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# Discrete non-linear inequalities and applications to boundary value problems

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**Abstract**

In this paper, we generalize some existing discrete Gronwall–Bellman–Ou-Iang-type inequalities to more general situations. These are in turn applied to study the boundedness, uniqueness, and continuous dependence of solutions of certain discrete boundary value problem for difference equations.


*Keywords:* Discrete Gronwall–Bellman–Ou-Iang-type inequalities; Boundary value problems

## 1. Introduction

In studying the boundedness behavior of the solutions of certain second order differential equations, Ou-Iang established the following Gronwall–Bellman-type integral inequality which is now known as Ou-Iang's inequality in the literature.

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**Theorem A.** (Ou-Iang [13]) *If $u$ and $f$ are non-negative functions on $[0,\infty)$ satisfying*

$$u^2(x) \leqslant k^2 + 2\int_0^x f(s)u(s)\,ds$$

*for all $x \in [0,\infty)$, where $k \geqslant 0$ is a constant, then*

$$u(x) \leqslant k + \int_0^x f(s)\,ds$$

*for all $x \in [0,\infty)$.*

Unlike many other types of integral inequalities, Ou-Iang-type inequalities or more generally, Gronwall–Bellman–Ou-Iang-type inequalities provide explicit bounds on the unknown function, and this special feature makes such inequalities especially important in many practical situations. In fact, over the years, such inequalities and their generalizations to various settings have proven to be very effective in the study of many qualitative as well as quantitative properties of solutions of differential equations. These include, among others, the global existence, boundedness, uniqueness, stability, and continuous dependence on initial data (see, for example, [1–3,5,6,8,10–12,14–17]). For example, in the process of establishing a connection between stability and the 2nd law of thermodynamics, Dafermos established the following result.

**Theorem B.** (Dafermos [7]) *If $u \in \mathcal{L}^\infty[0,r]$ and $f \in \mathcal{L}^1[0,r]$ are non-negative functions satisfying*

$$u^2(x) \leqslant M^2u^2(0) + 2\int_0^x \left[Nf(s)u(s) + Ku^2(s)\right] ds$$

*for all $x \in [0,r]$, where $M, N, K$ are non-negative constants, then*

$$u(r) \leqslant \left[Mu(0) + N\int_0^r f(s)\,ds\right] e^{Kr}.$$

More recently, Pachpatte established the following more general inequality:

**Theorem C.** (Pachpatte [16]) *Suppose $u, f, g$ are continuous non-negative functions on $[0,\infty)$ and $w$ a continuous non-decreasing function on $[0,\infty)$ with $w(r) > 0$ for $r > 0$. If*

$$u^2(x) \leqslant k^2 + 2\int_0^x \big(f(s)u(s) + g(s)u(s)w\big(u(s)\big)\big)\,ds$$

*for all $x \in [0,\infty)$, where $k \geqslant 0$ is a constant, then*

$$u(x) \leqslant \Omega^{-1}\left[\Omega\left(k + \int_0^x f(s)\,ds\right) + \int_0^x g(s)\,ds\right]$$

*for all* $x \in [0, x_1]$, *where*

$$\Omega(r) := \int_1^r \frac{ds}{w(s)}, \quad r > 0,$$

$\Omega^{-1}$ *is the inverse of* $\Omega$, *and* $x_1 \in [0, \infty)$ *is chosen in such a way that* $\Omega(k + \int_0^x f(s)\,ds) + \int_0^x g(s)\,ds \in \mathrm{Dom}(\Omega^{-1})$ *for all* $x \in [0, x_1]$.

On the other hand, Lipovan observed the following Gronwall–Bellman–Ou-Iang-type inequality which is a handy tool in the study of the global existence of solutions to certain integral equations and functional differential equations.

**Theorem D.** (Lipovan [9]) *Suppose* $u, f$ *are continuous non-negative functions on* $[x_0, X)$, $w$ *a continuous non-decreasing function on* $[0, \infty)$ *with* $w(r) > 0$ *for* $r > 0$, *and* $\alpha : [x_0, X) \to [x_0, X)$ *a continuous non-decreasing function with* $\alpha(x) \leqslant x$ *on* $[x_0, X)$. *If*

$$u(x) \leqslant k + \int_{\alpha(x_0)}^{\alpha(x)} f(s) w\big(u(s)\big)\,ds$$

*for all* $x \in [x_0, X)$, *where* $k \geqslant 0$ *is a constant, then*

$$u(x) \leqslant \Omega^{-1}\left[\Omega(k) + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\,ds\right]$$

*for all* $x \in [x_0, x_1)$, *where* $\Omega$ *is defined as in Theorem C, and* $x_1 \in [x_0, X)$ *is chosen in such a way that* $\Omega(k) + \int_{\alpha(x_0)}^{\alpha(x)} f(s)\,ds \in \mathrm{Dom}(\Omega^{-1})$ *for all* $x \in [x_0, x_1)$.

Very recently, in the process of studying the boundedness, uniqueness, and continuous dependence of the solutions of some boundary value problem, Cheung [5] established the following

**Theorem E.** (Cheung [5]) *Let* $I := [x_0, X) \subset \mathbb{R}$, $J := [y_0, Y) \subset \mathbb{R}$, *and* $\Delta := I \times J \subset \mathbb{R}^2$. *Suppose* $u \in C(\Delta, \mathbb{R}_+)$. *If* $k \geqslant 0$ *is a constant and* $a, b \in C(\Delta, \mathbb{R}_+)$, $\alpha, \gamma \in C^1(I, I)$, $\beta, \delta \in C^1(J, J)$, *and* $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are functions satisfying*

(i) $\alpha, \beta, \gamma, \delta$ *are non-decreasing with* $\alpha, \gamma \leqslant \mathrm{id}_I$ *and* $\beta, \delta \leqslant \mathrm{id}_J$;
(ii) $w$ *is non-decreasing with* $w(r) > 0$ *for* $r > 0$; *and*
(iii) *for any* $(x, y) \in \Delta$,

$$u^2(x, y) \leqslant k^2 + 2\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(s, t) u(s, t)\,dt\,ds$$

$$+2\int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} b(s,t)u(s,t)w\big(u(s,t)\big)\,dt\,ds,$$

*then*

$$u(x,y)\leqslant \Phi^{-1}\big[\Phi\big(k+A(x,y)\big)+B(x,y)\big]$$

*for all* $(x,y)\in[x_0,x_1]\times[y_0,y_1]$, *where*

$$A(x,y):=\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)\,dt\,ds,$$

$$B(x,y):=\int_{\gamma(x_0)}^{\gamma(x)}\int_{\delta(y_0)}^{\delta(y)} b(s,t)\,dt\,ds,$$

$$\Phi(r):=\int_1^r \frac{ds}{w(s)},\quad r>0,$$

$$\Phi(0):=\lim_{r\to 0^+}\Omega(r),$$

*and* $(x_1,y_1)\in\Delta$ *is chosen in such a way that* $\Phi(k+A(x,y))+B(x,y)\in \mathrm{Dom}(\Phi^{-1})$ *for all* $(x,y)\in[x_0,x_1]\times[y_0,y_1]$.

Among various generalizations of Ou-Iang's inequality, discretization is also an interesting direction. The point is, similar to the noteworthy contributions of the continuous versions of the inequality to the study of differential equations, one naturally expects that discrete versions of the inequality should also play an important role in the study of difference equations. In this respect, fewer results have been established. Recent results in this direction include the works of Pachpatte [16], Pang and Agarwal [18], and the following very recent result of Cheung [4].

**Theorem F.** (Cheung [4]) *Suppose* $u:\Omega\to\mathbb{R}_+$ *is a function on a 2-dimensional lattice* $\Omega$, $k\geqslant 0$ *is a constant,* $a,b:\Omega\to\mathbb{R}_+$, *and* $w\in C(\mathbb{R}_+,\mathbb{R}_+)$ *are functions satisfying*

(i) $w$ *is non-decreasing with* $w(r)>0$ *for* $r>0$; *and*
(ii) *for any* $(m,n)\in\Omega$,

$$u^2(m,n)\leqslant k^2+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u(s,t)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u(s,t)w\big(u(s,t)\big),$$

*then*

$$u(m,n)\leqslant \Phi^{-1}\big[\Phi\big(k+A(m,n)\big)+B(m,n)\big]$$

*for all* $(m,n) \in \Omega_{(m_1,n_1)}$, *where*

$$A(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t), \qquad B(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t),$$

$\Phi$ *is defined as in Theorem E, and* $(m_1,n_1) \in \Omega$ *is chosen such that* $\Phi(k + A(m,n)) + B(m,n) \in \operatorname{Dom} \Phi^{-1}$ *for all* $(m,n) \in \Omega_{(m_1,n_1)}$.

In this paper, we establish some new discrete Gronwall–Bellman–Ou-Iang-type inequalities with explicit bounds on unknown functions. The purpose here is three-fold. One, these generalize Theorem F together with other existing results of Cheung [5] and Pachpatte [16]. Two, these serve as discrete analogues of Gronwall–Bellman–Ou-Iang-type integral inequalities. Three, these furnish a handy tool for the study of qualitative as well as quantitative properties of solutions of difference equations. We illustrate this by applying our new inequalities to study the boundedness, uniqueness, and continuous dependence properties of the solutions of a discrete boundary value problem.

## 2. Discrete Gronwall–Bellman–Ou-Iang-type inequalities

Throughout this paper, $I := [m_0, M) \cap \mathbb{Z}$ and $J := [n_0, N) \cap \mathbb{Z}$ are two fixed lattices of integral points in $\mathbb{R}$, where $m_0, n_0 \in \mathbb{Z}$, $M, N \in \mathbb{Z} \cup \{\infty\}$. Let $\Omega := I \times J \subset \mathbb{Z}^2$, $\mathbb{R}_+ := [0, \infty)$, and for any $(s,t) \in \Omega$, the sub-lattice $[m_0, s] \times [n_0, t] \cap \Omega$ of $\Omega$ will be denoted as $\Omega_{(s,t)}$.

If $U$ is a lattice in $\mathbb{Z}$ (respectively $\mathbb{Z}^2$), the collection of all $\mathbb{R}$-valued functions on $U$ is denoted by $\mathcal{F}(U)$, and that of all $\mathbb{R}_+$-valued functions by $\mathcal{F}_+(U)$. For the sake of convenience, we extend the domain of definition of each function in $\mathcal{F}(U)$ and $\mathcal{F}_+(U)$ trivially to the ambient space $\mathbb{Z}$ (respectively $\mathbb{Z}^2$). So for example, a function in $\mathcal{F}(U)$ is regarded as a function defined on $\mathbb{Z}$ (respectively $\mathbb{Z}^2$) with support in $U$. As usual, the collection of all continuous functions of a topological space $X$ into a topological space $Y$ will be denoted by $C(X,Y)$.

If $U$ is a lattice in $\mathbb{Z}$, the difference operator $\Delta$ on $f \in \mathcal{F}(\mathbb{Z})$ or $\mathcal{F}_+(\mathbb{Z})$ is defined as

$$\Delta f(n) := f(n+1) - f(n), \quad n \in U,$$

and if $V$ is a lattice in $\mathbb{Z}^2$, the partial difference operators $\Delta_1$ and $\Delta_2$ on $u \in \mathcal{F}(\mathbb{Z}^2)$ or $\mathcal{F}_+(\mathbb{Z}^2)$ are defined as

$$\Delta_1 u(m,n) := u(m+1,n) - u(m,n), \quad (m,n) \in V,$$
$$\Delta_2 u(m,n) := u(m,n+1) - u(m,n), \quad (m,n) \in V.$$

For any $\varphi, \psi \in C(\mathbb{R}_+, \mathbb{R}_+)$ and any constant $\beta > 0$, define

$$\Phi_\beta(r) := \int_1^r \frac{ds}{\varphi(s^{1/\beta})}, \quad \Psi_\beta(r) := \int_1^r \frac{ds}{\psi(s^{1/\beta})}, \quad r > 0,$$
$$\Phi_\beta(0) := \lim_{r \to 0^+} \Phi_\beta(r), \qquad \Psi_\beta(0) := \lim_{r \to 0^+} \Psi_\beta(r).$$

Note that we allow $\Phi_\beta(0)$ and $\Psi_\beta(0)$ to be $-\infty$ here.

**Theorem 2.1.** *Suppose $u \in \mathcal{F}_+(\Omega)$. If $c \geqslant 0$, $\alpha > 0$ are constants and $b \in \mathcal{F}_+(\Omega)$, $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *$\varphi$ is non-decreasing with $\varphi(r) > 0$ for $r > 0$; and*
(ii) *for any $(m,n) \in \Omega$,*

$$u^\alpha(m,n) \leqslant c + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\varphi\big(u(s,t)\big), \tag{1}$$

*then*

$$u(m,n) \leqslant \big\{\Phi_\alpha^{-1}\big[\Phi_\alpha(c) + B(m,n)\big]\big\}^{1/\alpha} \tag{2}$$

*for all $(m,n) \in \Omega_{(m_1,n_1)}$, where*

$$B(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t),$$

*$\Phi_\alpha^{-1}$ is the inverse of $\Phi_\alpha$, and $(m_1,n_1) \in \Omega$ is chosen such that $\Phi_\alpha(c) + B(m,n) \in \mathrm{Dom}(\Phi_\alpha^{-1})$ for all $(m,n) \in \Omega_{(m_1,n_1)}$.*

**Proof.** It suffices to consider the case $c > 0$, for then the case $c = 0$ can be arrived at by continuity argument. Denote by $g(m,n)$ the right-hand side of (1). Then $g > 0$, $u \leqslant g^{1/\alpha}$ on $\Omega$, and $g$ is non-decreasing in each variable. Hence for any $(m,n) \in \Omega$,

$$\begin{aligned} \Delta_1 g(m,n) &= g(m+1,n) - g(m,n) = \sum_{t=n_0}^{n-1} b(m,t)\varphi\big(u(m,t)\big) \\ &\leqslant \sum_{t=n_0}^{n-1} b(m,t)\varphi\big(g^{1/\alpha}(m,t)\big) \leqslant \varphi\big(g^{1/\alpha}(m,n-1)\big) \sum_{t=n_0}^{n-1} b(m,t). \end{aligned} \tag{3}$$

Therefore, by the Mean-Value Theorem for integrals, for each $(m,n) \in \Omega$, there exists $g(m,n) \leqslant \xi \leqslant g(m+1,n)$ such that

$$\begin{aligned} \Delta_1(\Phi_\alpha \circ g)(m,n) &= \Phi_\alpha\big(g(m+1,n)\big) - \Phi_\alpha\big(g(m,n)\big) \\ &= \int_{g(m,n)}^{g(m+1,n)} \frac{ds}{\varphi(s^{1/\alpha})} = \frac{1}{\varphi(\xi^{1/\alpha})} \Delta_1 g(m,n). \end{aligned}$$

Since $\varphi$ is non-decreasing, $\varphi(\xi^{1/\alpha}) \geqslant \varphi(g^{1/\alpha}(m,n))$ and so by (3),

$$\Delta_1(\Phi_\alpha \circ g)(m,n) \leqslant \frac{1}{\varphi(g^{1/\alpha}(m,n))} \Delta_1 g(m,n) \leqslant \frac{\varphi(g^{1/\alpha}(m,n-1))}{\varphi(g^{1/\alpha}(m,n))} \sum_{t=n_0}^{n-1} b(m,t)$$

$$\leqslant \sum_{t=n_0}^{n-1} b(m,t)$$

for all $(m,n) \in \Omega$. Therefore,

$$\sum_{s=m_0}^{m-1} \Delta_1(\Phi_\alpha \circ g)(s,n) \leqslant \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) = B(m,n).$$

On the other hand, it is elementary to check that

$$\sum_{s=m_0}^{m-1} \Delta_1(\Phi_\alpha \circ g)(s,n) = \Phi_\alpha \circ g(m,n) - \Phi_\alpha \circ g(m_0,n),$$

thus

$$\Phi_\alpha \circ g(m,n) \leqslant \Phi_\alpha \circ g(m_0,n) + B(m,n) = \Phi_\alpha(c) + B(m,n).$$

Since $\Phi_\alpha^{-1}$ is increasing on $\operatorname{Dom} \Phi_\alpha^{-1}$, this yields

$$g(m,n) \leqslant \Phi_\alpha^{-1}\big[\Phi_\alpha(c) + B(m,n)\big]$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$. Hence the assertion. □

**Remarks.**

(i) When $\alpha = 1$, Theorem 2.1 reduces to Theorem 2.1 in [4].
(ii) In many cases the non-decreasing function $\varphi$ satisfies

$$\int_1^\infty \frac{ds}{\varphi(s^{1/\alpha})} = \infty.$$

For example, $\varphi =$ constant $> 0$, $\varphi(s) = s^\alpha$, $\varphi(s) = s^{\alpha/2}$, etc., are such functions. In such cases $\Phi_\alpha(\infty) = \infty$ and so we may take $m_1 = M$, $n_1 = N$. In particular, inequality (2) holds for all $(m,n) \in \Omega$.

**Theorem 2.2.** *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \geqslant 0$, $p > 1$ are constants and $a, b \in \mathcal{F}_+(\Omega)$, $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *$\varphi$ is non-decreasing with $\varphi(r) > 0$ for $r > 0$; and*
(ii) *for any $(m,n) \in \Omega$,*

$$u^p(m,n) \leqslant k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)u(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u(s,t)\varphi\big(u(s,t)\big), \tag{4}$$

*then*

$$u(m,n) \leqslant \big\{\Phi_{p-1}^{-1}\big[\Phi_{p-1}\big(k^{1-1/p} + A(m,n)\big) + B(m,n)\big]\big\}^{1/(p-1)} \tag{5}$$

*for all* $(m,n) \in \Omega_{(m_1,n_1)}$, *where*

$$A(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t), \qquad B(m,n) := \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t),$$

*and* $(m_1,n_1) \in \Omega$ *is chosen such that* $\Phi_{p-1}(k^{1-1/p} + A(m,n)) + B(m,n) \in \operatorname{Dom} \Phi_{p-1}^{-1}$ *for all* $(m,n) \in \Omega_{(m_1,n_1)}$.

**Proof.** Similar to the proof of Theorem 2.1, it suffices to consider the case $k > 0$. Denote by $f(s,t)$ the right-hand side of (4). Then $f > 0$, $u \leqslant f^{1/p}$ on $\Omega$, and $f$ is non-decreasing in each variable. Hence for any $(m,n) \in \Omega$,

$$\begin{aligned}
\Delta_1 f(m,n) &= f(m+1,n) - f(m,n) \\
&= \sum_{t=n_0}^{n-1} a(m,t)u(m,t) + \sum_{t=n_0}^{n-1} b(m,t)u(m,t)\varphi\big(u(m,t)\big) \\
&\leqslant \sum_{t=n_0}^{n-1} a(m,t) f^{1/p}(m,t) + \sum_{t=n_0}^{n-1} b(m,t) f^{1/p}(m,t)\varphi\big(f^{1/p}(m,t)\big) \\
&\leqslant f^{1/p}(m,n-1)\left[\sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)\varphi\big(f^{1/p}(m,t)\big)\right],
\end{aligned}$$

or

$$\frac{\Delta_1 f(m,n)}{f^{1/p}(m,n-1)} \leqslant \sum_{t=n_0}^{n-1} a(m,t) + \sum_{t=n_0}^{n-1} b(m,t)\varphi\big(f^{1/p}(m,t)\big).$$

Therefore, for any $(m,n) \in \Omega$,

$$\begin{aligned}
\sum_{s=m_0}^{m-1} \frac{\Delta_1 f(s,n)}{f^{1/p}(s,n-1)} &\leqslant \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\varphi\big(f^{1/p}(s,t)\big) \\
&= A(m,n) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\varphi\big(f^{1/p}(s,t)\big).
\end{aligned}$$

On the other hand, by the non-decreasing property of $f$ in each variable, it is easy to check that

$$\begin{aligned}
&\sum_{s=m_0}^{m-1} \frac{\Delta_1 f(s,n)}{f^{1/p}(s,n-1)} \\
&\quad = \frac{f(m,n)}{f^{1/p}(m-1,n-1)} - \frac{f(m-1,n)}{f^{1/p}(m-1,n-1)} + \frac{f(m-1,n)}{f^{1/p}(m-2,n-1)} \\
&\qquad - \frac{f(m-2,n)}{f^{1/p}(m-2,n-1)} + \cdots + \frac{f(m_0+1,n)}{f^{1/p}(m_0,n-1)} - \frac{f(m_0,n)}{f^{1/p}(m_0,n-1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{f(m,n)}{f^{1/p}(m-1,n-1)} \\
&\quad + \sum_{s=1}^{m-m_0-1} f(m-s,n)\left[\frac{1}{f^{1/p}(m-s-1,n-1)} - \frac{1}{f^{1/p}(m-s,n-1)}\right] \\
&\quad - \frac{f(m_0,n)}{f^{1/p}(m_0,n-1)} \\
&\geqslant \frac{f(m,n)}{f^{1/p}(m,n)} - \frac{f(m_0,n)}{f^{1/p}(m_0,n-1)} \\
&= f^{1-1/p}(m,n) - k^{1-1/p}
\end{aligned}
$$

for all $(m,n) \in \Omega$. Hence we have

$$
f^{1-1/p}(m,n) \leqslant k^{1-1/p} + A(m,n) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi\big(f^{1/p}(s,t)\big)
$$

for all $(m,n) \in \Omega$. In particular, since $A$ is non-decreasing in each variable, for any fixed $(\overline{m},\bar{n}) \in \Omega_{(m_1,n_1)}$,

$$
\begin{aligned}
f^{1-1/p}(m,n) &= \big[f^{1/p}(m,n)\big]^{p-1} \\
&\leqslant \big(k^{1-1/p} + A(\overline{m},\bar{n})\big) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\varphi\big(f^{1/p}(s,t)\big)
\end{aligned}
$$

for all $(m,n) \in \Omega_{(\overline{m},\bar{n})}$. Now by applying Theorem 2.1 to the function $f^{1/p}(m,n)$, we have

$$
u(m,n) \leqslant f^{1/p}(m,n) \leqslant \big\{\Phi_{p-1}^{-1}\big[\Phi_{p-1}\big(k^{1-1/p} + A(\overline{m},\bar{n})\big) + B(m,n)\big]\big\}^{1/(p-1)}
$$

for all $(m,n) \in \Omega_{(\overline{m},\bar{n})}$. In particular, this gives

$$
u(\overline{m},\bar{n}) \leqslant \big\{\Phi_{p-1}^{-1}\big[\Phi_{p-1}\big(k^{1-1/p} + A(\overline{m},\bar{n})\big) + B(\overline{m},\bar{n})\big]\big\}^{1/(p-1)}.
$$

Since $(\overline{m},\bar{n}) \in \Omega_{(m_1,n_1)}$ is arbitrary, this concludes the proof of the theorem. □

**Remarks.**

(i) When $p = 2$, Theorem 2.2 reduces to Theorem F.
(ii) Similar to the previous remark, in many cases $\Phi_{p-1}(\infty) = \infty$ and so in these situations, inequality (5) holds for all $(m,n) \in \Omega$.

In case $\Omega$ degenerates into a 1-dimensional lattice, Theorem 2.2 takes the following simpler form which is a generalization of a result of Pachpatte in [16].

**Corollary 2.3.** *Suppose $u \in \mathcal{F}_+(I)$. If $k \geqslant 0$, $p > 1$ are constants and $a, b \in \mathcal{F}_+(I)$, $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) $\varphi$ *is non-decreasing with* $\varphi(r) > 0$ *for* $r > 0$; *and*
(ii) *for any* $m \in I$,

$$u^p(m) \leqslant k + \sum_{s=m_0}^{m-1} a(s)u(s) + \sum_{s=m_0}^{m-1} b(s)u(s)\varphi\big(u(s)\big),$$

*then*

$$u(m) \leqslant \left\{\Phi_{p-1}^{-1}\left[\Phi_{p-1}\left(k^{1-1/p} + \sum_{s=m_0}^{m-1} a(s)\right) + \sum_{s=m_0}^{m-1} b(s)\right]\right\}^{1/(p-1)}$$

*for all* $m \in [m_0, m_1] \cap I$, *where* $m_1 \in I$ *is chosen such that* $\Phi_{p-1}(k^{1-1/p} + \sum_{s=m_0}^{m-1} a(s)) + \sum_{s=m_0}^{m-1} b(s) \in \operatorname{Dom} \Phi_{p-1}^{-1}$ *for all* $m \in [m_0, m_1] \cap I$.

**Proof.** It follows immediately from Theorem 2.2 by setting $\Omega = I \times \{n_0\}$ for some $n_0 \in \mathbb{Z}$, and extending the functions $a(s)$, $b(s)$, $u(s)$ to $a(s, n_0)$, $b(s, n_0)$ and $u(s, n_0)$ respectively in the obvious way. $\square$

Theorem 2.2 can easily be applied to generate other useful discrete inequalities in more general situations. For example, we have

**Theorem 2.4.** *Suppose* $u \in \mathcal{F}_+(\Omega)$. *If* $k \geqslant 0$, $p > q > 0$ *are constants and* $a, b \in \mathcal{F}_+(\Omega)$, $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are functions satisfying*

(i) $\varphi$ *is non-decreasing with* $\varphi(r) > 0$ *for* $r > 0$; *and*
(ii) *for any* $(m, n) \in \Omega$,

$$u^p(m,n) \leqslant k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^q(s,t)\varphi\big(u(s,t)\big), \tag{6}$$

*then*

$$u(m,n) \leqslant \big\{\Phi_{p-q}^{-1}\big[\Phi_{p-q}\big(k^{1-q/p} + A(m,n)\big) + B(m,n)\big]\big\}^{1/(p-q)} \tag{7}$$

*for all* $(m, n) \in \Omega_{(m_1,n_1)}$, *where* $A(m, n)$, $B(m, n)$ *are defined as in Theorem* 2.2, *and* $(m_1, n_1) \in \Omega$ *is chosen such that* $\Phi_{p-q}(k^{1-q/p} + A(m, n)) + B(m, n) \in \operatorname{Dom} \Phi_{p-q}^{-1}$ *for all* $(m, n) \in \Omega_{(m_1,n_1)}$.

**Proof.** For any $r > 0$, define

$$\psi(r) := \varphi\big(r^{1/q}\big). \tag{8}$$

Then clearly $\psi$ satisfies condition (i) of Theorem 2.2. By (6),

$$u^p(m,n) \leqslant k + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} a(s,t)u^q(s,t) + \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)u^q(s,t)\psi\big(u^q(s,t)\big)$$

for all $(m,n) \in \Omega$. Writing $v = u^q$, this becomes

$$v^{p/q}(m,n) \leqslant k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)v(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)v(s,t)\psi\big(v(s,t)\big).$$

Since $p/q > 1$, it follows from Theorem 2.2 that

$$\begin{aligned} v(m,n) &\leqslant \big\{\Psi_{p/q-1}^{-1}\big[\Psi_{p/q-1}\big(k^{1-1/(p/q)} + A(m,n)\big) + B(m,n)\big]\big\}^{1/(p/q-1)} \\ &= \big\{\Psi_{(p-q)/q}^{-1}\big[\Psi_{(p-q)/q}\big(k^{(p-q)/p} + A(m,n)\big) + B(m,n)\big]\big\}^{q/(p-q)} \end{aligned}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$. Now it is elementary to check by the definition of $\psi$ in (8) that

$$\Psi_{(p-q)/q}(r) = \Phi_{p-q}(r),$$

thus we have

$$v(m,n) \leqslant \big\{\Phi_{p-q}^{-1}\big[\Phi_{p-q}\big(k^{(p-q)/p} + A(m,n)\big) + B(m,n)\big]\big\}^{q/(p-q)}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, or

$$\begin{aligned} u(m,n) &= v^{1/q}(m,n) \\ &\leqslant \big\{\Phi_{p-q}^{-1}\big[\Phi_{p-q}\big(k^{(p-q)/p} + A(m,n)\big) + B(m,n)\big]\big\}^{1/(p-q)} \end{aligned}$$

for all $(m,n) \in \Omega_{(m_1,n_1)}$, where $(m_1,n_1) \in \Omega$ is chosen such that $\Phi_{p-q}(k^{(p-q)/p} + A(m,n)) + B(m,n) \in \operatorname{Dom} \Phi_{p-q}^{-1}$ for all $(m,n) \in \Omega_{(m_1,n_1)}$. □

An important special case of Theorem 2.4 is the following

**Corollary 2.5.** *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \geqslant 0$, $p > 1$ are constants and $a, b \in \mathcal{F}_+(\Omega)$, $\varphi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *$\varphi$ is non-decreasing with $\varphi(r) > 0$ for $r > 0$; and*
(ii) *for any $(m,n) \in \Omega$,*

$$\begin{aligned} u^p(m,n) \leqslant k &+ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t)u^{p-1}(s,t) \\ &+ \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)u^{p-1}(s,t)\varphi\big(u(s,t)\big), \end{aligned}$$

*then*

$$u(m,n) \leqslant \Phi_1^{-1}\big[\Phi_1\big(k^{1/p} + A(m,n)\big) + B(m,n)\big]$$

*for all $(m,n) \in \Omega_{(m_1,n_1)}$, where $A(m,n)$, $B(m,n)$ are defined as in Theorem 2.2, and $(m_1,n_1) \in \Omega$ is chosen such that $\Phi_1(k^{1/p} + A(m,n)) + B(m,n) \in \operatorname{Dom} \Phi_1^{-1}$ for all $(m,n) \in \Omega_{(m_1,n_1)}$.*

**Proof.** The assertion follows immediately from Theorem 2.4 by taking $q = p - 1 > 0$. $\square$

In particular, we have the following useful consequence.

**Corollary 2.6.** *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \geqslant 0$, $p > 1$ are constants and $a, b \in \mathcal{F}_+(\Omega)$ are functions such that for any $(m,n) \in \Omega$,*

$$u^p(m,n) \leqslant k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} a(s,t) u^{p-1}(s,t) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) u^p(s,t),$$

*then*

$$u(m,n) \leqslant \big(k^{1/p} + A(m,n)\big) \exp B(m,n)$$

*for all $(m,n) \in \Omega$, where $A(m,n)$, $B(m,n)$ are defined as in Theorem* 2.2.

**Proof.** Assume first that $k > 0$. Let $\varphi$ be the identity mapping of $\mathbb{R}_+$ onto itself. Then all conditions of Corollary 2.5 are satisfied. Note that in this cases $\Phi_1 = \ln$ and so $\Phi_1^{-1} = \exp$. In particular, $\Phi_1^{-1}$ is defined everywhere on $\mathbb{R}$. By Corollary 2.5, we have

$$u(m,n) \leqslant \exp\big[\ln\big(k^{1/p} + A(m,n)\big) + B(m,n)\big] = \big[k^{1/p} + A(m,n)\big] \exp B(m,n)$$

for all $(m,n) \in \Omega$. Finally, as this is true for all $k > 0$, by continuity, this should also hold for the case $k = 0$. $\square$

In case $\Omega$ degenerates into a 1-dimensional lattice, Corollary 2.6 takes the following simpler form which generalizes another result of Pachpatte in [17].

**Corollary 2.7.** *Suppose $u \in \mathcal{F}_+(I)$. If $k \geqslant 0$, $p > 1$ are constants and $a, b \in \mathcal{F}_+(I)$ are functions such that for any $m \in I$,*

$$u^p(m) \leqslant k + \sum_{s=m_0}^{m-1} a(s) u^{p-1}(s) + \sum_{s=m_0}^{m-1} b(s) u^p(s),$$

*then*

$$u(m) \leqslant \left[k^{1/p} + \sum_{s=m_0}^{m-1} a(s)\right] \prod_{s=m_0}^{m-1} \exp b(s)$$

*for all $m \in I$.*

**Proof.** Analogous to that of Corollary 2.3 and apply Corollary 2.6. $\square$

Another special situation of Corollary 2.6 is the following 2-dimensional discrete version of Ou-Iang's inequality.

**Corollary 2.8.** *Suppose $u \in \mathcal{F}_+(\Omega)$. If $k \geqslant 0$, $p > 1$ are constants and $b \in \mathcal{F}_+(\Omega)$ is a function such that for any $(m,n) \in \Omega$,*

$$u^p(m,n) \leqslant k + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t) u^p(s,t),$$

*then*

$$u(m,n) \leqslant k^{1/p} \exp B(m,n)$$

*for all $(m,n) \in \Omega$, where $B(m,n)$ is as defined in Theorem* 2.2.

**Proof.** This follows immediately from Corollary 2.6 by setting $a \equiv 0$. □

In case $\Omega$ degenerates into a 1-dimensional lattice, Corollary 2.8 takes the following simpler form which is a generalized 1-dimensional discrete analogue of Ou-Iang's inequality.

**Corollary 2.9.** *Suppose $u \in \mathcal{F}_+(I)$. If $k \geqslant 0$, $p > 1$ are constants and $b \in \mathcal{F}_+(I)$ is a function such that for any $m \in I$,*

$$u^p(m) \leqslant k + \sum_{s=m_0}^{m-1} b(s) u^p(s),$$

*then*

$$u(m) \leqslant k^{1/p} \prod_{s=m_0}^{m-1} \exp b(s)$$

*for all $m \in I$.*

**Proof.** It follows from Corollary 2.5 by setting $a \equiv 0$, or by imitating the proof of Corollary 2.3 and applying Corollary 2.8. □

**Remark.** It is evident that the results above can be generalized to obtain explicit bounds for functions satisfying certain discrete sum inequalities involving more retarded arguments. It is also clear that these results can be extended to functions on higher dimensional lattices in the obvious way. As details of these are rather algorithmic, they will not be carried out here.

## 3. Applications to boundary value problems

In this section, we shall illustrate how the results obtained in Section 2 can be applied to study the boundedness, uniqueness, and continuous dependence of the solutions of certain boundary value problems for difference equations involving 2 independent variables.

We consider the following

*Boundary Value Problem* (*BVP*):

$$\Delta_{12} z^p(m,n) = F\big(m,n,z(m,n)\big)$$

satisfying

$$z(m,n_0) = f(m), \qquad z(m_0,n) = g(n), \qquad f(m_0) = g(n_0) = 0,$$

where $p > 1$, $F \in \mathcal{F}(\Omega \times \mathbb{R})$, $f \in \mathcal{F}(I)$, and $g \in \mathcal{F}(J)$ are given.

Our first result deals with the boundedness of solutions.

**Theorem 3.1.** *Consider* (*BVP*). *If*

$$\big|F(m,n,v)\big| \leqslant b(m,n)|v|^p \tag{9}$$

*and*

$$\big|f(m)\big|^p + \big|g(n)\big|^p \leqslant k^p \tag{10}$$

*for some $k \geqslant 0$, where $b \in \mathcal{F}_+(\Omega)$, then all solutions of* (*BVP*) *satisfy*

$$\big|z(m,n)\big| \leqslant k \exp B(m,n), \quad (m,n) \in \Omega,$$

*where $B(m,n)$ is defined as in Theorem* 2.1. *In particular, if $B(m,n)$ is bounded on $\Omega$, then every solution of* (*BVP*) *is bounded on $\Omega$.*

**Proof.** Observe first that $z = z(m,n)$ solves (BVP) if and only if it satisfies the sum-difference equation

$$z^p(m,n) = f^p(m) + g^p(n) + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} F\big(s,t,z(s,t)\big). \tag{11}$$

Hence by (9) and (10),

$$\big|z(m,n)\big|^p \leqslant k^p + \sum_{s=m_0}^{m-1} \sum_{t=n_0}^{n-1} b(s,t)\big|z(s,t)\big|^p$$

for all $(m,n) \in \Omega$. An application of Corollary 2.8 to the function $|z(m,n)|$ gives the assertion immediately. $\square$

The next result is about uniqueness.

**Theorem 3.2.** *Consider* (*BVP*). *If*

$$\big|F(m,n,v_1) - F(m,n,v_2)\big| \leqslant b(m,n)\big|v_1^p - v_2^p\big|$$

*for some $b \in \mathcal{F}_+(\Omega)$, then* (*BVP*) *has at most one solution on $\Omega$.*

**Proof.** Let $z(m,n)$ and $\bar{z}(m,n)$ be two solutions of (BVP) on $\Omega$. By (11), we have

$$\begin{aligned}\left|z^p(m,n)-\bar{z}^p(m,n)\right| &\leqslant \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F\big(s,t,z(s,t)\big)-F\big(s,t,\bar{z}(s,t)\big)\right| \\ &\leqslant \sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1} b(s,t)\left|z^p(s,t)-\bar{z}^p(s,t)\right|.\end{aligned}$$

An application of Corollary 2.8 to the function $|z^p(s,t)-\bar{z}^p(s,t)|^{1/p}$ shows that

$$\left|z^p(s,t)-\bar{z}^p(s,t)\right|^{1/p} \leqslant 0 \quad \text{for all } (s,t)\in\Omega.$$

Hence $z=\bar{z}$ on $\Omega$. □

Finally, we investigate the continuous dependence of the solutions of (BVP) on the function $F$ and the boundary data $f$ and $g$. For this we consider the following variation of (BVP):

$(\overline{BVP})$: $\quad \Delta_{12} z^p(m,n) = \overline{F}\big(m,n,z(m,n)\big)$

with

$$z(m,n_0)=\overline{f}(m), \qquad z(m_0,n)=\overline{g}(n), \qquad \overline{f}(m_0)=\overline{g}(n_0)=0,$$

where $p>1$, $\overline{F}\in\mathcal{F}(\Omega\times\mathbb{R})$, $\overline{f}\in\mathcal{F}(I)$, and $\overline{g}\in\mathcal{F}(J)$ are given.

**Theorem 3.3.** *Consider* (*BVP*) *and* $(\overline{BVP})$. *If*

(i) $|F(m,n,v_1)-F(m,n,v_2)|\leqslant b(m,n)|v_1^p-v_2^p|$ *for some* $b\in\mathcal{F}_+(\Omega)$;
(ii) $|(f^p(m)-\overline{f}^p(m))+(g^p(n)-\overline{g}^p(n))|\leqslant\varepsilon/2$; *and*
(iii) *for all solutions* $\bar{z}(m,n)$ *of* $(\overline{BVP})$,

$$\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F\big(s,t,\bar{z}(s,t)\big)-\overline{F}\big(s,t,\bar{z}(s,t)\big)\right|\leqslant\frac{\varepsilon}{2}$$

*for all* $(m,n)\in\Omega$ *and* $v_1,v_2\in\mathbb{R}$, *then*

$$\left|z^p(m,n)-\bar{z}^p(m,n)\right|\leqslant\varepsilon\exp\big(pB(m,n)\big),$$

*where* $B(m,n)$ *is as defined in Theorem* 2.1. *Hence* $z^p$ *depends continuously on* $F$, $f$, *and* $g$. *In particular, if* $z$ *does not change sign, it depends continuously on* $F$, $f$ *and* $g$.

**Proof.** Let $z(m,n)$ and $\bar{z}(m,n)$ be solutions of (BVP) and $(\overline{\mathrm{BVP}})$, respectively. Then $z$ satisfies (11) and $\bar{z}$ satisfies the corresponding equation

$$\bar{z}^p(m,n)=\overline{f}^p(m)+\overline{g}^p(n)+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\overline{F}\big(s,t,\bar{z}(s,t)\big).$$

Hence

$$\begin{aligned}
&\left|z^p(m,n)-\bar{z}^p(m,n)\right| \\
&\quad\leqslant \left|\left(f^p(m)-\overline{f}^p(m)\right)+\left(g^p(n)-\bar{g}^p(n)\right)\right| \\
&\qquad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F\left(s,t,z(s,t)\right)-\overline{F}\left(s,t,\bar{z}(s,t)\right)\right| \\
&\quad\leqslant \frac{\varepsilon}{2}+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F\left(s,t,z(s,t)\right)-F\left(s,t,\bar{z}(s,t)\right)\right| \\
&\qquad+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}\left|F\left(s,t,\bar{z}(s,t)\right)-\overline{F}\left(s,t,\bar{z}(s,t)\right)\right| \\
&\quad\leqslant \varepsilon+\sum_{s=m_0}^{m-1}\sum_{t=n_0}^{n-1}b(s,t)\left|z^p(s,t)-\bar{z}^p(s,t)\right|
\end{aligned}$$

by assumptions (i), (ii) and (iii). Now by applying Corollary 2.8 to the function $|z^p(m,n)-\bar{z}^p(m,n)|^{1/p}$, we have

$$\left|z^p(m,n)-\bar{z}^p(m,n)\right|^{1/p}\leqslant \varepsilon^{1/p}\exp B(m,n)$$

for all $(m,n)\in\Omega$, or

$$\left|z^p(m,n)-\bar{z}^p(m,n)\right|\leqslant \varepsilon\exp\left(pB(m,n)\right).$$

Now when restricted to any compact sub-lattice, $B(m,n)$ is bounded, so

$$\left|z^p(m,n)-\bar{z}^p(m,n)\right|\leqslant \varepsilon\cdot K$$

for some $K>0$ for all $(m,n)$ in this compact sub-lattice. Hence $z^p$ depends continuously on $F$, $f$ and $g$. $\square$

**Remark.** The boundary value problem (BVP) is clearly not the only problem for which the boundedness, uniqueness, and continuous dependence of its solutions can be studied by using the results in Section 2. For example, one can arrive at similar results (with much more complicated computations) for the following variation of the (BVP):

$$\Delta_{12}z^p(m,n)=F\left(m,n,z(m,n),z(m,n)\cdot\varphi\left(\left|z(m,n)\right|\right)\right)$$

with

$$z(m,n_0)=f(m),\qquad z(m_0,n)=g(n),\qquad f(m_0)=g(n_0)=0,$$

where $p>1$, $F\in\mathcal{F}(\Omega\times\mathbb{R}^2)$, $f\in\mathcal{F}(I)$, $g\in\mathcal{F}(J)$, and $\varphi\in C(\mathbb{R}_+,\mathbb{R}_+)$ are given.

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