International Conference
on Industrial Engineering and Systems Management
IESM 2007
May 30 - June 2
BEIJING - CHINA

北京 二OO七

On Hybrid Re-manufacturing Systems: A Matrix Geometric Approach ⋆

Wai-Ki CHING [a], Tang LI [b], Jun-Gong XUE [c]

[a] *Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, China.*

[b] *Advanced Modeling and Applied Computing Laboratory, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, China.*

[c] *Department of Mathematics, Fudan University, Shanghai, China*

Abstract

In this paper, we propose a model for hybrid re-manufacturing system. The system consists of both manufacturing and re-manufacturing processes. There are two types of inventory to be managed in the system: the returns and the serviceable product. The serviceable product is controlled by an (s, S) continuous review policy. We assume the arrival processes of demands and returns follow a Poisson process and there is no rejection of returns from the system. The outside procurement for the serviceable product is also allowed when there is shortage. The hybrid system is then modeled by an Markovian queueing network. Matrix geometric method is applied to analyze the resulting queueing network and we derive the average system running cost in terms of the steady-state probability distribution. A quasi-optimization problem is then formulated for obtaining the optimal procurement size.

Key words: Inventory, Markov Process, Re-manufacturing Systems, Matrix Geometric Method, Steady-State Probability Distribution

1 Introduction

In the world of limited disposal capacities, there is always an environmental pressure in using re-manufacturing systems, a recycling process to reduce the amount of waste generated by the manufacturers. Moreover, it is also very popular nowadays that consumers are allowed to return a purchased product within a given period with a full refund. This will of course create a large amount of returns and therefore handling returns is a key issue in running a business. Take for example a major manufacturer of copy machines Xerox has been putting effort into re-manufacturing used equipment. Xerox reports an annual savings of several hundred million dollars due to the re-manufacturing and the re-use of equipment and parts. At the same time they also divert more than fifty thousand tones of waste from the waste stream, see for instance [4]. In this paper, we assume that a return is first repaired/tested and then it will be re-sold to the market. The result of introducing a re-manufacturing process in the business is that the

⋆ Research supported in part by HKU Strategic Research Theme Fund on Computational Physics and Numerical Methods, HKU CRGC Grant and NSFC under grant No. 10571031. This paper was not presented at any other revue. Corresponding author:Tang LI, Email:litang@hkusua.hku.hk Fax (852)2559-2225

Email addresses: `wching@hkusua.hku.hk` (Wai-Ki CHING), `litang@hkusua.hku.hk` (Tang LI), `xuej@fudan.edu.cn` (Jun-Gong XUE).

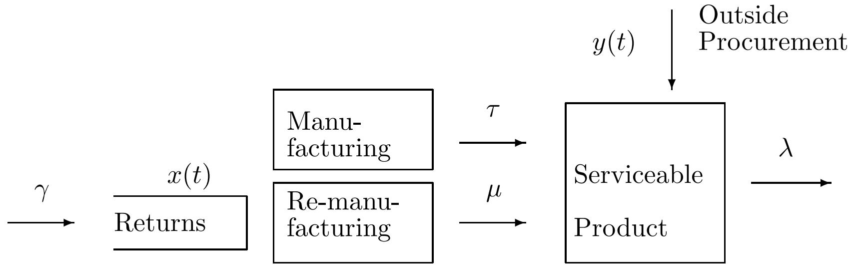

Fig. 1. The hybrid manufacturing system for returns and demands.

manufacturers have to take the returns into account in their production planning. Moreover, it will also complicate the inventory control process. Markovian queueing model is a useful tool for modeling manufacturing systems [2] and flexible manufacturing system [3]. Here we employ it to model the hybrid manufacturing system.

A simple Markovian model for handling returns has been proposed in [3] where the re-manufacturing time is assumed to be zero. The model is then further extended to the case of exponentially distributed re-manufacturing time with a buffer for holding the returns [10]. Models allow the disposal of returns when the buffer for returns is full have been studied in [7,9,11]. In [11], a fast direct method for solving the steady-state probability distribution of the hybrid manufacturing systems based on the Fast Fourier Transformation (FFT) has been introduced. In [4], Fleischmann illustrates the facets of the reverse logistics in the example of IBM. IBM's business activities as a leading manufacturer of IT equipment and services involve several groups of "reverse" goods flows. The total annual volume has an amount up to several ten thousand tonnes world-wide. Other reviews on different models in reverse logistics can also be found in the book by Muckstadt [8].

The remainder of this paper is organized as follows. In Section 2, we introduce the hybrid system by giving the basic assumptions of our model. We then gives the generator matrix of the hybrid system. In Section 3, we discuss the stability of the hybrid system using matrix geometric method. In Section 4, we give analyze the asymptotic behavior of the steady-state probability distribution. We express the average system running cost in terms of the steady-state probability distribution. In Section 5, we then formulate an quasi-optimization problem for solving the optimal procurement policy. Finally concluding remarks are given in Section 6 to address further research issues.

2 The Hybrid Re-manufacturing System

In this section, we propose a hybrid system which consists of both manufacturing and re-manufacturing processes. In the proposed model, there are two types of inventory to be managed: the serviceable product and the returns. Here we assume that the demands and the returns follow two independent Poisson processes with mean rates λ and γ respectively. The re-manufacturing process is then modeled by an M/M/1 queue. A return acts as a customer and the re-manufacturing process (with processing rate μ) acts as the server in the queue. The re-manufacturing process is stopped only when there is no more space for placing the serviceable product or there is no return available for re-manufacturing. Here we also assume that when the return level is zero, in order to satisfy the demands, the manufacturing part of the hybrid system can produce at a rate of τ with exponential production time. The serviceable product and the outside procurements are controlled by a (s, S) continuous review policy. This means that when the inventory level drops to s, an outside procurement order of size $(S - s)$ is placed and arrived at once. For simplicity of discussion, we assume that the procurement level s is -1. Therefore when the inventory level is -1, i.e. there is a backlog, the procurement can clear the backlog and bring the serviceable product inventory up to S. We assume that it is always possible to purchase the required procurement. The inventory levels of both the returns and the serviceable product are modelled as Markovian process. Figure 1 gives an illustration of the framework of the hybrid system.

We then give the generator matrix of the hybrid system. We let $x(t)$ and $y(t)$ be the inventory levels of the returns and the inventory levels of the serviceable products at time t respectively. Then $x(t)$ and $y(t)$

take integral values in $[0, \infty]$ and $[0, Q]$ respectively. The joint inventory process

$$\{(x(t), y(t)),\ t \geq 0\}$$

is a continuous time Markov chain taking values in the following state-space

$$S = \{(x, y) : x = 0, 1, \ldots, \quad y = 0, 1, \ldots, Q\}.$$

We order the joint inventory states lexicographically, according to x first and then y. The generator matrix of the joint inventory process can be written as follows:

$$P = \begin{pmatrix} B & A_0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & \cdots \\ 0 & 0 & A_2 & A_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1}$$

where

$$B = \begin{pmatrix} -(\gamma+\tau+\lambda) & \tau & & & \lambda \\ \lambda & -(\gamma+\tau+\lambda) & \tau & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & -(\gamma+\tau+\lambda) & \tau \\ 0 & & & \lambda & -(\gamma+\lambda) \end{pmatrix},$$

$$A_0 = \gamma I_{Q+1}, \qquad A_2 = \begin{pmatrix} 0 & \mu & & 0 \\ & 0 & \mu & \\ & & \ddots & \ddots \\ & & & \ddots & \mu \\ 0 & & & & 0 \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} -(\gamma+\lambda+\mu) & & & & \lambda \\ \lambda & -(\gamma+\lambda+\mu) & & & \\ & \lambda & \ddots & & \\ & & \ddots & \ddots & \\ & & & \lambda & -(\gamma+\lambda+\mu) & \\ 0 & & & & \lambda & -(\gamma+\lambda) \end{pmatrix}.$$

3 The Stability of the Hybrid System

In this section, we study the stability of the hybrid system. Let

$$A = A_0 + A_1 + A_2 = \begin{pmatrix} -(\lambda+\mu) & \mu & & & \lambda \\ \lambda & -(\lambda+\mu) & \mu & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & -(\lambda+\mu) & \mu \\ 0 & & & \lambda & -\lambda \end{pmatrix}$$

and $\boldsymbol{\nu}$ be the stationary distribution of A, i.e.,

$$\boldsymbol{\nu} A = 0, \quad \boldsymbol{\nu}\mathbf{e} = 1.$$

It is straight-forward to check that

$$\boldsymbol{\nu} = c\left(1, 1+\frac{\mu}{\lambda}, 1+\frac{\mu}{\lambda}+\left(\frac{\mu}{\lambda}\right)^2, \ldots, \sum_{k=0}^{Q}\left(\frac{\mu}{\lambda}\right)^k\right),$$

with

$$c^{-1} = \sum_{k=0}^{Q}(Q+1-k)\left(\frac{\mu}{\lambda}\right)^k = \frac{Q+1-(Q+2)\rho+\rho^{Q+2}}{(1-\rho)^2} \tag{2}$$

where

$$\rho = \frac{\mu}{\lambda}.$$

Proposition 1 *If*

$$\frac{\gamma}{\mu} < \frac{f(Q)}{f(Q+1)} \tag{3}$$

then the hybrid system is stable. Here

$$f(Q) = Q-(Q+1)\rho+\rho^{Q+1}.$$

Proof : To make the queue stable, we need the following condition:

$$\boldsymbol{\nu} A_2\mathbf{e} > \boldsymbol{\nu} A_0\mathbf{e},$$

i.e.,

$$\gamma < \mu\left(1 - c\sum_{k=0}^{Q}\left(\frac{\mu}{\lambda}\right)^k\right).$$

Using (2), the result follows.

Proposition 2 *If* $\gamma < \min\{\mu, \lambda\}$ *then there exists* Q_0 *such that the hybrid system is stable for* $Q \geq Q_0$.

Proof : We note that

$$f'(Q) = 1-\rho+\rho^{Q+1}\ln(\rho)$$

and

$$f''(Q) = \rho^{Q+1}(\ln(\rho))^2 > 0.$$

Therefore $f(Q)$ is a convex function in Q.

For $Q \geq 0$ and $\rho > 0$, we have $f'(Q) \geq 0$. Thus $f(Q)$ is a monotonic increasing function in Q.

For $\rho \leq 1$, we have

$$\lim_{Q\to\infty}\frac{f(Q)}{f(Q+1)} = 1.$$

Thus for large enough Q, we have (3) holds. When $\rho > 1$, we have

$$\lim_{Q\to\infty}\frac{f(Q)}{f(Q+1)} = \frac{1}{\rho} = \frac{\lambda}{\mu}.$$

Again for large enough Q, we have (3) holds as $\gamma < \lambda$.

4 The Steady-state Probability Distribution

In this section, we first give an analysis on the steady-state probability distribution. We then give the average running cost of the hybrid system. For the hybrid system, we are interested in the inventory costs of holding the returns and the serviceable product.

In the following, we derive the expected inventory levels for both returns and serviceable product. Under stable condition (3), there exists a unique steady-state distribution

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots,)$$

for the underlying Markov chain of the hybrid system. According to the matrix geometric method [1,5,6],

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 R^k,$$

where R is the minimal nonnegative solution to the matrix equation

$$A_0 + RA_1 + R^2 A_2 = 0, \tag{4}$$

and $\boldsymbol{\pi}_0$ satisfies

$$\boldsymbol{\pi}_0(B + RA_2) = 0$$

and is normalized as

$$\boldsymbol{\pi}_0(I - R)^{-1}\mathbf{e} = 1.$$

We denote η_Q the Perron eigenvalue of R and $\mathbf{x}_Q$ the normalized left Perron eigenvector, i.e.,

$$\mathbf{x}_Q^T R = \eta_Q \mathbf{x}_Q^T \quad \text{and} \quad \mathbf{x}_Q^T \mathbf{e} = 1.$$

Pre-multiplying $\mathbf{x}_Q^T$ to both sides of (4) we get

$$\mathbf{x}_Q^T A(\eta_Q) = 0,$$

where $A(z)$ is defined as

$$A(z) = A_0 + zA_1 + z^2 A_2 = \begin{pmatrix} b(z) & c(z)/\lambda z & & & & \lambda z \\ \lambda z & b(z) & c(z)/\lambda z & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda z & b(z) & c(z)/\lambda z \\ 0 & & & & \lambda z & b(z) + \mu z \end{pmatrix},$$

where

$$b(z) = \gamma - (\gamma + \lambda + \mu)z \quad \text{and} \quad c(z) = \lambda\mu z^3.$$

For $z > 0$, we denote $\chi(z)$ the Perron eigenvalue of $A(z)$, i.e., the eigenvalue with maximal real part. Then η_Q is the unique solution to the equation

$$\chi(z) = 0, \quad 0 < z < 1, \tag{5}$$

which gives that η_Q is the unique root to the equation

$$\det A(z) = 0, \qquad 0 < z < 1, \tag{6}$$

where $\det B$ denotes the determinant of B. It is straight-forward to show that

$$\det A(z) = (-1)^{Q+2}(\lambda z)^{Q+1} + \det S_{Q+1}(z),$$

where $S_k(z)$ is a tri-diagonal matrix of dimension k of the form

$$S_k(z) = \begin{pmatrix} b(z) & c(z)/\lambda z & & & & 0 \\ \lambda z & b(z) & c(z)/\lambda z & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \lambda z & b(z) & c(z)/\lambda z \\ 0 & & & & \lambda z & b(z)+\mu z \end{pmatrix}.$$

Taking the advantage of the relation

$$\det S_{k+2}(z) = b(z)\det S_{k+1}(z) - c(z)\det S_k(z)$$

with

$$\det S_0(z) = 1, \quad \det S_1(z) = b(z) + \mu z,$$

we have

$$\det A(z) = (-1)^{Q+2}(\lambda z)^{Q+1} + c_1\left(\frac{b(z)+\sqrt{b^2(z)-4c(z)}}{2}\right)^{Q+1}$$
$$+c_2\left(\frac{b(z)-\sqrt{b^2(z)-4c(z)}}{2}\right)^{Q+1},$$

where

$$c_1 = \frac{2\mu z + b(z) + \sqrt{b^2(z)-4c(z)}}{2\sqrt{b^2(z)-4c(z)}}$$

and

$$c_2 = \frac{-2\mu z - b(z) + \sqrt{b^2(z)-4c(z)}}{2\sqrt{b^2(z)-4c(z)}}.$$

We know that $\det A(z)$ is a polynomial with respect to z.

We end this section by the following technical remarks on computation:

(i) Although one can calculate η_Q by solving equation (6), where $\det A(z)$ can be computed in $O(Q)$ flops, we compute it in an alternative way. For $z > 0$, $\chi(z)$ is convex and in the interval $[0, 1]$, $\chi(z)$ first decreases as z increases and after reaching the minimum, it grows as z increase to 1. We thus can use bisection method to solve (5) to obtain η_Q. At each step we need to determine whether $\chi(z) < 0$.

(ii) We note that the off-diagonal of $A(z)$ is nonnegative. Therefore $\chi(z) < 0$ if and only if $-A(z)$ is an M-matrix [6]. To determine whether $\chi(z) < 0$, we need not compute $\chi(z)$, instead we do LU factorization (without pivoting) of $A(z)$. If all the pivots are nonnegative, then $\chi(z) < 0$. Otherwise, if we encounter positive pivots in the LU factorization, we quit the factorization and know that $\chi(z) > 0$.

(iii) Taking advantage of the structure of $A(z)$, one can perform the factorization in $O(Q)$ steps. Moreover, because the sign structure of $A(z)$, one can also develop high-accuracy algorithm for this factorization without pivoting.

5 The Optimization Problem

In this section, we formulate a quasi-optimization problem for solving the optimal procurement policy. We denote the unit inventory cost of the returns by s_{re} and the unit inventory cost of the serviceable by s_{se}. We are interested in minimizing the inventory costs by choosing a proper Q.

The expected costs for holding the returns is given by

$$s_{re}\sum_{i=1}^{\infty} i \times \boldsymbol{\pi}_i \mathbf{e} = s_{re}\boldsymbol{\pi}_0(\sum_{i=1}^{\infty} i \times R^i)\mathbf{e} = s_{re}\boldsymbol{\pi}_0 R(I-R)^{-2}\mathbf{e} \tag{7}$$

and the expected costs of holding the serviceable is

$$s_{se}\boldsymbol{\pi}_0(I-R)^{-1}D\mathbf{e},$$

where

$$D = \mathrm{Diag}(0,1,2,\ldots,Q)$$

is a diagonal matrix. Consequently, the optimization problem can be formulated as

$$\begin{cases} & \min_{Q}\left\{s_{re}\boldsymbol{\pi}_0 R(I-R)^{-2}\mathbf{e} + s_{se}\boldsymbol{\pi}_0(I-R)^{-1}D\mathbf{e}\right\} \\ \text{Subject to} & \\ & \gamma/\mu < \dfrac{f(Q)}{f(Q+1)}. \end{cases}$$

However, this optimization problem is very difficult to solve. We thus consider the quasi-optimization problem. We observe that $\boldsymbol{\pi}_0(I-R)^{-1}$ can be viewed as one step of inverse iteration with the initial guess $\boldsymbol{\pi}_0$ and that

$$\boldsymbol{\pi}_0(I-R)^{-1}\mathbf{e} = 1.$$

One can then approximate $\boldsymbol{\pi}_0(I-R)^{-1}$ by $\mathbf{x}_Q^T$ and the expected inventory cost can be estimated as

$$s_{re}\frac{\eta_Q}{1-\eta_Q} + s_{se}\mathbf{x}_Q^T D\mathbf{e}.$$

Therefore we can explicitly write down $\mathbf{x}_Q^T$. Let

$$\mathbf{x}_Q^T = (x_0, x_1, x_2, \ldots, x_Q).$$

Then using the relation

$$(\lambda z)x_{k+2} + b(\eta_Q)x_{k+1} + (c(\eta_Q)/\lambda\eta_Q)x_k = 0,$$

we have

$$x_k = c_3 h_1^k(\eta_Q) + c_4 h_2^k(\eta_Q),$$

where

$$h_1(z) = \frac{-b(z)+\sqrt{b^2(z)-4c(z)}}{2\lambda z}, \quad h_2(z) = \frac{-b(z)-\sqrt{b^2(z)-4c(z)}}{2\lambda z}$$

and

$$c_3 = \alpha\frac{h_1(\eta_Q)}{2\sqrt{b^2(\eta_Q)-4c(\eta_Q)}}, \qquad c_4 = -\alpha\frac{h_2(\eta_Q)}{2\sqrt{b^2(\eta_Q)-4c(\eta_Q)}},$$

where α is chosen such that

$$\sum_{k=0}^{Q} x_k = 1.$$

Now the inventory cost for storing the serviceable product can be expressed as follows:

$$f_Q = c_3\frac{1-(Q+1)h_1^{Q+1}(\eta_Q)+Qh_1^{Q+2}(\eta_Q)}{(1-h_1(\eta_Q))^2} + c_4\frac{1-(Q+1)h_2^{Q+1}(\eta_Q)+Qh_2^{Q+2}(\eta_Q)}{(1-h_2(\eta_Q))^2}.$$

In short, f_Q is a rational function of η_Q. We are now to solve the quasi-optimization problem:

$$\begin{cases} & \min_{Q}\left\{\dfrac{s_{re}\eta_Q}{(1-\eta_Q)} + s_{se}f_Q\right\} \\ \text{Subject to} & \\ & \eta_Q \text{ is the solution to } \det A(z)=0,\ 0<z<1. \end{cases}$$

We note that the objective function of this problem is a rational function of η_Q, and the constraint is that η_Q is the root to a polynomial equation. Thus given Q, η_Q and $\mathbf{x}_Q$ are very easy to calculate. Hence the optimal value of Q can also be easily obtained by performing a linear search.

Finally we give a numerical example as follows: Suppose that $\tau = 1.0$, $\mu = 0.6$, $\gamma = 0.1$, $\lambda = 0.3$ and $s_{re} = s_{se} = 1$. The optimal Q here is 1.

Table 1
The cost for different Q.

Q	1	2	3	4	5	6	7	8	9	10
η_Q	0.2500	0.0973	0.0879	0.1250	0.0833	0.0825	0.082	0.1414	0.0814	0.0813
Cost	0.3333	0.5402	0.7472	1.4956	0.5903	0.5130	0.5405	2.5318	0.5262	0.5212

6 Concluding Remarks

In this paper, we consider a hybrid system. The hybrid system consists of both manufacturing and re-manufacturing processes. We model the re-manufacturing process by a M/M/1 queue and we discuss the stability of the hybrid system. We give the generator matrix of the joint inventory process of manufacturing and re-manufacturing. The steady-state probability distribution is then obtained by using the matrix geometric method. Finally we formulate a quasi-optimization problem for solving the optimal procurement policy.

The followings are some future research issues: (i) Extension of the model to handle the case when there are more than one product in the system. For example, the hybrid system can handle several different products and they may make of more than one part. In the IBM example mentioned in Section one taken from [4], it encompass the following three categories: used machines, unused machines and the rotable spare parts. The first category can be further divided into three parts: lease returns, trade-in offers and environmental take-back. While the second category can be divided into retailer stock rotation and canceled orders. (ii) The re-manufacturing or the manufacturing process can follow a more general processing time distribution rather than the exponential distribution. For example, the re-manufacturing process can be a G/G/1 queue. It is interesting to further examine the above cases.

References

[1] A. Alfa. Waiting time distribution of the map/d/k system in discrete time - a more efficient algorithm. *Operations Research Letters*, 31, 03.

[2] W. Ching. *Iterative Methods for Queuing and Manufacturing Systems.* Springer, London, 01.

[3] Yuen W. Ching, W. and W. Loh. An inventory model with returns and lateral transshipments. *J. Operat. Res. Soc.*, 54:636–641, 03.

[4] M. Fleischmann. *Quantitative Models for Reverse Logistics, Lecture Notes in Economics and Mathematical System, 501.* Springer, 01.

[5] G. Latouche and V. Ramaswami. *Introduction to Matrix Analytic Methods in Stochastic Modeling.* SIAM, 89.

[6] G. Latouche and P. Taylor. *Matrix-Analytic Methods Theory and Applications.* World Scientific, 02.

[7] H. Lee. A multi-echelon inventory model for repairable items with emergency lateral transshipments. *Manag. Sci.*, 33:1302–1316, 87.

[8] J. Muckstadt. *Analysis and Algorithms for Service Parts Supply Chains.* Springer, 05.

[9] R. Dekker, M. Salomon, E. van der Laan and A. Ridder. An (s,q) inventory model with re-manufacturing and disposal. *Inter. J. Prod. Econ.*, 46:339–350, 01.

[10] W. Ching and W. Yuen. Iterative methods for re-manufacturing systems. *Inter. J. Appl. Math.*, 9:335–347, 02.

[11] W. Yuen, W. Ching and M. Ng. A direct method for solving block-toeplitz with near-circulant-block systems with applications to hybrid manufacturing systems. *Journal of Numerical Linear Algebra with Applications*, 12:957–966, 05.