The circumference of a graph with no $K_{3,t}$-minor

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Abstract

It was shown by Chen and Yu that every 3-connected planar graph $G$ contains a cycle of length at least $|G|^{\log_3 2}$, where $|G|$ denotes the number of vertices of $G$. Thomas made a conjecture in a more general setting: there exists a function $\beta(t) > 0$ for $t \geq 3$, such that every 3-connected graph $G$ with no $K_{3,t}$-minor, $t \geq 3$, contains a cycle of length at least $|G|^{\beta(t)}$. We prove that this conjecture is true with $\beta(t) = \log_{8t+1} 2$. We also show that every 2-connected graph with no $K_{2,t}$-minor, $t \geq 3$, contains a cycle of length at least $|G|/t^{t-1}$.

1 Introduction and notation

Over the past seven decades the Hamilton cycle problem has attracted tremendous research effort. Significant work has been done in characterizing those graphs that contain Hamilton cycles, and the earliest such results are concerned with planar graphs. In 1931, Whitney [22] proved that every 4-connected planar triangulation contains a Hamilton cycle. Tutte generalized this result to all 4-connected planar graphs [21], and Thomassen [20] strengthened this by showing that every 4-connected planar graph is in fact Hamilton connected.

With an attempt to generalize Tutte’s theorem to other surfaces, Grünbaum [6] and Nash-Williams [12] independently conjectured that every 4-connected toroidal

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graph contains a Hamilton cycle. While this conjecture remains open, it has been shown that every 4-connected toroidal graph contains a Hamilton path [19], and that 5-connected toroidal graphs are Hamiltonian [18]. Moreover, Thomas and Yu [17] proved that 4-connected projective-planar graphs are also Hamiltonian. Generalizing to other surfaces, Yu [23] managed to show that every “locally planar” 5-connected triangulation of a surface contains a Hamilton cycle.

Notice that the above results are all concerning graphs with fairly high connectivity. If we relax the condition, then the situation changes dramatically: there are many 3-connected planar graphs that contain no Hamilton cycles, as exemplified in [8]. On the other hand, all cubic, bipartite, 3-connected, planar graphs are conjectured to be Hamiltonian by Barnette (see [10]).

When a graph $G$ contains no Hamilton cycle, one may ask how long a cycle it contains. The length of a longest cycle in $G$, denoted by $c(G)$, is called the circumference of $G$. A good lower bound on $c(G)$ has also been the subject of extensive research. While studying paths in polytopes, Moon and Moser [11] implicitly conjectured that if $G$ is a 3-connected planar graph then $c(G) \geq \alpha |G| \log_3 2$, where $\alpha$ is a constant and $|G|$ denotes the number of vertices of $G$. (Grünbaum and Walther [7] made the same conjecture for the class of 3-connected cubic planar graphs.) Jackson and Wormald [9] gave the first polynomial lower bound on $c(G)$ for 3-connected planar graphs. This bound was improved by Gao and Yu [5] and further refined by Chung [4]. In [3], Chen and Yu fully established the Moon-Moser conjecture and showed that the same is true (within a constant factor) for 3-connected graphs embeddable in the torus or the Klein bottle. Based on these results, Böehme, Mohar, and Thomassen [2] proved that if $G$ is a 3-connected graph of orientable genus $g$ then $c(G) \geq \epsilon(g)|G| \log_3 2$, where $\epsilon(g)$ is a constant dependent on $g$. Furthermore, $\epsilon(g)$ can be replaced by an absolute constant if $G$ is also “locally planar” [15].

It is well known that a planar graph contains no $K_{3,3}$-minors. As a different generalization of the Chen-Yu result [3] on planar graphs, one may ask whether there is a similar result for 3-connected graphs with no $K_{3,t}$-minors. It is worthwhile pointing out that graphs containing no $K_{3,t}$-minors form an important class in the theory of graph minors. As discovered by Robertson and Seymour [14], in order to embed a graph in a given surface one must exclude large $K_{3,t}$-minors. In [13], Oporowski, Oxley, and Thomas showed that if $G$ is a 3-connected graph with no $K_{3,t}$-minor, then it contains a large wheel. Inspired by this, Thomas and Seymour [16] made the following two conjectures.

(1.1) **Conjecture** (by Thomas). There exists a function $\beta(t) > 0$ for $t \geq 3$ such that, for any integer $t \geq 3$ and any 3-connected graph $G$ with no $K_{3,t}$-minor, $c(G) \geq |G|^{\beta(t)}$.

(1.2) **Conjecture** (by Seymour and Thomas). There exist a constant $\beta > 0$ and a function $\alpha(t) > 0$ for $t \geq 3$ such that, for any integer $t \geq 3$ and any 3-connected graph $G$ with no $K_{3,t}$-minor, $c(G) \geq \alpha(t)|G|^\beta$.

To prove the above conjectures, one reasonable approach is to find a structural
description of 3-connected graphs with no $K_{3,t}$-minors. In this direction, Böhme, Maharry and Mohar [1] have obtained structural information about 7-connected graphs that contain no $K_{3,t}$-minors. However, a complete characterization of all 3-connected graphs seems to be very difficult to obtain. Alternatively, one might try to show the existence of a function $g(t) > 0$ such that any 3-connected graph containing no $K_{3,t}$-minor is embeddable in a surface of genus $g(t)$, in order to apply the result in [2]. Unfortunately this is not true, as there exist 3-connected graphs with no large $K_{3,t}$-minors but with arbitrarily large genus. (For example, let $C = v_0v_1\ldots v_{k-1}v_0$ be a cycle of length $k$. Let $G$ be obtained from $C$ be replacing each $v_i$ by a complete graph with three vertices $x_{i,1}$, $x_{i,2}$, and $x_{i,3}$ such that $x_{i,j}$ and $x_{\ell,m}$ are adjacent if and only if $\ell = i - 1$ or $\ell = i + 1$, where the subscripts are taken modulo $k$. It was verified [1] that $G$ contains no $K_{3,7}$-minor and the orientable genus of $G$ is at least $k$.) One might therefore hope for some collection of simple reductions on 3-connected graphs with no $K_{3,t}$-minors which can be used to produce graphs embeddable in a surface of genus $g(t)$.

In this paper we approach the conjectures by direct construction of long cycles. Our main result is the following, which establishes Conjecture (1.1). (We shall actually prove a slightly stronger technical result, as stated in Section 2.)

\textbf{(1.3) Theorem.} For any integer $t \geq 3$ and for any 3-connected graph $G$ with no $K_{3,t}$-minor, $c(G) \geq |G|^{r(t)}$, where $r(t) = \log_{8t+1} 2$.

From the graphs constructed by Moon and Moser [11] (also see [3]), we see that the exponent $\beta(t)$ cannot exceed $\log_3 2$. We feel that $\beta(t)$ can be improved to $\log_2 2$. Yet, it is still unknown what the best bound is and whether the method used in this paper can be further extended to establish Conjecture (1.2).

The remainder of this paper is organized as follows. In section 2, we state the main theorem, introduce some terminology, and exhibit some useful properties of the function $f(x) = x^{\log_2 2}$. In Section 3, we study graphs with weighted edges, and establish a result about paths in weighted graphs and a result about the circumference of 2-connected graphs with no $K_{2,t}$-minors. (We shall use weighted graphs to store information when performing certain reduction operations in the proof.) In Sections 4 - 6, we complete the proof of the technical result stated in Section 2. There are three statements in the technical result: (a), (b), and (c). The proof of the technical result is by induction on the number of vertices. The induction step for (a) is done in Section 4, and the induction step for (b) is done in Section 5. The induction step for (c) is done in Section 6, and the inductive proof will be completed in Section 6.

\section{The technical theorem}

The main goals of this section are to state a technical theorem which implies (1.3) and to prove some properties of the function $f(x) = x^{\log_2 2}$ which will be frequently used. First, we introduce notation and terminology necessary for stating
We only consider simple graphs. We use $A := B$ to rename $B$ as $A$. For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively, and we let $|G| := |V(G)|$. For graphs $H$ and $G$, $H \subseteq G$ means that $H$ is a subgraph of $G$.

Let $G$ be a graph and let $U \subseteq V(G)$. Then $G[U]$ denotes the subgraph of $G$ induced by $U$. The set $U$ is said to be a connected set of $G$ if $G[U]$ is connected. Let $G - U := G[V(G) - U]$, and if $U = \{u\}$ then let $G - u := G - U$. We say that $U$ is a $k$-cut of $G$ if $G - U$ is not connected and $|U| = k$. If $\{u\}$ is a 1-cut of $G$, then $u$ is called a cutvertex of $G$. Let $N_G(U) := \{x \in V(G) - U : x$ is adjacent to some vertex in $U\}$, and let $N_G(u) := N_G(\{u\})$. For convenience, let $N_G(H) := N_G(V(H))$ for any subgraph $H$ of $G$. If there is no danger of confusion, we will simply drop the subscript $G$.

Let $G$ be a graph. For two distinct vertices $x, y$ of $G$, an $x$-$y$ path in $G$ is a path between $x$ and $y$ in $G$. If $P$ is a path, we use $\ell(P)$ to denote the length of $P$, which is the number of edges of $P$. For any distinct vertices $x, y$ of a path $P$, we use $P[x, y]$ to denote the subpath of $P$ between $x$ and $y$ (inclusive), and define $P[x, y] := P[x, y] - y$, $P(x, y) := P[x, y] - x$, and $P(x, y) := P[x, y] - \{x, y\}$. An edge of $G$ with ends $u$ and $v$ is often denoted by $uv$, or $vu$, or $\{u, v\}$. Let $S$ be a set of 2-element subsets of $V(G)$. Then we use $G + S$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \cup S$. (Note that each edge of $G$ is a 2-element subset of $V(G)$.) If $S = \{\{u_i, v_i\} : i = 1, \ldots, k\}$, then we also write $G + \{u_iv_i : i = 1, \ldots, k\}$ instead of $G + S$. If $S = \{\{u, v\}\}$, then we let $G + uv := G + S$.

A graph $H$ is a minor of a graph $G$ if there exist disjoint connected sets $V_x$ of $G$, indexed by $x \in V(H)$, such that, for any distinct $x, y \in V(H)$, $xy \in E(H)$ if and only if $G$ has an edge with one end in $V_x$ and the other in $V_y$. These sets form a representation of $H$ in $G$. If $H$ is a minor of $G$, then we say that $G$ contains an $H$-minor. When there is no danger of confusion, we will not make an effort to distinguish between the edges of $H$ and the edges of $G$. That is, we may view the edges of $H$ as edges of $G$. Let $G$ be a graph and let $U$ be a connected subgraph of $G$; then we use $G/U$ to denote the graph obtained from $G$ by contracting $U$ (and deleting resulting multiple edges and loops).

The graph $K_{3,t}$ is the complete bipartite graph with one part of size 3 and the other of size $t$. Let $G$ be a graph and $\{x, y, z\} \subseteq V(G)$. We say that a $K_{3,t}$-minor $H$ of $G$ is rooted at $\{x, y, z\}$ if $H$ has a representation in $G$ such that $x \in V_1$, $y \in V_2$, $z \in V_3$, where $V_1, V_2, V_3$ are connected sets of $G$ representing vertices of $H$ in the partition set of size three. We define $\mu(G; x, y, z)$ to be the largest integer $t$ such that $H$ has a $K_{3,t}$-minor rooted at $\{x, y, z\}$.

We can now state the aforementioned technical theorem.

(2.1) Theorem. Let $t \geq 3$ be an integer, let $r(t) := \log_{8t+1} 2$, and let $G$ be a 3-connected graph with no $K_{3,t}$-minor. Then the following statements hold.

(a) For any distinct vertices $x, y, z$ of $G$ such that $xz, yz \in E(G)$, $G - z$ contains an $x$-$y$ path of length at least $\left(\frac{|G|-1}{3}\right)^{r(t)}$, where $\mu := \mu(G; x, y, z)$. 


(b) For any $xy \in E(G)$, $G$ contains an $x$-$y$ path of length at least $|G|^{r(t)}$.

(c) For any two distinct edges $xy$, $f$ of $G$, $G$ contains an $x$-$y$ path through $f$ which has length at least $(\frac{|E|}{\text{t}})^{r(t)} + 1$.

Note that (1.3) is an immediate consequence of (b) of Theorem (2.1). In order to prove Theorem (2.1), we need the following property of the function $f(x) = x^{\log_b 2}$.

(2.2) **Lemma.** For any integer $b \geq 4$ and for any $m \geq n > 0$,

$$m^{\log_b 2} + n^{\log_b 2} \geq (m + (b - 1)n)^{\log_b 2}.$$ 

**Proof.** By dividing both sides of the above inequality by $m^{\log_b 2}$, it suffices to show that, for any $s$ with $0 \leq s \leq 1$,

$$1 + s^{\log_b 2} \geq (1 + (b - 1)s)^{\log_b 2}.$$ 

Let $f(s) = 1 + s^{\log_b 2} - (1 + (b - 1)s)^{\log_b 2}$. Clearly, $f(0) = f(1) = 0$. Differentiating with respect to $s$, we have

$$f'(s) = \log_b 2 \cdot (s^{(\log_b 2) - 1} - (b - 1)(1 + (b - 1)s)^{(\log_b 2) - 1}).$$

A simple calculation shows that $f'(s) = 0$ has a unique solution. Therefore, since $f(0) = f(1) = 0$, either 0 is the absolute maximum of $f(s)$ over $[0,1]$ or 0 is the absolute minimum of $f(s)$ over $[0,1]$. That is, either $f(s) \geq 0$ for all $s \in [0,1]$ or $f(s) \leq 0$ for all $s \in [0,1]$. Note that $0 < \frac{1}{b} < 1$ (since $b \geq 4$) and

$$f\left(\frac{1}{b}\right) = (1 + \frac{1}{2}) - (1 + \frac{b - 1}{b})^{\log_b 2}$$

$$= \frac{3}{2} - \frac{(2b - 1)^{\log_b 2}}{2}$$

$$> \frac{3}{2} - \frac{(2b)^{\log_b 2}}{2}$$

$$= \frac{3}{2} - 2^{\log_b 2}.$$ 

Since $b \geq 4$, $2^{\log_b 2} \leq 2^{\log_4 2} = \sqrt{2} < 3/2$. Thus, $f\left(\frac{1}{b}\right) > 0$ for $b \geq 4$. Therefore, we have $f(s) \geq 0$ for all $s \in [0,1]$. 

(2.3) **Corollary.** Let $a \geq 1$ and $b \geq 4$ be integers, and let $m > 0$ and $n > 0$. If $m \geq \frac{n}{a}$, then

$$m^{\log_b 2} + n^{\log_b 2} \geq (m + \frac{b - 1}{a}n)^{\log_b 2}.$$ 

**Proof.** Since $m \geq \frac{n}{a} > 0$ and by Lemma (2.2), we have $m^{\log_b 2} + \left(\frac{n}{a}\right)^{\log_b 2} \geq (m + (b - 1)\frac{n}{a})^{\log_b 2}$. Since $a \geq 1$, $m^{\log_b 2} + n^{\log_b 2} \geq m^{\log_b 2} + \left(\frac{n}{a}\right)^{\log_b 2}$. So (2.3) holds.

By repeatedly applying (2.3), we obtain the following
We now state and prove the main result of this section.

**Theorem.** Suppose $m, n_1, \ldots, n_k$ are positive numbers such that $m \geq \frac{n_i}{a}$ for all $1 \leq i \leq k$. Then, for any integer $b \geq 4$,

$$m \log_b 2 + \sum_{i=1}^{k} n_i \log_b 2 \geq (m + \frac{b - 1}{a} \sum_{i=1}^{k} n_i) \log_b 2.$$

\section{Circumferences of 2-connected graphs}

In this section, we prove a result about long paths in weighted graphs, which will be useful for proving Theorem (2.1). We will also see that a similar argument can be used to prove an interesting result about the circumference of a 2-connected graph with no $K_{2,t}$-minor.

For convenience, we introduce the concept of bridge. Let $G$ be a graph and $H$ a subgraph of $G$. An $H$-bridge of $G$ is a subgraph of $G$ which is induced by either (i) an edge in $E(G) - E(H)$ with both ends in $V(H)$ or (ii) the edges in a component $D$ of $G - V(H)$ and edges of $G$ from $D$ to $H$. The $H$-bridges satisfying (ii) are said to be non-trivial. If $U \subseteq V(G)$, we may view $U$ as a subgraph of $G$ with vertex set $U$ and no edges. Hence, we will also speak of $U$-bridges or bridges of $G$ associated with $U$.

In the proof of (2.1), we need to replace certain bridges of a graph associated with 2-cuts by edges, and each such edge will be assigned a weight which records the number of vertices in the corresponding bridge. The following concepts will allow us to relate subgraphs in the weighted graph to $K_{3,t}$-minors and rooted $K_{3,t}$-minors in the original graph.

Let $G$ be a graph and let $S \subseteq E(G)$. An $S$-link of size $m$ in $G$ consists of two disjoint connected subgraphs $A, B$ of $G$ and a subset $S'$ of $S$ such that $|S'| = m$ and each edge in $S'$ has one end in $V(A)$ and the other in $V(B)$, and we denote it by $(A,B;S')$. If, in addition, $x \in V(A)$ and $y \in V(B)$ or $x \in V(B)$ and $y \in V(A)$, then $(A,B;S')$ is said to be between $x$ and $y$.

Again, let $G$ be a graph and let $S \subseteq E(G)$. Let $P$ be a path in $G$. For any $e \in E(P)$, an $(S;P)$-ladder with top $e$ and $m$ rungs is an $S$-link $(A,B;S')$ of size $m$ such that $e \notin S'$, one component of $P-e$ is contained in $A$, and the other component of $P-e$ is contained in $B$. The edges in $S'$ are called the rungs of the ladder. We use $\alpha(P;e)$ to denote the maximum number $t$ such that $G$ has an $(S;P)$-ladder with top $e$ and $t$ rungs.

Let $\mathbb{R}^+$ denote the set of non-negative real numbers. For any function $\omega : E(G) \mapsto \mathbb{R}^+$ and a subgraph $H$ of $G$, we define $\omega(H) := \sum_{e \in E(H)} \omega(e)$. We can now state and prove the main result of this section.

**Theorem.** Let $t \geq 2$ be an integer, $G$ be 2-connected graph, $\omega : E(G) \mapsto \mathbb{R}^+$, $S = \{e \in E(G) : \omega(e) > 0\}$, and $x, y \in V(G)$ be distinct. If $G$ does not contain any
\[ \sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq \omega(G). \]

**Proof.** Note that \( \omega(G) = \omega(S) \). We will apply induction on \(|G| + |S| \). If \(|S| = 0 \) then \( \omega(G) = 0 \), and hence, any \( x-y \) path \( P \) in \( G \) gives the desired path. If \(|S| = 1 \) then since \( G \) is 2-connected, \( G \) has an \( x-y \) path \( P \) containing the edge in \( S \), and clearly \( \sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq \omega(G) \). So we may assume \(|S| \geq 2 \).

Suppose \(|G| = 3 \). Then \( G \) is a triangle. Let \( P, Q \) denote the \( x-y \) paths in \( G \), and assume without loss of generality that \( \omega(P) \geq \omega(Q) \). Therefore, since \( t \geq 2 \), \( t \omega(P) \geq \omega(G) \). If \( S \subseteq E(P) \), then \( \sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq \omega(G) \). So assume that \( S \cap E(Q) \neq \emptyset \). Then for any \( e \in E(P) \), \( \alpha(P;e) \geq 1 \). Hence, \( \sum_{e \in E(P)} t^{\alpha(P;e)} \omega(e) \geq t \omega(P) \geq \omega(G) \). Therefore, we may assume that \(|G| \geq 4 \). We distinguish between two cases.

**Case 1.** \( \{x, y\} \) is a 2-cut of \( G \) or some edge in \( S \) is incident with both \( x \) and \( y \).

In this case, there exist subgraphs \( G_1 \) and \( G_2 \) of \( G \) such that \( G_1 \cup G_2 = G \), \( V(G_1) \cap V(G_2) = \{x, y\} \), and either \(|G_1| \geq 3 \leq |G_2| \) or, for some \( i \in \{1, 2\} \), \( G_i \) is induced by an edge in \( S \). (See Figure 1(a) for an illustration.) Without loss of generality, we may assume that \( \omega(G_1) \geq \omega(G_2) \). Then, since \( t \geq 2 \), \( t \cdot \omega(G_1) \geq \omega(G) \).

First, let us assume that \( G_1 \) is induced by an edge \( f \in S \). Then \( f \) is incident with both \( x \) and \( y \). Let \( P = G_1 \). Since \( G \) is 2-connected and \(|S| \geq 2 \), there exists an edge \( g \in S \setminus \{f\} \) and an \( x-y \) path \( R \) in \( G_2 \) containing \( g \). Let \( (A, B; \{g\}) \) be an \( (S; P) \)-ladder with top \( f \) and one rung. Thus, \( \alpha(P; f) \geq 1 \). So \( \sum_{e \in E(P)} t^{\alpha(P; e)} \omega(e) \geq t \cdot \omega(f) = t \cdot \omega(G_1) \geq \omega(G) \).

Now assume that \(|G_1| \geq 3 \). Let \( G^* := G_1 + xy \) and let \( S^* = S \cap E(G_1) \). Define \( \omega^* : E(G^*) \to \mathbb{R}^+ \) as follows: for any \( e \in E(G_1) \), \( \omega^*(e) = \omega(e) \); and if \( xy \notin E(G_1) \) then \( \omega^*(xy) = 0 \). Note that \( G^* \) is 2-connected and \(|G^*| + |S^*| < |G| + |S| \). So by the induction hypothesis, there is an \( x-y \) path \( P \) in \( G^* \) such that

\[ \sum_{e \in E(P)} t^{\alpha^*(P;e)} \omega^*(e) \geq \omega^*(G^*) = \omega(S^*) = \omega(G_1), \]

where \( \alpha^*(P; e) \) denotes the greatest integer \( m \) such that \( G^* \) has an \((S^*; P)-ladder\) with top \( e \) and \( m \) rungs.

If \( S \cap E(G_2) = \emptyset \), then \( \omega(G_1) = \omega(G) \) and \( \alpha(P; e) = \alpha^*(P; e) \) for all \( e \in E(P) \). Hence,

\[ \sum_{e \in E(P)} t^{\alpha(P; e)} \omega(e) = \sum_{e \in E(P)} t^{\alpha^*(P; e)} \omega^*(e) \geq \omega(G_1) = \omega(G). \]

So we may assume that \( S \cap E(G_2) \neq \emptyset \). Then, since \( G \) is 2-connected, \( G_2 \) has an \( x-y \) path \( R \) containing an edge \( f \in S \). For any \((S^*; P)-ladder\) \((A, B; S^*)\) in \( G^* \) between \( x \) and \( y \) with top \( e \) and \( m \) rungs, we can form an \((S; P)-ladder\)
with top $e$ and $m + 1$ rungs by adding to $A$ the component of $R - f$ containing $x$, adding to $B$ the component of $R - f$ containing $y$, and adding $f$ to $S^*$. Hence, $\alpha(P; e) \geq \alpha^*(P; e) + 1$ for all $e \in E(P)$. So

$$\sum_{e \in E(P)} t^{\alpha(P; e)} \omega(e) \geq \sum_{e \in E(P)} t^{\alpha^*(P; e) + 1} \omega^*(e) \geq t \cdot \omega(G_1) \geq \omega(G).$$

![Diagram](image.png)

Figure 1: Two cases in the proof of (3.1).

**Case 2.** $\{x, y\}$ is not a 2-cut of $G$, and no edge in $S$ is incident with both $x$ and $y$.

Then $y$ is contained in a unique block of $G - x$, say $Y$. Let $X$ be a $(Y \cup \{x\})$-bridge of $G$ with $\omega(X)$ maximum, and let $u$ be the unique vertex in $V(X) \cap V(Y)$. (See Figure 1(b).) Since we are in Case 2, $u \neq y$. Because $G$ has no $S$-link of size $t$, there are at most $t - 1$ $(Y \cup \{x\})$-bridges of $G$ that contain edges in $S$. So $t\omega(X) \geq \omega(G) - \omega(Y)$. Let $S_X = S \cap E(X)$ and $S_Y = S \cap E(Y)$. Clearly $|X| + |S_X| < |G| + |S| > |Y| + |S_Y|$. Define $\omega_X : E(X) \to \mathbb{R}^+$ such that, for any $e \in E(X)$, $\omega_X(e) = \omega(e)$, and define $\omega_Y : E(X) \to \mathbb{R}^+$ such that, for any $e \in E(Y)$, $\omega_Y(e) = \omega(e)$. So $\omega_X(X) = \omega(X)$ and $\omega_Y(Y) = \omega(Y)$. In the next two paragraphs, we will find an $x-u$ path $P_x$ in $X$ and a $u-y$ path $P_y$ in $Y$.

If $|X| = 2$, then let $P_x := X$, which is an $x-u$ path. If $|X| \geq 3$, then by the induction hypothesis, $X$ has an $x-u$ path $P_x$ such that $\sum_{e \in E(P_x)} t^{\alpha_X(P_x; e)} \omega_X(e) \geq \omega_X(X) = \omega(X)$, where $\alpha_X(P_x; e)$ is the greatest integer $m$ such that $X$ has an $(S_X; P_x)$-ladder with top $e$ and $m$ rungs.

If $|Y| = 2$, then let $P_y := Y$, which is a $u-y$ path. If $|Y| \geq 3$, then by the induction hypothesis, $Y$ has a $u-y$ path $P_y$ such that $\sum_{e \in E(P_y)} t^{\alpha_Y(P_y; e)} \omega_Y(e) \geq \omega_Y(Y) = \omega(Y)$, where $\alpha_Y(P_y; e)$ is the greatest integer $m$ such that $Y$ has an $(S_Y; P_y)$-ladder with top $e$ and $m$ rungs.

Let $P := P_x \cup P_y$. For any $e \in E(P_y)$, $\alpha(P; e) \geq \alpha_Y(P_y; e)$, and so, $\sum_{e \in E(P_y)} t^{\alpha(P; e)} \omega(e) \geq \sum_{e \in E(P_y)} t^{\alpha_Y(P_y; e)} \omega_Y(e) \geq \omega(Y)$. If $S_X \cup S_Y = S$ then $\sum_{e \in E(P_x)} t^{\alpha(P; e)} \omega(e) \geq \sum_{e \in E(P_x)} t^{\alpha_X(P_x; e)} \omega_X(e) \geq \omega(X) = \omega(G) - \omega(Y)$. If $S_X \cup S_Y \neq S$ then it is easy to see that $\alpha(P; e) \geq \alpha_X(P_x; e) + 1$ for all $e \in E(P_x)$, and so, $\sum_{e \in E(P_x)} t^{\alpha(P; e)} \omega(e) \geq \sum_{e \in E(P_x)} t^{\alpha_X(P_x; e) + 1} \omega_X(e) \geq t \cdot \omega(X) \geq \omega(G) - \omega(Y)$.
Therefore,

\[
\sum_{e \in E(P)} t^{\alpha (P; e)} \omega (e) \geq \sum_{e \in E(P_x)} t^{\alpha_X (P_x; e)} \omega_X (e) + \sum_{e \in E(P_y)} t^{\alpha_Y (P_y; e)} \omega_Y (e) \\
\geq \omega (G) - \omega (Y) + \omega (Y) \\
= \omega (G).
\]

\[\square\]

In Theorem (3.1), if \( G \) does not contain any \( S \)-link of size \( t \) between \( x \) and \( y \), then \( \alpha (P; e) \leq t - 1 \) for all \( e \in E(P) \). Hence, we have following corollary.

**Corollary.** Let \( G \) be a 2-connected graph, let \( \omega : E(G) \mapsto \mathbb{R}^+ \), and let \( S = \{ e : \omega (e) > 0 \} \). Let \( x, y \in V(G) \) be distinct, and assume that \( G \) does not contain any \( S \)-link of size \( t \) between \( x \) and \( y \). Then there is an \( x \)-\( y \) path \( P \) in \( G \) such that

\[
\sum_{e \in E(P)} \omega (e) \geq \frac{\omega (G)}{t^{t-1}}.
\]

Next we use an argument similar to the proof of (3.1) to derive a result on the circumference of 2-connected graphs.

**Proposition.** Let \( t \geq 2 \) be an integer, and let \( G \) be a 2-connected graph with no \( K_{2,t} \)-minors. Then, for any distinct vertices \( x, y \) of \( G \), there is an \( x \)-\( y \) path in \( G \) of length at least \( |G|/t^{t-1} \). In particular, \( c(G) \geq |G|/t^{t-1} \).

**Proof.** We will prove the following stronger result from which (3.3) follows.

(*) Let \( G \) be a 2-connected graph containing no \( K_{2,t} \)-minors, let \( x, y \in V(G) \) be distinct, and let \( \mu := \mu (G; x, y) \) denote the largest integer \( m \) such that \( G \) has a \( K_{2,m} \)-minor rooted at \( \{ x, y \} \). Then \( G \) contains an \( x \)-\( y \) path of length at least \( |G|/t^\mu \).

Since \( t^\mu \geq 2 \), (*) holds when \( |G| \leq 3 \). So assume that \( |G| \geq 4 \) and (*) holds for all graphs with less than \( |G| \) vertices. We consider two cases (see Figure 1 for an illustration).

**Case 1.** \( \{ x, y \} \) is a 2-cut of \( G \).

In this case, there exist subgraphs \( G_1, G_2 \) of \( G \) such that \( V(G_1) \cap V(G_2) = \{ x, y \} \), \( E(G_1) \cap E(G_2) = \emptyset \), and \( |G_1| \geq 3 \leq |G_2| \). Without loss of generality, we may assume that \( |G_1| \geq |G_2| \). Since \( G \) contains no \( K_{2,t} \)-minor, \( t|G_1| \geq |G| \). Since \( G \) is 2-connected and \( G_2 - \{ x, y \} \neq \emptyset \), \( |G_1 + xy| < |G| \) and \( \mu (G; x, y) \geq \mu (G_1 + xy; x, y) + 1 \). Clearly, \( G_1 + xy \) contains no \( K_{2,t} \)-minors. Hence, by applying induction to \( G_1 + xy \), we conclude that \( G_1 + xy \) contains an \( x \)-\( y \) path \( P \) of length at least \( |G_1 + xy|/t^\mu \). Note that we can always choose \( P \) to be a path in \( G \).
Case 2. \( \{x, y\} \) is not a 2-cut of \( G \).

Then \( y \) is contained in a unique block of \( G - x \), say \( Y \). Let \( X \) be a \( (Y \cup \{x\}) \)-bridge of \( G \) with \( |X| \) maximum, and let \( u \) be the unique vertex in \( V(X) \cap V(Y) \).

Since we are in Case 2, \( u \neq y \). Since \( G \) contains no \( K_{2,t} \)-minor, there are at most \( t - 1 \) \((Y \cup \{x\})\)-bridges in \( G \). So \( t|X| \geq |G| - |Y| \). Note that \( |X| < |G| > |Y| \). Next, we find an \( x-u \) path \( P_x \) in \( X \) and a \( u-y \) path \( P_y \) in \( Y \).

If \( |X| = 2 \), then let \( P_x := X \). In this case, it follows from the choice of \( X \) that all \((Y \cup \{x\})\)-bridges of \( G \) are trivial. So \( \ell(P_x) = 1 \) and \( |G| = |Y| + 1 \). Now assume that \( |X| \geq 3 \). Then \( X + xu \) is a 2-connected graph containing no \( K_{2,t} \)-minor. By applying induction to \( X + xu \), we find an \( x-u \) path \( P_x \) in \( X + xu \) of length at least \( \frac{|X + xu|}{|X + xu; x, u|} \). We can always choose \( P_x \) to be a path in \( G \).

If \( |Y| = 2 \), then let \( P_y := Y \). In this case, \( \ell(P_y) \geq \frac{|Y|}{t^\mu} \). Now assume that \( |Y| \geq 3 \). By applying induction to \( Y, u, y \), we find a \( u-y \) path \( P_y \) of length at least \( \frac{|Y|}{t^\mu} \). We can always choose \( P_y \) to be a path in \( G \).

Let \( P := P_x \cup P_y \); then \( P \) is an \( x-y \) path in \( G \) and \( \ell(P) = \ell(P_x) + \ell(P_y) \).

If \( |X| = 2 \), then \( |Y| \geq 3 \) and \( \ell(P) \geq 1 + |Y|/t^\mu \geq |G|/t^\mu \). So assume that \( |X| \geq 3 \). Note that \( \mu(X + xu; x, u) \leq \mu(G; x, y) \), and if \( G \) has at least two non-trivial \((Y \cup \{x\})\)-bridges then \( \mu(X + xu; x, u) + 1 \leq \mu(G; x, y) \). So \( \ell(P_x) \geq \frac{|X + xu|}{\mu(X + xu; x, u)} \geq \frac{|G| - (|Y| - 1)}{t^\mu} \).

Therefore, \( \ell(P) \geq \frac{|G| - |Y| + 1}{t^\mu} + |Y|/t^\mu \geq |G|/t^\mu \).

To prove (2.1), we also need to consider 2-connected graphs which are obtained from 3-connected graphs by contracting connected subgraphs.

(3.4) Lemma. Let \( G \) be a 3-connected graph, let \( H \) be an induced subgraph of \( G \) such that \( U := G - V(H) \) is connected, and let \( H^* := G/U \). Then (1) \( H^* \) is a minor of \( G \), and (2) if \( H \) is 2-connected then \( H^* \) is 3-connected.

Proof. Since \( U \) is connected, \( H^* \) is a minor of \( G \). Now assume that \( H \) is 2-connected. Then \( |H| \geq 3 \), and so, \( H^* \) is 2-connected (since \( G \) is 3-connected). Suppose for a contradiction that \( H^* \) is not 3-connected. Let \( T \) be a 2-cut of \( H^* \), and let \( u \) denote the vertex of \( H^* \) resulting from the contraction of \( U \). If \( u \in T \), then \( T - \{u\} \) is a 1-cut of \( H \), contradicting the assumption that \( H \) is 2-connected. Thus, \( u \notin T \). Hence \( H^* - T \) has a component, say \( D \), not containing \( u \). Then \( D \) is also a component of \( G - T \), contradicting the assumption that \( G \) is 3-connected.

4 Paths avoiding a vertex

Here we prove the following lemma which will serve as the induction step for proving (a) of Theorem (2.1).

(4.1) Lemma. Suppose \( n \geq 5 \) and Theorem (2.1) holds for graphs with at most \( n - 1 \) vertices. Then (a) of Theorem (2.1) holds for graphs with \( n \) vertices.
Proof. Let $t \geq 3$ be an integer, let $G$ be a 3-connected graph with no $K_{3,t}$-minor, and let $|G| = n$. Let $\{x, y, z\} \subseteq V(G)$, and assume that $\{zx, zy\} \subseteq E(G)$. For convenience, we let $b := 8t^t + 1$, $r := \log_b 2$, and $H := G - z$.

**Claim 1.** We may assume that $H$ is not 3-connected.

Suppose $H$ is 3-connected. Since $|H| < n$, (2.1) holds for $H$. In particular, (b) of (2.1) holds for $H$. Therefore, $H$ has an $x$-$y$ path of length at least $|H|^r = (|G| - 1)^r > (\frac{|G| - 1}{\mu})^r$ (because $\mu \geq 1$ and $t \geq 3$). Hence (a) of (2.1) holds for $G$.

**Claim 2.** We may assume that $\{x, y\}$ is not a 2-cut of $H$.

Suppose on the contrary that $\{x, y\}$ is a 2-cut of $H$. Let $H_1, H_2, \ldots, H_s$ be the non-trivial $\{x, y\}$-bridges of $H$. Note that $s \geq 2$. (See Figure 2.) Without loss of generality, we may assume that $|H_1| \geq |H_i|$ for all $1 \leq i \leq s$. Since $G$ is 3-connected, $z$ has a neighbor in $H_i - \{x, y\}$ for each $1 \leq i \leq s$. Since $G$ has no $K_{3,t}$-minor, $s \leq t - 1$. By the choice of $H_1$, $|H_1| \geq |H|/s$.

First, let us assume that $H_1$ is 2-connected. See Figure 2(a). Since $G$ is 3-connected, $U := G - V(H_1)$ is connected. Let $H_1^* := G/U$ and let $u$ denote the vertex of $H_1^*$ resulting from the contraction of $U$. Note that $ux, uy \in E(H_1^*)$. Since $H_1$ is 2-connected, it follows from (3.4) that $H_1^*$ is 3-connected and contains no $K_{3,t}$-minor. Let $\mu_1 := \mu(H_1^*; x, y, u)$. Recall that $\mu = \mu(G; x, y, z)$. Since $z$ has a neighbor in $V(H_i) - \{x, y\}$ for each $2 \leq i \leq s$, $\mu \geq \mu_1 + (s - 1) \geq \mu_1 + 1$. Since $|H_1^*| < n$, (2.1) holds for $H_1^*$. In particular, (a) of (2.1) holds for $H_1^*$. So $H_1 = H_1^* - u$ contains an $x$-$y$ path $P$ such that

$$\ell(P) \geq \left(\frac{|H_1^*| - 1}{\mu_1}\right)^r \geq \left(\frac{|H_1|}{\mu_1}\right)^r \geq \left(\frac{|H|}{\mu}\right)^r = \left(\frac{|G| - 1}{\mu}\right)^r.$$ 

Hence (a) of (2.1) holds for $G$.

![Figure 2: Two cases in the proof of Claim 2.](image-url)
Now assume that $H_1$ is not 2-connected. Since $H$ is 2-connected, the blocks of $H_1$ can be labeled as $F_0, \ldots, F_k$ and the cutvertices of $H_1$ can be labeled as $x_1, \ldots, x_k$ such that (i) for each $0 \leq i \leq k-1$, $V(F_i) \cap V(F_{i+1}) = \{x_{i+1}\}$, (ii) for any $1 \leq i, j \leq k-1$ with $|i-j| \geq 2$, $V(F_i) \cap V(F_j) = \emptyset$, and (iii) $x_0 := x \in V(F_0) - \{x_1\}$ and $x_{k+1} := y \in V(F_k) - \{x_k\}$. Since $G$ is 3-connected, $W_i := G - V(F_i)$ is connected for each $0 \leq i \leq k$. Let $F_i^* := G/W_i$ and let $w_i$ denote the vertex of $F_i^*$ resulting from the contraction of $W_i$. Then $w_i, x_i, w_i, x_{i+1} \in E(F_i^*)$. Let $\mu_i := \mu(F_i^*, x_i, x_{i+1}, w_i)$ if $|F_i| \geq 3$, and let $\mu_i = 1$ if $|F_i| = 2$. Then $\mu \geq \mu_i + (s - 1) \geq \mu_i + 1$. If $|F_i| = 2$ then let $P_i := F_i$ and it is easy to see that $\ell(P_i) = 1 \geq \frac{|F_i|^r}{1}$ (because $t \geq 3$ and $\mu_i = 1$ in this case). If $|F_i| \geq 3$, then it follows from (3.4) that $F_i^*$ is 3-connected and has no $K_{3, \ell}$-minor. Since $|F_i^*| < n$, (2.1) holds for $F_i^*$. In particular, (a) of (2.1) holds for $F_i^*$. So $F_i = F_i^* - w_i$ has an $x_i-x_{i+1}$ path $P_i$ such that $\ell(P_i) \geq \frac{|F_i^*|-1}{|F_i^*|} r = \frac{|F_i^*|-1}{|F_i^*|} r$. Let $P := \bigcup_{i=0}^{k} P_i$; then $P$ is an $x-y$ path in $H_1$. Since $\sum_{0 \leq i \leq k} (|F_i| - 1) = |H_1| - 1$ and by (2.4),

$$\ell(P) \geq \sum_{i=0}^{k} (\frac{|F_i|}{\mu_i})^r \geq \sum_{i=0}^{k} (\frac{|F_i|}{\mu_i})^r \geq (\frac{|H_1| - 1}{\mu_i})^r \geq (\frac{|G| - 1}{\mu_i})^r.$$  

So (a) of (2.1) holds for $G$. This completes the proof of Claim 2.

Let $U := \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$ denote a maximal collection of 2-cuts of $H$ satisfying the following three properties:

(C1) for each $1 \leq i \leq k$, $\{x, y\}$ is contained in a $\{u_i, v_i\}$-bridge $B_i$ of $H$;

(C2) for each $1 \leq i \leq k$, for any 2-cut $T$ of $H$ with $T \neq \{u_i, v_i\}$, and for any $T$-bridge $B$ of $H$ containing $\{x, y\}$, $B \not\subseteq B_i$; and

(C3) $(H - V(B_i)) \cap (H - V(B_j)) = \emptyset$ for $1 \leq i \neq j \leq k$.

Let $X := \{B_i\}_{i=1}^{k} + \{xy, u_i v_i : i = 1, \ldots, k\}$ and let $G_i := (G - (V(B_i) - \{u_i, v_i\})) + \{zu_i, zv_i, u_i v_i\}$.

Claim 3. (1) $X$ is a minor of $G$, (2) either $X$ is a triangle or $X$ is 3-connected, and (3) for each $1 \leq i \leq k$, $G_i$ is a minor of $G$ and $G_i$ is 3-connected.

We may view $X$ as obtained from $G$ by contracting connected subgraphs $G_i - \{z, u_i\}$ to $v_i$ and by contracting $z$ to $x$ (because $xz, yz \in E(G)$). So $X$ is a minor of $G$.

Clearly, $X$ is 2-connected. If $|X| = 3$ then $X$ is a triangle. Now assume $|X| \geq 4$. Suppose that $T$ is a 2-cut of $X$, and let $B_T$ denote a non-trivial $T$-bridge of $X$ not containing $\{x, y\}$. Then $T$ may be viewed as a 2-cut of $H$, and by (C2), $E(B_T) \cap \{u_i v_i : i = 1, \ldots, k\} = \emptyset$. Hence, $U \cup \{T\}$ contradicts the maximality of $U$.

Since $H$ is 2-connected, $H - (V(G_i) - \{u_i, v_i\})$ has disjoint paths $P_u, P_v$ from $\{x, y\}$ to $\{u_i, v_i\}$, with $u_i \in V(P_u)$ and $v_i \in V(P_v)$. So $H - (V(G_i) - \{u_i, v_i\})$ is the disjoint union of connected graphs $P'_u, P'_v$ such that $P' \subseteq P'_u$ and $P_v \subseteq P'_v$. Since $H - (V(G_i) - \{u_i, v_i\})$ is connected, there is an edge of $G$ between $P'_u$ and $P'_v$. Therefore, we may view $G_i$ as obtained from $G$ by contracting $P'_u$ and $P'_v$ to $u_i$ and
Figure 3: The description of $\mathcal{U} = \{\{u_1, v_1\}, \ldots, \{u_k, v_k\}\}$, $X$ and $G_i$.

$v_i$, respectively. So $G_i$ is a minor of $G$. Clearly $G_i$ is 3-connected. This completes the proof of Claim 3.

**Claim 4.** We may assume that $|X| < \frac{|H|}{t}$.

Suppose $|X| \geq \frac{|H|}{t}$. By Claim 3, either $X$ is a triangle or $X$ is 3-connected. If $X$ is a triangle, then let $P_X$ be the $x$-$y$ path in $X$ of length 2. Then $\ell(P_X) = 2 > 3' = |X|^r$. Now assume that $X$ is 3-connected. By (1) of Claim 3, $X$ is a minor of $G$. So $X$ contains no $K_{3,t}$-minor. Since $|X| < n$, (2.1) holds for $X$. In particular, (b) of (2.1) holds for $X$. Recall that $xy \in E(X)$. Hence, $X$ contains an $x$-$y$ path $P_X$ such that $\ell(P_X) \geq |X|^r$.

In any case, $X$ contains an $x$-$y$ path $P_X$ such that

$$\ell(P_X) \geq |X|^r \geq \left(\frac{|H|}{t}\right)^r \geq \left(\frac{|H|}{t \mu}\right)^r \geq \left(\frac{|G| - 1}{t \mu}\right)^r.$$

Clearly, $P_X$ can be extended to the desired $x$-$y$ path $P$ in $H$ by replacing each edge $u_i v_i$ in $E(P_X)$ with a $u_i$-$v_i$ path in $G_i - z$ of length at least 2. So we have Claim 4.

Next, we define $\omega : E(X) \to \mathbb{R}^+$ as follows: $\omega(e) = 0$ if $e \in E(X) - \{u_i v_i : i = 1, \ldots, k\}$, and $\omega(u_i v_i) = |G_i| - 3$ for $i = 1, \ldots, k$. Let $S := \{u_i v_i : i = 1, \ldots, k\} = \{e \in E(X) : \omega(e) > 0\}$. Since $G$ contains no $K_{3,t}$-minor, $X$ contains no $S$-link of size $t$ between $x$ and $y$. By Theorem (3.1), we have the following.

**Claim 5.** $X$ contains an $x$-$y$ path $P_X$ such that

$$\sum_{e \in E(P_X)} t^\alpha(P_X; e) \omega(e) \geq \omega(S) = |H| - |X|,$$

where $\alpha(P_X; e)$ is the greatest integer $m$ such that $X$ has an $(S; P_X)$-ladder with top $e$ and $m$ rungs.
Let $\mu_i := \mu(G_i; u_i, v_i, z)$ for each $1 \leq i \leq k$. From an $(S; P_X)$-ladder, if we replace each rung $u_iv_i$ by $G_i$, we see that $G$ has a $K_{3, p'}$-minor rooted at $\{x, y, z\}$, where $p = \mu + \alpha(P_X; u_i v_i)$. So we have

Claim 6. $t - 1 \geq \mu \geq \mu_i + \alpha(P_X; u_i v_i)$ for each $1 \leq i \leq k$.

By Claims 4 and 5, $\omega(S) > 0$. Hence $E(P_X) \cap S \neq \emptyset$. Without loss of generality, we may assume that $u_1 v_1 \in E(P_X)$ and $\frac{\omega(u_1 v_1)}{t^{\alpha}} \geq \frac{\omega(u_i v_i)}{t^{\alpha}}$ for all $u_i v_i \in E(P_X)$. We distinguish two cases.

Case 1. $|X| \geq \sum_{e \in E(P_X) \setminus \{u_1 v_1\}} t^{\alpha} \omega(e)$.

By Claim 5, we have

\[
\omega(u_1 v_1) \geq \frac{1}{t^{\alpha}} |H| - |X| - \sum_{e \in E(P_X) \setminus \{u_1 v_1\}} t^{\alpha} \omega(e) \\
\geq \frac{1}{t^{\alpha}} |H| - 2|X| \quad \text{(by Case 1)} \\
> \frac{1}{t^{\alpha}} |H| - \frac{2|H|}{t} \quad \text{(by Claim 4)}.
\]

Therefore, since $\alpha(P_X; u_1 v_1) \leq t - 1$, we have

(1) $\omega(u_1 v_1) \geq \frac{t-2}{t^{\alpha} t^{\alpha} t^{\alpha} + 1} |H| \geq \frac{t-2}{t^{\alpha} |H|}$.

By Claim 3, $G_1$ is 3-connected and has no $K_{3, t'}$-minor. Since $|G_1| < n$, (2.1) holds for $G_1$. In particular, (a) of (2.1) holds. Recall that $\{zu_1, zv_1\} \subseteq E(G_1)$. Hence, $G_1 - z$ contains a $u_1 v_1$ path $P_1$ (other than $u_1 v_1$ so that $P_1 \subseteq G$) such that

(2) $\ell(P_1) \geq \frac{(|G_1| - 1)^r}{t^{\alpha} t^{\alpha} t^{\alpha} + 1}$.

By Claim 3, either $X$ is a triangle or $X$ is a 3-connected minor of $G$. If $X$ is a triangle, then $u_1 v_1 \neq xy$ (since $\{x, y\}$ is not a 2-cut of $H$), and so, $X$ has an $x-y$ path $Q_X$ of length 2 and through $u_1 v_1$. Because $t \geq 3$, $\ell(Q_X) = 2 > 3^r + 1 > \frac{|X|}{t^{\alpha}} + 1$. If $X$ is a 3-connected minor of $G$, then (2.1) holds for $X$. In particular, (c) of (2.1) holds for $X$. Recall that $\{xy, u_1 v_1\} \subseteq E(X)$. Hence $X$ has an $x-y$ path $Q_X$ through $u_1 v_1$ such that $\ell(Q_X) \geq \frac{|X|}{t^{\alpha}} + 1$.

In any case, $X$ has an $x-y$ path $Q_X$ through $u_1 v_1$ such that

(3) $\ell(Q_X) \geq \frac{|X|}{t^{\alpha}} + 1$.

Let $P := (Q_X - u_1 v_1) \cup P_1$. Clearly, $P$ is an $x-y$ path in $H$.

If $\frac{\omega(u_1 v_1)}{t^{\alpha}} \leq \frac{|X|}{t^{\alpha}}$, then

\[
\ell(P) \geq \ell(P_1) + (\ell(Q_X) - 1) \\
\geq (\frac{\omega(u_1 v_1)}{t^{\alpha}})^r + \frac{|X|}{t^{\alpha}} \quad \text{(by (2) and (3))} \\
\geq (b - 1) \frac{\omega(u_1 v_1)}{t^{\alpha}} + \frac{|X|}{t^{\alpha}} \quad \text{(by (2.3) and since $\frac{\omega(u_1 v_1)}{t^{\alpha}} \leq \frac{|X|}{t^{\alpha}}$)} \\
\geq \frac{3\omega(u_1 v_1)}{t^{\alpha}} \quad \text{(because $b \geq 4$ and $\frac{\omega(u_1 v_1)}{t^{\alpha}} \leq \frac{|X|}{t^{\alpha}}$)}
\]
Recall that

\[ \ell \geq \left( \frac{3(t-2)|H|}{t \alpha(P_X;u_1v_1) t \mu} \right)^r \quad \text{(by (1))} \]

\[ \geq \left( \frac{|H|}{t \mu} \right)^r \quad \text{(since } t \geq 3) \]

\[ \geq \left( \frac{H}{\mu} \right)^r \quad \text{(by Claim 6)} \]

\[ = \left( \frac{|G| - 1}{\mu} \right)^r. \]

Hence (a) of (2.1) holds for \( G \).

So we may assume \( \frac{\omega(u_1v_1)}{\mu} \geq \frac{|X|}{r} \). Since we are in Case 1, \( t^\alpha(P_X;u_1v_1) \omega(u_1v_1) + |X| \geq \sum_{e \in E(P_X)} t^\alpha(P_X;e) \omega(e) \). By Claim 5, \( t^\alpha(P_X;u_1v_1) \omega(u_1v_1) + |X| \geq |H| - |X| \).

Thus, we have

\[ (4) \quad |X| \geq \frac{1}{2}(|H| - t^\alpha(P_X;u_1v_1) \omega(u_1v_1)). \]

Then

\[ \ell(P) \geq \ell(P_1) + \left( \ell(QX) - 1 \right) \]

\[ \geq \left( \frac{\omega(u_1v_1)}{t \mu_1} \right)^r + \left( \frac{|X|}{t} \right)^r \quad \text{(by (2) and (3))} \]

\[ \geq \left( \frac{\omega(u_1v_1)}{t \mu_1} + (b-1) \left( \frac{|X|}{t} \right)^r \right) \quad \text{(Since } \frac{\omega(u_1v_1)}{\mu_1} \geq \frac{|X|}{r} \text{ and by (2.3))} \]

\[ \geq \left( \frac{\omega(u_1v_1)}{t \mu_1} + 2b \right) + \left( \frac{|X|}{t} \right)^r \quad \text{(by (2))} \]

\[ \geq \left( \frac{|H|}{t \mu_1 + \alpha(P_X;u_1v_1)} \right)^r \quad \text{(by Claim 6 and } b = 8t^{t+1}) \]

\[ \geq \left( \frac{|H|}{t \mu_1 + \alpha(P_X;u_1v_1)} \right)^r \quad \text{(by Claim 6)} \]

\[ = \left( \frac{|G| - 1}{\mu} \right)^r. \]

Hence (a) of (2.1) holds for \( G \).

**Case 2.** \( |X| \leq \sum_{e \in E(P_X) - \{u_1v_1\}} t^\alpha(P_X;e) \omega(e) \).

For each \( 1 \leq i \leq k \), \( |G_i| < n \). By Claim 3, \( G_i \) is 3-connected and contains no \( K_{3,t} \)-minor. So (2.1) holds for \( G_i \). In particular, (a) of (2.1) holds for \( G_i \). Recall that \( \{zu_i, zv_i\} \subset E(G_i) \). Hence \( G_i - z \) contains a \( u_i - v_i \) path \( P_i \) such that

\[ \ell(P_i) \geq \left( \frac{|G_i| - 1}{\mu_i} \right)^r \geq \left( \frac{\omega(u_i v_i)}{\mu_i} \right)^r \]

Let \( P := (P_X - S) \cup ( \bigcup_{u_i v_i \in E(P_X)} P_i ) \). Clearly, \( P \) is an \( x-y \) path in \( H \) and

\[ \ell(P) \geq \sum_{u_i v_i \in E(P_X)} \ell(P_i) \]

\[ \geq \sum_{u_i v_i \in E(P_X)} \left( \frac{\omega(u_i v_i)}{\mu_i} \right)^r. \]

\[ 15 \]
It is easy to see that (b) of (2.1) holds when \( n \) and let 
\[
G \geq (\omega(u_1v_1)_{\mu_1}) + (b-1) \sum_{u_iv_i \in E(P_X) - \{u_1v_1\}} \omega(u_iv_i)^{r}.
\]

The final inequality above follows from the assumption that \( \omega(u_1v_1)^{r} \geq \frac{\omega(u_nv_n)^{r}}{t^{\mu_1}} \) for all \( u_1v_1 \in E(P_X) \) (see before Case 1) and by applying Lemma (2.4). By Claim 6, \( t^{\mu_1+\alpha(P_X;u_1v_1)} \leq t^{\mu} \). Hence,
\[
\ell(P) \geq (\omega(u_1v_1)_{\mu_1}) + \frac{b-1}{t^{\mu}} \sum_{u_iv_i \in E(P_X) - \{u_1v_1\}} t^{\alpha(P_X;u_1v_i)} \omega(u_iv_i)^{r}
\]
\[
\geq (\omega(u_1v_1)_{\mu_1}) + \frac{b-1}{2t^{\mu}} (|X| + \sum_{u_iv_i \in E(P_X) - \{u_1v_1\}} t^{\alpha(P_X;u_1v_i)} \omega(u_iv_i)^{r}) \quad \text{(by Case 2)}
\]
\[
\geq (\omega(u_1v_1)_{\mu_1}) + \frac{1}{t^{\mu_1+\alpha(P_X;u_1v_1)}} (|H| - t^{\alpha(P_X;u_1v_1)} \omega(u_1v_1)^{r}) \quad \text{(by Claim 5)}
\]
\[
\geq (\omega(u_1v_1)_{\mu_1}) + \frac{|H|}{t^{\mu_1+\alpha(P_X;u_1v_1)}}
\]
\[
\geq (\frac{|H|}{t^{\mu}})^{r} \quad \text{(by Claim 6)}.
\]

Hence (a) of (2.1) holds for \( G \).

\[\square\]

5 Paths in 3-connected graphs

We now prove the following result which will serve as the induction step for part (b) in the proof of Theorem (2.1).

(5.1) Lemma. Suppose \( n \geq 5 \) and Theorem (2.1) holds for graphs with at most \( n-1 \) vertices. Then (b) of Theorem (2.1) holds for graphs with \( n \) vertices.

Proof. Let \( t \geq 3 \) be an integer, let \( G \) be a 3-connected graph with no \( K_{3,t} \)-minor, and let \( |G| = n \). Let \( xy \in E(G) \). For convenience, we let \( b := 8t^{t+1} \) and \( r := \log_2. \)

It is easy to see that (b) of (2.1) holds when \( n \leq 8t^{t+1} \). So we may assume that \( n \geq 8t^{t+1} \). Therefore, \( \frac{n}{4(t-1)t^{t-r}} > 1 \).

To find the desired \( x-y \) path in (b) of (2.1), we start from \( x \) and “extend” our path to \( y \). At a certain point, the remaining graph is no longer 3-connected, and we are forced to choose one out of several parts of the graph. While our choice may be “good” at certain stage, it may become undesirable at some later stage. In that case, we need to come back and modify our choice. This is a very complicated process, and the following concept of “magic minor” will help us explain things in a precise and concise way.

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Let \( H_0 \) be an induced subgraph of \( G \) and let \( x_0, y_0 \in V(H_0) \) be distinct such that \( H_0 + x_0 y_0 \) is 2-connected. We say that \( (H_0, x_0, y_0) \) is a magic minor of \((G, x, y)\) if the following conditions hold:

(M1) \( G - (V(H_0) - \{x_0, y_0\}) \) contains vertex disjoint paths \( X_0, Y_0 \) from \( x, y \) to \( x_0, y_0 \), respectively;

(M2) \( U_0 := G - V(H_0) \) is connected and \( H^*_0 \) is 3-connected, where \( H^*_0 := G/U_0 \) if \( H_0 \) is 2-connected and \( H^*_0 := (G/U_0) + x_0 y_0 \) otherwise;

(M3) \( U_0 \) is the disjoint union of \( \Lambda_0 \) and \( \Omega_0 \) such that \( V(X_0) \subseteq \Lambda_0 \cup \{x_0\} \), \( V(Y_0) \subseteq \Omega_0 \cup \{y_0\} \), both \( G[\Lambda_0] \) and \( G[\Omega_0] \) are connected, and \( N(V(H_0) - y_0) \subseteq \Lambda_0 \cup \{y_0\} \);

and

(M4) \(|H_0| \geq n/2\) and, for any \( a \geq \frac{n}{2r} \) with \( a \geq 1 \),

\[
a^r + \ell(X_0) + \ell(Y_0) \geq (a + 4(n - |H_0|))^r.
\]

We say that \((H_0, x_0, y_0)\) is a minor of \((G, x, y)\) if (M1), (M2), and (M3) hold.

Let \( \mathcal{M} \) denote the set of all magic minors of \((G, x, y)\). Then

(1) \( \mathcal{M} \neq \emptyset \) and we may choose \((H_0, x_0, y_0) \in \mathcal{M}\) such that \(|H_0|\) is minimum.

Let \( H_0 := G - x \), let \( y_0 := y \), and let \( x_0 \) be a neighbor of \( x \) other than \( y \). Then \( G - (V(H_0) - \{x_0, y_0\}) \) consists of vertex disjoint paths \( X_0 \) and \( Y_0 \) with \( V(X_0) = \{x, x_0\} \) and \( V(Y_0) = \{y, y_0\} \). So (M1) holds. Clearly, \( U_0 := G - V(H_0) \) consists of only one vertex (namely \( x \)), and so, is connected. Since \( G \) is 3-connected, \( H_0 \) is 2-connected. So \( H^*_0 := G/U_0 = G \) is 3-connected, and (M2) holds. Set \( \Lambda_0 = \{x\} \) and \( \Omega_0 = \emptyset \). It is easy to see that (M3) holds. Obviously, \(|H_0| = n - 1 \geq n/2\). Also, for any \( a \geq \frac{n}{2r} \) with \( a \geq 1 \),

\[
a^r + \ell(X_0) + \ell(Y_0) = a^r + 1 \geq (a + b)^r \geq (a + 4)^r \text{ by Lemma (2.2) (because } a \geq 1 \text{ and } b \geq 4\).
\]

Since \(|H_0| = n - 1 \), \((a + 4(n - |H_0|))^r \geq (a + 4)^r \).

Hence (M4) also holds. Therefore \((H_0, x_0, y_0) \in \mathcal{M}\), and so, we have (1).

Next we recursively define minors of \((G, x, y)\) starting from \((H_0, x_0, y_0)\). Suppose we have already defined a minor \((H_i, x_i, y_i) \) (for some \( i \geq 0 \)) of \((G, x, y, z)\). That is, \((m0) \) \( H_i \) is an induced subgraph of \( G \) and \( H_i + x_i y_i \) is 2-connected, \((m1) \) \( G - (V(H_i) - \{x_i, y_i\}) \) contains vertex disjoint paths from \( x, y \) to \( x_i, y_i \), respectively, \((m2) \) \( U_i := G - V(H_i) \) is connected and \( H^*_i := G/U_i \) if \( H_i \) is 2-connected and \( H^*_i := (G/U_i) + x_i y_i \) otherwise, \((m3) \) \( U_i \) is the disjoint union of \( \Lambda_i \) and \( \Omega_i \), such that both \( G[\Lambda_i] \) and \( G[\Omega_i] \) are connected, and \( N(V(H_i) - y_i) \subseteq \Lambda_i \cup \{y_i\} \), and \((m4) \) \(|H_i| \geq n/2\).

According to the rules (R1), (R2) and (R3) below, we define the following:

\[
(H_{i+1}, x_{i+1}, y_{i+1}), \ U_{i+1}, u_{i+1}, \text{ and } H^*_{i+1}; \ (F_{i+1}, x'_{i+1}, y'_{i+1}), \ W_{i+1}, w_{i+1}, \text{ and } F^*_{i+1};
\]

\[
(H_{i+1,j}, x_{i+1}, y_{i+1,j}), \ U_{i+1,j}, u_{i+1,j}, \text{ and } H^*_{i+1,j}; \ (F_{i+1,j}, x'_{i+1,j}, y'_{i+1,j}), \ W_{i+1,j}, w_{i+1,j}, \text{ and } F^*_{i+1,j};
\]

and \( \Lambda_{i+1} \) and \( \Omega_{i+1} \). See Figure 4 for an illustration.

(R1) Suppose \( \{x_i, y_i\} \) is a 2-cut of \( H_i \). See Figure 4(a). Let \( B_i \) denote an \( \{x_i, y_i\} \)-bridge of \( H_i \) with the maximum number of vertices, and let \( H_{i+1} := G[V(B_i)] \).
Let $H_{i+1,j}$, $j = 1, \ldots, s_{i+1}$, denote the non-trivial $\{x_i, y_i\}$-bridges of $H_i$ different from $B_i$. Let $x_{i+1} = x_i$, $y_{i+1} = y_i$, and $y_{i+1,j} = y_j$ for $1 \leq j \leq s_{i+1}$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$. In this case, $F_{i+1}$ and $F_{i+1,j} = \emptyset$ are not defined.

(R2) Suppose $\{x_i, y_i\}$ is not a 2-cut of $H_i$. See Figure 4(b) and Figure 4(c). Let $B_i$ denote the unique block of $H_i - x_i$ containing $y_i$. Let $B_{i,x}$ be a $(B_i \cup \{x_i\})$-bridge of $H_i$ with the maximum number of vertices, and let $z_i \in V(B_i) \cap V(B_{i,x})$. Let $B_{i,y}$ be a maximum $\{y_i, z_i\}$-bridge of $B_i$. (Possibly $B_{i,y} = B_i$.) If $|B_{i,x}| \geq |B_{i,y}|$, then let $H_{i+1} := G[V(B_{i,x})]$, $x_{i+1} := x_i$, and $y_{i+1} := z_i$; let $H_{i+1,j}$, $j = 1, \ldots, s_{i+1}$, denote the non-trivial $(B_i \cup \{x_i\})$-bridges of $H_i$ different from $B_{i,x}$, let $y_{i+1,j}$ denote the vertex in $V(H_{i+1,j}) \cap V(B_i)$; let $F_{i+1} := G[V(B_{i,y})]$, $x_{i+1} := z_i$, and $y_{i+1,j} := y_i$; let $F_{i+1,j}$, $j = 1, \ldots, t_{i+1}$, denote the non-trivial $\{y_i, z_i\}$-bridges of $B_i$ different from $B_{i,y}$, and let $y_{i+1,j} = y_i$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i \cup V(F_{i+1}) - \{y_{i+1}\}$. See Figure 4(b). If $|B_{i,x}| < |B_{i,y}|$, then let $H_{i+1} := G[V(B_{i,y})]$, $x_{i+1} := z_i$, and $y_{i+1,j} := y_i$; let $H_{i+1,j}$, $j = 1, \ldots, s_{i+1}$, denote the non-trivial $\{z_i, y_i\}$-bridges of $B_i$ different from $B_{i,y}$, and let $y_{i+1,j} = y_i$; let $F_{i+1} := G[V(B_{i,x})]$, $x_{i+1} := x_i$, and $y_{i+1,j} := z_i$; let $F_{i+1,j}$, $j = 1, \ldots, t_{i+1}$, denote the non-trivial $(B_i \cup \{x_i\})$-bridges of $H_i$ different from $B_{i,x}$, and let $y_{i+1,j} = z_i$. Set $\Lambda_{i+1} := \Lambda_i \cup (V(H_i) - V(H_{i+1}))$ and $\Omega_{i+1} := \Omega_i$. See Figure 4(c).

(R3) Let $U_{i+1} := G - V(H_{i+1})$, let $H_{i+1} := G/U_{i+1}$ if $H_{i+1}$ is 2-connected and let $H_{i+1} := (G/U_{i+1}) + x_{i+1}y_{i+1}$ otherwise, and let $u_{i+1}$ denote the vertex of $H_{i+1}$ resulting from the contraction of $U_{i+1}$. Let $U_{i+1,j} := G - V(H_{i+1,j})$, let $H_{i+1,j} := (G/U_{i+1,j}) + x_{i+1}y_{i+1}$ otherwise, and let $u_{i+1,j}$ denote the vertex of $H_{i+1,j}$ resulting from the contraction of $U_{i+1,j}$. Let $W_{i+1} := G - V(F_{i+1})$, let $F_{i+1} := G/W_{i+1}$ if $F_{i+1}$ is 2-connected and let $F_{i+1} := (G/W_{i+1}) + x_{i+1}y_{i+1}$ otherwise, and let $w_{i+1}$ denote the vertex of $F_{i+1}$ resulting from the contraction of $W_{i+1}$.

Next we derive some useful properties (assuming the graphs involved are defined).

(2) $U_{i+1}$, $U_{i+1,j}$ and $W_{i+1}$ are connected subgraphs of $G$, $H_{i+1}$ and $F_{i+1}$ are induced subgraphs of $G$, $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected, $H_{i+1}^*$, $H_{i+1,j}$ and $F_{i+1}^*$ are 3-connected minors of $G$, $\{u_{i+1,i+1} + x_{i+1}y_{i+1}\} \subseteq E(H_{i+1}^*)$, $\{y_{i+1,j}x_{i+1}, y_{i+1,j}u_{i+1,j}\} \subseteq E(H_{i+1,j}^*)$, and $\{w_{i+1}x_{i+1}^*, w_{i+1}y_{i+1}^*\} \subseteq E(F_{i+1}^*)$. Moreover, $U_{i+1}$ is the disjoint union of $\Lambda_{i+1}$ and $\Omega_{i+1}$, both $G[\Lambda_{i+1}]$ and $G[\Omega_{i+1}]$ are connected, and $N(V(H_{i+1}) - y_{i+1}) \subseteq \Lambda_{i+1} \cup \{y_{i+1}\}$.

Since $G$ is 3-connected and $U_i$ is connected (see (m2)), it follows from (R1), (R2), and (R3) that $U_{i+1}$, $U_{i+1,j}$, and $W_{i+1}$ are connected. Since $H_i$ is an induced subgraph of $G$ (by (m0)), it follows from (R1) and (R2) that $H_{i+1}$ and $F_{i+1}$ are induced subgraphs of $G$. Since $H_i + x_iy_i$ is 2-connected (by (m0)) and $|H_i| \geq n/2$ (by (m4)), we see that $|H_{i+1}| \geq 3$ and $H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected. If $H_{i+1}$ is 2-connected then $H_i^*$ is 3-connected by (3.4). If $H_{i+1}$ is not 2-connected then, since
$H_{i+1} + x_{i+1}y_{i+1}$ is 2-connected, $H'_{i+1} = (G/U_i) + x_{i+1}y_{i+1} = (G + x_{i+1}y_{i+1})/U_i$ is 3-connected. Similarly, we can show that $F_{i+1}^*$ (if $F_{i+1} \neq \emptyset$) and $H_{i+1,j}^*$ are 3-connected. The properties enjoyed by $\Lambda_{i+1}$ and $\Omega_{i+1}$ follow instantly from (m3) and the construction of $\Lambda_{i+1}$ and $\Omega_{i+1}$. The rest of (2) follows from (R3).

From (R1) and (R2), we have (3) and (4) below.

(3) $H_i - (V(H_{i+1}) - \{x_{i+1}, y_{i+1}\})$ contains vertex disjoint paths from $x_{i+1}, y_{i+1}$ to $x_i, y_i$, respectively, and $H_i - (V(H_{i+1,j}) - \{x_{i+1,j}, y_{i+1,j}\})$ contains vertex disjoint paths from $x_{i+1,j}, y_{i+1,j}$ to $x_i, y_i$, respectively. Also if $F_{i+1}$ is defined, then $H_i - (V(F_{i+1}) - \{x'_{i+1}, y'_{i+1}\})$ contains vertex disjoint paths from $x'_{i+1}, y'_{i+1}$ to $x_i, y_i$, respectively.

(4) $H_{i+1}$ and $F_{i+1}$ intersect at $z_i \in \{x_{i+1}, y_{i+1}\}$, $|V(H_{i+1,j}) \cap V(F_{i+1})| \leq 1$ and $V(H_{i+1,j}) \cap V(F_{i+1}) \subseteq \{x_{i+1}, y_{i+1}\}$, and $H_{i+1} - \{x_{i+1}, y_{i+1}\}$ and $H_{i+1,j} - \{x_{i+1,j}, y_{i+1,j}\}$; $j = 1, \ldots, s_{i+1}$, are disjoint.

By (m0), $H_i$ is an induced subgraph of $G$. Since $G$ is 3-connected and has no $K_{3,t}$-minor, $s_{i+1} \leq t - 2$ and $t_{i+1} \leq t - 2$. Because $|H_{i+1}| \geq |H_{i+1,j}|$ for $j = 1, \ldots, s_{i+1}$ and $|F_{i+1}| \geq |F_{i+1,j}|$ for $j = 1, \ldots, t_{i+1}$, it follows from (R1) and (R2) that

(5) $(t - 1)|H_{i+1}| + (t - 1)|F_{i+1}| \geq |H_i|.$

Now suppose $\{(H_i, x_i, y_i) : i = 0, \ldots, k\}$ is a maximal sequence constructed recursively starting from $(H_0, x_0, y_0)$ by rules (R1) and (R2), subject to the following two conditions:
(S1) \(|H_k| \geq \frac{n}{2}\), and
(S2) for each \(1 \leq s \leq k\),
\[
\sum_{i=1}^{s} \sum_{j=1}^{s_i} |H_{i,j}| \leq \frac{1}{2}(n - |H_s|).
\]

By (R1), (R2) and (R3), we can construct from \((H_k, x_k, y_k)\) the following:
\((H_{k+1}, x_{k+1}, y_{k+1})\), \((H_{k+1,j}, x_{k+1}, y_{k+1,j})\) for \(j = 1, \ldots, s_{k+1}\), \((F_{k+1}, x_{k+1}', y_{k+1}')\), \(U_{k+1}, W_{k+1}, H_{k+1}', F_{k+1}^s, u_{k+1}, w_{k+1}, \Lambda_{k+1}\), and \(\Omega_{k+1}\).

By (2) and (3) and since \((H_0, x_0, y_0)\) is a minor of \((G, x, y)\), \((H_i, x_i, y_i)\) is a minor of \((G, x, y)\) for all \(1 \leq i \leq k + 1\). Also (2)-(5) hold for \(i = 1, \ldots, k\).

Note that, for each \(1 \leq s \leq k + 1\), the vertices of \(G\) outside \(H_s\) are either outside \(H_0\), or in \(H_{i,j}\) for some \(1 \leq i \leq s\) and \(1 \leq j \leq s_i\), or in \(F_i\) for some \(1 \leq i \leq s\), or in \(F_{i,j}\) for some \(1 \leq i \leq s\) and \(1 \leq j \leq t_i\). Also note that \(n - |H_0|\) is the number of vertices of \(G\) outside \(H_0\), \(\sum_{i=1}^{s} \sum_{j=1}^{s_i} |H_{i,j}|\) is the number of vertices of \(G\) in \(H_{i,j}\) for \(1 \leq i \leq s\) and \(1 \leq j \leq s_i\), \((t - 1) \sum_{1 \leq i \leq s} |F_i|\) is at least the number of vertices of \(G\) in \(F_i\) or \(F_{i,j}\) for \(1 \leq i \leq s\) and \(1 \leq j \leq t_i\), and \(n - |H_s|\) is the number of vertices of \(G\) outside \(H_s\). Hence, we have

\[
(6) \text{ For each } 1 \leq s \leq k + 1, \quad \sum_{i=1}^{s} \sum_{j=1}^{s_i} |H_{i,j}| + (t - 1) \sum_{i=1}^{s} |F_i| + (n - |H_0|) \geq n - |H_s|.
\]

Since \(|H_{k+1}| < |H_0|\) and by (1), \((H_{k+1}, x_{k+1}, y_{k+1})\) is not a magic minor of \((G, x, y)\). By (2) and (3), \((H_{k+1}, x_{k+1}, y_{k+1})\) is a minor of \((G, x, y)\). Thus the maximality of \(k\) implies that either (S1) or (S2) fails with respect to \((H_{k+1}, x_{k+1}, y_{k+1})\); that is,

\[
(7) \text{ For each } 1 \leq k, \quad |H_{k+1}| < \frac{n}{2}, \quad |H_{k+1}| \geq \frac{n}{2} \quad \text{and} \quad \sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| > \frac{1}{2}(n - |H_{k+1}|).
\]

Since \(|H_{k+1}^*| < n > |F_i^*|\) for \(1 \leq i \leq k + 1\), it follows from (2) that (2.1) holds for \(H_{k+1}^*\) and \(F_i^*\). In particular, (a) of Theorem (2.1) holds for \(H_{k+1}^*\) and \(F_i^*\). Recall from (2) that \(\{u_{k+1}x_{k+1}, u_{k+1}y_{k+1}\} \subseteq E(H_{k+1}^*)\), and \(\{w_ix_i', w_iy_i'\} \subseteq E(F_i^*)\). Hence, since \(\mu \geq t - 1\), we have the following.

\[
(8) \text{ For each } 1 \leq i \leq k + 1, \quad H_{k+1} = H_{k+1}^* - u_{k+1} \text{ contains an } x_{k+1}y_{k+1} \text{ path } Q_{k+1} \text{ such that } \ell(Q_{k+1}) \geq (\frac{|H_{k+1}|}{t_{k+1}})^r, \quad \text{and for each } 1 \leq i \leq k + 1, \quad F_i = F_i^* - w_i \text{ contains an } x_{i}'y_{i}' \text{ path } R_i \text{ such that } \ell(R_i) \geq (\frac{|F_i|}{t_i})^r.
\]

Recall that \(|H_i| \geq \frac{n}{2}\) for \(1 \leq i \leq k\). It follows from (5) (with \(i = k\)) and (S1) that

\[
(9) \text{ For } 1 \leq i \leq k, \quad |H_{k+1}| \geq \frac{|H_i|}{2(t - 1)} \geq \frac{n}{4(t - 1)}, \quad \text{and hence, } |H_{k+1}| \geq \frac{n}{4(t - 1)}(t - 1) \geq \frac{|F_i|}{2(t - 1)^t} \text{ for } 1 \leq i \leq k \quad \text{and } \frac{|H_{k+1}|}{t_{k+1}} \geq \frac{n}{4(t - 1)^t} \geq \frac{|H_{i,j}|}{2(t - 1)^t} \text{ for } 1 \leq i \leq k \text{ and } 1 \leq j \leq s_i.
\]

So by (5) (with \(i = k\)) and (S1), we have
(10) $\sum_{i=1}^{k+1} |F_i| \geq \frac{1}{t} (t |H_{k+1}| + (t - 1)|F_{k+1}|) \geq \frac{1}{t} |H_k| \geq \frac{m}{2t}$.

Let $Q_k := Q_{k+1} \cup R_{k+1}$. Then, by (4) and (8), $Q_k$ is an $x_k$-$y_k$ path in $H_k$ and

$$\ell(Q_k) = \ell(Q_{k+1}) + \ell(R_{k+1}) \geq \left( \frac{|H_{k+1}|}{t} \right)^r + \left( \frac{|F_{k+1}|}{t} \right)^r \quad \text{(by (8))}$$

$$\geq \left( \frac{|H_{k+1}|}{t} + \frac{b - 1}{t} |F_{k+1}| \right)^r \quad \text{(since $|H_{k+1}| \geq |F_{k+1}|$ and by (2.2))}$$

$$\geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1)|F_{k+1}| \right)^r \quad \text{(because $b - 1 \geq 4(t - 1)\max_t$).}$$

Similarly, let $Q_{k-1} := Q_{k+1} \cup R_{k+1} \cup R_k = Q_k \cup R_k$. Then, by (4) and (8), $Q_{k-1}$ is an $x_{k-1}$-$y_{k-1}$ path in $H_{k-1}$. By the above inequality, we have

$$\ell(Q_{k-1}) = \ell(Q_k) + \ell(R_k) \geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1)|F_{k+1}| \right)^r + \left( \frac{|F_k|}{t} \right)^r \quad \text{(by (8))}$$

$$\geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1)|F_{k+1}| + \frac{b - 1}{2(t - 1)\max_t} \right)^r \quad \text{(by (9) and by (2.3))}$$

$$\geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1)|F_{k+1}| + 4(t - 1)|F_k| \right)^r \quad \text{(because $b - 1 \geq 8(t - 1)\max_t$).}$$

Continuing in this fashion, let $Q_0 := Q_{k+1} \cup (\bigcup_{i=1}^{k+1} R_i)$. Then by (4) and (8), $Q_0$ is an $x_0$-$y_0$ path in $H_0$ and

$$\ell(Q_0) \geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1) \sum_{i=1}^{k+1} |F_i| \right)^r.$$

Let $P := X_0 \cup Q_0 \cup Y_0$. Recall $X_0$ and $Y_0$ from (M1). Then $P$ is an $x$-$y$ path in $G$, and

$$\ell(P) = \ell(Q_0) + \ell(X_0) + \ell(Y_0) \geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1) \sum_{i=1}^{k+1} |F_i| \right)^r + \ell(X_0) + \ell(Y_0)$$

$$\geq \left( \frac{|H_{k+1}|}{t} + 4(t - 1) \sum_{i=1}^{k+1} |F_i| + 4(n - |H_0|) \right)^r \quad \text{(by (M4) and (10)).}$$

(11) We may assume that $\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| > \frac{1}{2} (n - |H_{k+1}|)$.

For, suppose $\sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| \leq \frac{1}{2} (n - |H_{k+1}|)$. Then by (7), $|H_{k+1}| < n/2$.

By (6),

$$(t - 1) \sum_{i=1}^{k+1} |F_i| + (n - |H_0|) \geq n - |H_{k+1}| - \sum_{i=1}^{k+1} \sum_{j=1}^{s_i} |H_{i,j}| \geq \frac{1}{2} (n - |H_{k+1}|).$$
Hence
\[
\ell(P) \geq \left( \frac{|H_{k+1}|}{t^{r-1}} \right) + 4(t - 1) \sum_{i=1}^{k+1} |F_i| + 4(n - |H_0|)^r \geq (2(n - |H_{k+1}|)^r \geq n^r.
\]
So (b) of (2.1) holds for \( G \). Hence, we have (11).

Statement (11) suggests that we route the desired path through \( H_{i,j}, 1 \leq i \leq k + 1 \) and \( 1 \leq j \leq s_i \). Since \( |H_{i,j}| < n \) and by (2), (2.1) holds for \( H_{i,j}^* \). In particular, (a) of (2.1) holds for \( H_{i,j}^* \). Recall that \( \{y_{i,j}, u_{i,j}, y_j, x_i\} \subseteq E(H_{i,j}^*) \). Hence, \( H_{i,j}^* - y_{i,j} \) has a \( u_{i,j} - x_i \) path \( Q_{i,j}^* \) of length at least \( (\frac{|H_{i,j}^*| - 1}{t^{r-1}})^r \) (because \( k \geq t - 1 \)).

Since \( G[\Lambda_i] \) is connected, \( V(P[x, x_i]) \subseteq \Lambda_i \), and \( N(V(H_i) - y_i) \subseteq \Lambda_i \cup \{y_i\} \), we can extend \( Q_{i,j}^* \) in \( G[\Lambda_i \cup \{x_i\}] \) to obtain a path \( Q_{i,j} \) in \( G \) such that (i) \( Q_{i,j} \) is an \( x_{i,j} - x_i \) path in \( G \) for some \( x_i \in V(P[x, x_i]) \), (ii) \( V(Q_{i,j}) \cap V(P) = \{x_{i,j}, x_i\} \), and (iii) subject to (i) and (ii), \( |P[x_{i,j}, x_i]| \) is maximum. (Note that (ii) holds since \( V(P[y_i, y_i]) \subseteq \Omega_i \) and \( \Lambda_i \cap \Omega_i = \emptyset \).) Then we have (12) and (13) below.

(12) For any \( (i_1, j_1) \neq (i_2, j_2) \), \( E(P[x_{i_1}, x_{i_1, j_1}]) \cap E(P[x_{i_2}, x_{i_2, j_2}]) = \emptyset \) implies that \( V(Q_{i_1, j_1}) \cap V(Q_{i_2, j_2}) \subseteq V(P) \).

(13) \( \ell(Q_{i,j}) \geq |Q_{i,j}^*| \geq (\frac{|H_{i,j}^*|}{t^{r-1}})^r \).

Next, we show that some \( Q_{i,j} \)'s can be used to construct our desired path. For convenience, we define an auxiliary graph \( A \) with vertex set \( V(A) := \{Q_{i,j} : 1 \leq i \leq k + 1 \) and \( 1 \leq j \leq s_i \} \) such that \( Q_{i_1, j_1} \) and \( Q_{i_2, j_2} \) are adjacent in \( A \) if and only if \( E(P[x_{i_1}, x_{i_1, j_1}]) \cap E(P[x_{i_2}, x_{i_2, j_2}]) \neq \emptyset \). By definition,

(14) \( A \) is an interval graph, and therefore is perfect.

Let \( \theta \) be the cardinality of a maximum clique of \( A \).

(15) We claim that \( \theta \leq t - 1 \).

For, let \( C \) be a clique of \( A \) with \( V(C) = \{Q_{i_1, j_1}, Q_{i_2, j_2}, \ldots, Q_{i_t, j_t}\} \). Without loss of generality, we may assume that \( i_1 \leq i_2 \leq \ldots \leq i_t \). Then, \( x_{i_s, j_s} \in V(P[x, x_{i_1}]) \) for all \( s = 2, \ldots, t \). Note that (from (3)) all \( y_{i, j} \)'s are contained in a connected subgraph \( Y \) of \( G - V(P[x, x_{k+1}] \) containing \( P[y_{k+1}, y_0] \). Since \( x_{i_s} \in V(P[x_{i_1}, x_{k+1}]) \) for all \( 2 \leq s \leq t \), we can produce a \( K_{3,t} \)-minor in \( G \) by contracting \( Y \), \( P[x_{i_1}, x_{i_1}] \), and \( P[x_{i_1}, x_{k+1}] \). But this is a contradiction. So \( \theta \leq t - 1 \).

It follows from (14) and (15) that the chromatic number \( \chi(A) \leq \theta \leq t - 1 \). Therefore, there is an independent set \( I \) of \( A \) such that

(16) \( \sum_{Q_{i,j} \in I} |H_{i,j}| \geq \frac{1}{t^r} \sum_{i=1}^{k+1} s_i \sum_{j=1}^{s_i} |H_{i,j}| \).

Hence by (11), we have

(17) \( \sum_{Q_{i,j} \in I} |H_{i,j}| \geq \frac{n - |H_{k+1}|}{2^{(r-1)}} \).

Since \( I \) is an independent set in \( A \) and by (12), two distinct members \( Q_{i_1, j_1} \) and \( Q_{i_2, j_2} \) of \( I \) have one vertex in common if and only if either \( x_{i_1, j_1} = x_{i_2} \) or \( x_{i_2, j_2} = x_{i_1} \). So no three members of \( I \) share a common vertex of \( P \). Thus, \( (\bigcup_{Q_{i,j} \in I} Q_{i,j}) \cup \)
$P[x, x_{k+1}]$ contains an $x$-$x_{k+1}$ path $X_{k+1}$ which contains $\bigcup_{Q_{i,j} \in I} Q_{i,j}$. Let $Y_{k+1} = P[y_{k+1}, y_0] \cup Y_0$.

Note that $X_{k+1}$ and $Y_{k+1}$ are vertex disjoint paths in $G - (V(H_{k+1}) - \{x_{k+1}, y_{k+1}\})$ from $x, y$ to $x_{k+1}, y_{k+1}$, respectively. So (M1) holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. By (2), $H_{k+1}$ is an induced subgraph of $G$, $U_{k+1} := G - V(H_{k+1})$ is connected, and $H_{k+1}^*$ is a 3-connected minor of $G$. So (M2) holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. Recall (2), $U_{k+1}$ is the disjoint union of $\Lambda_{k+1}$ and $\Omega_{k+1}$, both $G[\Lambda_{k+1}]$ and $G[\Omega_{k+1}]$ are connected, and $N(V(H_{k+1}) - y_{k+1}) \subseteq \Lambda_{k+1} \cup \{y_{k+1}\}$. From the construction of $\Lambda_{k+1}$ and $\Omega_{k+1}$, it can be seen that $V(X_{k+1}) \subseteq \Lambda_{k+1} \cup \{x_{k+1}\}$, $V(Y_{k+1}) \subseteq \Omega_{k+1} \cup \{y_{k+1}\}$. Hence (M3) also holds for $(H_{k+1}, x_{k+1}, y_{k+1})$. For any $a \geq \frac{n}{2t^r}$, we have $a \geq 1$ (since $n \geq 8t^{t+1}$) and

$$a^r + \ell(X_{k+1}) + \ell(Y_{k+1})$$

$$\geq a^r + \sum_{Q_{i,j} \in I} \ell(Q_{i,j})$$

$$\geq a^r + \sum_{Q_{i,j} \in I} \left(\frac{|H_{i,j}|}{t^r-1}\right)^r \quad \text{(by (13))}$$

$$\geq (a + \sum_{Q_{i,j} \in I} \frac{|H_{i,j}|}{t^r-1}^r) \quad \text{(by (9) and (2.4))}$$

$$\geq (a + \sum_{Q_{i,j} \in I} |H_{i,j}|^r) \quad \text{(by (16))}$$

$$\geq (a + 4(n - |H_{k+1}|))^r \quad \text{(by (11)).}$$

Since $|H_{k+1}| < |H_0|$, it follows from (1) that $(H_{k+1}, x_{k+1}, y_{k+1})$ is not a magic minor of $(G, x, y)$. Hence (M4) does not hold for $(H_{k+1}, x_{k+1}, y_{k+1})$. Therefore,

$$(18) \ |H_{k+1}| < \frac{n}{2^r}.$$
6 Paths through a given edge

In this section, we first prove a result which serves as the induction step for part (c) in the proof of Theorem (2.1). We then complete the proof of (2.1).

(6.1) Lemma. Suppose $n \geq 5$ and (a) of Theorem (2.1) holds for graphs with at most $n$ vertices. Then (c) of Theorem (2.1) holds for graphs with $n$ vertices.

**Proof.** Let $t \geq 3$ be an integer, let $G$ be a 3-connected graph with no $K_{3,t}$-minor, and let $|G| = n$. Let $xy, f \in E(G)$. For convenience, we let $b := 8t^{t+1}$ and $r := \log_2 2$.

First, assume that $f$ is incident with one of $\{x, y\}$. By symmetry, we may assume that $f$ is incident with $y$. Let $y'$ denote the other end of $f$. Since $f \neq xy$, $y' \neq x$. By applying (4.1) to $G, x, y', y$ (as $G, x, y, z$, respectively), we see that (a) of (2.1) holds for $G, x, y, y'$. That is, $G - y$ contains an $x-y'$ path $P'$ such that $\ell(P') \geq \left(\frac{|G| - 1}{t^r}\right)^r \geq \left(\frac{|G| - 1}{t^r}\right)^r$. We can extend $P'$ to an $x-y$ path $P$ through $f$ in $G$ such that $\ell(P) \geq \left(\frac{|G| - 1}{t^r}\right)^r + 1 \geq \left(\frac{|G|}{t^r}\right)^r + 1$ (since $t \geq 3$). Hence (c) of (2.1) holds for $G$.

Therefore, we may assume that $f$ is incident with neither $x$ nor $y$. Since $G$ is 3-connected, $G$ contains an $x-y$ path $Q$ through $f$. Let $Q_x$ and $Q_y$ be the components of $Q - f$ containing $x$ and $y$, respectively.

Let $X$ denote the minimal union of blocks of $G - V(Q_y)$ containing $Q_x$. Then the blocks of $X$ can be labeled as $X_0, X_1, \ldots, X_p$ and the cutvertices of $X$ can be labeled as $x_1, \ldots, x_p$ such that

- (X1) $V(X_i) \cap V(X_{i+1}) = \{x_i\}$,
- (X2) $V(X_i) \cap V(X_j) = \emptyset$ if $j \geq i + 2$, and
- (X3) $x_0 := x \in V(X_0) - \{x_1\}$, $x_{p+1} \in V(X_p) - \{x_p\}$, and $x_{p+1}$ is incident with $f$.

See Figure 5. Since $G$ is 3-connected, $U_i := G - V(X_i)$ is connected for each $0 \leq i \leq p$. By (3.4), $X_i^* := G/\overline{U_i}$ is either a triangle or a 3-connected minor of $G$. Let $u_i$ denote the vertex of $X_i^*$ resulting from the contraction of $U_i$. Since $xy, f \in E(G)$, $u_i x_i, u_i x_{i+1} \in E(X_i^*)$. Since $|Q_y| \geq 2, |X_i^*| < n$.

Since $G$ is 3-connected, $Y := G - V(X)$ has all its cutvertices contained in $V(Q_y)$. So the blocks of $Y$ can be labeled as $Y_0, Y_1, \ldots, Y_q$ and the cutvertices of $Y$ can be labeled as $y_1, \ldots, y_q$ such that

- (Y1) $V(Y_i) \cap V(Y_{i+1}) = \{y_i\}$,
- (Y2) $V(Y_i) \cap V(Y_j) = \emptyset$ if $j \geq i + 2$, and
- (Y3) $y_0 := y \in V(Y_0) - \{y_1\}$, $y_{q+1} \in V(Y_q) - \{y_q\}$, and $y_{q+1}$ is incident with $f$. 

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Thus $y \in E(G)$, for each $0 \leq i \leq q$, $W_i := G - V(Y_i)$ is connected. By (3.4), $Y_i^* := G/W_i$ is either a triangle or a 3-connected minor of $G$. Let $w_i$ denote the vertex of $Y_i^*$ resulting from the contraction of $W_i$. Since $xy, f \in E(G)$, $w_iy_i, w_iy_{i+1} \in E(Y_i^*)$. Because $|X| \geq 2$, $|Y_i^*| < n$.

If $|X_i| = 2$ then let $P_i := X_i$, and so $\ell(P_i) = 1 \geq (|X_i|/t)^r$ (since $t \geq 3$). If $|X_i| \geq 3$ then, since $|X_i^t| < n$, (2.1) holds for $X_i^t$. In particular, (a) of (2.1) holds for $X_i^t$. Thus $X_i := X_i^t - u_i$ has an $x_i-x_{i+1}$ path $P_i$ such that $\ell(P_i) \geq (|X_i^t| - 1)r = (|X_i|/t)^r$.

If $|Y_i| = 2$ then let $Q_i := Y_i$, and so $\ell(Q_i) = 1 \geq (|Y_i|/t)^r$ (since $t \geq 3$). If $|Y_i| \geq 3$ then, since $|Y_i^t| < n$, (2.1) holds for $Y_i^t$. In particular, (a) of (2.1) holds for $Y_i^t$. Thus $Y_i := Y_i^t - w_i$ has an $y_i-y_{i+1}$ path $Q_i$ such that $\ell(Q_i) \geq (|Y_i^t| - 1)r = (|Y_i|/t)^r$.

Now let $P := ((\bigcup_{i=1}^p P_i) \cup (\bigcup_{i=1}^q Q_i)) + f$. Then $P$ is an $x$-$y$ path in $G$ through $f$ and

$$\ell(P) = \sum_{i=1}^p \ell(P_i) + \sum_{i=1}^q \ell(Q_i) + 1$$

$$\geq \sum_{i=1}^p \left(\frac{|X_i|}{t^{r-1}}\right)^r + \sum_{i=1}^q \left(\frac{|Y_i|}{t^{r-1}}\right)^r + 1$$

$$\geq \left(\frac{|G|}{t^{r-1}}\right)^r + 1$$

$$> \left(\frac{|G|}{t^r}\right)^r + 1.$$ 

Thus (c) of (2.1) holds for $G$. \qed

Proof of Theorem (2.1). For convenience, we let $b := 8t^{t+1}$ and $r := \log_2 2$. We apply induction on $n := |G|$. First assume that $n = 4$. Then $G$ is isomorphic to the complete graph on 4 vertices. It is easy to see that $G - z$ contains an $x$-$y$ path $P$ such that $\ell(P) = 2$. Since $t \geq 3$, $\ell(P) = 2 \geq (3/(t^r)) = (\frac{n-1}{t^r})r$. So (a) holds.

Figure 5: The blocks $X_0, \ldots, X_p$ and $Y_0, \ldots, Y_q$. 
Clearly, $G$ contains an $x$-$y$ path $Q$ such that $\ell(Q) = 3$. Hence $\ell(Q) = 3 \geq 4^r = n^r$, and (b) holds. Finally, $G$ contains an $x$-$y$ path $R$ through $f$ such that $\ell(R) = 3$. Hence $\ell(R) = 3 \geq (\frac{4}{3})^r + 1 = (\frac{n}{3})^r + 1$.

So we may assume that $n \geq 5$ and (2.1) holds for graphs with at most $n - 1$ vertices. By Lemmas (4.1), (5.1) and (6.1), we see that (2.1) also holds for $G$. □

References


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