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Option Valuation Under Multivariate Markov Chain Model via Esscher Transform

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In this paper, we develop an option valuation model in the context of a discrete-time multivariate Markov chain model using the Esscher transform. The multivariate Markov chain provides a flexible way to incorporate the dependency of the underlying asset price processes and price multi-state options written on several dependent underlying assets. In our model, the price of an individual asset can take finitely many values. The market described by our model is incomplete in general, and, hence there are more than one equivalent martingale pricing measures. We adopt the conditional Esscher transform to determine an equivalent martingale measure for option valuation. We also document consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model.

Keywords: Option Pricing; Conditional Esscher Transform; Multi-State Options; Dependent Assets; Multivariate Markov chain Model.

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1. Introduction

Option valuation has long been a very important topic in financial economics. Since the seminal work of Black and Scholes [2] and Merton [25], there has been an explosive growth in the amount of literature on the theory and practice of option pricing models. The key economic insight behind the Black-Scholes-Merton model is the concept of perfect hedging and pricing by no-arbitrage principle. The Black-Scholes-Merton option pricing formula is preference-free; that is, the formula does not depend on the real-world expected return of the underlying asset, which is replaced by the risk-free interest rate, for instance, the interest rates of U.S. Treasury Bill. Pricing is done in a risk-neutral world in which the expected return on each asset is the same as the risk-free interest rate. Cox, Ross and Rubinstein [10] was the first one to establish the relationship between the risk-neutral valuation and the no-arbitrage principle. Harrison and Kreps [18], Harrison and Pliska ([19], [20]) established a solid mathematical foundation for the relationship between the no-arbitrage principle and the notion of risk-neutral valuation using the language of probability theory. They showed that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure. If the securities market is complete, there is a unique martingale measure and, hence, the unique price of any contingent claim is given by its expected discounted payoff at expiry under the martingale measure. However, in an incomplete market, there are infinitely many equivalent martingale measures and, hence, a range of no-arbitrage prices for a contingent claim. This makes the pricing and hedging of derivative instruments more complicated.

The Cox-Ross-Rubinstein (CRR) model introduced in the seminal paper by Cox, Ross and Rubinstein [10] describes the price dynamics of an underlying asset as a binomial lattice in which the price of the asset at a particular time period can take one of the two possible values, namely, "up" and "down". In fact, the idea of using binomial lattice for pricing derivatives was first suggested by W.F. Sharpe and developed by Cox, Ross and Rubinstein [10]. The CRR model can provide some important insights into the concept of risk-neutral valuation and a simple and accurate approximation to the continuous-time option pricing model. However, the assumption that the price of the asset at a particular time period can only take two possible values is not realistic. Boyle [3] developed a trinomial model for option valuation. The trinomial model assumed that the price of the underlying asset at a particular tine period can take three possible values, "up", "middle" and "down". The market described by the Boyle model is complete. He [21] further extended the lattice models and proposed a discrete-time multinomial model for the valuation of the option written on several correlated risky assets. The rationale of He's approach is to adopt a sequence of discrete-time multinomial processes to approximate a given continuous-time multivariate diffusion processes for the correlated underlying assets. The He approach can preserve the market completeness. The price of an option can be determined uniquely under the no-arbitrage principle. In the finance literature, there are other important discrete-time models for approximating multivariate diffusion processes, for instances, Boyle [3], Cheyette [8], Boyle, Evnine and Gibbs [4], Madan, Milne an Shefrin [23] and Ho, Stapleton and Subrahmanyam [22]. See Boyle [5] for a comprehensive account on various discrete-time models.

In this paper, we develop an option valuation model in the context of a discrete-time multivariate Markov chain model using a well-known tool in actuarial science, namely, the Esscher transform. The multivariate Markov chain model provides a flexible way to incorporate the dependency of the underlying asset price processes in a discrete framework. In our model, the price of an individual asset can take finitely many values. The market described by our model is incomplete in general, and, hence there are more than one equivalent martingale pricing measures. We adopt the conditional Esscher transform in Buhlmann et al. ([6], [7]) to determine an equivalent martingale measure for option valuation. The pioneering work on the Esscher transform for option valuation was done by Gerber and Shiu [17]. The Gerber-Shiu approach provided market practitioners with a convenient and flexible way to consider various parametric models for option valuation. Their work highlights the interplay between financial and insurance pricing problems in an incomplete market. This is an important issue in actuarial science and finance as pointed out by Bühlmann et al. [6] and Embrechts [16]. Some other works on the use of Esscher transform and its variants for option valuation include Bühlmann et al. [7], Pafumi [26], Yao [30], McLeish and Reesor [24] and Siu et al. [29], etc. In particular, the paper by Bühlmann et al. [7] considered the use of the Esscher transform for option valuation in a discrete-time economy. Our model can incorporate the dependency of the price dynamics for individual assets described by a Markovian version of the multinomial model. It provides market practitioners with a flexible way to price multi-state options written on several dependent underlying assets. We also document consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model. In particular, we investigate whether misspecification of the level of the dependency of the underlying asset prices can have significant impact on the option prices.

The rest of the paper is organized as follows. In Section 2, we introduce the multivariate Markov chain model for modelling the dependency of the price dynamics of the underlying asset. Section 3 presents the Esscher transform for determining an equivalent martingale pricing measure. Section 4 documents consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model. Finally, concluding remarks are given in Section 5.

2. Asset Price Dynamics

We consider a discrete-time financial model with one primary risk-free asset and n underlying risky assets. Suppose $\mathcal{T}$ represents the time index set $\{0, 1, 2, \dots, \infty\}$

on which all economic activities take place. Let r_t denote the risk-free interest rate over the time interval $[t-1, t]$. We assume that the price dynamics of the risk-free asset B is governed by:

$$B_t = B_{t-1} e^{r_t} \ , t \in \mathcal{T} \backslash \{0\} \ . \tag{2.1}$$

Fix a complete probability space $(\Omega, F, \mathcal{P})$, where $\mathcal{P}$ is a real-world physical probability measure. For each $j = 1, 2, \ldots, n$, let $\{S_{jt}\}_{t \in \mathcal{T}}$ denote the price dynamics of the j^{th} risky asset at time t. Let $R_t^{(j)}$ denote the rate of return of the j^{th} risky asset from time $t-1$ to time t; that is,

$$S_{jt} = S_{j,t-1} \exp(R_t^{(j)}) \ , t \in \mathcal{T} \backslash \{0\} \ . \tag{2.2}$$

We assume that the return processes $R := (R^{(1)}, R^{(2)}, \ldots, R^{(n)})$ of the n risky assets are governed by a multivariate Markov Chain model by Ching, Fung and Ng [9]. The multivariate Markov chain model can incorporate the dependency of the movements of the returns of the risky assets. For each $j = 1, 2, \ldots, n$, we suppose that $R^{(j)} := \{R_t^{(j)}\}_{t \in \mathcal{T} \backslash \{0\}}$ denotes a stochastic process on $(\Omega, F, \mathcal{P})$ with a common state space $L = (L_0, L_1, \ldots, L_{m-1})$. Note that L_i $(i = 0, 1, \ldots, m-1)$ represents one possible state of the return of a risky asset. In general, we can consider the case that the state spaces are different for different risky assets. This makes the notations more complicated without adding much interest.

On $(\Omega, F, \mathcal{P})$, we define n categorical time series $Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)}$ with common time index set $\mathcal{T}$. Let $\mathcal{S}$ denote a set of unit basis vectors $\{e_0, e_1, \ldots, e_{m-1}\}$ in R^m, where $e_i = (0, \ldots, 0, \overbrace{1}^{i^{th} entry}, 0, \ldots, 0)^T \in R^m$. For each $j = 1, 2, \ldots, n$, $Y^{(j)} := \{Y_t^{(j)}\}_{t \in \mathcal{T}}$ represents a discrete-time and finite-state stochastic process with state space $\mathcal{S}$. Note that $\{Y_t^{(j)}\}_{t \in \mathcal{T}}$ represents the underlying state process for the return dynamics of the j^{th} risky asset. The event $\{\omega \in \Omega | Y_t^{(j)}(\omega) = e_i\}$ means that the return of the j^{th} risky asset is in the i^{th} state at time t. Define the space $\hat{\mathcal{S}}$ as

$$\{s \in R^m | s = \sum_{i=0}^{m-1} \alpha_i e_i, \quad 0 \leq \alpha_i \leq 1, \quad \sum_{i=0}^{m-1} \alpha_i = 1\}.$$

For each $t \in \mathcal{T}$, we assume that $R_t^{(j)}$ is a function $R^{(j)}(t, Y_t^{(j)})$ of both time t and the state $Y_t^{(j)}$ and that

$$R_t^{(j)} = < L, Y_t^{(j)} > \ , \tag{2.3}$$

where $< x, y >$ represents the inner product of two vectors x and y in $\mathcal{R}^m$.

For each $j = 1, 2, \ldots, n$, we denote the dynamics of the discrete probability distributions for $Y^{(j)}$ as $\{X_t^{(j)}\}_{t \in \mathcal{T}}$, where $X_t^{(j)} \in \hat{\mathcal{S}}$, for each $t \in \mathcal{T}$. In particular, for each $t \in \mathcal{T}$, the i^{th} entity of the probability vector $X_t^{(j)}$ represents the probability that the return of the j^{th} asset is in the i^{th} state at time t. Suppose that the return of the j^{th} risky asset is in the i^{th} state at time t; that is, $Y_t^{(j)} = e_i$. This means that the probability of the return of the j^{th} risky asset being in the i^{th} state at time t is equal to one. Hence,

$$X_t^{(j)} = e_i = (0, \ldots, 0, \underbrace{1}_{i\text{th entry}}, 0 \ldots, 0)^T.$$

Let $P^{(jk)}$ be a transition probability matrix from the states in the return dynamics of the k^{th} risky asset to the states in the return dynamics of the j^{th} risky asset. For each $j = 1, 2, \ldots, n$, $P^{(jj)}$ represents the transition probability matrix for the return dynamics of the j^{th} risky asset $Y^{(j)}$. Write $X_t^{(k)}$ for the state probability distribution of the return of the k^{th} risky asset at time t. Then, we assume that the dynamics of the probability distributions of the return dynamics for the j^{th} risky asset are governed by the following equation:

$$X_{t+1}^{(j)} = \sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X_t^{(k)}, \quad \text{for} \quad j = 1, 2, \cdots, n \tag{2.4}$$

where

$$\lambda_{jk} \geq 0, \quad 1 \leq j, k \leq n \quad \text{and} \quad \sum_{k=1}^{n} \lambda_{jk} = 1, \quad \text{for} \quad j = 1, 2, \cdots, n. \tag{2.5}$$

The interpretation of Equation (2.4) is that the state probability distribution of the return dynamics of the j^{th} risky asset $Y^{(j)}$ at time $t+1$ depends on the weighted average of $P^{(jk)} X_t^{(k)}$ at time t. It can be shown that the conditional probability distribution for the return dynamics of the j^{th} risky asset at time $t+1$ depends on the returns of all risky assets in the model at time t. We can write Equation (2.4) in the following matrix form:

$$X_{t+1} \equiv \begin{pmatrix} X_{t+1}^{(1)} \\ X_{t+1}^{(2)} \\ \vdots \\ X_{t+1}^{(n)} \end{pmatrix} = \begin{pmatrix} \lambda_{11} P^{(11)} & \lambda_{12} P^{(12)} & \cdots & \lambda_{1n} P^{(1n)} \\ \lambda_{21} P^{(21)} & \lambda_{22} P^{(22)} & \cdots & \lambda_{2n} P^{(2n)} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_{n1} P^{(n1)} & \lambda_{n2} P^{(n2)} & \cdots & \lambda_{nn} P^{(nn)} \end{pmatrix} \begin{pmatrix} X_t^{(1)} \\ X_t^{(2)} \\ \vdots \\ X_t^{(n)} \end{pmatrix} \equiv Q X_t$$

or

$$X_{t+1} = Q X_t.$$

3. Option Valuation by the conditional Esscher transform

We employ a multivariate version of the conditional Esscher transform proposed by Bühlmann et al. ([6], [7]) to determine an equivalent martingale measure in the context of the multivariate Markov chain model.

First, suppose $\mathcal{F} := \{\mathcal{F}_t\}_{t \in \mathcal{T}}$ denote the $\mathcal{P}$-augmentation of the natural filtration $\mathcal{F}^Y := \{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ generated by the n-dimensional process $Y := (Y^{(1)}, Y^{(2)}, \ldots, Y^{(n)})$, where $\mathcal{F}_t^Y := \sigma\{Y_1, Y_2, \ldots, Y_t\}$, for each $t \in \mathcal{T}$.

For each $t \in \mathcal{T}\backslash\{0\}$ and $j = 1, 2, \ldots, n$, the joint conditional distribution $P^{(j)}_{t|t-1}$ of $Y^{(j)}_t$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}$ is given by:

$$P^{(j)}_{t|t-1} := (p^{(j0)}_{t|t-1}, p^{(j1)}_{t|t-1}, \ldots, p^{(j,m-1)}_{t|t-1}) \ , \tag{3.1}$$

where

$$p^{(ji)}_{t|t-1} := \mathcal{P}(\{Y^{(j)}_t = e_i\}|\mathcal{F}_{t-1}) \ . \tag{3.2}$$

Lemma 3.1. *Let $[V]^i$ denote the i^{th} element of the column vector V. Then, for each $j = 1, 2, \ldots, n$ and $i = 0, 1, \ldots, m-1$,*

$$p^{(ji)}_{t|t-1} = [\sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X^{(k)}_{t-1}]^i|_{X_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} \ . \tag{3.3}$$

Proof. Since Y is a Markov processs with respect to $\mathcal{F}$ under $\mathcal{P}$, by Elliott, Aggoun and Moore [15], it can be shown that $p^{(ji)}_{t|t-1}$ is given by:

$$\begin{aligned} p^{(ji)}_{t|t-1} &:= E(< Y^{(j)}_t, e_i > |\mathcal{F}_{t-1}) \\ &= E(< Y^{(j)}_t, e_i >)|_{Y_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} \ , \end{aligned}$$

where

(1) $E(\cdot)$ denotes the expectation operator with respect to the measure $\mathcal{P}$.
(2) $f(Y_{t-1})|_{Y_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})}$ represents the value of the function f of the vector Y_{t-1} evaluated at $(e_{i_1}, e_{i_2}, \ldots, e_{i_n})$, for some $i_1, i_2, \ldots, i_n \in \{0, 1, \ldots, m-1\}$.

From Equation (2.4) in Section 2,

$$X^{(j)}_t = \sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X^{(k)}_{t-1}, \quad \text{for} \quad j = 1, 2, \cdots, n \ .$$

For each $j = 1, 2, \ldots, n$ and $i = 0, 1, \ldots, m-1$,

$$\begin{aligned} p^{(ji)}_{t|t-1} &= E(< Y^{(j)}_t, e_i >)|_{Y_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} = \mathcal{P}(\{Y^{(j)}_t = e_i\})|_{Y_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} \\ &= [X^{(j)}_t]^i|_{Y_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} = [X^{(j)}_t]^i|_{X_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} \\ &= [\sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X^{(k)}_{t-1}]^i|_{X_{t-1}=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})} \ . \end{aligned}$$

□

Let $M_{S_t|\mathcal{F}_{t-1}}(Z)$ denote the moment generating function of the joint conditional distribution of the random vector $S_t := (S_{1t}, S_{2t}, \ldots, S_{nt})^*$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}$, where $Z := (Z_1, Z_2, \ldots, Z_n)^* \in \mathcal{R}^n$; that is,

$$M_{S_t|\mathcal{F}_{t-1}}(Z) := E(e^{Z^* S_t}|\mathcal{F}_{t-1}) \ , \tag{3.4}$$

where Z^* represents the transpose of the vector Z.

Since S_t can take values in a finite state space, $M_{S_t|\mathcal{F}_{t-1}}(Z) < \infty$, for some $Z \in \mathcal{R}^n$, for each $t \in \mathcal{T}$. Let $\{\Theta_t\}_{t \in \mathcal{T}\backslash\{0\}}$ denote an n-dimensional stochastic process which is predictable with respect to $\mathcal{F}$; that is, Θ_t is measurable with respect to $\mathcal{F}_{t-1}$, for each $t \in \mathcal{T} \backslash \{0\}$. Then, we define a sequence $\{\Lambda_t\}_{t \in \mathcal{T}}$ with $\Lambda_0 = 1$ and

$$\Lambda_t = \prod_{k=1}^{t} \frac{e^{\Theta_k^* S_k}}{M_{S_k|\mathcal{F}_{k-1}}(\Theta_k)} \ , \ t \in \mathcal{T} \backslash \{0\}. \tag{3.5}$$

Lemma 3.2. *$\{\Lambda_t\}_{t \in \mathcal{T}}$ is a $(\mathcal{F}, \mathcal{P})$-martingale.*

Proof. For each $t \in \mathcal{T} \backslash \{0\}$,

$$\begin{aligned} & E\left(\frac{\Lambda_{t+1}}{\Lambda_t} \middle| \mathcal{F}_t \right) \\ &= \frac{E(e^{\Theta_{t+1}^* S_{t+1}} | \mathcal{F}_t)}{M_{S_{t+1}|\mathcal{F}_t}(\Theta_{t+1})} \\ &= 1 \ , \quad \mathcal{P} - \text{a.s.} \end{aligned}$$

Hence, the result follows. □

Define a probability measure $\mathcal{P}_\Theta \sim \mathcal{P}$ on (Ω, F) by the following multivariate conditional Esscher transform:

$$\left. \frac{d\mathcal{P}_\Theta}{d\mathcal{P}} \right|_{\mathcal{F}_t} = \Lambda_t \ . \tag{3.6}$$

Note that Θ_t is the vector of conditional Esscher parameters given $\mathcal{F}_{t-1}$, for each $t \in \mathcal{T} \backslash \{0\}$. Let $P_{t|t-1}^{(i_1, i_2, \ldots, i_n)}$ denote the joint conditional distribution of $Y_t := (Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(n)})$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}$; that is,

$$\begin{aligned} P_{t|t-1}^{(i_1, i_2, \ldots, i_n)} &:= \mathcal{P}(Y_t^{(1)} = e_{i_1}, Y_t^{(2)} = e_{i_2}, \ldots, Y_t^{(n)} = e_{i_n} | \mathcal{F}_{t-1}) \\ &= \mathcal{P}(R_t^{(1)} = L_{i_1}, R_t^{(2)} = L_{i_2}, \ldots, R_t^{(n)} = L_{i_n} | \mathcal{F}_{t-1}) \ . \end{aligned} \tag{3.7}$$

Note that $(Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(n)})$ are assumed to be conditionally independent given $\mathcal{F}_{t-1}$. Hence,

$$P_{t|t-1}^{(i_1, i_2, \ldots, i_n)} := \prod_{j=1}^{n} p_{t|t-1}^{(j i_j)} \ . \tag{3.8}$$

Suppose $P_{t|t-1}^{(i_1, i_2, \ldots, i_n)}(\Theta)$ denotes the joint conditional distribution of $Y_t := (Y_t^{(1)}, Y_t^{(2)}, \ldots, Y_t^{(n)})$ given $\mathcal{F}_{t-1}$ under $\mathcal{P}_\Theta$; that is,

$$\begin{aligned} P_{t|t-1}^{(i_1, i_2, \ldots, i_n)}(\Theta) &:= \mathcal{P}_\Theta(Y_t^{(1)} = e_{i_1}, Y_t^{(2)} = e_{i_2}, \ldots, Y_t^{(n)} = e_{i_n} | \mathcal{F}_{t-1}) \\ &= \mathcal{P}_\Theta(R_t^{(1)} = L_{i_1}, R_t^{(2)} = L_{i_2}, \ldots, R_t^{(n)} = L_{i_n} | \mathcal{F}_{t-1}) \ . \end{aligned} \tag{3.9}$$

Then, it can be shown that

$$P^{(i_1,i_2,\ldots,i_n)}_{t|t-1}(\Theta) = \frac{P^{(i_1,i_2,\ldots,i_n)}_{t|t-1}\exp(\sum_{j=1}^{n} S_{j,t-1}\Theta^{(j)}_t e^{L_{i_j}})}{E(e^{\Theta^*_t S_t}|\mathcal{F}_{t-1})}$$
$$= \frac{\prod_{j=1}^{n} p^{(ji_j)}_{t|t-1}\exp(\sum_{j=1}^{n} S_{j,t-1}\Theta^{(j)}_t e^{L_{i_j}})}{\sum_{i_1=0}^{m-1}\sum_{i_2=0}^{m-1}\cdots\sum_{i_n=0}^{m-1}\prod_{j=1}^{n} p^{(ji_j)}_{t|t-1}\exp(\sum_{j=1}^{n} S_{j,t-1}\Theta^{(j)}_t e^{L_{i_j}})}\ , \tag{3.10}$$

where $\Theta_t := (\Theta^{(1)}_t, \Theta^{(2)}_t, \ldots, \Theta^{(n)}_t)$.

By employing the modern language of probability theory, Harrison and Kreps [18] established the relationship between the absence of arbitrage opportunities and the existence of an equivalent martingale measure under which all discounted asset price processes are martingale. This result is known as the fundamental theorem of asset pricing and further extended by Harrison and Pliska ([19], [20]), Dybvig and Ross [13], Back and Pliska [1], Schachermayer [27] and Delbaen and Schachermayer [11]. Back and Pliska [1] showed that the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure in a discrete-time and infinite-state-space setting. Delbaen and Schachermayer [11] pointed out that the equivalence between the absence of arbitrage opportunities and the existence of an equivalent martingale measure is not always true in the continuous-time setting. They adopted “essentially equivalent” instead of ”equivalent” to describe the relationship.

When there is no arbitrage and the market is complete, there exists a unique equivalent martingale measure. In this case, the no-arbitrage price of a contingent claim can be determined uniquely by the expectation of the discounted payoff of the claim with respect to the equivalent martingale measure. However, when the market is incomplete, there are infinitely many equivalent martingale measures, and, hence a range of no-arbitrage prices for the claim. A crucial issue is how to pick an equivalent martingale measure for pricing the claim in this case. Gerber and Shiu [17] pioneered the use of the Esscher transform for determining an equivalent martingale measure for option valuation. They provided a pertinent solution to choose an appropriate equivalent martingale measure in an incomplete market. We employ the conditional Esscher transform by Bühlmann et al. (1996) [6] to determine an equivalent martingale measure in the sequel.

First, we present an expression for the moment generating function $M_S(t, U; \Theta)$ of the joint conditional distribution of the random vector S_t given $\mathcal{F}_{t-1}$ under $\mathcal{P}_\Theta$ in the following lemma.

Lemma 3.3.

$$M_S(t, U; \Theta) := E^{\Theta}(e^{U^* S_t}|\mathcal{F}_{t-1}) = \frac{M_{S_t|\mathcal{F}_{t-1}}(\Theta_t + U)}{M_{S_t|\mathcal{F}_{t-1}}(\Theta_t)}\ , \tag{3.11}$$

where $E^{\Theta}(\cdot)$ represents the expectation operator with respect to the probability measure $\mathcal{P}_\Theta$ and U^* is the transpose of the vector $U := (U_1, U_2, \ldots, U_n) \in \mathcal{R}^n$.

Proof. By the Bayes rule,

$$\begin{aligned} M_S(t,U;\Theta) &:= E^{\Theta}(e^{U^* S_t}|\mathcal{F}_{t-1}) \\ &= \frac{E(\Lambda_t e^{U^* S_t}|\mathcal{F}_{t-1})}{E(\Lambda_t|\mathcal{F}_{t-1})} \\ &= E\left(\frac{\Lambda_t}{\Lambda_{t-1}} e^{U^* S_t}\middle|\mathcal{F}_{t-1}\right) \\ &= \frac{E(e^{(\Theta_t+U)^* S_t}|\mathcal{F}_{t-1})}{E(e^{\Theta_t^* S_t}|\mathcal{F}_{t-1})} \\ &= \frac{M_{S_t|\mathcal{F}_{t-1}}(\Theta_t+U)}{M_{S_t|\mathcal{F}_{t-1}}(\Theta_t)} \ . \end{aligned}$$

□

By employing the multivariate conditional Esscher transform, we determine an equivalent martingale pricing measure under which the discounted price processes of the n risky assets at the risk-free interest rate $\{\frac{S_{jt}}{B_t}\}$, for $i = 1,2,\ldots,n$, are martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$. The following proposition presents a sufficient condition on the sequence Θ for $\mathcal{P}_\Theta$ to be an equivalent martingale measure.

Proposition 3.1. *Let $I_j := (0,0,\ldots,1,\ldots,0,0) \in \mathcal{R}^n$, where the 1 in n-dimensional vector I_j is in the j^{th} position, for each $j = 1,2,\ldots,n$. Suppose Θ_t satisfies the following system of n coupled non-linear equations:*

$$E[e^{\Theta_t^* S_t}(e^{I_j^* S_t} - e^{r_t})|\mathcal{F}_{t-1}] = 0 \ , \quad \text{for each } j = 1,2,\ldots,n \text{ and } t \in \mathcal{T} \ . \tag{3.12}$$

Then, the discounted price processes $\{\frac{S_{jt}}{B_t}\}$ are martingales with respect to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ under $\mathcal{P}_\Theta$, for each $j = 1,2,\ldots,n$.

Proof. We only need to show that for each $t \in \mathcal{T}$ and $j = 1,2,\ldots,n$,

$$E^{\Theta}(B_{t+1}^{-1} S_{j,t+1}|\mathcal{F}_t) = B_t^{-1} S_{jt} \ , \mathcal{P}\text{-a.s.} \tag{3.13}$$

The general case that for any $t,k \in \mathcal{T}$ and each $j = 1,2,\ldots,n$,

$$E^{\Theta}(B_{t+k}^{-1} S_{j,t+k}|\mathcal{F}_t) = B_t^{-1} S_{jt} \ , \mathcal{P}\text{-a.s.} \ , \tag{3.14}$$

can be shown easily by induction.
By the Bayes rule and the condition (3.12), for each $j = 1,2,\ldots,n$,

$$\begin{aligned} E^{\Theta}(B_{t+1}^{-1} S_{j,t+1}|\mathcal{F}_t) &= B_t^{-1} S_{jt} E\left[\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right) e^{I_j^* S_{t+1} - r_{t+1}}\middle|\mathcal{F}_t\right] \\ &= B_t^{-1} S_{jt} \frac{E(e^{(\Theta_t+I_j)^* S_{t+1} - r_{t+1}}|\mathcal{F}_t)}{E(e^{\Theta^* S_{t+1}}|\mathcal{F}_t)} \\ &= B_t^{-1} S_{jt} \ . \end{aligned} \tag{3.15}$$

Hence, the result follows. □

Proposition 3.2. *Suppose for each $t \in \mathcal{T}$, $\Theta^*_t(S_t e^{-r_t} - S_{t-1})$ either equals zero with probability one or has both signs with positive probability. Then, there exist a Θ_t satisfying the following system of n coupled non-linear equations:*

$$E[e^{\Theta^*_t S_t}(e^{I^*_j S_t} - e^{r_t})|\mathcal{F}_{t-1}] = 0 \ , \quad \textit{for each } j = 1, 2, \ldots, n \textit{ and } t \in \mathcal{T} \ ,$$

Proof. The proof is adapted to the argument in Bülhmann et al. [7]. First, we consider the target function $T_{t-1}(\Theta)$ defined as follows:

$$T_{t-1}(\Theta) := \ln E(e^{\Theta^*(S_t e^{-r_t} - S_{t-1})}|\mathcal{F}_t) \ , \quad \Theta \in \mathcal{R}^n \ .$$

Suppose that $T_{t-1}(\Theta)$ exists and is finite in the neighborhood of some Θ. Then, the condition on $\Theta^*_t(S_t e^{-r_t} - S_{t-1})$ in the proposition implies that the minimum of $T_{t-1}(\Theta)$ is attained at an interior point $\hat{\Theta} := (\hat{\Theta}_1, \hat{\Theta}_2, \ldots, \hat{\Theta}_n)$. Since $T_{t-1}(\hat{\Theta})$ is the minimum value of $T_{t-1}(\Theta)$,

$$\left.\frac{dT_{t-1}(\Theta)}{d\Theta_j}\right|_{\Theta = \hat{\Theta}} = \frac{E[e^{\hat{\Theta}^*(S_t e^{-r_t} - S_{t-1})}(S_{jt} e^{-r_t} - S_{j,t-1})|\mathcal{F}_{t-1}]}{E(e^{\hat{\Theta}^*(S_t e^{-r_t} - S_{t-1})}|\mathcal{F}_{t-1})} = 0 \ ,$$

for each $j = 1, 2, \ldots, n$.
This is equivalent to the martingale conditions:

$$E^{\hat{\Theta}}(S_{j,t} e^{-r_t}|\mathcal{F}_{t-1}) = S_{j,t-1} \ , \quad \mathcal{P} - \text{a.s.} \ ,$$

for each $j = 1, 2, \ldots, n$. □

By Proposition 3.2, the above martingale conditions are equivalent to the the following system of n coupled non-linear equations:

$$E[e^{\hat{\Theta}^*_t S_t}(e^{I^*_j S_t} - e^{r_t})|\mathcal{F}_{t-1}] = 0 \ , \quad \text{for each } j = 1, 2, \ldots, n \text{ and } t \in \mathcal{T} \ ,$$

It can also be shown that the martingale conditions can be written as follows:

$$\sum_{i_1=0}^{m-1} \sum_{i_2=0}^{m-1} \cdots \sum_{i_n=0}^{m-1} \left\{ \prod_{j=1}^{n} p^{(ji_j)}_{t|t-1} \exp\left(\sum_{j=1}^{n} S_{j,t-1} \Theta^{(j)}_t e^{L_{i_j}} \right) [\exp(S_{k,t-1} e^{L_{i_k}}) - \exp(r_t)] \right\} = 0 \ , \tag{3.16}$$

where $k = 1, 2, \ldots, n$.

Now, we consider a European-style contingent claim V written on the n risky assets with maturity at time T and payoff function $V(S_{1T}, S_{2T}, \ldots, S_{nT}, T)$ at time T. Then, a price of the contingent claim V_t at time t is given by:

$$V_t = E^{\Theta}\left[\exp\left(\sum_{k=t+1}^{T} r_k \right) V(S_{1T}, S_{2T}, \ldots, S_{nT}, T) \middle| \mathcal{F}_t \right] \ , \tag{3.17}$$

which is the conditional expectation of the discounted payoff of the claim V given the $\mathcal{F}_t$ under $\mathcal{P}_{\Theta}$.

Note that this pricing by the Esscher transform is not unique. There are other possible ways to determine a price of the option, such as, the minimum variance hedging in Duffie and Richardson [12] and Schweizer [28].

4. Consequences for option prices of the dependency of asset prices

In this section, we also document consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model. In particular, we investigate whether misspecification of the level of the dependency of the underlying asset prices can have significant impact on the option prices. We shall consider a financial model with one risk-free asset B and two risky assets S_1 and S_2. The state space of the return process $R^{(j)}$ of the j^{th} risky asset $(j = 1, 2)$ is given by $\{L_0, L_1, L_2 \ldots, L_{m-1}\}$. We shall consider the case that $m = 3$ and, hence, the state space is $\{L_0, L_1, L_2\}$. In practice, the returns of a risky asset take real values instead of the categorical values $\{L_0, L_1, L_2\}$. The discrete state space can only serve as a proxy for the "actual" state space of the returns. Similar procedure has been adopted in Elliott and Rishell [14] for approximating the state space of a short rate process. We assume that $R^{(j)}_t = L_i = (i - 1)/20$, for $i = 0, 1, 2$. In practice, we can adjust the coefficient $1/20$ and the number of points m in the discrete state space to obtain a desirable and accurate approximation result. First, we note that the dependency of the asset prices in the multivariate Markov chain model is described by the parameters λ_{ij}. In particular, λ_{ii} $(i = 1, 2)$ describes the intra-dependency of the price dynamics of the i^{th} underlying asset while λ_{ij} describes the inter-dependency of the price dynamics of the i^{th} asset on the price dynamics of the j^{th} asset. If $\lambda_{ii} = 1$, all of the weights are given to the intra-transition probability matrix $P^{(ii)}$. If $\lambda_{ij} = 1$, all of the weights are given to the inter-transition probability matrix $P^{(ij)}$.

First, we investigate the situation that the "true" model is described by the multivariate Markov chain model with a "Strong" level of the inter-dependency of the price dynamics of the two underlying assets while the "assumed" model used for the evaluation of option prices is described by the multivariate Markov chain model with the level of inter-dependency of the price dynamics of the two underlying assets ranged from "Strong" to "Weak". We suppose that in the "true" model, the model parameters are given as follows:

$$P^{(11)} = \begin{bmatrix} 0.4069 & 0.3995 & 0.5642 \\ 0.3536 & 0.5588 & 0.0470 \\ 0.2395 & 0.0416 & 0.3887 \end{bmatrix}, \qquad P^{(12)} = \begin{bmatrix} 0.2016 & 0.2737 & 0.2056 \\ 0.2970 & 0.1303 & 0.4917 \\ 0.5014 & 0.5959 & 0.3027 \end{bmatrix},$$

$$P^{(21)} = \begin{bmatrix} 0.2554 & 0.2814 & 0.4571 \\ 0.7321 & 0.3558 & 0.2542 \\ 0.0126 & 0.3628 & 0.2887 \end{bmatrix}, \qquad P^{(22)} = \begin{bmatrix} 0.5102 & 0.5239 & 0.1434 \\ 0.3736 & 0.3925 & 0.4204 \\ 0.1162 & 0.0835 & 0.4361 \end{bmatrix},$$

and

$$\Lambda = \begin{bmatrix} 0.0000 & 1.0000 \\ 0.5000 & 0.5000 \end{bmatrix}.$$

In this case, 100% is allocated to the inter-transition probability matrix $P^{(12)}$ under the "true" model. This represents a "Strong" inter-dependency effect.

For the "assumed" model, we assume the same intra-transition probability matrices and inter-transition probability matrices with those in the "true" model. Also, the parameters λ_{21} and λ_{22} are the same as those in the "true" model. The only difference is that λ_{11} increases from 0.0000 to 1.0000; in other words, λ_{12} decreases from 1.0000 to 0.0000. This means that the inter-dependent effect decreases from maximum to minimum and that the intra-dependent effect increases from minimum to maximum.

We consider a European-style exchange option written on the two risky assets, which provides a buyer with the right, but not the obligation, to exchange the first risky asset for the second risky asset at the maturity of the option. The payoff function of the exchange option at the maturity time $T = 3$ is given by $V(S_{1T}, S_{2T}, T) := \max(S_{2T} - S_{1T}, 0)$. We then adopt the option pricing formula in Equation (22) to determine the price of the exchange option. We first assume that the initial prices of the two risky assets at time zero are both equal to 100; that is, the exchange option is at-the-money. We also suppose that the compound risk-free interest rate is constant and equals 2.5%. Note that the price of the exchange option obtained from the "true" model is 0.4776. Figure 1 displays the plot of the prices for the exchange option obtained from the "assumed" models against different levels of inter-dependence λ_{12}.

Now, we consider the case that the exchange option is out-of-the-money; that is, $S_{10} = 120$ and $S_{20} = 100$. We assume that the parameters in the "true" model and the "assumed" models remain the same as before. In this case, the price of the exchange option obtained from the "true" model is 5.6903e-005. Figure 2 presents the plot of the prices for the exchange option obtained from the "assumed" models against different levels of inter-dependence λ_{12}.

Finally, we consider the case that the exchange option is in-the-money; that is, $S_{10} = 80$ and $S_{20} = 100$. We assume that the parameters in the "true" model and the assumed models remain the same as before. In this case, the price of the exchange option obtained from the "true" model is 17.9539. Figure 3 displays the plot of the prices for the exchange option obtained from the "assumed" models against different levels of inter-dependence λ_{12}.

From Figures 1-3, we can see that the option prices change significiantly as the level of dependency λ_{12} varies for all cases, namely, at-the-money, in-the-money and out-of-the-money. Given the configuration for the transition probability matrices, the option prices implied by the "assumed" model increase as the "assumed" level of the dependency does for all cases. In other words, if the "true" model has a "Strong" level of the dependency, say $\lambda_{12} = 1$, the underpricing of the "assumed" model becomes more pronounced when the "assumed" level of the dependency decreases from $\lambda_{12} = 1$ to $\lambda_{12} = 0$.

5. Conclusion

We have developed an option valuation model in the context of a discrete-time multivariate Markov chain model using the conditional Esscher transform introduced by Bühlmann et al. [6]. This model can provide market practitioners with a flexible way to incorporate the dependency of the underlying asset price processes in a discrete framework. It also allows the price of an individual risky asset taking finitely many values. We have documented consequences for option prices of the dependency of the underlying asset prices described by the multivariate Markov chain model.

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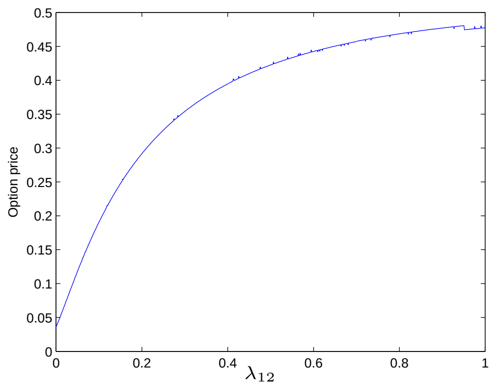

Fig. 1. At-the-money with various value of λ_{12}

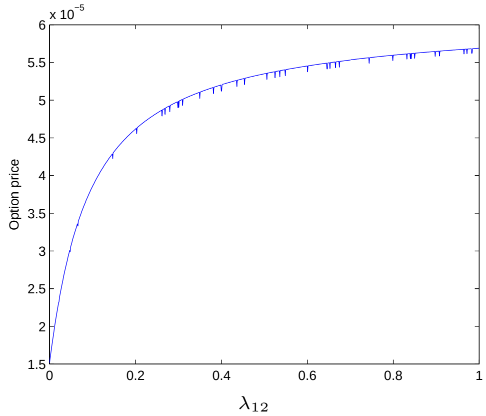

Fig. 2. Out-of-money with various value of λ_{12}

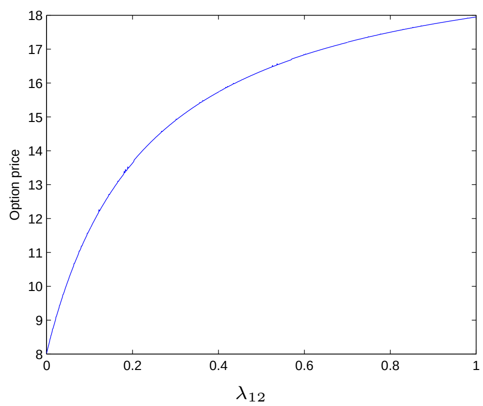

Fig. 3. In-the-money with various value of λ_{12}