Meromorphic solutions of higher order Briot–Bouquet differential equations

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Abstract

For differential equations $P(y^{(k)}, y) = 0$, where P is a polynomial, we prove that all meromorphic solutions having at least one pole are elliptic functions, possibly degenerate.

1. *Introduction*

According to a theorem of Weierstrass, meromorphic functions y in the complex plane $\mathbb{C}$ that satisfy an algebraic addition theorem

$$Q(y(z+\zeta), y(z), y(\zeta)) \equiv 0, \quad \text{where } Q \not\equiv 0 \text{ is a polynomial,} \tag{1·1}$$

are elliptic functions, possibly degenerate [**17**, **1**].

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More precisely, let us denote by W the class of meromorphic functions in $\mathbb{C}$ that consists of doubly periodic functions, rational functions and functions of the form $R(e^{az})$ where R is rational and $a \in \mathbb{C}$. Then each function $y \in W$ satisfies an identity of the form (1·1), and conversely, every meromorphic function[1]that satisfies such an identity belongs to W.

One way to prove this result is to differentiate (1·1) with respect to ζ and then set $\zeta = 0$. Then we obtain a Briot–Bouquet differential equation

$$P(y', y) = 0.$$

The fact that every meromorphic solution of such an equation belongs to W was known to Abel and Liouville, but probably it was stated for the first time in the work of Briot and Bouquet [**5**, **6**].

Here we consider meromorphic solutions of higher order Briot–Bouquet equations

$$P(y^{(k)}, y) = 0, \quad \text{where } P \text{ is a polynomial.} \tag{1·2}$$

Picard [**18**] proved that for $k = 2$, all meromorphic solutions belong to the class W. This work was one of the first applications of the famous Picard's theorems on omitted values.

In the end of 1970-s Hille [**12**, **13**, **14**, **15**] considered meromorphic solutions of (1·2) for arbitrary k. The result of Picard was already forgotten, and Hille stated it as a conjecture. Then Bank and Kaufman [**4**] gave another proof of Picard's theorem.

These investigations were continued in [**8**]. To state the main results from [**8**] we assume without loss of generality that the polynomial P in (1·2) is irreducible. Let F denote the compact Riemann surface defined by the equation

$$P(p, q) = 0. \tag{1·3}$$

Then every meromorphic solution y of (1·2) defines a holomorphic map $f : \mathbb{C} \to F$. According to another theorem of Picard, a Riemann surface which admits a non-constant holomorphic map from $\mathbb{C}$ has to be of genus 0 or 1, ([**19**], see also [**2**]). The following theorems were proved in [**8**]:

Theorem A. *If F is of genus* 1*, then every meromorphic solution of* (1·2) *is an elliptic function.*

Theorem B. *If k is odd, then every meromorphic solution of* (1·2) *having at least one pole, belongs to the class W.*

The main result of the present paper is the extension of Theorem B to the case of even k.

Theorem 1. *If y is a meromorphic solution of an equation* (1·2) *and y has at least one pole, then $y \in W$.*

This can be restated in the following way. *Let y be a meromorphic function in the plane which is not entire and does not belong to W. Then y and $y^{(k)}$ are algebraically independent.*

It is easy to see that for every function y of class W and every natural integer k there exists an equation of the form (1·2) which y satisfies.

[1] A "meromorphic function" in this paper means a function meromorphic in the complex plane, unless some other domain is specified. See [**17**, **20**] for discussion of the equation (1·1) in more general classes of functions.

It is not true that all meromorphic solutions of higher order Briot–Bouquet equations belong to W, a simple counterexample is $y''' = y$. We don't know whether non-linear irreducible counterexamples exist.

In the process of proving of Theorem 1 we will establish an estimate of the degrees of possible meromorphic solutions in terms of the polynomials P. Here by degree of a function of class W we mean the degree of a rational function y, or the degree of R in $y(z) = R(e^{az})$, or the number of poles in the fundamental parallelogram of an elliptic function y. Thus our result permits in principle the determination of all meromorphic solutions having at least one pole of a given equation (1·2).

Our method of proof is based on the so-called "finiteness property" of certain autonomous differential equations: there are only finitely many formal Laurent series with a pole at zero that satisfy these equations. The idea seems to occur for the first time in [**12**, p. 274] but the argument given there contains a mistake. This mistake was corrected in [**8**]. Later the same method was applied in [**7**] and [**10**] to study meromorphic solutions of other differential equations.

2. *Preliminaries*

We will use the following refined version of Wiman–Valiron theory which is due to Bergweiler, Rippon and Stallard.

Let y be a meromorphic function and G a component of the set $\{z : |y(z)| > M\}$ which contains no poles (so G is unbounded). Set

$$M(r) = M(r, G, y) = \max\{|y(z)| : |z| = r,\, z \in G\},$$

and

$$a(r) = d \log M(r)/d \log r = rM'(r)/M(r). \tag{2·1}$$

This derivative exists for all r except possibly a discrete set. According to a theorem of Fuchs [**11**],

$$a(r) \to \infty, \quad r \to \infty,$$

unless the singularity of y at ∞ is a pole. For every $r > r_0 = \inf\{|z| : z \in G\}$ we choose a point z_r with the properties $|z| = r,\ |y(z_r)| = M(r)$.

THEOREM C. *For every $\tau > 1/2$, there exists a set $E \subset [r_0, +\infty)$ of finite logarithmic measure, such that for $r \in [r_0, \infty)\backslash E$, the disk*

$$D_r = \{z : |z - z_r| < ra^{-\tau}(r)\}$$

is contained in G and we have

$$y^{(k)}(z) = \left(\frac{a(r)}{z}\right)^k \left(\frac{z}{z_r}\right)^{a(r)} y(z)(1 + o(1)), \quad r \to \infty, \quad z \in D_r. \tag{2·2}$$

When y is entire, this is a classical theorem of Wiman. Wiman's proof used power series, so it cannot be extended to the situation when y is not entire. A more flexible proof, not using power series is due to Macintyre [**16**]; it applies, for example to functions analytic and unbounded in $|z| > r_0$. The final result stated above was recently established in [**3**].

3. *Proof of Theorem 1*

In what follows, we always assume that the polynomial P in (1·2) is irreducible.

To state a result of [**8**] which we will need, we introduce the following notation. Let A be the field of meromorphic functions on F. The elements of A can be represented as rational functions $R(p,q)$ whose denominators are co-prime with P. In particular, p and q in (1·3) are elements of A. For $\alpha \in A$ and a point $x \in F$, we denote by $\mathrm{ord}_x\alpha$ the order of α at the point x. Thus if $\alpha(x) = 0$ then $\mathrm{ord}_x\alpha$ is the multiplicity of the zero x of α, if $\alpha(x) = \infty$ then $-\mathrm{ord}_x\alpha$ is the multiplicity of the pole, and $\mathrm{ord}_x\alpha = 0$ at all other points $x \in F$.

Let $I \subset F$ be the set of poles of q. For $x \in I$ we set $\kappa(x) = \mathrm{ord}_x p/\mathrm{ord}_x q$.

Theorem D. *Suppose that an irreducible equation* (1·2) *has a transcendental meromorphic solution y. Let $f : \mathbb{C} \to F$ be the holomorphic map defined by $z \mapsto (y^{(k)}(z), y(z))$. Then:*
a) Every pole of p belongs to I.
b) For every $x \in I$, the number $\kappa(x)$ is either 1 *or* $1 + k/n$*, where n is a positive integer.*
c) If $\kappa(x) = 1 + k/n$ for some $x \in I$, then the equation $f(z) = x$ has infinitely many solutions, and all these solutions are poles of order n of y.
d) If $\kappa(x) = 1$ for some $x \in I$, then the equation $f(z) = x$ has no solutions.

Picard's theorem on omitted values implies that $\kappa(x) = 1$ can happen for at most two points $x \in I$. For the convenience of the reader we include a proof of Theorem D in the Appendix.

The numbers $\kappa(x)$ can be easily determined from the Newton polygon of P. Thus Theorem D gives several effective necessary conditions for the equation (1·2) to have meromorphic or entire solutions.

Remark. The proof of Theorem D in [**8**] uses Theorem C which was stated in [**8**] but not proved. One can also give an alternative proof of Theorem D, using Nevanlinna theory instead of Theorem C, by the arguments similar to those in [**9**].

Lemma 1. *Suppose that y is a meromorphic solution of* (1·2). *If $\kappa(x) = 1$ for some $x \in I$ then y has order one, normal type.*

Proof. In view of Theorem A and Theorem D, d), we conclude that the genus of F is zero. Therefore, we can find $t = R(p,q)$ in A which has a single simple pole at x. Then $w = R(y^{(k)}, y)$ is an entire function by Theorem D, d). As t has a simple pole at x, the element $1/t \in A$ is a local parameter at x, and in a neighborhood of x we have

$$q = at^m + \dots \quad \text{and} \quad p = bt^m + \dots,$$

where $-m = \mathrm{ord}_x p = \mathrm{ord}_x q$ as $\kappa(x) = 1$, and the dots stand for the terms of degree smaller than m. Substituting $p = y^{(k)}$ and $q = y$ and differentiating the first equation k times we obtain for w a differential equation of the form

$$\frac{d^k}{dz^k} w^m + \dots = (b/a) w^m, \tag{3·1}$$

where the dots stand for the terms of degree smaller than m. Now we use a standard argument of Wiman–Valiron theory. Applying Theorem C to the entire function w^m, with $G = \mathbb{C}$ and $z = z_r$, we compare the asymptotic relations (2·2) and (3·1) to conclude that $a(r) \sim cr$, where $c \neq 0$ is a constant. This implies $\log M(r) \sim cr$, which means that w is of order 1, normal type. So y is also of order 1, normal type, because w and y satisfy a polynomial relation of the form $P(y, w) = 0$, where P is a polynomial with constant coefficients.

LEMMA 2. *Suppose that y is a meromorphic solution of* (1·2). *If $\kappa(x_1) = \kappa(x_2) = 1$ for two different points x_1 and x_2 in I, then y is a rational function of e^{az}, where $a \in \mathbb{C}$.*

Proof. As in the previous lemma, the genus of F is zero. Let $t = R(p, q)$ be a function in A with a single simple pole at x_1 and a single simple zero at x_2. Then $w = R(y^{(k)}, y)$ is an entire function of order 1, normal type (by Lemma 1) omitting 0 and ∞ (by Theorem D, d). So $w(z) = e^{az}$ for some $a \in \mathbb{C}$. Since t is a generator of A, by Lüroth's theorem, both p and q are rational functions of t and the lemma follows.

LEMMA 3. *Suppose that k is even, the Riemann surface F is of genus zero, y is a non-constant meromorphic solution of* (1·2), *and $\kappa(x) = 1$ for at most one point $x \in I$. Then the Abelian differential pdq is exact, that is $pdq = ds$ for some $s \in A$.*

Proof. It is sufficient to show that under the assumptions of Lemma 3, the integral of pdq over every closed path in F is zero. As F is of genus zero, we only have to consider residues of pdq. By Theorem D, a), all poles of our differential belong to the set I.

Consider first a point $x \in I$ with $\kappa(x) = 1 + k/n$. By Theorem D, c), we have a meromorphic solution y with a pole of order n at zero, such that the corresponding function f has the property $f(0) = x$. In a neighborhood of x we have a Puiseaux expansion

$$pdq = \sum_{j=J}^{\infty} c_j q^{-j/m} dq$$

with some positive integer m. We substitute $p = y^{(k)}$, $q = y$ and obtain

$$y^{(k)} y' = \sum_{j \neq -m} c_j y^{-j/m} y' + r y^{-1} y', \tag{3·2}$$

where $r = c_m$ is the residue of pdq at x. Now we notice that for even k,

$$y^{(k)} y' = \frac{d}{dz} \left\{ y^{(k-1)} y' - y^{(k-2)} y'' + \ldots \pm \frac{1}{2} (y^{(k/2)})^2 \right\}. \tag{3·3}$$

Using this, we integrate (3·2) over a small circle around 0 in the z-plane, described m times anticlockwise. We obtain that $2\pi i m r = 0$, so $r = 0$.

Now we consider a point $x \in I$ with $\kappa(x) = 1$. By the assumptions of the lemma, there is at most one such point. Then the residue of pdq at x is zero because the sum of all residues of a differential on a compact Riemann surface is zero. This proves the lemma.

Using (3·3) and Lemma 3, if the assumptions of Lemma 3 are satisfied, we can rewrite our differential equation

$$y^{(k)} = p(y) \tag{3·4}$$

as

$$y^{(k-1)} y' - y^{(k-2)} y'' + \ldots \pm \frac{1}{2} (y^{(k/2)})^2 = s(y) + c, \tag{3·5}$$

where $s \in A$ is an integral of the exact differential pdq, and c is a constant that depends on the particular solution y. We have the relation $p(y) = ds/dy$.

Lemma 4. *For a given differential equation of the form* (3·5), *there are only finitely many formal Laurent series with a pole at zero that satisfy the equation.*

Proof. By making a linear change of the independent variable, we may assume that

$$s(y) = y^{2+k/n} + \dots.$$

Then

$$p(y) = (2 + k/n)y^{1+k/n} + \dots.$$

Now we substitute a Laurent series with undetermined coefficients

$$y(z) = \sum_{j=0}^{\infty} c_j z^{-n+j} \tag{3·6}$$

to the equation (3·4), which is a consequence of (3·5). With even k we have:

$$\begin{aligned} y^{(k)}(z) = &\frac{(k+n-1)!}{(n-1)!} c_0 z^{-n-k} + \frac{(k+n-2)!}{(n-2)!} c_1 z^{-n-k-1} \\ &+ \dots + k! c_{n-1} z^{-k-1} \\ &+ k! c_{n+k} + \frac{(k+1)!}{1!} c_{n+k+1} z + \frac{(k+1)!}{2!} c_{n+k+2} z^2 + \dots; \end{aligned}$$

and

$$\begin{aligned} y^{1+k/n}(z) = z^{-k-n} \Big[&c_0^{1+k/n} + \Big((1+k/n) c_0^{k/n} c_1 + (\dots)_1 \Big) z \\ &+ \Big((1+k/n) c_0^{k/n} c_2 + (\dots)_2 \Big) z^2 + \dots \\ &+ \Big((1+k/n) c_0^{k/n} c_j + (\dots)_j \Big) z^j + \dots \Big]. \end{aligned}$$

In the last formula, the symbol $(\dots)_j$ stands for a finite sum of products of the coefficients of the series (3·6) which contain no coefficients c_i with $i \geq j$. Substituting to (3·4) and comparing the coefficients at z^{-k-n} we obtain

$$\frac{(k+n-1)!}{(n-1)!} c_0 = (2 + k/n) c_0^{1+k/n}.$$

This equation has finitely many non-zero roots c_0. We have

$$(2 + k/n) c_0^{k/n} = \frac{(k+n-1)!}{(n-1)!}. \tag{3·7}$$

Further we obtain

$$\frac{(k+n-2)!}{(n-1)!} c_1 = (2 + k/n) c_0^{k/n} (1 + k/n) c_1 + (\dots)_1. \tag{3·8}$$

Substituting here the value of $(2 + k/n)c_0^{k/n}$ from (3·7), we see that the coefficient at c_1 is different from zero, because

$$\frac{(k+n-2)!}{(n-2)!} \neq \frac{(k+n-1)!}{(n-1)!} \frac{k+n}{n}.$$

Thus c_1 is uniquely determined from (3·8). The situation is analogous for all coefficients

c_j with $j < n+k$. These coefficients are uniquely determined from the equation (3·4) once c_0 is chosen.

Now we consider the coefficients c_{n+k+j} with $j \geq 0$. We have

$$\frac{(k+j)!}{j!}c_{n+k+j} = (2+k/n)c_0^{k/n}\frac{n+k}{n}c_{n+k+j} + (\ldots)_{n+k+j}.$$

Again we substitute the value of $(2+k/n)c_0^{k/n}$ from (3·7) and conclude that the coefficient at c_{n+k+j} equals

$$\frac{(k+j)!}{j!} - \frac{(k+n)!}{n!}.$$

This coefficient is zero for a single value of j, namely $j = n$. Thus c_{2n+k} cannot be determined from the equation (3·4), but once c_0 and c_{2n+k} are chosen, the rest of the coefficients of the series (3·6) are determined uniquely.

To determine c_{2n+k} we invoke the equation (3·5):

$$y^{(k-1)}y' - y^{(k-2)}y'' + \ldots \pm \frac{1}{2}(y^{(k/2)})^2 = y^{2+k/n} + \ldots, \tag{3·9}$$

where the dots stand for the terms of lower degrees. We have

$$\begin{aligned}
y'(z) &= -nc_0z^{-n-1} + \ldots + c_{2n+k}(n+k)z^{n+k-1} + \ldots, \\
y'' &= n(n+1)c_0z^{-n-2} + \ldots + c_{2n+k}(n+k)(n+k-1)z^{n+k-2} + \ldots, \\
\ldots \quad & \ldots, \\
y^{(k-1)} &= -n(n+1)\ldots(n+k-2)c_0z^{-n-k+1} + \ldots \\
&\quad + c_{2n+k}(n+k)(n+k-1)\ldots(n+2)z^{n+1} + \ldots.
\end{aligned}$$

Substituting this to our equation (3·9) we write the condition that the constant terms in both sides of (3·9) are equal. This condition is a polynomial equation in $c, c_0, \ldots, c_{2n+k}$ (it is linear with respect to c_{2n+k}) and the coefficient at c_{2n+k} in this equation equals

$$c_0\sum_{m=0}^{k-1}\frac{(n+m)!(n+k)!}{(n+m+1)!(n-1)!}.$$

This expression is not zero because each term of the sum is positive. Thus c_{2n+k} is determined uniquely, and this completes the proof of the lemma.

Remark. It follows from this proof that the only meromorphic solutions of the differential equations

$$y^{(k)} = y^m$$

are exponential polynomials when $m = 1$ and functions $c(z-z_0)^{-n}$ where $m = 1 + k/n$, $z_0 \in \mathbb{C}$ and c is an appropriate constant.

The rest of the proof of Theorem 1 is a repetition of the argument from [**8**].

By Theorems A and B, we may assume that F is of genus zero, and k is even. In view of Lemmas 2 and 3, it is enough to consider the case that the differential pdq is exact. Then every solution of (1·2) also satisfies (3·5) with some constant c.

Assume that y is a transcendental meromorphic solution of (3·5), having at least one pole. By Theorem D, d), c), y has infinitely many poles z_j, $j = 1, 2, 3, \ldots$. The functions $y(z-z_j)$ satisfy the assumptions of Lemma 4, therefore some of them are equal. We conclude that y is a periodic function. By making a linear change of the independent variable we may assume that the smallest period is $2\pi i$.

Consider the strip $D = \{z : 0 \leq \Im z < 2\pi\}$.

Case 1. y has infinitely many poles in D. Applying Lemma 4 again, we conclude that y has a period in D, so y is doubly periodic.

Case 2. y is bounded in $D \cap \{z : |\Re z| > C\}$ for some $C > 0$. Since y is $2\pi i$-periodic, we have $y(z) = R(e^z)$ where R is meromorphic in $\mathbb{C}^*$. As R is bounded in some neighborhoods of 0 and ∞, we conclude that R is rational.

Case 3. y has finitely many poles in D and is unbounded in $D \cap \{z : |\Re z| > C\}$ for every $C > 0$. As y is $2\pi i$-periodic, we write $y = R(e^z)$ where R is meromorphic in $\mathbb{C}^*$. Now R has finitely many poles and is unbounded either in a neighborhood of 0 or in a neighborhood of ∞. Suppose that it is unbounded in a neighborhood of ∞. Then the set $\{z : |R(z)| > M\}$, where M is large enough has an unbounded component G containing no poles of R. On this component G, the function R satisfies a differential equation

$$\sum_{m=1}^{k} \binom{k}{m} w^m \frac{d^m R}{dw^m} = (c + o(1))R^\kappa,$$

where c is some constant and $\kappa = 1$ or κ is one of the numbers $1 + k/n$ from Theorem D. Applying Theorem C in G as we did in the proof of Lemma 1, we obtain that $\kappa = 1$ and that R has a pole at infinity. Similar argument works for the singularity at 0, so R is rational, and this completes the proof.

4. *Appendix*

Proof of Theorem D. Statement a) is a special case of [**9**, Th. 10], but we give a simple independent proof using Theorem C. Proving it by contradiction, suppose that p has a pole at a point $x \in F$ such that $q(x) = b \in \mathbb{C}$. Let $D_\epsilon \subset \mathbb{C}$ be a disk of radius ϵ centered at b, and $V_\epsilon \subset F$ a component of $q^{-1}(D_\epsilon)$ containing x. We assume that the disk D_ϵ is so small that V_ϵ contains no other poles of p, except the pole at x. Let y be a meromorphic solution of our equation (1·2) and consider the map $f : \mathbb{C} \to F$ given by $f(z) = (y^{(k)}(z), y(z))$. The image of this map is dense in F and the point x is evidently omitted by f. Let $G_\epsilon \subset \mathbb{C}$ be a component of the preimage $f^{-1}(D_\epsilon)$. Consider the meromorphic function $w = 1/(y - a)$. It is holomorphic and unbounded in G_ϵ, and $|w(z)| = 1/\epsilon$ for $z \in \partial G_\epsilon$. We conclude that G_ϵ is unbounded. Now we apply Theorem C to w in G_ϵ.

Set $M(r) = \max\{|w(z)| : |z| = r, z \in G_\epsilon\}$ and let $a(r)$ be defined as in (2·1). For any $r > r_0 = \inf\{|z| : z \in G_\epsilon\}$, we choose a point z_r with $|z| = r$ and $|w(z_r)| = M(r)$. By Theorem C, we have

$$|w^{(j)}(z_r)| = \left(\frac{a(r)}{r}\right)^j |w(z_r)|(1 + o(1)) = \frac{a(r)^j}{r^j} M(r)(1 + o(1)) \tag{4·1}$$

where $r \to \infty$ outside a set of finite logarithmic measure.

From Lemma 6.10 of [**3**], we have for every $\beta > 0$,

$$(a(r))^\beta = o(M(r)), \tag{4·2}$$

as $r \to \infty$ outside a set of finite logarithmic measure.

Differentiating the equation $y = 1/w + a$ we obtain

$$y^{(k)} = \frac{1}{w} Q\left(\frac{w'}{w}, \frac{w''}{w}, \cdots, \frac{w^{(k)}}{w}\right), \tag{4·3}$$

where Q is a polynomial. On the other hand, from the Puiseaux expansion at the point x we obtain

$$y^{(k)} = (c + o(1))w^{\alpha}, \quad w \to \infty, \tag{4·4}$$

where $c \neq 0$ is a constant and $\alpha > 0$. Combining (4·3) and (4·4) we obtain

$$Q\left(\frac{w'}{w}, \dots, \frac{w^{(k)}}{w}\right) = (c + o(1))w^{1+\alpha}.$$

Inserting to this asymptotic relation $z = z_r$ and using (4·1) and (4·2) we obtain a contradiction which proves a).

Consider now a point $x \in I$. From the Puiseaux expansion we obtain

$$y^{(k)} = (c + o(1))y^{\kappa(x)}, \quad y \to \infty. \tag{4·5}$$

If x has a preimage under the map f, then this preimage is a pole z_0 of y. If this pole is of order n we have $y(z) \sim c_1(z-z_0)^{-n}$ and $y^{(k)}(z) \sim c_2(z-z_0)^{-n-k}$ as $z \to z_0$. Substituting to (4·5) we conclude that $\kappa(x) = 1 + k/n$. Thus if x has at least one preimage under f then $\kappa(x) = 1 + k/n$ with a positive integer n, and every preimage of x is a pole of order n of y. This implies d).

Now suppose that a point $x \in I$ has only finitely many preimages. Let $U_\epsilon = \{z \in \overline{\mathbb{C}} : |z| > 1/\epsilon\}$ be a neighborhood of infinity, and $V_\epsilon \subset F$ a component of the preimage $q^{-1}(U_\epsilon)$. We may assume that $\epsilon > 0$ is so small that V_ϵ does not contain other poles of q except x. Let G_ϵ be a component of the preimage $f^{-1}(V_\epsilon)$. If G_ϵ is bounded then $f : G_\epsilon \to U_\epsilon$ is a ramified covering of a finite degree, and f takes the value x somewhere in G. As we assume that f is transcendental but x has only finitely many preimages, there should exist an unbounded component G_ϵ. Choosing a smaller ϵ if necessary, we achieve that this unbounded component G_ϵ contains no f-preimages of x. Then y is a holomorphic function in G_ϵ, $|y(z)| = 1/\epsilon$, $z \in \partial G_\epsilon$, and y is unbounded in G_ϵ. Applying Theorem C to the function y in G_ϵ we obtain the asymptotic relation (2·2). Putting $z = z_r$ in this relation, taking (4·2) into account, and comparing with (4·5) we conclude that $\kappa = 1$ in (4·5). This implies c). Thus in any case $\kappa = 1 + k/n$ or $\kappa = 1$, which proves b).

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