symmetric or the contact type, the projective second fundamental form of $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ can be described in terms of the quadratic polynomial $\mathcal{Q}$, as follows. Pick $\alpha \in \widetilde{\mathcal{C}}_o$ and write $\widetilde{\mathcal{Q}}$ for the $N$-valued symmetric bilinear form on $P_\alpha$ such that $\widetilde{\mathcal{Q}}(\xi, \xi) = \mathcal{Q}(\xi)$. At $\alpha \in \widetilde{\mathcal{C}}_o$ the Euclidean second fundamental form $\widetilde{\sigma}$ of $\widetilde{\mathcal{C}}_o$ in $T_o(S)$ is given by $\widetilde{\sigma}_\alpha(\xi, \eta) = \mathcal{Q}(\xi, \eta) + P_\alpha \in T_o(S)/P_\alpha$. At $[\alpha] \in \mathcal{C}_o$ we have $T_{[\alpha]}(\mathcal{C}_o) = P_\alpha/\mathbb{C}\alpha$. Since $\widetilde{\mathcal{C}}_o$ is invariant under homotheties $\widetilde{\sigma}_\alpha$ descends to the projective second fundamental form $\sigma_{[\alpha]} : P_\alpha/\mathbb{C}\alpha \times P_\alpha/\mathbb{C}\alpha \to T_o(S)/P_\alpha$ given simply by $\sigma_{[\alpha]}(\overline{\xi},\overline{\eta}) = \mathcal{Q}(\xi,\eta) + P_\alpha \in T_o(S)/P_\alpha \cong N_{\mathcal{C}_o|\mathbb{P}T_o(S),[\alpha]}$, where $\overline{\xi}$ denotes $\xi + \mathbb{C}\alpha$, etc. In the symmetric case since $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is linearly nondegenerate the image of $\widetilde{\sigma}_\alpha$ spans $N$. In the contact case let $\widetilde{\mathcal{L}}$ be the symmetric cubic form such that $\widetilde{\mathcal{L}}(\xi,\xi,\xi) = \mathcal{L}(\xi)$. Let $Q_\alpha \subset W_o$ be the linear subspace obtained by adjoining the image of $\widetilde{\sigma}_\alpha$ to $P_\alpha$. In other words, $Q_\alpha \supset P_\alpha$ and $Q_\alpha/P_\alpha$ is the image of $\widetilde{\sigma}_\alpha$. Then, $Q_\alpha \subset W_o$ is of codimension 1. The Euclidean third fundamental form $\widetilde{\kappa}_\alpha : S^3P_\alpha \to W_o/Q_\alpha$ is defined by $\widetilde{\kappa}_\alpha(\xi,\xi,\xi) = \mathcal{L}(\xi) \bmod Q_\alpha$, and $W_o$ is obtained from $Q_\alpha$ by adjoining the image of $\widetilde{\kappa}_\alpha$. We have a filtration $\mathbb{C}\alpha \subset P_\alpha \subset Q_\alpha \subset W_o \subset T_o(S)$, which will be needed in the study of parallel transport of the third fundamental form in §3. Note that passing to projectivizations at $[\alpha] \in \mathcal{C}_o$ the normal space $N_{\mathcal{C}_o|\mathbb{P}W_o(S),[\alpha]}$ is obtained by successively adjoining the image of the second fundamental form $\sigma_{[\alpha]}$ and the projective third fundamental form $\kappa_{[\alpha]}$ defined in an analogous way by $\widetilde{\kappa}_\alpha$.

## §2 Characterization in terms of varieties of minimal rational tangents in the symmetric case

(2.1) In this article we are going to prove the following characterization theorems in terms of varieties of minimal rational tangents for certain rational homogeneous manifolds, including irreducible Hermitian symmetric spaces of the compact type and Fano contact homogeneous manifolds of Picard number 1 other than odd-dimensional projective spaces.

**Main Theorem.** *Let $S$ be a rational homogeneous manifold of Picard number 1 which is either a Hermitian symmetric space or a Fano contact homogeneous manifold. For a reference point $o \in S$, let $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ denote the variety of minimal rational tangents on $S$. Let $X$ be a Fano manifold of Picard number 1 and $\mathcal{K}$ be a minimal rational component on $X$. Suppose the variety of $\mathcal{K}$-rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety. Then, $X$ is biholomorphically isomorphic to $S$.*

For the case where $S$ is the projective space the Main Theorem follows Theorem 1.1.2 from Cho–Miyaoka–Shepherd-Barron [2]. To prove the Main Theorem, in this section we will first deal with the case of irreducible Hermitian symmetric spaces of the compact type and of rank $\geq 2$. In the sequel this will be referred to simply as the symmetric case (of the Main Theorem). By the contact case we mean the case of Fano contact homogeneous manifolds of Picard number 1 other than odd-dimensional projective spaces.

In the symmetric case the model manifold $S$ carries a canonical G-structure, where G $\subset GL(T_o(S))$ is some special complex reductive linear group. To be more precise $S = G/P$, where $G = \mathrm{Aut}(S)$ is a simple complex Lie group, and $P \subset G$ is a maximal parabolic subgroup, equivalently the isotropy subgroup, at any reference point $o \in S$. Let $L \subset P$ be a Levi subgroup, and $K^{\mathbb{C}} = \{df(o) \in \mathrm{End}_o(S) : f \in L\}$. Then, $S$ carries canonically a $K^{\mathbb{C}}-$structure. If we fix a compact real form $G_c \subset G$ and denote by $K \subset G_c$ the isotropy subgroup at $o \in S$, then $K^{\mathbb{C}}$ is a complexification of $K$, hence the notation.

Let $E \subset X$ be the bad locus of $(X, \mathcal{K})$. By the hypothesis of the Main Theorem, over $X - E$ the varieties of minimal rational tangents correspond to the highest weight orbits of a holomorphic $K^{\mathbb{C}}-$structure. By Proposition 1.3.1 we conclude that $X \cong S$ provided that $E = \emptyset$. By Proposition 1.3.3 on Hartogs extension it is sufficient to prove that $E \subset X$ may be taken to be of codimension at least 2. In general, let $H \subset E$ be an irreducible component which is a hypersurface in $X$. To prove the Main Theorem in the symmetric case it remains to show that for any such irreducible hypersurface $H \subset X$ there exists some point $y \in H$ such that the hypothesis on varieties of minimal rational tangents holds true on some open neighborhood of $y$ in $X$. This is precisely what we are going to prove, as follows:

**Proposition 2.1.** *Let $S$ be an irreducible Hermitian symmetric space of the compact type and of rank $\geq 2$. In the notations of the Main Theorem let $H \subset E$ be an irreducible component of codimension 1 in $X$. Let $C$ be a standard minimal rational curve on $X$ such that $C \cap (X - E) \neq \emptyset$. Then $C \cap H$ is nonempty, and, for every point $y \in C \cap H$ there exists an open neighborhood $U_y$ of $y$ in $X$ such that for every $z \in U_y$, the variety of minimal rational tangents $\mathcal{C}_z \subset \mathbb{P}T_z(X)$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety.*

(2.2) The proof of Proposition 2.1 breaks up into two parts. The first part concerns the behavior of second fundamental forms of varieties of minimal rational tangents along the tautological lifting of a standard rational curve, which is a general result for minimal rational curves on uniruled projective manifolds. Here and in what follows we will be considering complex submanifolds of the projective space and the second fundamental form will mean the projective second fundamental form. The second part concerns families of projective subvarieties whose typical fiber is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for an irreducible Hermitian symmetric space $S$ of the compact type and of rank $\geq 2$.

Let $V$ and $V'$ be two complex Euclidean spaces of the same dimension, and $A \subset \mathbb{P}V$, $A' \subset \mathbb{P}V'$ be two local complex submanifolds of the same dimension. Let $a \in A$, $a = [\alpha]$; $a' \in A'$, $a' = [\alpha']$. Write $T_a(A) = \mathbb{P}E/\mathbb{C}\alpha$, $E \subset V$ (resp. $T_{a'}(A') = \mathbb{P}E'/\mathbb{C}\alpha'$, $E' \subset V'$) being a vector subspace containing $\alpha$ (resp. $\alpha'$). We say that the second fundamental form $\sigma_a$ of $A \subset \mathbb{P}V$ at $a \in A$ is isomorphic to the second fundamental form $\sigma_{a'}$ of $A' \subset \mathbb{P}V'$ at $a' \in A'$ if and only if there exists a linear isomorphism $\varphi : V \cong V'$ such that $\varphi(\alpha) = \alpha'$, $\varphi(E) = E'$, and such that $\varphi$ satisfies the following additional property (†)

(†) Let $\overline{\varphi} : V/E \to V'/E'$ be the linear map induced by $\varphi$ (where $\varphi(E) = E'$), and denote by $\widetilde{\sigma}_\alpha$ (resp. $\widetilde{\sigma}_{\alpha'}$) the second fundamental form of $\widetilde{A}$ at $\alpha$ (resp. of $\widetilde{A}'$ at $\alpha'$). Then, for any $\xi, \eta \in E$ we have

$$\widetilde{\sigma}_{\alpha'}\big(\varphi(\xi), \varphi(\eta)\big) = \overline{\varphi}\big(\widetilde{\sigma}_\alpha(\xi, \eta)\big) .$$

For the first part of the proof of Proposition 2.1, let $X$ denote more generally a uniruled projective manifold, and $\mathcal{K}$ denote a minimal rational component on $X$, which will be implicit in what follows. Denote by $\mu : \mathcal{U} \to X$ the universal family of minimal rational curves and by $\nu : \overline{\mathcal{C}} \to X$ the compactified total space of varieties of minimal rational tangents. Let $B \subset X$ be the largest subvariety such that $\nu|_{X-B} : \overline{\mathcal{C}}|_{X-B} \to X - B$ is flat. $B$ is of codimension at least 2 in $X$. Let $f : \mathbb{P}^1 \to X$ be a parametrized standard rational curve, $f(\mathbb{P}^1) := C$. $C$ lifts canonically to $\widetilde{C} \subset \mathcal{U}$, whose image under the tangent map gives the tautological lifting $\widehat{C} \subset \mathcal{C}$. Suppose $C \subset X - B$. At each of the finitely many points $\widetilde{x}_k$ of $\widetilde{C} \cap \mu^{-1}(x)$ there is an open neighborhood $U_k$ such that $\tau_x$ embeds $U_k$ holomorphically onto a smooth submanifold $\mathcal{C}_x^k$, which is the germ of some irreducible component of $\mathcal{C}_x$ at $[\alpha_k] = \tau_x(\widetilde{x}_k)$.

For $t \in \mathbb{P}^1$ we write $\mathcal{C}_t$ for $(f^*\mathcal{C})_t$, $[\alpha(t)]$ for $\widehat{C} \cap \mathcal{C}_t$, and $V_t$ for $f^*T_{f(t)}(X)$. We have $\mathcal{C}_t \subset \mathbb{P}V_t$. For every $t \in \mathbb{P}^1$ we have a germ of smooth projective submanifold $\mathcal{C}_t^o \subset \mathcal{C}_t \subset \mathbb{P}V_t$ at $[\alpha(t)]$ corresponding to one of the germs $\mathcal{C}_x^k, x = f(t)$, in the last paragraph, chosen in such a way that the union of $\mathcal{C}_t^o$ is a germ of smooth complex submanifold along the smooth curve $\widetilde{C}$. Write $T_{[\alpha(t)]}$ for $T_{[\alpha(t)]}(\mathcal{C}_t^o)$. In an implicit form we introduced in Mok [17, §3.2, p.2651ff.] the notion of parallel transport of the second fundamental form along the tautological lifting $\widehat{C}$ of a standard rational curve $C$. By this we mean that the second fundamental form can be interpreted in a natural way as a holomorphic section of a vector bundle which is holomorphically trivial over $\widehat{C}$. By a straightforward adaptation of [17, Lemma 3.2.1] we have

**Proposition 2.2.** *For every $t \in \mathbb{P}^1$ the second fundamental form $\sigma_{[\alpha(t)]} : S^2T_{[\alpha(t)]} \to N_{\mathcal{C}_t^o|\mathbb{P}V_t,[\alpha(t)]}$ is isomorphic to the second fundamental $\sigma_{[\alpha_o]} : S^2T_{[\alpha_o]} \to N_{\mathcal{C}_o|\mathbb{P}T_o(S),[\alpha_o]}$ for the model manifold $S$.*

*Proof.* Write $\pi : \mathbb{P}V \to \mathbb{P}^1$ for the canonical projection, where $V = f^*T_X$, and $T_\pi$ for its relative tangent bundle. Write $\lambda = \pi|_{f^*\mathcal{C}}$, and recall that $T_{[\alpha(t)]} = T_{[\alpha(t)]}(\mathcal{C}_t^o)$. Write $N_{[\alpha(t)]} = T_{\pi,[\alpha(t)]}/T_{[\alpha(t)]}$. Putting together $T_{[\alpha(t)]}$, $t \in \mathbb{P}^1$, we obtain a holomorphic vector bundle on $\widehat{C}$ which we write as $T_\lambda|_{\widehat{C}}$, by abuse of notations. Likewise, putting together $N_{[\alpha(t)]}$, $t \in \mathbb{P}^1$, we obtain a holomorphic vector bundle on $\widehat{C}$ which we write as $N_\lambda|_{\widehat{C}}$. We compute the degrees of holomorphic line bundles entering into the picture. On the projective space $\mathbb{P}V_t$ for a nonzero vector $\alpha(t) \in V_t$ we have the isomorphism $T_{[\alpha(t)]}(\mathbb{P}V_t) \cong V_t/\mathbb{C}\alpha(t)$, which depends on the choice of the vector $\alpha(t)$, not just on $[\alpha(t)]$. The isomorphism $T_{[\alpha(t)]}(\mathbb{P}V_t) \otimes L_{[\alpha(t)]} \cong \pi^*V_t/L_{[\alpha(t)]}$, where $L_{[\alpha(t)]} = \mathbb{C}\alpha(t)$ is the tautological line at $[\alpha(t)]$, is then canonically determined. Varying over $\widehat{C}$ we obtain a canonical isomorphism $T_\pi \otimes L \cong \pi^*V_t/L$ over $\widehat{C}$. Since $L|_{\widehat{C}} \cong T_{\widehat{C}}$ canonically, and $C$ is a standard rational curve, it follows that $\pi^*V|_{\widehat{C}} \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^q$, and we have

$$T_\pi|_{\widehat{C}} \cong \pi^*V|_{\widehat{C}}/T_{\widehat{C}} \otimes T^*_{\widehat{C}} \cong \big(\big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^q\big) \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p \oplus \big(\mathcal{O}(-2)\big)^q.$$

Since at $[\alpha(t)], T_{[\alpha(t)]} \otimes L_{[\alpha(t)]} \cong P_{\alpha(t)}/\mathbb{C}\alpha(t)$, where $P_{\alpha(t)} \subset V_t$ is generated by vectors belonging to the positive components of the Grothendieck decomposition of $\pi^*V_t$, over $\widehat{C}$ we have $T_\lambda|_{\widehat{C}} \cong \big(\mathcal{O}(1)\big)^p \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p$ and $N_\lambda|_{\widehat{C}} \cong \mathcal{O}^q \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-2)\big)^q$. As a consequence, over $\widehat{C}$

$$\mathrm{Hom}\big(S^2T_\lambda|_{\widehat{C}}, N_\lambda|_{\widehat{C}}\big) \cong \mathrm{Hom}\Big(\big(\mathcal{O}(-2)\big)^{\frac{p(p+1)}{2}}, \big(\mathcal{O}(-2)\big)^q\Big) \cong \mathcal{O}^{\frac{qp(p+1)}{2}}$$

is a holomorphically trivial vector bundle. It follows that at any $t \in \mathbb{P}^1$ the second fundamental form $\sigma_{[\alpha(t)]} : S^2T_{[\alpha(t)]} \to N_{[\alpha(t)]}$ must be isomorphic to the second fundamental $\sigma_{[\alpha_o]} : S^2T_{[\alpha_o]} \to N_{\mathcal{C}_o|\mathbb{P}T_o(S),[\alpha_o]}$ for the model space $S$, as desired. $\square$

(2.3) We apply now the notion of parallel transport of the second fundamental form along the tautological lifting of a standard rational curve, as given in (2.2), to the special case where the variety of minimal rational tangents at a general point is isomorphic to that of an irreducible Hermitian symmetric space of the compact type and of rank $\geq 2$. We are going to prove

**Proposition 2.3.** *Let $V$ be a holomorphic vector bundle over $\mathbb{P}^1$, $\pi : \mathbb{P}V \to \mathbb{P}^1$ be the canonical projection, and $\mathcal{C} \subset \mathbb{P}V$ be an irreducible subvariety. Assume that for some finite set $A \subset \mathbb{P}^1$ and for $t \in \mathbb{P}^1 - A$, $\mathcal{C}_t := \nu^{-1}(t) \subset \mathbb{P}V_t$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety for a fixed irreducible Hermitian symmetric space $S$ of the compact type and of rank $\geq 2$. For $s \in A$, assume that there exists $[\gamma_s^o] \in \mathcal{C}_s$ and a nonsingular germ of irreducible branch $\Omega \subset \mathcal{C}$ at $[\gamma_s^o]$ such that $\pi|_\Omega$ is a submersion and such that, for $\gamma_s \in \Omega \cap \mathcal{C}_s$, the second fundamental form $\sigma_{[\gamma_s]} : S^2T_{\pi,[\gamma_s]} \to N_{\mathcal{C}_s|\mathbb{P}V_s,[\gamma_s]}$ is isomorphic to the second fundamental form $\sigma_{[\alpha_o]} : S^2T_{[\alpha_o]} \to N_{\mathcal{C}_o|\mathbb{P}T_o(S),[\alpha_o]}$ for the model space $S$. Then, for every $t \in \mathbb{P}^1$, $\mathcal{C}_t \subset \mathbb{P}V_t$ is isomorphic to the model $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety, and $\pi|_\mathcal{C} : \mathcal{C} \to \mathbb{P}^1$ is a holomorphic fiber subbundle of $\pi : \mathbb{P}V \to \mathbb{P}^1$ with typical fiber isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$.*

For the proof of Proposition 2.3 the set of irreducible Hermitian symmetric spaces $S$ of the compact type and of rank $\geq 2$ will be broken up into 3 cases as a consequence of the classification given in (1.4), as follows:

(A) The variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is itself an irreducible Hermitian symmetric space of rank 2.

(B) $S$ is a Lagrangian Grassmannian $G^{III}(n+1, n+1)$ of rank $n+1 \geq 2$, in which case $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is isomorphic to the Veronese embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\frac{n(n+3)}{2}}$.

(C) $S$ is the Grassmannian $G(p, q)$ of $p$-planes in $\mathbb{C}^{p+q}$, where $2 \leq p \leq q$, in which case $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is isomorphic to the Segre embedding of $\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ into $\mathbb{P}^{pq-1}$.

To study of varieties of minimal rational tangents we will make use of the following simple lemma on limits of projective embeddings.

**Lemma 2.3.** *Let $W$ be a holomorphic vector bundle over the unit disk $\Delta$. Let $\mathcal{Z}$ be a projective manifold and $\Psi$ be a meromorphic mapping from $\mathcal{Z} \times \Delta$ into $\mathbb{P}W$ respecting canonical projections onto $\Delta$ such that*

(a) *at $t \in \Delta$, $t \neq 0$, $\Psi$ restricts to a holomorphic embedding $\Psi_t : \mathcal{Z} \times \{t\} \mapsto \mathbb{P}W_t$ defined by a complete linear system;*

(b) *there is some $p \in \mathcal{Z} \times \{0\}$ such that $\Psi$ is holomorphic at $p$, and $\Psi(U_p) \subset \mathbb{P}W_o$ is linearly nondegenerate for some open neighborhood $U_p$ of $p$ in $\mathcal{Z} \times \{0\}$.*

*Then, $\Psi$ is a holomorphic embedding.*

*Proof of Lemma.* There exists a proper subvariety $A \subset \mathcal{Z} \times \{0\}$ such that $\Psi$ is holomorphic outside $A$. Let $L$ be the relative tautological line bundle on $\mathbb{P}W$. Then $\Psi^* L^{-1}$ is defined as a holomorphic line bundle $\Lambda$ on $(\mathcal{Z} \times \Delta) - A$. Since $\Psi$ is meromorphic there exists a torsion-free coherent sheaf $\mathcal{F}$ of rank 1 on $\mathcal{Z} \times \Delta$ agreeing with $\Lambda$ outside $A$. The double dual $\mathcal{F}^{**}$ is then a locally-free rank-1 sheaf on $\mathcal{Z} \times \Delta$, corresponding to a holomorphic line bundle on $\mathcal{Z} \times \Delta$, to be denoted by the same symbol $\Lambda$. For $t \neq 0$ the embeddings $\Psi_t : \mathcal{Z} \times \{t\} \to \mathbb{P}W_t$ are isomorphic to each other up to projective linear transformation, by the hypothesis (a) in the Lemma. Fix $t_o \in \Delta,\ t_o \neq 0$. Then, we have

$$\Psi(z,t) = A_t(\Psi(z,t_o))$$

for some projective linear isomorphism $A_t : \mathbb{P}W_{t_o} \to \mathbb{P}W_t$. Under the assumption that $\Psi(U_p) \subset \mathbb{P}W_0$ is linearly nondegenerate, obviously $A_t$ converges to a projective linear isomorphism $A_0 : \mathbb{P}W_{t_o} \to \mathbb{P}W_0$ as $t$ approaches 0. $\square$

*Proof of Proposition 2.3.*
Case (A). For $S$ belonging to Case (A), $\mathcal{C}_o \subset \mathbb{P}T_o(S)$, as an irreducible Hermitian symmetric space of rank 2, carries naturally an integrable G-structure for some reductive linear group G. Consider the total space $\mathcal{C}(\mathcal{C}_o) \subset \mathbb{P}T(\mathcal{C}_o)$ of variety of minimal rational tangents on $\mathcal{C}_o$. $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is the first canonical embedding, and any minimal rational curve on $\mathcal{C}_o$ is a projective line on $\mathbb{P}T_o(S)$. By the quadratic expansion of $\widetilde{\mathcal{C}_o}$ in Lemma 1.4.1 a nonzero minimal rational tangent is then precisely $\overline{\beta} := \beta + \mathbb{C}\alpha \in T_o(S)/\mathbb{C}\alpha$ such that $\mathbb{P}(\mathbb{C}\alpha + \mathbb{C}\beta)$ is a line on $\mathcal{C}_o$. This is the case if and only if $\sigma_{[\alpha]}(\overline{\beta}, \overline{\beta}) = 0$ for the second fundamental form $\sigma_{[\alpha_o]}$ of $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ at $[\alpha]$. Thus, $\mathcal{C}(\mathcal{C}_o)$ is completely determined by the second fundamental forms $\sigma_{[\alpha]}$, $[\alpha] \in \mathcal{C}_o$.

Pick $t_o \neq A$, and let $\ell \subset \mathbb{P}V_{t_o}$ be a line lying on $\mathcal{C}_{t_o}$. Let $\mathcal{H}$ be the irreducible component of the Chow space of $\mathcal{C}$ which contains the point $[\ell]$ corresponding to $\ell$. Let now $t \in A$, and $\Omega$ be as in the statement of the Proposition, and assume without loss of generality that $\pi(\Omega) \cap A = s$. Write $\omega = \pi|_\Omega$. For each $[\alpha] \in \Omega$ lying above $t \neq s$ we have the variety of $\mathcal{H}$-tangents $\mathcal{C}_{[\alpha]}(\mathcal{C}_t)$ which agrees with the set of all $\overline{\beta} = \beta + \mathbb{C}\alpha$ such that $\sigma_{[\alpha]}(\overline{\beta}, \overline{\beta}) = 0$. The latter equation makes sense when $\alpha$ is replaced by $\gamma_s \in \Omega \cap \mathcal{C}_s$. It defines therefore a subvariety $\mathcal{S} \subset \mathbb{P}T_\omega$, where $T_\omega$ denotes the relative tangent bundle of $\omega : \Omega \to \mathbb{P}^1$, such that for $[\alpha] \in \Omega$ lying above $s \notin A$, the fiber $\mathcal{S}_{[\alpha]}$ of $\mathcal{S}$ at $[\alpha]$ agrees with $\mathcal{C}_{[\alpha]}(\mathcal{C}_t)$. By the assumption on second fundamental forms, for $\gamma_s \in \Omega \cap \mathcal{C}_s$, $\mathcal{S}_{[\gamma_s]} \subset \mathbb{P}T_{\omega,[\gamma_s]}$ is isomorphic to $\mathcal{C}_{[\alpha_o]}(\mathcal{C}_o) \subset \mathbb{P}T_{[\alpha_o]}(\mathcal{C}_o)$ of the model space $\mathcal{C}_o$ (which is an irreducible Hermitian symmetric space of rank 2) as a projective subvariety. Since the family $\mathcal{S}_{[\alpha]}$ must be continuous in $[\alpha]$ for $[\alpha] \in \Omega$ and holomorphic outside $\Omega \cap \mathcal{C}_s$ we conclude that $\mathcal{S} \subset \mathbb{P}T_\omega$ is a holomorphic fiber subbundle such that $\mathcal{S}|_{\Omega \cap \mathcal{C}_t}$ defines a G-structure corresponding to the irreducible Hermitian symmetric manifold $\mathcal{C}_o$ on $\Omega \cap \mathcal{C}_t \neq \emptyset$ whenever $t$ is close to $s$. The latter is integrable for $t \neq s$, hence also at $s$ by continuity. It follows that, shrinking $\Omega$ if necessary there is a holomorphic mapping $f : \Omega \to \mathcal{C}_o$ whose restriction $f_t : \Omega \cap \mathcal{C}_t \to \mathcal{C}_o$ is a developing map (i.e., $f_t$ preserves G-structures) whenever $\Omega \cap \mathcal{C}_t \neq \emptyset$. It follows that for some disk $D$ containing $s$ there is a holomorphic map $\Phi : \mathcal{C}_o \times (D - \{s\}) \to \mathbb{P}V$ respecting canonical projections such that $\Phi$

induces a biholomorphism of $\mathcal{C}_o$ onto $\mathcal{C}_t \subset \mathbb{P}V_t$ for $t \in D - \{s\}$, and such that $\Phi$ extends holomorphically across some nonempty open subset $U_s$ of $\mathcal{C}_o \times \{s\}$. By Hartogs extension of meromorphic functions $\Phi$ extends to a meromorphic map from $\mathcal{C}_o \times D$ into $\mathbb{P}V|_D$ respecting canonical projections, to be denoted by the same symbol $\Phi$. Denote by $\Phi_t$ the restriction of $\Phi$ to $\mathcal{C}_o \times \{t\}$.

It remains to prove that $\Phi : \mathcal{C}_o \times D \to \mathbb{P}V|_D$ is a biholomorphic map onto its image. We claim that this is indeed the case. In fact, if $T_{[\gamma_s^o]}(\Omega \cap \mathcal{C}_s) = F/\mathbb{C}\gamma_s^o$ and the image of the second fundamental form $\sigma_{[\gamma_s^o]}$ is $Q/F$, then the linear span of $\Phi_s(U_s)$ contains $\mathbb{P}Q$. On the model space $\mathcal{C}_o$ we know that the second fundamental form $\sigma_{[\alpha_o]}$ is surjective. By the hypothesis in the Proposition this remains true at $\gamma_s^o$, and it follows that $Q = V_s$, proving the claim. By Lemma 2.3, $\Phi : \mathcal{C}_o \times D \to \mathbb{P}V\big|_D$ is a holomorphic embedding. The proof of the Proposition is complete for Case (A).

Case (B). For this case where $S$ is the Lagrangian Grassmannian we note the special feature that the second fundamental form $\sigma_{[\alpha_o]} : S^2T_{[\alpha]} \to N_{\mathcal{C}_o|\mathbb{P}T_o(S),[\alpha_o]}$ is an isomorphism. In the same set-up as in Case (A) let now $\ell \subset \mathcal{C}_{t_o} \cong \dot{\mathbb{P}}^n$ be the image of a line under the Veronese embedding. Thus $\ell \subset \mathcal{C}_{t_o} \cong \mathbb{P}^n \subset \mathbb{P}V_{t_o} \cong \mathbb{P}^N, N = \frac{n(n+1)}{2}$, is a degree-2 curve on $\mathbb{P}V_{t_o}$ whose linear span is a 2-plane. For a point $[\gamma_s] \in \Omega$, an element of $\mathcal{H}$ passing through $[\gamma_s]$ is either a degree-2 curve, the union of two distinct lines, or a line counted with multiplicity 2. Consider such an element $[\Gamma]$ of $\mathcal{H}$ whose germ at $[\gamma_s]$ lies on $\Omega$. If $\Gamma$ contains a line then $\sigma_{\gamma_s}(\overline{\beta}, \overline{\beta}) = 0$ for $\overline{\beta} \in T_{[\gamma_s]}(\Gamma)$, contradicting the fact that $\sigma_{[\gamma_t]}$ is an isomorphism. It follows that any such $[\Gamma]$ is a degree-2 curve whose linear span is precisely a 2-plane, so that $\Gamma$ is smooth. Let $\lambda : D \to \Omega$ be a holomorphic section such that $\lambda(s) = [\gamma_s^o]$. Since the second fundamental form $\sigma_{[\gamma_s]}$ is surjective, $\Omega$ is linearly nondegenerate in $\mathbb{P}V_s$. Considering the Chow space of marked $\mathcal{H}$-rational curves at $\lambda(t)$, $t \in D$ we obtain a projective space $\mathcal{U}_{\lambda(t)} \cong \mathbb{P}^{n-1}$ at each $\lambda(t)$, such that for $t \neq s, \mathcal{C}_t$ can be reconstructed from the universal family corresponding to $\mathcal{U}_\lambda$ by blowing down the distinguished section corresponding to the marking at $\lambda(s)$. The same construction works at $\lambda(s)$, giving therefore a holomorphic mapping from $\mathbb{P}^n \times D$ into $\mathcal{C}|_D \subset \mathbb{P}V|_D$. Finally, this mapping must be a biholomorphic map onto its image because $\Omega$ is linearly nondegenerate in $\mathbb{P}V_s$.

Case (C). This is the case of the Grassmannian $G(p, q)$ where $\mathcal{C}_o \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$ is reducible. Write $a = p-1, b = q-1$. Following Case (A) we want to construct a meromorphic map $\Phi : \mathcal{C}_o \times D \to \mathcal{C}|_D$ such that $\Phi_s$ is a biholomorphism over $t \neq s$ and such that the image of $\Phi$ includes the germ of $\Omega$ at $[\gamma_s^o]$. (In particular, $\Phi_s$ is of maximal rank.) In place of G-structures we consider product decompositions. The product decomposition $\mathcal{C}_s \cong \mathbb{P}^a \times \mathbb{P}^b, t \neq s$, arises from holomorphic foliations given at $[\alpha] \in \mathcal{C}_t$ by the solutions of $\sigma_{[\alpha]}(\overline{\beta}, \overline{\beta}) = 0$, whose solution set is a union of two transversal and complementary vector spaces, defining the two holomorphic foliations. By the hypothesis on second fundamental forms these foliations extend to $\Omega$, over which we may assume that there are two distinct holomorphic foliations $\mathcal{F}'$ and $\mathcal{F}''$, with leaves that can be extended to projective $a$-planes resp. $b$-planes. Let $\mathcal{P}' \subset \mathcal{C}|_D$ resp. $\mathcal{P}'' \subset \mathcal{C}|_D$ be two regular families of projective $a$-planes resp. $b$-planes over $D$ such that

$\mathcal{P}'_t$ resp. $\mathcal{P}''_t$ contains the leaf of $\mathcal{F}'_t$ resp. $\mathcal{F}''_t$ passing through $\lambda(t)$, where $\lambda$ is as in Case (B). Shrinking $\Omega$ if necessary we have a holomorphic map $f$ from $\Omega$ into $\mathcal{P}' \times_\pi \mathcal{P}''$ which associates any point $[\alpha]$ to $(\xi'([\alpha]), \xi''([\alpha]))$, where $\xi'([\alpha])$ resp. $\xi''([\alpha])$ is the intersection of the leaf $\mathcal{F}''_{[\alpha]}$ resp. $\mathcal{F}'_{[\alpha]}$ passing through $[\alpha]$ with $\mathcal{P}'_{\pi([\alpha])}$ resp. $\mathcal{P}''_{\pi([\alpha])}$. Inverting $f$, and noting that $f_s$ extends to a biholomorphism over $t \neq s$ we obtain a meromorphic map $\Phi$ from $\mathcal{C}_o \times D$ into $\mathcal{C}|_D$ respecting canonical projections onto $D$ such that $\Phi$. The fact that $\Phi$ is a biholomorphism follows as in Case (A). $\square$

(2.4) We now deduce the Main Theorem for irreducible Hermitian symmetric spaces.

*Proof of the Main Theorem in the symmetric case.* Let $S$ be an irreducible Hermitian symmetric space of the compact type of rank $\geq 2$. On the Fano manifold $X$ of Picard number 1 denote by $\mathcal{C}$ the total space of varieties of $\mathcal{K}$-rational tangents. By Proposition 2.3 we conclude that there exists a subvariety $Z \subset X$ of codimension at least 2 such that $\mathcal{C}|_{X-Z}$ is a holomorphic fiber subbundle of $\mathbb{P}T_{X-Z}$ where each fiber is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for the model space $S$. By Proposition 1.3.2 on Hartogs extension the $S$-structure extends across $Z$ and we conclude that the uniruled projective manifold $X$ admits an $S$-structure. By Proposition 1.3.2, $X$ is biholomorphically isomorphic to $S$. $\square$

## §3 Characterization in terms of varieties of minimal rational tangents in the contact case

(3.1) To prove the Main Theorem for the contact case we will first of all need to establish the analogues of Proposition 2.2 and Proposition 2.3 involving the third fundamental form. Let $V$ be a complex Euclidean space and $A \subset \mathbb{P}V$ be a local complex submanifold. Denote by $\rho : V - \{o\} \to \mathbb{P}V$ for the canonical projection and write $\widetilde{A} = \rho^{-1}(A)$. Suppose the image of the second fundamental form $\sigma$ of $A$ in $\mathbb{P}V$ has the same rank on $A$. Let $Q \subset \widetilde{A} \times V = T_V|_{\widetilde{A}}$ be obtained by adjoining at each point $\alpha \in \widetilde{A}$ the image of the Euclidean second fundamental form $\widetilde{\sigma}$ of $\widetilde{A}$ in $V$. Then, $Q \subset \widetilde{A} \times V$ is a holomorphic vector subbundle and we can define the Euclidean third fundamental form $\widetilde{\kappa} : S^3 T_{\widetilde{A}} \to \widetilde{A} \times V/Q$, which descends to the (projective) third fundamental form $\kappa_{[\alpha]} : S^3 T_A \to T_{\mathbb{P}V}/\overline{Q}$, where $\overline{Q}_{[\alpha]} := Q_\alpha \bmod \mathbb{C}\alpha \subset T_{[\alpha]}(\mathbb{P}V)$. In what follows we will be considering $A \subset \mathbb{P}V$ and call $\kappa$, where defined, the third fundamental form.

Let $W \subset T_S$ be the meromorphic distribution spanned at a general point $x \in X$ by its variety of minimal rational tangents. $W$ is holomorphic outside its singularity set $\mathrm{Sing}(W) \subset X$ of codimension $\geq 2$. Let $f : \mathbb{P}^1 \to X$ be a parametrized standard $\mathcal{K}$-rational curve such that $C := f(\mathbb{P}^1)$ does not entirely lie on the bad locus $E$ of $(X, \mathcal{K})$. By Lemma 1.2 we may choose $C$ such that $C \cap \mathrm{Sing}(W) = \emptyset$. In this section we write $\pi : \mathbb{P}(f^*W) \to \mathbb{P}^1$ for the canonical projection, and $\lambda := \pi|_{f^*\mathcal{C}}$. Otherwise we adopt the same notations as in (2.2). Recall from [(1.4), after the statement of Lemma (1.4.2)] that for the model $\mathcal{C}_o \subset \mathbb{P}W_o \subset \mathbb{P}T_o(S)$ there is at $\alpha_o \in \widetilde{\mathcal{C}}_o$ the filtration $\mathbb{C}\alpha_o \subset P_{\alpha_o} \subset Q_{\alpha_o} \subset W_o \subset T_x(S)$. On $f^*\mathcal{C}$ this leads to a filtration $L \subset P \subset Q \subset f^*W \subset f^*T_X$, where $L$ is the pull-back of the tautological line bundle. Restricted to the tautological lifting $\widehat{C} \subset f^*\mathcal{C}$ we have $L|_{\widehat{C}} \cong \mathcal{O}(2)$, $P|_{\widehat{C}} \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p$. By the proof of Proposition 2.2, $Q/P \otimes \mathcal{O}(-2)$ is the image of $S^2\big((P/T_{\mathbb{P}^1}) \otimes \mathcal{O}(-2)\big) \cong \big(\mathcal{O}(-2)\big)^{\frac{p(p+1)}{2}}$ in $f^*T_W \otimes \mathcal{O}(-2)$ under $\sigma$. Since $Q \subset f^*W$ is of co-rank 1 by the cubic expansion in Lemma 1.4.2, we have a short exact sequence $O \to \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \to Q \to \mathcal{O}^p \to 0$, so that $Q \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^p$. Let $\kappa_{[\alpha(t)]} : S^3T_{[\alpha(t)]} \to N_{[\alpha(t)]} = T_{\pi,[\alpha(t)]}/\overline{Q}_{[\alpha(t)]}$ be the relative third fundamental form along $\widehat{C}$. In an implicit form in the proof of [17, Lemma (3.3.1)] we introduced the notion of parallel transport (cf. (2.2) here) of the third fundamental form on tautological liftings of standard rational curves. By a straightforward adaptation of the argument there we have

**Proposition 3.1.** *Let $S$ be a Fano contact homogeneous manifold of Picard number 1 other than an odd-dimensional projective space, $\mathcal{C}_o \subset \mathbb{P}W_o \subset \mathbb{P}T_o(S)$ be the variety of minimal rational tangents at a reference point $o \in S$. For $[\alpha_o] \in \mathcal{C}_o$ write $T_{[\alpha_o]}$ for $T_{[\alpha_o]}(\mathcal{C}_o) = P_{\alpha_o}/\mathbb{C}\alpha_o$. Let $Q_{\alpha_o} \subset W_o$ be the vector subspace of codimension 1, $Q_{\alpha_o} \supset P_{\alpha_o}$, obtained by adjoining the image of the second fundamental form $\sigma_{[\alpha_o]} : S^2T_{[\alpha_o]} \to N_{\mathcal{C}_o|\mathbb{P}W_o,[\alpha_o]}$. On the Fano manifold $X$ of Picard number 1 let $f : \mathbb{P}^1 \to X$ be a general standard $\mathcal{K}$-rational curve such that $f(\mathbb{P}^1) \cap \mathrm{Sing}(W) = \emptyset$. Then, at any $t \in \mathbb{P}^1$ the third fundamental form $\kappa_{[\alpha(t)]} : S^3T_{[\alpha(t)]} \to T_{\pi,[\alpha(t)]}/\overline{Q}_{[\alpha(t)]}$ is isomorphic to the third fundamental form $\kappa_{[\alpha_o]} : S^3T_{[\alpha_o]} \to T_{[\alpha_o]}(\mathbb{P}W_o)/\overline{Q}_{[\alpha_o]}$ for the model manifold $S$.*

*Proof.* We will show that for a general standard rational curve $\mathcal{C}$ lying on $X - \mathrm{Sing}(W)$, $f^*W \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^p \oplus \mathcal{O}(-1)$ and apply an analogue of the argument for the proof of Proposition 2.2. From $Q \subset f^*T_W \subset f^*T_X$ we obtain $f^*T_W/Q \subset f^*T_X/Q \cong \mathcal{O}^2$, so that $f^*T_W/Q \cong \mathcal{O}(-k)$ for some $k \geq 0$. From the short exact sequence $0 \to \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^p \to f^*T_W \to \mathcal{O}(-k) \to 0$ it follows that $f^*T_W \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^p \oplus \mathcal{O}(-k)$. We have $f^*T_X/f^*T_W \cong \mathcal{O}(k)$. Consider now the Frobenius form $\varphi : \Lambda^2 W \to T_X/W$ over $X - Z$. If $k = 0$, then, for $z \in \mathbb{P}^1$, $\alpha \in T_{\mathbb{P}',z}$, $\varphi\big(df(\alpha), \eta\big) = 0$ for any $\eta \in W_{f(z)}$. Thus, if $k = 0$ for a general standard rational curve (lying on $X - Z$), by varying the rational curve $C$ it follows that for a general point $x \in X - Z$, $\varphi$ vanishes at $x$. This contradicts with Proposition 2.2.1, according to which $W \subsetneqq T_X$ cannot be integrable as $X$ is of Picard number 1.

Consider now the third fundamental form $\kappa$ along the tautological lifting $\widehat{C}$ of $C$, which can be interpreted in analogy with Proposition 2.2 as a section of $\mathrm{Hom}\big(S^3T_\lambda|_{\widehat{C}}, N_\lambda|_{\widehat{C}}\big) = \mathrm{Hom}\big(S^3\big(P \otimes \mathcal{O}(-2)\big), f^*T_W/Q \otimes \mathcal{O}(-2)\big)$. We have $P \otimes \mathcal{O}(-2) \cong \big(\mathcal{O}(-1)\big)^p$. Thus

$$\kappa \in \Gamma\left(\mathbb{P}^1, \mathrm{Hom}\Big(\big(\mathcal{O}(-3)\big)^{\frac{p(p+1)(p+2)}{6}}, \mathcal{O}(-2-k)\Big)\right).$$

If $k \geq 2$, then $\kappa \equiv 0$.

Since $\kappa$ is known to be nonzero at a general point of $\mathbb{P}^1$, we conclude that $k = 1$. It follows now immediately that $\kappa$ is a nonzero holomorphic section of a trivial holomorphic vector bundle $\mathcal{O}^{\frac{p(p+1)(p+2)}{6}}$. As a consequence, $\kappa$ is nowhere zero, and all the third fundamental forms $\kappa(z)$, $z \in \mathbb{P}^1$, are isomorphic to that of $\mathcal{C}_o \subset \mathbb{P}W_o$ at $[\alpha_o] \in \mathcal{C}_o$ for the model manifold $S$, as desired. $\square$

(3.2) As an analogue to Proposition 2.3 we have

**Proposition 3.2.** *Let $S$ be a Fano contact homogeneous manifold of Picard number 1 other than an odd-dimensional projective space. Denote by $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ the variety of minimal rational tangents of $S$ at a reference point $o \in S$, and write $W_o \subset T_o(S)$ for the linear span of $\widetilde{\mathcal{C}}_o$. Let $V$ be a holomorphic vector bundle over $\mathbb{P}^1$, $\pi : \mathbb{P}V \to \mathbb{P}^1$ be the canonical projection, and $\mathcal{C} \subset \mathbb{P}V$ be an irreducible subvariety. Assume that for some finite set $A \subset \mathbb{P}^1$ and for $t \in \mathbb{P}^1 - A$, $\mathcal{C}_t := \nu^{-1}(t) \subset \mathbb{P}V_t$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety. For $s \in A$, assume that there exists $[\gamma_s^o] \in \mathcal{C}_s$ and a germ of irreducible branch $\Omega \subset \mathcal{C}$ at $[\gamma_s^o]$ such that $\pi|_\Omega$ is a submersion and such that, for $\gamma_s \in \Omega \cap \mathcal{C}_s$, the second fundamental form $\sigma_{[\gamma_s]} : S^2T_{\pi,[\gamma_s]} \to N_{\mathcal{C}_t|\mathbb{P}V_t,[\gamma_s]}$ is isomorphic to the second fundamental form $\sigma_{[\alpha_o]} : S^2T_{[\alpha_o]} \to N_{\mathcal{C}_o|\mathbb{P}T_o(S),[\alpha_o]}$ for the model manifold $S$. For $[\alpha] \in \Omega$ denote by $\overline{Q}_{[\alpha]} \subset T_{\pi,[\alpha]}$, $\overline{Q}_{[\alpha]} \supset T_{[\alpha]}$ such that $\overline{Q}_{[\alpha]}/T_{[\alpha]}$ is the image of the second fundamental form $\sigma_{[\alpha]}$. Assume that the third fundamental form $\kappa_{[\alpha]} : S^3T_{[\alpha]} \to T_{\pi,[\alpha]}/\overline{Q}_{[\alpha]}$ is isomorphic to the third fundamental form of $\kappa_{[\alpha_o]} : S^3T_{[\alpha_o]} \to T_{[\alpha_o]}(\mathbb{P}W_o)/\overline{Q}_{[\alpha_o]}$ of the model manifold $S$. Then, for every $t \in \mathbb{P}^1$, $\mathcal{C}_t \subset \mathbb{P}V_t$ is isomorphic to the model $\mathcal{C}_o \subset \mathbb{P}W_o$ as a projective subvariety, and $\pi|_\mathcal{C} : \mathcal{C} \to \mathbb{P}^1$ is a holomorphic fiber subbundle of $\pi : \mathbb{P}V \to \mathbb{P}^1$ with typical fiber isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$.*

Let $S$ be a Fano homogeneous manifold of Picard number 1, $S \not\cong \mathbb{P}^{2n-1}$, and write $W \subset T_S$ for the contact distribution. Then, $\mathcal{C}_o \subset \mathbb{P}W_o$ is linearly nondegenerate. For the proof of Proposition 3.2 the set of such manifolds $S$ will be broken up into 5 cases as a consequence of the classification given in (1.4), as follows:

(D) The variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}W_o$ is itself an irreducible Hermitian symmetric space of rank 3.

(E) $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ is reducible and $\mathcal{C}_o \cong \mathbb{P}^1 \times Z$, where $Z$ is a hyperquadric of dimension $\geq 3$.

(F) $W_o$ can be identified with $E \otimes F$, where $E$ is 2-dimensional and $F$ is 3-dimensional. Here $\mathcal{C}_o \subset \mathbb{P}(E \otimes F)$ is the image of $\mathbb{P}E \times \Gamma$ under the Segre embedding, where $\Gamma \subset \mathbb{P}F \cong \mathbb{P}^2$ is a smooth quadric curve.

(G) $\mathcal{C}_o \subset W_o$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$, embedded by the Segre embedding.

(H) $\mathcal{C}_o \subset W_o$ is up to isomorphism the embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$ defined by $\Gamma(\mathbb{P}^1, \mathcal{O}(3))^* \cong \mathbb{C}^4$.

In what follows we will refer to Cases (A), (B), (C) in the proof of Proposition 2.3 in the symmetric case.

*Proof of Proposition 3.2.*
Case (D) This is the analogue of Case (A) in Proposition 2.3. $\mathcal{C}_o \subset W_o$ is the first canonical embedding. Write $[\alpha]$ for a point on $\mathcal{C}_o$. By Lemma 1.4.2 the space of lines on $\mathcal{C}_o$ passing through $[\alpha] \in \mathcal{C}_o$, given by $\mathbb{P}(\mathbb{C}\beta + \mathbb{C}\alpha)$, $[\beta] \neq [\alpha]$, corresponds precisely to the conditions $\sigma_{[\alpha]}(\overline{\beta}, \overline{\beta}) = 0$, $\kappa_{[\alpha]}(\overline{\beta}, \overline{\beta}, \overline{\beta}) = 0$; $\overline{\beta} := \beta + \mathbb{C}\alpha$; for the second and third fundamental forms.

For $s \in A$ from the hypothesis on second and third fundamental forms for $[\gamma_s] \in \Omega \cap \mathcal{C}_s$ the argument for Case (A) of Proposition 2.3 using G-structures works for Case (D). Here for the proof in the last step to show that $\Phi : \mathcal{C}_o \times D \to \mathbb{P}W$ is a holomorphic embedding using Lemma 2.3, we make use of the cubic expansion of the variety of minimal rational tangents $\mathcal{C}_o \subset \mathbb{P}W_0$ as in Lemma 1.4.2. In particular for the model space $S$, and $[\alpha_o] \in \mathcal{C}_o$, we have the filtration $\mathbb{C}\alpha_o \subset T_{\alpha_o}(\widetilde{\mathcal{C}}_o) := P_{\alpha_o} \subset Q_{\alpha_o} \subset W_o$, where $\widetilde{\mathcal{C}}_o \cup \{o\} \subset W_o$ is the affine cone over $\mathcal{C}_o$, $Q_{\alpha_o}$ is obtained from $P_{\alpha_o}$ by adjoining the image of the second fundamental form $\sigma_{\alpha_o}$, and $W_o$ is obtained from $Q_{\alpha_o}$ by adjoining the image of the third fundamental form $\kappa_{\alpha_o}$.

Case (E)  Writing $\mathbb{P}^1 = \mathbb{P}E$, and denoting by $Z \subset \mathbb{P}F$ the first canonical embedding, the embedding $\mathbb{P}^1 \times Z \subset W_o$, $\mathbb{P}^1 \times Z = \mathbb{P}E \times Z \subset \mathbb{P}E \times \mathbb{P}F \subset \mathbb{P}(E \otimes F)$, where the last inclusion is the Segre embedding. As in Case (C) of Proposition 2.3 we proceed to recuperate the two natural foliations $\mathcal{C}_o$ from the second fundamental form. Consider $\mathcal{C}_o \subset \mathbb{P}F_o(S)$ as a projective manifold uniruled by projective lines. At $[\alpha] \in \mathcal{C}_o$ denote by $\mathcal{C}_{[\alpha]}(\mathcal{C}_o) \subset \mathbb{P}T_{[\alpha]}(\mathcal{C}_o)$ the variety of rational tangents defined by such lines. The variety $\mathcal{C}_{[\alpha]}(\mathcal{C}_o)$ has two connected components. One component is a holomorphic section of $\mathbb{P}T_{\mathcal{C}_o}$ over $\mathcal{C}_o$ and defines the foliation arising from $\mathbb{P}E \cong \mathbb{P}^1$. The other component is isomorphic to the variety of minimal rational tangents $\mathcal{C}_o(Z)$, the linear span of which reproduces the foliation defined by the factor $Z$ in $\mathcal{C}_o = \mathcal{C}_o(S)$. Thus both foliations are determined by the solution set of $\sigma_{[\alpha]}(\overline{\beta}, \overline{\beta}) = 0$. The proof of Case (D) is completed by an obvious combination of the argument on product structures as in Case (C) and the use of G-structures as in Case (A) of Proposition 2.3. For this we note that for $s \in A$ and at a point $[\gamma_s] \in \Omega \cap \mathcal{C}_s$ the second foliation spans linearly a projective space of the same dimension as $\mathbb{P}F$, by the hypothesis on the second fundamental form $\sigma_{[\gamma_s]}$. In notations analogous to Case (C) the latter allows us to apply Lemma 2.3 to show that the leaf of the second foliation $\mathcal{F}''$ passing through $[\gamma^o_s]$, $s \notin A$, can be completed to a projective subvariety $\mathcal{P}''_s \subset \mathcal{C}_s$ with $\mathcal{P}''_s \cong Z$, such that $\mathcal{P}''_t, t \in D$, defines a regular family of projective manifolds biholomorphic to $Z$, and such that their linear spans $\mathbb{P}F_t, t \in D$, define a regular family of projective spaces biholomorphic to $\mathbb{P}F$. The last step is an application of Lemma 2.3 as in Case (D).

Case (F)  Fix $s \in A$. As in Case (E) we proceed to obtain foliations on $\Omega$. Here there is a difference. The only lines on $\mathbb{P}E \times \Gamma \subset \mathbb{P}(E \otimes F)$ arise from the linear factor $\mathbb{P}E$. The solution set of $\sigma_{[\alpha]}(\overline{\beta}, \overline{\beta})$ defines therefore only one of the two foliations. For the other foliation we note that $Q_{[\alpha]} = T_{[\alpha]}(\mathbb{P}E) \oplus T_{[\alpha]}(\mathbb{P}F)$. For an element $\overline{\beta} = \overline{\xi} + \overline{\eta} \in T_{[\alpha]}$, $\overline{\xi} \in T_{[\alpha]}(\mathbb{P}E)$, $\overline{\eta} \in T_{[\alpha]}(\Gamma)$, we have $\kappa_{[\alpha]}(\overline{\xi} + \overline{\eta}, \overline{\xi} + \overline{\eta}, \overline{\xi} + \overline{\eta}) = \xi \otimes s(\eta, \eta) \mod Q_\alpha$, where $s(\eta, \eta) \mod T_\alpha = \sigma_{[\alpha]}(\overline{\eta}, \overline{\eta})$. It follows that $\kappa_{[\alpha]}(\overline{\beta}, \overline{\beta}, \overline{\beta}) = 0$ if and only if $\overline{\xi} = 0$. Thus the foliation given by the quadric factor $\Gamma$ can be recuperated from the vanishing of the third fundamental form as in the above. At $s \in A$ from the hypothesis on second and third fundamental forms it follows that leaves passing through $[\gamma_s]$ consists of a line and a degree-2 curve lying in some 2-plane, which must then be smooth, and that furthermore the linear span of $\Omega \cap \mathcal{C}_s$ must be isomorphic to $\mathbb{P}^5$. The rest of the argument follows from the proof of Case (C) for Grassmannians in Proposition 2.3.

Case (G) The proof is a straightforward modification of Case (C) of Proposition 2.3.

Case (H) It is analogous to Case (B) of Proposition 2.3 for $n = 1$. Here it suffices to note that a rational curve of degree 3 in $\mathbb{P}^3$ is isomorphic to the embedding of $\mathbb{P}^1 \subset \mathbb{P}^3$ defined by $\mathcal{O}(3)$, provided the image is linearly nondegenerate. At $s \in A$ from the nondegeneracy of both $\sigma_{[\gamma_s^o]}$ and $\kappa_{[\gamma_s^o]}$ it follows that the linear span of $\mathcal{C}_s$ must be isomorphic to $\mathbb{P}^3$. $\square$

The proofs of Propositions 2.3 and 3.2 are based on the reconstruction of varieties of minimal rational tangents from local differential projective-geometric invariants which have been proven to be invariant under a form of parallel transport. While the proofs rely on classification into cases (A)-(H) depending on the form of the variety of minimal rational tangents $\mathcal{C}_o$ (which is always a Hermitian symmetric manifold realized by some equivariant embedding as a projective subvariety), the arguments split essentially into those for the cases where $\mathcal{C}_o$ is irreducible, for which geometric structures can be used (with the case of Veronese embeddings requiring special treatment), and adaptations to the cases where $\mathcal{C}_o$ splits into a product, in which case deformations of foliations corresponding to product structures had to be dealt with. The situation is one in which we have deformations of varieties of minimal rational tangents as projective subvarieties. In many cases there are stronger forms of abstract characterizations of equivariant holomorphic embeddings of Hermitian symmetric spaces in terms of local differential projective-geometric invariants, as given by Landsberg [13, 14] and Hwang-Yamaguchi [10], where the abstract complex submanifold of the projective space need only be defined as a germ and *a fortiori* need not be connected to the model space by deformation. We have chosen nonetheless to verify of all the cases without resorting to the deeper cited results for two reasons, first of all to give self-contained proofs in line with arguments used elsewhere in the article, secondly because abstract characterizations in terms of local differential projective-geometric invariants are either unavailable or not readily applicable in some cases.

(3.3) We are now ready to deduce the Main Theorem for the contact case, thus completing the proof of the Main Theorem.

*Proof of the Main Theorem for the contact case.* Let $S$ be a Fano contact homogeneous manifold of Picard number 1 other than an odd-dimensional projective space, and denote by $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ its variety of minimal rational tangents at a reference point $o \in S$. Let $X$ be a Fano manifold of Picard number 1 equipped with a minimal rational component $\mathcal{K}$ such that the associated variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is congruent to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety. By Proposition 3.2 the analogue of Proposition 2.1 holds true in the contact case. In other words, for some subvariety $Z \subset X$ of codimension $\geq 2$ and for $x \in X - Z$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ as a projective subvariety.

Let $W \subset T_X$ be the meromorphic distribution on $S$ spanned at a general point $x$ by varieties of minimal rational tangents. Since for $x \in X - Z$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is congruent to the model $\mathcal{C}_o \subset \mathbb{P}T_o(S)$, $W \subset T_X$ must be a holomorphic vector subbundle over $X - Z$. Since $Z \subset X$ is of codimension $\geq 2$, by Proposition 1.2.3 there exists a standard $\mathcal{K}$-rational curve $C$ lying on $X - Z$. Write $f : \mathbb{P}^1 \to C \subset X$ for a parametrization of $C$. Recall that there is a filtration $T_{\mathbb{P}^1} \subset P \subset Q \subset f^*W$ where $P$ is the positive part of the Grothendieck decomposition of $f^*T_X$, $Q \subset f^*W$ is obtained by adjoining the image of the relative second fundamental forms of $f^*\mathcal{C} \to \mathbb{P}^1$ in $\mathbb{P}(f^*T_X)$. From the proof of Proposition 3.1 we have $P \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p$, $Q \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^p$, $f^*W \cong \mathcal{O}(2) \oplus \big(\mathcal{O}(1)\big)^p \oplus \mathcal{O}^p \oplus \mathcal{O}(-1)$.

To apply the proof of Proposition 1.3.3 (from Hong [3]) we consider now the Frobenius form $\varphi : \Lambda^2 W \to T_{X-Z}/W$ over $X - Z$. We are going to prove that $\varphi$ is everywhere nondegenerate on $X-Z$. On $\mathcal{C}_o \subset \mathbb{P}W_o$ the variety of minimal rational tangents $\mathcal{T} \subset \mathbb{P}(\Lambda^2 W_o)$ spans a hypersurface. Applying Proposition 1.1.2 it follows that at each point $x \in X-Z$ either $\varphi(x)$ vanishes, or it is nondengenerate. Furthermore, from the Grothendieck decomposition of $f^*W$ it follows readily that $\varphi(x)$ vanishes at a point $x \in C$ if and only if it vanishes identically on $C$. By Proposition 1.1.1, $W$ is not integrable, so that $\varphi(x) \neq 0$ at some point $x \in X - Z$. If $\varphi$ vanishes at some point $y \in X - Z$, it must vanish on $A - Z \neq \emptyset$ for some proper subvariety $A \subset Z$ which is saturated in the sense that every standard $\mathcal{K}$-rational curve intersecting $A - Z$ at some point must lie entirely on $A$. Since $X$ is of Picard number 1, obviously such a subvariety $A \subset X$ must be of codimension at least 2. On the other hand, since $\varphi$ is nondegenerate at a general point of $X - Z$ by taking determinants $A$ must be a hypersurface, leading to a contradiction.

Since there exists a standard $\mathcal{K}$-rational curve lying on $X-Z$ by the proof of Proposition 1.3.3 there exists a developing map $f : X - Z \to S$ which preserves varieties of minimal rational tangents. By Theorem 1.2 on Cartan-Fubini extension it follows that $f$ extends to biholomorphic map $F : X \to S$, as desired. $\square$

Remarks. Since $f$ is already holomorphic on $X - Z$ we do not need the full force of Theorem 1.2. It suffices to show first of all that $f$ preserves the tautological foliation on total spaces of varieties of minimal rational curves (Hwang-Mok [7, Theorem 3.1.4, p.368]). As a consequence $f$ maps any minimal rational curve on $X - Z$ biholomorphically onto a line and defines hence a birational map from $X$ onto $S$. Finally, by Hwang-Mok ([8, Proposition 4.4, p.574]), the birational map $f : X \to S$ preserving varieties of minimal rational tangents must be a biholomorphism.

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