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Some new nonlinear inequalities and applications to boundary value problems

Wing-Sum Cheung*, [1]

Department of Mathematics, University of Hong Kong, Hong Kong

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Abstract

In this paper, we establish some new nonlinear integral inequalities of the Gronwall–Bellman–Ou-Iang-type in two variables. These on the one hand generalizes and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of differential equations. We illustrate this by applying our new results to certain boundary value problem.

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1. Introduction

The celebrated Gronwall–Bellman inequality [3,9] states that if u and f are non-negative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \leqslant c + \int_a^t f(s)u(s)\,\mathrm{d}s, \quad t \in [a, b],$$

* Tel.: +852 2859 1996; fax: +852 2559 2225.
E-mail address: wscheung@hku.hk.

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for some constant $c \geqslant 0$, then

$$u(t) \leqslant c \exp\left(\int_a^t f(s)\,\mathrm{d}s\right), \quad t \in [a, b]. \tag{1}$$

Since (1) provides an explicit bound to the unknown function and hence furnishes a handy tool to the study of many qualitative as well as quantitative properties of solutions of differential and integral equations, it has become one of the very few classic and most influential results in the theory and applications of inequalities. Because of its fundamental importance, over the years, many generalizations and analogous results of (1) have been established. Such inequalities are in general known as Gronwall–Bellman-type inequalities in the literature (see, e.g., [1,2,4–7,12–14,17,18]). Among various branches of Gronwall–Bellman-type inequalities, a very useful one is originated from Ou-Iang. In his study of the boundedness of certain second order differential equations he established the following result which is generally known as Ou-Iang's inequality.

Theorem A (*Ou-Iang [15]*). *If u and f are non-negative functions defined on* $[0, \infty)$ *such that*

$$u^2(x) \leqslant k^2 + 2\int_0^x f(s)u(s)\,\mathrm{d}s$$

for all $x \in [0, \infty)$, *where* $k \geqslant 0$ *is a constant*, *then*

$$u(x) \leqslant k + \int_0^x f(s)\,\mathrm{d}s$$

for all $x \in [0, \infty)$.

In view of the many important applications of Ou-Iang's inequality (see, e.g., [1,2,10,13,14]), many have devoted much time and effort in its generalizations and in turn, further applications. For example, Dafermos established the following generalization of Ou-Iang's inequality in the process of establishing a connection between stability and the second law of thermodynamics.

Theorem B (*Dafermos [8]*). *If* $u \in \mathscr{L}^\infty[0, r]$ *and* $f \in \mathscr{L}^1[0, r]$ *are non-negative functions satisfying*

$$u^2(x) \leqslant M^2 u^2(0) + 2\int_0^x [Nf(s)u(s) + Ku^2(s)]\,\mathrm{d}s$$

for all $x \in [0, r]$, *where* M, N, K *are non-negative constants*, *then*

$$u(r) \leqslant [Mu(0) + N\int_0^r f(s)\,\mathrm{d}s]\mathrm{e}^{Kr}.$$

More recently, Pachpatte established the following further generalizations of Theorem B.

Theorem C (*Pachpatte [16]*). *If u, f, g are continuous non-negative functions on* $[0, \infty)$ *satisfying*

$$u^2(x) \leqslant k^2 + 2\int_0^x [f(s)u(s) + g(s)u^2(s)]\,\mathrm{d}s$$

for all $x \in [0, \infty)$, *where* $k \geqslant 0$ *is a constant, then*

$$u(x) \leqslant \left(k + \int_0^x f(s)\,\mathrm{d}s\right) \exp\left(\int_0^x g(s)\,\mathrm{d}s\right)$$

for all $x \in [0, \infty)$.

Theorem D (*Pachpatte [16]*). *Suppose u, f, g are continuous non-negative functions on* $[0, \infty)$ *and w a continuous non-decreasing function on* $[0, \infty)$ *with $w(r) > 0$ for $r > 0$. If*

$$u^2(x) \leqslant k^2 + 2\int_0^x (f(s)u(s) + g(s)u(s)w(u(s)))\,\mathrm{d}s$$

for all $x \in [0, \infty)$, *where* $k \geqslant 0$ *is a constant, then*

$$u(x) \leqslant \Omega^{-1}\left[\Omega\left(k + \int_0^x f(s)\,\mathrm{d}s\right) + \int_0^x g(s)\,\mathrm{d}s\right]$$

for all $x \in [0, x_1]$, *where*

$$\Omega(r) := \int_1^r \frac{\mathrm{d}s}{w(s)}, \quad r > 0,$$

Ω^{-1} *is the inverse of* Ω, *and* $x_1 \in [0, \infty)$ *is chosen in such a way that* $\Omega(k + \int_0^x f(s)\,\mathrm{d}s) + \int_0^x g(s)\,\mathrm{d}s \in \mathrm{Dom}(\Omega^{-1})$ *for all* $x \in [0, x_1]$.

On the other hand, Bainov–Simeonov and Lipovan observed the following Gronwall–Bellman-type inequalities which are handy in the study of the global existence of solutions to certain integral equations and functional differential equations.

Theorem E (*Bainov and Simeonov [1]*). *Let* $I = [0, a]$, $J = [0, b]$, *where* $a, b \leqslant \infty$. *Let* $c \geqslant 0$ *be a constant*, $\varphi \in C([0, \infty), [0, \infty))$ *be non-decreasing with* $\varphi(r) > 0$ *for* $r > 0$, *and* $b \in C(I \times J, [0, \infty))$. *If* $u \in C(I \times J, [0, \infty))$ *satisfies*

$$u(x, y) \leqslant c + \int_0^x \int_0^y b(s, t)\varphi(u(s, t))\,\mathrm{d}t\,\mathrm{d}s$$

for all $(x, y) \in I \times J$, *then*

$$u(x, y) \leqslant \Phi^{-1}\left[\Phi(c) + \int_0^x \int_0^y b(s, t)\,\mathrm{d}t\,\mathrm{d}s\right]$$

for all $(x, y) \in [0, x_1] \times [0, y_1]$, *where*

$$\Phi(r) := \int_1^r \frac{\mathrm{d}s}{\varphi(s)}, \quad r > 0,$$

Φ^{-1} is the inverse of Φ, and $(x_1, y_1) \in I \times J$ is chosen in such a way that $\Phi(c) + \int_0^x \int_0^y b(s,t)\,\mathrm{d}t\,\mathrm{d}s \in \mathrm{Dom}(\Phi^{-1})$ for all $(x, y) \in [0, x_1] \times [0, y_1]$.

Theorem F (*Lipovan [11]*). *Suppose u, f are continuous non-negative functions on $[x_0, X)$, w a continuous non-decreasing function on $[0,\infty)$ with $w(r) > 0$ for $r > 0$, and $\alpha : [x_0, X) \to [x_0, X)$ a continuous non-decreasing function with $\alpha(x) \leqslant x$ on $[x_0, X)$. If*

$$u(x) \leqslant k + \int_{\alpha(x_0)}^{\alpha(x)} f(s) w(u(s))\,\mathrm{d}s$$

for all $x \in [x_0, X)$, where $k \geqslant 0$ is a constant, then

$$u(x) \leqslant \Omega^{-1}\left[\Omega(k) + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\,\mathrm{d}s\right]$$

for all $x \in [x_0, x_1)$, where Ω is defined as in Theorem D, and $x_1 \in [x_0, X)$ is chosen in such a way that $\Omega(k) + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\,\mathrm{d}s \in \mathrm{Dom}(\Omega^{-1})$ for all $x \in [x_0, x_1)$.

The purpose of this paper is to establish some new Gronwall–Bellman–Ou-Iang type inequalities with explicit bounds on unknown functions along the line of Theorems A–F. These results on the one hand generalize the inequalities given in Theorems A–F and on the other hand furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of differential and integral equations. We illustrate this by applying our new inequalities to study the boundedness, uniqueness, and continuous dependence properties of the solutions of a boundary value problem.

2. Gronwall–Bellman–Ou-Iang-type inequalities

Throughout this paper, $x_0, y_0 \in \mathbb{R}$ are two fixed numbers. Let $\mathbb{R}_+ := [0,\infty)$, $I := [x_0, X) \subset \mathbb{R}$, $J := [y_0, Y) \subset \mathbb{R}$, and $\Delta := I \times J \subset \mathbb{R}^2$. Note that here we allow X or Y to be $+\infty$. As usual, $C^i(U, V)$ will denote the set of all i-times continuously differentiable functions of U into V, and $C^0(U, V) := C(U, V)$. Partial derivatives of a function $z(x, y)$ are denoted by z_x, z_y, z_{xy}, etc. The identity function will be denoted as *id* and so in particular, id_U is the identity function of U onto itself.

For any $\varphi, \psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ and any constant $\beta > 0$, define

$$\Phi_\beta(r) := \int_1^r \frac{\mathrm{d}s}{\varphi(s^{1/\beta})}, \qquad \Psi_\beta(r) := \int_1^r \frac{\mathrm{d}s}{\psi(s^{1/\beta})}, \quad r > 0,$$
$$\Phi_\beta(0) := \lim_{r\to 0^+} \Phi_\beta(r), \qquad \Psi_\beta(0) := \lim_{r\to 0^+} \Psi_\beta(r).$$

Note that we allow $\Phi_\beta(0)$ and $\Psi_\beta(0)$ to be $-\infty$ here.

Theorem 2.1. *Let $c \geqslant 0$ and $p > 0$ be constants. Let $b \in C(\Delta, \mathbb{R}_+)$, $\gamma \in C^1(I, I)$, $\delta \in C^1(J, J)$, and $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be functions satisfying*

(i) *γ, δ are non-decreasing and $\gamma \leqslant id_I$, $\delta \leqslant id_J$; and*

(ii) *φ is non-decreasing with $\varphi(r) > 0$ for $r > 0$.*

If $u \in C(\Delta, \mathbb{R}_+)$ satisfies

$$u^p(x, y) \leqslant c + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t)\varphi(u(s, t))\,\mathrm{d}t\,\mathrm{d}s \tag{2}$$

for all $(x, y) \in \Delta$, then

$$u(x, y) \leqslant \{\Phi_p^{-1}[\Phi_p(c) + B(x, y)]\}^{1/p} \tag{3}$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$, where

$$B(x, y) := \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t)\,\mathrm{d}t\,\mathrm{d}s,$$

Φ_p^{-1} is the inverse of Φ_p, and $(x_1, y_1) \in \Delta$ is chosen in such a way that $\Phi_p(c) + B(x, y) \in \mathrm{Dom}(\Phi_p^{-1})$ for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$.

Proof. It suffices to consider the case $c > 0$, for the case $c = 0$ can then be arrived at by continuity argument. Denote by $g(x, y)$ the RHS of (2). Then $g > 0$, $u \leqslant g^{1/p}$ on Δ, and g is non-decreasing in each variable. Hence for any $(x, y) \in \Delta$,

$$\begin{aligned} g_x(x, y) &= \gamma'(x) \int_{\delta(y_0)}^{\delta(y)} b(\gamma(x), t)\varphi(u(\gamma(x), t))\,\mathrm{d}t \\ &\leqslant \gamma'(x) \int_{\delta(y_0)}^{\delta(y)} b(\gamma(x), t)\varphi(g(\gamma(x), t))\,\mathrm{d}t \\ &\leqslant \gamma'(x)\varphi(g(\gamma(x), \delta(y))) \int_{\delta(y_0)}^{\delta(y)} b(\gamma(x), t)\,\mathrm{d}t \\ &\leqslant \gamma'(x)\varphi(g(x, y)) \int_{\delta(y_0)}^{\delta(y)} b(\gamma(x), t)\,\mathrm{d}t. \end{aligned}$$

By the definition of Φ_p,

$$\begin{aligned} (\Phi_p \circ g)_x(x, y) &= \left.\frac{\mathrm{d}\Phi_p}{\mathrm{d}r}\right|_{g(x,y)} \cdot g_x(x, y) \\ &\leqslant \frac{1}{\varphi(g(x, y))} \cdot \gamma'(x)\varphi(g(x, y)) \int_{\delta(y_0)}^{\delta(y)} b(\gamma(x), t)\,\mathrm{d}t \\ &= \left(\int_{\delta(y_0)}^{\delta(y)} b(\gamma(x), t)\,\mathrm{d}t\right)\gamma'(x). \end{aligned}$$

Upon integration with respect to x on $[x_0, x]$, this gives

$$\Phi_p(g(x,y)) - \Phi_p(g(x_0,y)) \leqslant \int_{x_0}^{x} \left(\int_{\delta(y_0)}^{\delta(y)} b(\gamma(\xi), t)\,\mathrm{d}t \right) \gamma'(\xi)\,\mathrm{d}\xi$$
$$= \int_{\gamma(x_0)}^{\gamma(x_1)} \int_{\delta(y_0)}^{\delta(y_1)} b(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

or

$$\Phi_p(g(x,y)) \leqslant \Phi_p(c) + B(x,y) \quad \text{for all } (x,y) \in \Delta.$$

As Φ_p^{-1} is increasing on $\mathrm{Dom}(\Phi_p^{-1})$, this yields

$$u(x,y) \leqslant g^{1/p}(x,y) \leqslant \{\Phi_p^{-1}[\Phi_p(c) + B(x,y)]\}^{1/p}$$

for all $(x,y) \in [x_0, x_1] \times [y_0, y_1]$. □

Remarks. (i) In many cases the non-decreasing function φ satisfies $\int_1^\infty (\mathrm{d}s/\varphi(s^{1/p})) = \infty$. Examples of such functions are $\varphi \equiv 1$, $\varphi(s) = s^p$, $\varphi(s) = s^{p/2}$, etc. In such cases, $\Phi_p(\infty) = \infty$ and so we may take $x_1 = X$, $y_1 = Y$. In particular, inequality (3) holds for all $(x,y) \in \Delta$.

(ii) Theorem 2.1 reduces to Theorem 2.1 of Cheung [6] when $p = 1$, and reduces further to Theorem E if we set $\gamma(x) = x$, $\delta(y) = y$.

(iii) Theorem 2.1 is also a generalization of the main result in Lipovan [11, Theorem F] to the case of two independent variables. In fact, if we set $p = 1$ and $\delta(y) = \delta(y_0)$ for all $y \in J$, Theorem 2.1 reduces to Theorem F. If we further require $\gamma(x) = x$ for all $x \in I$, Theorem 2.1 further reduces to the famous Bihari's inequality [4].

Theorem 2.2. *Let $k \geqslant 0$ and $p > 1$ be constants. Let $a, b \in C(\Delta, \mathbb{R}_+)$, $\alpha, \gamma \in C^1(I, I)$, $\beta, \delta \in C^1(J, J)$, and $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be functions satisfying*

(i) *$\alpha, \beta, \gamma, \delta$ are non-decreasing with $\alpha, \gamma \leqslant id_I$ and $\beta, \delta \leqslant id_J$; and*
(ii) *φ is non-decreasing with $\varphi(r) > 0$ for $r > 0$.*

If $u \in C(\Delta, \mathbb{R}_+)$ satisfies

$$u^p(x,y) \leqslant k + \frac{p}{p-1} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t) u(s,t)\,\mathrm{d}t\,\mathrm{d}s + \frac{p}{p-1} \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s,t) u(s,t) \varphi(u(s,t))\,\mathrm{d}t\,\mathrm{d}s \tag{4}$$

for all $(x,y) \in \Delta$, then

$$u(x,y) \leqslant \{\Phi_{p-1}^{-1}[\Phi_{p-1}(k^{1-1/p} + A(x,y)) + B(x,y)]\}^{1/(p-1)} \tag{5}$$

for all $(x,y) \in [x_0, x_1] \times [y_0, y_1]$, where

$$A(x,y) := \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

$$B(x,y) := \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s,t)\,\mathrm{d}t\,\mathrm{d}s,$$

and $(x_1, y_1) \in \Delta$ *is chosen in such a way that* $\Phi_{p-1}(k^{1-1/p} + A(x,y)) + B(x,y) \in \mathrm{Dom}(\Phi_{p-1}^{-1})$ *for all* $(x,y) \in [x_0, x_1] \times [y_0, y_1]$.

Proof. It suffices to consider the case $k > 0$, for the case $k = 0$ can then be arrived at by continuity argument. So assume $k > 0$. Denote by $f(x,y)$ the RHS of (4). Then $f > 0$, $u \leqslant f^{1/p}$ on Δ, and f is non-decreasing in each variable. Hence for any $(x,y) \in \Delta$,

$$\begin{aligned} f_x(x,y) &= \frac{p}{p-1}\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(\alpha(x),t)u(\alpha(x),t)\,\mathrm{d}t \\ &\quad + \frac{p}{p-1}\gamma'(x)\int_{\delta(y_0)}^{\delta(y)} b(\gamma(x),t)u(\gamma(x),t)\varphi(u(\gamma(x),t))\,\mathrm{d}t \\ &\leqslant \frac{p}{p-1}\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} a(\alpha(x),t)f^{1/p}(\alpha(x),t)\,\mathrm{d}t \\ &\quad + \frac{p}{p-1}\gamma'(x)\int_{\delta(y_0)}^{\delta(y)} b(\gamma(x),t)f^{1/p}(\gamma(x),t)\varphi(f^{1/p}(\gamma(x),t))\,\mathrm{d}t \\ &\leqslant \frac{p}{p-1}\alpha'(x)f^{1/p}(\alpha(x),\beta(y))\int_{\beta(y_0)}^{\beta(y)} a(\alpha(x),t)\,\mathrm{d}t \\ &\quad + \frac{p}{p-1}\gamma'(x)f^{1/p}(\gamma(x),\delta(y))\int_{\delta(y_0)}^{\delta(y)} b(\gamma(x),t)\varphi(f^{1/p}(\gamma(x),t))\,\mathrm{d}t \\ &\leqslant \frac{p}{p-1}\alpha'(x)f^{1/p}(x,y)\int_{\beta(y_0)}^{\beta(y)} a(\alpha(x),t)\,\mathrm{d}t \\ &\quad + \frac{p}{p-1}\gamma'(x)f^{1/p}(x,y)\int_{\delta(y_0)}^{\delta(y)} b(\gamma(x),t)\varphi(f^{1/p}(\gamma(x),t))\,\mathrm{d}t. \end{aligned}$$

Since $f^{1/p} > 0$,

$$\begin{aligned} \frac{p-1}{p}\frac{f_x(x,y)}{f^{1/p}(x,y)} &\leqslant \left(\int_{\beta(y_0)}^{\beta(y)} a(\alpha(x),t)\,\mathrm{d}t\right)\alpha'(x) \\ &\quad + \left(\int_{\delta(y_0)}^{\delta(y)} b(\gamma(x),t)\varphi(f^{1/p}(\gamma(x),t))\,\mathrm{d}t\right)\gamma'(x). \end{aligned}$$

Upon integration with respect to x on $[x_0, x]$, this gives

$$\begin{aligned} & f^{1/p}(x,y) - f^{1/p}(x_0,y) \\ & \quad \leqslant \int_{x_0}^{x} \left(\int_{\beta(y_0)}^{\beta(y)} a(\alpha(\xi),t)\,\mathrm{d}t \right) \alpha'(\xi)\,\mathrm{d}\xi \\ & \qquad + \int_{x_0}^{x} \left(\int_{\delta(y_0)}^{\delta(y)} b(\gamma(\xi),t)\varphi(f^{1/p}(\gamma(\xi),t))\,\mathrm{d}t \right) \gamma'(\xi)\,\mathrm{d}\xi \\ & \quad = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s,t)\,\mathrm{d}t\,\mathrm{d}s + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s,t)\varphi(f^{1/p}(s,t))\,\mathrm{d}t\,\mathrm{d}s, \end{aligned}$$

or

$$f^{1-1/p}(x,y) \leqslant k^{1-1/p} + A(x,y) + \int_{\gamma(y_0)}^{\gamma(y)} \int_{\delta(y_0)}^{\delta(y)} b(s,t)\varphi(f^{1/p}(s,t))\,\mathrm{d}t\,\mathrm{d}s$$

for all $(x, y) \in \Delta$. Hence for any fixed $(\overline{x}, \overline{y}) \in [x_0, x_1] \times [y_0, y_1]$, since A is non-decreasing in each variable, we have

$$f^{1-1/p}(x,y) \leqslant k^{1-1/p} + A(\overline{x},\overline{y}) + \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s,t)\varphi(f^{1/p}(s,t))\,\mathrm{d}t\,\mathrm{d}s$$

for all $(x, y) \in [x_0, \overline{x}] \times [y_0, \overline{y}]$. Applying Theorem 2.1 to the function $f^{1-1/p}(x, y)$, we have

$$f^{1-1/p}(x,y) \leqslant \Phi_{1-1/p}^{-1}[\Phi_{1-1/p}(k^{1-1/p} + A(\overline{x},\overline{y})) + B(x,y)]$$

for all $(x, y) \in [x_0, \overline{x}] \times [y_0, \overline{y}]$. In particular, this gives

$$\begin{aligned} u(\overline{x},\overline{y}) & \leqslant f^{1/p}(\overline{x},\overline{y}) = [f^{1-1/p}(\overline{x},\overline{y})]^{1/(p-1)} \\ & \leqslant \{\Phi_{1-1/p}^{-1}[\Phi_{1-1/p}(k^{1-1/p} + A(\overline{x},\overline{y})) + B(\overline{x},\overline{y})]\}^{1/(p-1)}. \end{aligned}$$

Since $(\overline{x}, \overline{y}) \in [x_0, x_1] \times [y_0, y_1]$ is arbitrary, this concludes the proof of the theorem. □

Remarks. (i) Similar to (i) of the previous remark, in many cases $\Phi_{p-1}(\infty) = \infty$ and so in such cases, inequality (5) holds for all $(x, y) \in \Delta$.

(ii) Similar to (ii) of the previous remark, if we set $\beta(y) = \beta(y_0)$ and $\delta(y) = \delta(y_0)$ for all $y \in J$ in Theorem 2.2, we easily arrive at the following one-dimensional result.

Corollary 2.3. *Let $k \geqslant 0$ and $p > 1$ be constants. Let $a, b \in C(I, \mathbb{R}_+)$, $\alpha, \gamma \in C^1(I, I)$, and $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be functions satisfying*

(i) *α, γ are non-decreasing with $\alpha, \gamma \leqslant id_I$; and*
(ii) *φ is non-decreasing with $\varphi(r) > 0$ for $r > 0$.*

If $u \in C(I, \mathbb{R}_+)$ satisfies

$$u^p(x) \leqslant k + \frac{p}{p-1} \int_{\alpha(x_0)}^{\alpha(x)} a(s)u(s)\,\mathrm{d}s + \frac{p}{p-1} \int_{\gamma(x_0)}^{\gamma(x)} b(s)u(s)\varphi(u(s))\,\mathrm{d}s$$

for all $x \in I$, *then*

$$u(x) \leqslant \{\Phi_{p-1}^{-1}[\Phi_{p-1}(k^{1-1/p} + A(x)) + B(x)]\}^{1/(p-1)} \tag{6}$$

for all $x \in [x_0, x_1]$, *where*

$$A(x) := \int_{\alpha(x_0)}^{\alpha(x)} a(s)\,\mathrm{d}s,$$

$$B(x) := \int_{\gamma(x_0)}^{\gamma(x)} b(s)\,\mathrm{d}s,$$

and $x_1 \in I$ *is chosen in such a way that* $\Phi_{p-1}(k^{1-1/p} + A(x)) + B(x) \in \mathrm{Dom}(\Phi_{p-1}^{-1})$ *for all* $x \in [x_0, x_1]$.

Remarks. (i) Same as before, in case $\Phi_{p-1}(\infty) = \infty$, inequality (6) holds for all $x \in I$.

(ii) Corollary 2.3 generalizes Theorem D. In fact, if we impose the conditions $p=2$, $x_0=0$, and $\alpha(x) = \gamma(x) = x$ for all $x \in I$, Corollary 2.3 reduces to Theorem D.

Theorem 2.2 can easily be applied to generate other useful nonlinear integral inequalities in more general situations. For example, we have the following:

Theorem 2.4. *Let* $k \geqslant 0$ *and* $p > q > 0$ *be constants. Let* $a, b \in C(\Delta, \mathbb{R}_+)$, $\alpha, \gamma \in C^1(I, I)$, $\beta, \delta \in C^1(J, J)$, *and* $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be functions satisfying*

(i) $\alpha, \beta, \gamma, \delta$ *are non-decreasing with* $\alpha, \gamma \leqslant id_I$ *and* $\beta, \delta \leqslant id_J$; *and*
(ii) φ *is non-decreasing with* $\varphi(r) > 0$ *for* $r > 0$.

If $u \in C(\Delta, \mathbb{R}_+)$ *satisfies*

$$\begin{aligned} u^p(x, y) \leqslant k &+ \frac{p}{p-q} \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s, t) u^q(s, t)\,\mathrm{d}t\,\mathrm{d}s \\ &+ \frac{p}{p-q} \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t) u^q(s, t) \varphi(u(s, t))\,\mathrm{d}t\,\mathrm{d}s \end{aligned} \tag{7}$$

for all $(x, y) \in \Delta$, *then*

$$u(x, y) \leqslant \{\Phi_{p-q}^{-1}[\Phi_{p-q}(k^{1-q/p} + A(x, y)) + B(x, y)]\}^{1/(p-q)}$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$, *where* $A(x, y)$ *and* $B(x, y)$ *are defined as in Theorem 2.2, and* $(x_1, y_1) \in \Delta$ *is chosen in such a way that* $\Phi_{p-q}(k^{1-q/p} + A(x, y)) + B(x, y) \in \mathrm{Dom}(\Phi_{p-q}^{-1})$ *for all* $(x, y) \in [x_0, x_1] \times [y_0, y_1]$.

Proof. For any $r > 0$, define

$$\psi(r) := \varphi(r^{1/q}). \tag{8}$$

Then clearly ψ satisfies condition (ii) of Theorem 2.2. By (7),

$$u^p(x,y) \leqslant k + \frac{p}{p-q}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)u^q(s,t)\,\mathrm{d}t\,\mathrm{d}s + \frac{p}{p-q}\int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} b(s,t)u^q(s,t)\psi(u^q(s,t))\,\mathrm{d}t\,\mathrm{d}s$$

for all $(x,y) \in \Delta$. Writing $v = u^q$, this becomes

$$v^{p/q}(x,y) \leqslant k + \frac{\frac{p}{q}}{\frac{p}{q}-1}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)v(s,t)\,\mathrm{d}t\,\mathrm{d}s + \frac{\frac{p}{q}}{\frac{p}{q}-1}\int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} b(s,t)v(s,t)\psi(v(s,t))\,\mathrm{d}t\,\mathrm{d}s$$

for all $(x,y) \in \Delta$. Since $p/q > 1$, it follows from Theorem 2.2 that

$$\begin{aligned} v(x,y) &\leqslant \{\Psi_{p/q-1}^{-1}[\Psi_{p/q-1}(k^{1-q/p} + A(x,y)) + B(x,y)]\}^{1/(p/q-1)} \\ &= \{\Psi_{(p-q)/q}^{-1}[\Psi_{(p-q)/q}(k^{(p-q)/p} + A(x,y)) + B(x,y)]\}^{q/(p-q)} \end{aligned}$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, where $(x_1,y_1) \in \Delta$ is chosen such that $\Psi_{(p-q)/q}(k^{(p-q)/p} + A(x,y)) + B(x,y) \in \mathrm{Dom}(\Psi_{(p-q)/q}^{-1})$ for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$. Now it is elementary to check by the definition of ψ in (8) that

$$\Psi_{(p-q)/q}(r) = \Phi_{p-q}(r),$$

thus we have

$$v(x,y) \leqslant \{\Phi_{p-q}^{-1}[\Phi_{p-q}(k^{(p-q)/p} + A(x,y)) + B(x,y)]\}^{q/(p-q)}$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, or

$$u(x,y) = v^{1/q}(x,y) \leqslant \{\Phi_{p-q}^{-1}[\Phi_{p-q}(k^{(p-q)/p} + A(x,y)) + B(x,y)]\}^{1/(p-q)}$$

for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$, where $(x_1,y_1) \in \Delta$ is chosen such that $\Phi_{p-q}(k^{(p-q)/p} + A(x,y)) + B(x,y) \in \mathrm{Dom}(\Phi_{p-q}^{-1})$ for all $(x,y) \in [x_0,x_1]\times[y_0,y_1]$. $\square$

An important special case of Theorem 2.4 is the following:

Corollary 2.5. *Let $k \geqslant 0$ and $p > 1$ be constants. Let $a, b \in C(\Delta, \mathbb{R}_+)$, $\alpha, \gamma \in C^1(I,I)$, $\beta, \delta \in C^1(J,J)$, and $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be functions satisfying*

(i) *$\alpha, \beta, \gamma, \delta$ are non-decreasing with $\alpha, \gamma \leqslant id_I$ and $\beta, \delta \leqslant id_J$; and*
(ii) *φ is non-decreasing with $\varphi(r) > 0$ for $r > 0$.*

If $u \in C(\Delta, \mathbb{R}_+)$ *satisfies*

$$u^p(x, y) \leqslant k + p \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s, t) u^{p-1}(s, t)\, \mathrm{d}t\, \mathrm{d}s + p \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t) u^{p-1}(s, t) \varphi(u(s, t))\, \mathrm{d}t\, \mathrm{d}s$$

for all $(x, y) \in \Delta$, *then*

$$u(x, y) \leqslant \Phi_1^{-1}[\Phi_1(k^{1/p} + A(x, y)) + B(x, y)]$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$, *where* $A(x, y)$ *and* $B(x, y)$ *are defined as in Theorem* 2.2, *and* $(x_1, y_1) \in \Delta$ *is chosen in such a way that* $\Phi_1(k^{1/p} + A(x, y)) + B(x, y) \in \mathrm{Dom}(\Phi_1^{-1})$ *for all* $(x, y) \in [x_0, x_1] \times [y_0, y_1]$.

Proof. The assertion follows immediately from Theorem 2.4 by taking $q = p - 1 > 0$. □

In particular, we have the following useful consequence.

Corollary 2.6. *Let* $k \geqslant 0$ *and* $p > 1$ *be constants. Let* $a, b \in C(\Delta, \mathbb{R}_+)$, $\alpha, \gamma \in C^1(I, I)$, *and* $\beta, \delta \in C^1(J, J)$ *be functions such that* $\alpha, \beta, \gamma, \delta$ *are non-decreasing with* $\alpha, \gamma \in id_I$ *and* $\beta, \delta \leqslant id_J$. *If* $u \in C(\Delta, \mathbb{R}_+)$ *satisfies*

$$u^p(x, y) \leqslant k + p \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s, t) u^{p-1}(s, t)\, \mathrm{d}t\, \mathrm{d}s + p \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t) u^p(s, t)\, \mathrm{d}t\, \mathrm{d}s$$

for all $(x, y) \in \Delta$, *then*

$$u(x, y) \leqslant (k^{1/p} + A(x, y)) \exp B(x, y) \tag{9}$$

for all $(x, y) \in \Delta$, *where* $A(x, y)$ *and* $B(x, y)$ *are as defined in Theorem* 2.2.

Proof. Assume first that $k > 0$. Let $\varphi = id$ on $\mathbb{R}_+$. Then all conditions of Corollary 2.5 are satisfied. Note that in this cases $\Phi_1 = \ln$ and so $\Phi_1^{-1} = \exp$ is defined everywhere on $\mathbb{R}$. By Corollary 2.5, we have

$$u(x, y) \leqslant \exp[\ln(k^{1/p} + A(x, y)) + B(x, y)] = (k^{1/p} + A(x, y)) \exp B(x, y)$$

for all $(x, y) \in \Delta$. Now as this inequality holds for all $k > 0$, by continuity argument it also holds for $k = 0$. □

Remark. Corollary 2.6 generalizes the results of Pachpatte (Theorem C), Dafermos (Theorem B), and Ou-Iang (Theorem A).

Corollary 2.7. *Let $k \geqslant 0$ and $p > 1$ be constants. Let $b \in C(\Delta, \mathbb{R}_+)$, $\gamma \in C^1(I, I)$, and $\delta \in C^1(J, J)$ be functions such that γ, δ are non-decreasing with $\gamma \leqslant id_I$ and $\delta \leqslant id_J$. If $u \in C(\Delta, \mathbb{R}_+)$ satisfies*

$$u^p(x, y) \leqslant k + p \int_{\gamma(x_0)}^{\gamma(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t) u^p(s, t) \, \mathrm{d}t \, \mathrm{d}s$$

for all $(x, y) \in \Delta$, then

$$u(x, y) \leqslant k^{1/p} \exp B(x, y) \tag{10}$$

for all $(x, y) \in \Delta$, where $B(x, y)$ is as defined in Theorem 2.2.

Proof. This follows immediately from Corollary 2.6 by taking $a \equiv 0$. □

Remark. Corollary 2.7 generalizes the Corollary in Lipovan [11] to the case of two independent variables. In fact, if we set $p = 2$ and $\delta(y) = \delta(y_0)$ for all $y \in J$, Corollary 2.7 reduces to the said corollary. In particular, if we further require $\gamma(x) = x$ for all $x \in I$, Corollary 2.7 further reduces to the famous Gronwall–Bellman inequality [3,9].

Remark. It is evident that the results above can easily be generalized to obtain explicit bounds for functions satisfying certain integral inequalities involving more retarded arguments. It is also clear that these results can be extended to functions of more than 2 variables in the obvious way. Details of these are rather algorithmic and so will not be given here.

3. Applications to boundary value problems

We shall in this section illustrate how the results in Section 2 can be applied to study the boundedness, uniqueness, and continuous dependence of the solutions of certain initial boundary value problems for hyperbolic partial differential equations. Consider the following:

Boundary value problem (*BVP*):

$$z^{p-1} z_{xy} + (p-1) z^{p-2} z_x z_y = F(x, y, z(\rho(x), \lambda(y)))$$

satisfying

$$z(x, y_0) = f(x), \quad z(x_0, y) = g(y), \quad f(x_0) = g(y_0) = 0,$$

where $p \geqslant 2$, $F \in C(\Delta \times \mathbb{R}, \mathbb{R})$, $f \in C^1(I, \mathbb{R})$, $g \in C^1(J, \mathbb{R})$, $\rho \in C^1(I, I)$, $\lambda \in C^1(J, J)$, $0 < \rho', \lambda' \leqslant 1$, $\rho(x_0) = x_0$, $\lambda(y_0) = y_0$.

Remark. Setting $\rho(x) = x - h(x)$ and $\lambda(y) = y - k(y)$, (BVP) becomes an initial boundary value problem with delay.

Our first result deals with the boundedness of solutions.

Theorem 3.1. *Consider* (*BVP*). *If*

$$|F(x, y, v)| \leqslant b(x, y)|v|^p \tag{11}$$

and

$$|f^p(x) + g^q(y)| \leqslant k, \tag{12}$$

where $b \in C(\Delta, \mathbb{R}_+)$ *and* $k \geqslant 0$ *is a constant*, *then all solutions* $z(x, y)$ *of* (*BVP*) *satisfy*

$$|z(x, y)| \leqslant k^{1/p} \exp \overline{B}(x, y), \quad (x, y) \in \Delta,$$

where

$$\overline{B}(x, y) := MN \int_{\rho(x_0)}^{\rho(x)} \int_{\lambda(y_0)}^{\lambda(y)} \overline{b}(\sigma, \tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma,$$

$$\overline{b}(\sigma, \tau) := b(\rho^{-1}(\sigma), \lambda^{-1}(\tau)),$$

$$M := \max \left\{ \frac{1}{\rho'(x)} : x \in I \right\},$$

$$N := \max \left\{ \frac{1}{\lambda'(y)} : y \in J \right\}.$$

In particular, *if* $\overline{B}$ *is bounded on* Δ, *then every solution* z *of* (*BVP*) *is bounded on* Δ.

Proof. First observe that $z = z(x, y)$ solves (BVP) if and only if it satisfies the integral equation

$$z^p(x, y) = f^p(x) + g^q(y) + p \int_{x_0}^{x} \int_{y_0}^{y} F(s, t, z(\rho(s), \lambda(t)))\, \mathrm{d}t\, \mathrm{d}s. \tag{13}$$

Hence by (11) and (12),

$$|z(x, y)|^p \leqslant k + p \int_{x_0}^{x} \int_{y_0}^{y} b(s, t)|z^p(\rho(s), \lambda(t))|\, \mathrm{d}t\, \mathrm{d}s.$$

By a change of variables $\sigma = \rho(s)$, $\tau = \lambda(t)$, we have

$$\begin{aligned} |z(x, y)|^p &\leqslant k + p \int_{\rho(x_0)}^{\rho(x)} \int_{\lambda(y_0)}^{\lambda(y)} b(\rho^{-1}(\sigma), \lambda^{-1}(\tau))|z^p(\sigma, \tau)|(\rho^{-1})'(\sigma)(\lambda^{-1})'(\tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma \\ &\leqslant k + pMN \int_{\rho(x_0)}^{\rho(x)} \int_{\lambda(y_0)}^{\lambda(y)} \overline{b}(\sigma, \tau)|z^p(\sigma, \tau)|\, \mathrm{d}\tau\, \mathrm{d}\sigma. \end{aligned}$$

An application of Corollary 2.7 to the function $|z(x, y)|$ now gives the assertion immediately. □

The next result is about uniqueness.

Theorem 3.2. *Consider* (*BVP*). *If*

$$|F(x, y, v_1) - F(x, y, v_2)| \leqslant b(x, y)|v_1^p - v_2^p|,$$

where $b \in C(\Delta, \mathbb{R}_+)$, *then* (*BVP*) *has at most one solution on* Δ.

Proof. Let $z(x, y)$ and $\overline{z}(x, y)$ be two solutions of (BVP). By (13), we have

$$\begin{aligned} &z^p(x, y) - \overline{z}^p(x, y) \\ &\quad = p \int_{x_0}^{x} \int_{y_0}^{y} [F(s, t, z(\rho(s), \lambda(t))) - F(s, t, \overline{z}(\rho(s), \lambda(t)))] \, \mathrm{d}t \, \mathrm{d}s. \end{aligned}$$

By assumption, we then have

$$|z^p(x, y) - \overline{z}^p(x, y)| \leqslant p \int_{x_0}^{x} \int_{y_0}^{y} b(s, t)|z^p(\rho(s), \lambda(t)) - \overline{z}^p(\rho(s), \lambda(t))| \, \mathrm{d}t \, \mathrm{d}s.$$

By a change of variables $\sigma = \rho(s)$, $\tau = \lambda(t)$, this yields

$$\begin{aligned} |z^p(x, y) - \overline{z}^p(x, y)| &\leqslant pMN \int_{\rho(x_0)}^{\rho(x)} \int_{\lambda(y_0)}^{\lambda(y)} \overline{b}(\sigma, \tau)|z^p(\sigma, \tau) - \overline{z}^p(\sigma, \tau)| \, \mathrm{d}\tau \, \mathrm{d}\sigma \\ &= pMN \int_{\rho(x_0)}^{\rho(x)} \int_{\lambda(y_0)}^{\lambda(y)} \overline{b}(\sigma, \tau)[|z^p(\sigma, \tau) - \overline{z}^p(\sigma, \tau)|^{1/p}]^p \, \mathrm{d}\tau \, \mathrm{d}\sigma. \end{aligned}$$

By applying Corollary 2.7 to the function $|z^p(x, y) - \overline{z}^p(x, y)|^{1/p}$, we conclude that

$$|z^p(x, y) - \overline{z}^p(x, y)|^{1/p} \leqslant 0 \quad \text{for all } (x, y) \in \Delta$$

and hence $z = \overline{z}$ on Δ. □

Finally, we shall investigate the continuous dependence of the solutions of (BVP) on the function F and the boundary data. For this we consider a variation of (BVP):

($\overline{BVP}$):

$$z^{p-1} z_{xy} + (p-1) z^{p-2} z_x z_y = \overline{F}(x, y, z(\rho(x), \lambda(y)))$$

satisfying

$$z(x, y_0) = \overline{f}(x), \quad z(x_0, y) = \overline{g}(y), \quad \overline{f}(x_0) = \overline{g}(y_0) = 0,$$

where $p \geqslant 2$, $\overline{F} \in C(\Delta \times \mathbb{R}, \mathbb{R})$, $\overline{f} \in C^1(I, \mathbb{R})$, $\overline{g} \in C^1(J, \mathbb{R})$, $\rho \in C^1(I, I)$, $\lambda \in C^1(J, J)$, $0 < \rho', \lambda' \leqslant 1$, $\rho(x_0) = x_0$, $\lambda(y_0) = y_0$.

Theorem 3.3. *Consider* (*BVP*) *and* ($\overline{BVP}$). *If*

(i) $|F(x, y, v_1) - F(x, y, v_2)| \leqslant b(x, y)|v_1^p - v_2^p|$ *for some* $b \in C(\Delta, \mathbb{R}_+)$;
(ii) $|(f(x) - \overline{f}(x)) + (g(y) - \overline{g}(y))| \leqslant \frac{\varepsilon}{2}$; *and*

(iii) *for all solutions* $\bar{z}(x, y)$ *of* $(\overline{BVP})$,

$$\int_{x_0}^{x}\int_{y_0}^{y} |F(s, t, \bar{z}(\rho(s), \lambda(t))) - \overline{F}(s, t, \bar{z}(\rho(s), \lambda(t)))|\,\mathrm{d}t\,\mathrm{d}s \leqslant \frac{\varepsilon}{2},$$

then

$$|z^p(x, y) - \bar{z}^p(x, y)| \leqslant \varepsilon \exp(p\overline{B}(x, y)),$$

where $\overline{B}(x, y)$ *is as defined in Theorem* 3.1. *Hence* $z^p(x, y)$ *depends continuously on* F, f, *and* g. *In particular*, *if* $z(x, y)$ *does not change sign*, *it depends continuously on* F, f *and* g.

Proof. Let $z = z(x, y)$ and $\bar{z} = \bar{z}(x, y)$ be solutions of (BVP) and $(\overline{\mathrm{BVP}})$, respectively. Then z satisfies (13) and $\bar{z}$ satisfies

$$\bar{z}^p(x, y) = \bar{f}^p(x) + \bar{g}^q(y) + p\int_{x_0}^{x}\int_{y_0}^{y} \overline{F}(s, t, \bar{z}(\rho(s), \lambda(t)))\,\mathrm{d}t\,\mathrm{d}s.$$

Hence

$$\begin{aligned} z^p(x, y) - \bar{z}^p(x, y) = {} & [(f^p(x) - \bar{f}^p(x)) + (g^q(y) - \bar{g}^q(y))] \\ & + p\int_{x_0}^{x}\int_{y_0}^{y} [F(s, t, z(\rho(s), \lambda(t))) \\ & - \overline{F}(s, t, \bar{z}(\rho(s), \lambda(t)))]\,\mathrm{d}t\,\mathrm{d}s \end{aligned}$$

and so by assumption (ii),

$$\begin{aligned} & |z^p(x, y) - \bar{z}^p(x, y)| \\ & \quad \leqslant \frac{\varepsilon}{2} + p\int_{x_0}^{x}\int_{y_0}^{y} |F(s, t, z(\rho(s), \lambda(t))) - \overline{F}(s, t, \bar{z}(\rho(s), \lambda(t)))|\,\mathrm{d}t\,\mathrm{d}s \\ & \quad \leqslant \frac{\varepsilon}{2} + p\int_{x_0}^{x}\int_{y_0}^{y} |F(s, t, z(\rho(s), \lambda(t))) - F(s, t, \bar{z}(\rho(s), \lambda(t)))|\,\mathrm{d}t\,\mathrm{d}s \\ & \qquad + p\int_{x_0}^{x}\int_{y_0}^{y} |F(s, t, \bar{z}(\rho(s), \lambda(t))) - \overline{F}(s, t, \bar{z}(\rho(s), \lambda(t)))|\,\mathrm{d}t\,\mathrm{d}s. \end{aligned}$$

Now by assumption (i) and by a change of variables $\sigma = \rho(s)$, $\tau = \lambda(t)$,

$$\begin{aligned} & p\int_{x_0}^{x}\int_{y_0}^{y} |F(s, t, z(\rho(s), \lambda(t))) - F(s, t, \bar{z}(\rho(s), \lambda(t)))|\,\mathrm{d}t\,\mathrm{d}s \\ & \quad \leqslant p\int_{x_0}^{x}\int_{y_0}^{y} b(s, t)|z^p(\rho(s), \lambda(t)) - \bar{z}^p(\rho(s), \lambda(t))|\,\mathrm{d}t\,\mathrm{d}s \\ & \quad = p\int_{\rho(x_0)}^{\rho(x)}\int_{\lambda(y_0)}^{\lambda(y)} b(\rho^{-1}(\sigma), \lambda^{-1}(\tau))|z^p(\sigma, \tau) - \bar{z}^p(\sigma, \tau)|(\rho^{-1})'(\sigma)(\lambda^{-1})'(\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma \\ & \quad \leqslant pMN\int_{\rho(x_0)}^{\rho(x)}\int_{\lambda(y_0)}^{\lambda(y)} \bar{b}(\sigma, \tau)|z^p(\sigma, \tau) - \bar{z}^2(\sigma, \tau)|\,\mathrm{d}\tau\,\mathrm{d}\sigma, \end{aligned}$$

while by assumption (iii),

$$p \int_{x_0}^{x} \int_{y_0}^{y} |F(s, t, \bar{z}(\rho(s), \lambda(t))) - \overline{F}(s, t, \bar{z}(\rho(s), \lambda(t)))| \, dt \, ds \leqslant \frac{\varepsilon}{2},$$

thus

$$|z^p(x, y) - \bar{z}^p(x, y)| \leqslant \varepsilon + pMN \int_{\rho(x_0)}^{\rho(x)} \int_{\lambda(y_0)}^{\lambda(y)} \bar{b}(\sigma, \tau) |z^p(\sigma, \tau) - \bar{z}^p(\sigma, \tau)| \, d\tau \, d\sigma.$$

By applying Corollary 2.7 to the function $|z^p(\sigma, \tau) - \bar{z}^p(\sigma, \tau)|^{1/p}$, we have

$$|z^p(x, y) - \bar{z}^p(x, y)|^{1/p} \leqslant \varepsilon^{1/p} \exp \overline{B}(x, y)$$

for all $(x, y) \in \Delta$, or

$$|z^p(x, y) - \bar{z}^p(x, y)| \leqslant \varepsilon \exp(p\overline{B}(x, y)).$$

Now when restricted to any compact set, $\overline{B}(x, y)$ is bounded and so

$$|z^p(x, y) - \bar{z}^p(x, y)| \leqslant \varepsilon \cdot K$$

for some $K > 0$ for all (x, y) lying in the compact set. Hence z^p depends continuously on F, f and g. □

Remark. The initial boundary value problem (BVP) considered in this section is clearly not the only problem for which the boundedness, uniqueness, and continuous dependence of its solutions can be studied by using the main results in Section 2. For example, one can arrive at similar results (much more complicated computations are involved though) for the following variation of our (BVP):

$$z^{p-1} z_{xy} + (p-1) z^{p-2} z_x z_y = F(x, y, z(\rho(x), \lambda(y)), z(\mu(x), \nu(y)) \cdot w(\mu(x), \nu(y)))$$

satisfying

$$z(x, y_0) = f(x), \quad z(x_0, y) = g(y), \quad f(x_0) = g(y_0) = 0,$$

where $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ is non-decreasing with $w(r) > 0$ for $r > 0$. Details of the computations will be omitted here.

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