DIRECT IMAGES OF BUNDLES UNDER FROBENIUS MORPHISM

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Abstract. Let X be a smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$ and $F : X \to X_1$ be the relative Frobenius morphism. For any vector bundle W on X, we prove that instability of F_*W is bounded by instability of $W \otimes \mathrm{T}^\ell(\Omega^1_X)$ $(0 \le \ell \le n(p-1))$(Corollary 4.9). When X is a smooth projective curve of genus $g \ge 2$, it implies F_*W being stable whenever W is stable.

Dedicated to Professor Zhexian Wan on the occasion of his 80th birthday.

1. Introduction

Let X be a smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. Fix an ample divisor H on X, by a semistable (resp. stable) torsion free sheaf, we mean a H-slope semistable (resp. H-slope stable) sheaf in this paper. For a torsion free sheaf $\mathcal{F}$ on X, there is a unique filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ $(1 \le i \le k)$ are semistable torsion free sheaves and

$$\mu_{\max}(\mathcal{F}) := \mu(\mathcal{F}_1) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \cdots > \mu(\mathcal{F}_k/\mathcal{F}_{k-1}) := \mu_{\min}(\mathcal{F}).$$

The instability of $\mathcal{F}$ was defined as $\mathrm{I}(\mathcal{F}) = \mu_{\max}(\mathcal{F}) - \mu_{\min}(\mathcal{F})$, which measures how far from $\mathcal{F}$ being semi-stable. In particular, $\mathcal{F}$ is semistable if and only if $\mathrm{I}(\mathcal{F}) = 0$. On the other hand, there are sub-bundles $\mathrm{T}^\ell(\Omega^1_X) \subset (\Omega^1_X)^{\otimes \ell}$, $0 \le \ell \le n(p-1)$, which are the associated bundles of Ω^1_X through some elementary (perhaps interesting) representations of $\mathrm{GL}(n)$. These representations do not appear in characteristic zero.

Let $F : X \to X_1$ be the relative Frobenius morphism, for any vector bundle W on X, let $\mathrm{I}(W, X)$ be the maximal value of $\mathrm{I}(W \otimes \mathrm{T}^\ell(\Omega^1_X))$

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where $0 \leq \ell \leq n(p-1)$. Then one of our results in this paper shows (Corollary 4.9): When $K_X \cdot \mathrm{H}^{n-1} \geq 0$, we have

$$\mathrm{I}(F_*W) \leq p^{n-1}\mathrm{rk}(W)\,\mathrm{I}(W,X)\,.$$

In particular, if the bundles $W \otimes \mathrm{T}^{\ell}(\Omega^1_X)$, $0 \leq \ell \leq n(p-1)$, are semistable, then F_*W is semistable. In fact, when $K_X \cdot \mathrm{H}^{n-1} > 0$, we can show that the stability of $W \otimes \mathrm{T}^{\ell}(\Omega^1_X)$, $0 \leq \ell \leq n(p-1)$, implies the stability of F_*W (Theorem 4.8).

The main theorem has an immediate corollary that when X is a smooth projective curve of genus $g \geq 2$, the stability of W implies stability of F_*W. This is in fact our original motivation stimulated by a question raised by Herbert Lange at a conference. When W is a line bundle, it is due to Lange and Pauly ([6, Proposition 1.2]). The present version is based on our earlier preprint ([8]), where the theorem was completely proved only for curves. It should be pointed out, in case of curves, Mehta and Pauly have proved independently that semi-stability of W implies semi-stability of F_*W in a different method. However, their method was not able to prove the stability of F_*W when W is stable. In fact, they asked the question: Is stability also preserved by F_* ? (cf. [7, Section 7] for the discussions).

To describe the idea of proof, let us compare it to its opposite case, a Galois étale G-cover $f : Y \to X$. Recall that for a semi-stable bundle W on Y, to prove semistability of f_*W, one uses the fact that $f^*(f_*W)$ decomposes into pieces of W^{σ} ($\sigma \in G$). To imitate this idea for $F : X \to X_1$, we need a similar decomposition of $V = F^*(F_*W)$. Indeed, use the canonical connection $\nabla : V \to V \otimes \Omega^1_X$, Joshi-Ramanan-Xia-Yu have defined in [4] for $\dim(X) = 1$ a canonical filtration

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_{\ell} \subset V_{\ell-1} \subset \cdots V_1 \subset V_0 = V$$

such that $V_{\ell}/V_{\ell+1} \cong W \otimes (\Omega^1_X)^{\otimes \ell}$. It is this filtration and its generalization that we are going to use for the study of F_*W.

As the first step, we generalize the canonical filtration to higher dimensional X. Its definition can be generalized straightforwardly by using the canonical connection $\nabla : V \to V \otimes \Omega^1_X$. The study of its graded quotients are much involved. We show (Theorem 3.7) that there exists a canonical filtration

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W)$$

such that ∇ induces injective morphisms $V_{\ell}/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_{\ell}) \otimes \Omega^1_X$ of vector bundles and the isomorphisms $V_{\ell}/V_{\ell+1} \cong W \otimes \mathrm{T}^{\ell}(\Omega^1_X)$, where $\mathrm{T}^{\ell}(\Omega^1_X) \subset (\Omega^1_X)^{\otimes \ell}$ are subbundles given by representations of $\mathrm{GL}(n)$ (cf. Definition 3.4). In characteristic zero, $\mathrm{T}^{\ell}(\Omega^1_X) = \mathrm{Sym}^{\ell}(\Omega^1_X)$. In characteristic $p > 0$, $\mathrm{T}^\ell(\Omega^1_X) \cong \mathrm{Sym}^\ell(\Omega^1_X)$ only for $\ell < p$. In general, there is a resolution of $\mathrm{T}^\ell(\Omega^1_X)$ (Proposition 3.5) by symmetric powers of Ω^1_X and exterior powers of $F^*(\Omega^1_X)$ (After [8] appeared, Indranil Biswas told me that a similar filtration was defined and studied in Proposition 4.1 of their preprint [1]. However, since their map (4.7) was wrong, the Proposition 4.1 (also Proposition 4.2 consequently) of [1] was wrong. After we pointed out these gaps, they have corrected these mistakes in [2]).

To prove the main theorem, we also need to compare sub-sheaves of $V_\ell/V_{\ell+1}$ to sub-sheaves of $V_{n(p-1)-\ell}/V_{n(p-1)-\ell+1}$ which are ∇-invariant (Proposition 4.7). It is reduced to consider the (graded) K-algebra

$$R = \frac{K[y_1, y_2, \cdots, y_n]}{(y_1^p, y_2^p, \ldots, y_n^p)} = \bigoplus_{\ell=0}^{n(p-1)} R^\ell$$

with a D-module structure, where

$$\mathrm{D} = \frac{K[\partial_{y_1}, \cdots, \partial_{y_n}]}{(\partial_{y_1}^p, \cdots, \partial_{y_n}^p)} = K[t_1, t_2, \cdots, t_n] = \bigoplus_{\ell=0}^{n(p-1)} \mathrm{D}_\ell$$

which acts on R through the partial derivations ∂_{y_1}, ∂_{y_2}, ..., ∂_{y_n}. For any subspace $V \subset R^\ell$, let $\mathbb{L}(\mathrm{D}_{2\ell-n(p-1)} \cdot V)$ be the linear subspace spanned by $\mathrm{D}_{2\ell-n(p-1)} \cdot V \subset R^{n(p-1)-\ell}$. Then we are reduced to ask if

$$\dim(V) \leq \dim \mathbb{L}(\mathrm{D}_{2\ell-n(p-1)} \cdot V) \quad \text{when } \frac{n(p-1)}{2} \leq \ell \leq n(p-1) \ ?$$

Our Lemma 4.5 and Proposition 4.7 give an affrmative answer to it.

When X is a smooth projective curve of genus $g \geq 1$, the proof of theorem is very elementary and simple, which does not need the more involved arguments of higher dimensional case and shows the idea of proof best. Thus, although it is a direct corollary of the general case (Theorem 4.8), we still put its proof in an independent section. It is also convenient for a reader who is only interested in the proof for curves.

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2. The case of curves

Let k be an algebraically closed field of characteristic $p > 0$ and X be a smooth projective curve over k. Let $F : X \to X_1$ be the relative k-linear Frobenius morphism, where $X_1 := X \times_k k$ is the base change of X/k under the Frobenius $\mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k)$. Let W be a vector bundle on X and $V = F^*(F_*W)$. It is known ([5, Theorem 5.1]) that V has a canonical connection $\nabla : V \to V \otimes \Omega^1_X$ with zero p-curvature. In [4, Section 5], the authors defined a canonical filtration

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_\ell \subset V_{\ell-1} \subset \cdots V_1 \subset V_0 = V \tag{2.1}$$

where $V_1 = \ker(V = F^*F_*W \twoheadrightarrow W)$ and

$$V_{\ell+1} = \ker(V_\ell \xrightarrow{\nabla} V \otimes \Omega^1_X \to V/V_\ell \otimes \Omega^1_X). \tag{2.2}$$

The following lemma belongs to them (cf. [4, Theorem 5.3]).

Lemma 2.1. (i) *$V_0/V_1 \cong W$, $\nabla(V_{\ell+1}) \subset V_\ell \otimes \Omega^1_X$ for $\ell \geq 1$.*

(ii) *$V_\ell/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_\ell) \otimes \Omega^1_X$ is an isomorphism for $1 \leq \ell \leq p-1$.*

(iii) *If $g \geq 2$ and W is semistable, then the canonical filtration* (2.1) *is nothing but the Harder-Narasimhan filtration.*

Proof. (i) follows by the definition, which and (ii) imply (iii). To prove (ii), let $I_0 = F^*F_*\mathcal{O}_X$, $I_1 = \ker(F^*F_*\mathcal{O}_X \twoheadrightarrow \mathcal{O}_X)$ and

$$I_{\ell+1} = \ker(I_\ell \xrightarrow{\nabla} I_0 \otimes \Omega^1_X \twoheadrightarrow I_0/I_\ell \otimes \Omega^1_X) \tag{2.3}$$

which is the canonical filtration (2.1) in the case $W = \mathcal{O}_X$.

(ii) is clearly a local problem, we can assume $X = \mathrm{Spec}\,(k[[x]])$ and $W = k[[x]]^{\oplus r}$. Then $V_0 := V = F^*(F_*W) = I_0^{\oplus r}$, $V_\ell = I_\ell^{\oplus r}$ and

$$V_\ell/V_{\ell+1} = (I_\ell/I_{\ell+1})^{\oplus r} \xrightarrow{\oplus\nabla} (I_{\ell-1}/I_\ell \otimes \Omega^1_X)^{\oplus r} = V_{\ell-1}/V_\ell \otimes \Omega^1_X. \tag{2.4}$$

Thus it is enough to show that

$$I_\ell/I_{\ell+1} \xrightarrow{\nabla} (I_{\ell-1}/I_\ell \otimes \Omega^1_X \tag{2.5}$$

is an isomorphism. Locally, $I_0 = k[[x]] \otimes_{k[[x^p]]} k[[x]]$ and

$$\nabla : k[[x]] \otimes_{k[[x^p]]} k[[x]] \to I_0 \otimes_{\mathcal{O}_X} \Omega^1_X, \tag{2.6}$$

where $\nabla(g \otimes f) = g \otimes f' \otimes \mathrm{d}x$. The $\mathcal{O}_X$-module

$$I_1 := \ker(k[[x]] \otimes_{k[[x^p]]} k[[x]] \twoheadrightarrow k[[x]]) \tag{2.7}$$

has a basis $\{x^i \otimes 1 - 1 \otimes x^i\}_{1 \leq i \leq p-1}$. Notice that I_1 is also an ideal of the $\mathcal{O}_X$-algebra $I_0 = k[[x]] \otimes_{k[[x^p]]} k[[x]]$, let $\alpha = x \otimes 1 - 1 \otimes x$, then $\alpha^i \in I_1$.

It is easy to see that α, α^2, $\dots$, α^{p-1} is a basis of the $\mathcal{O}_X$-module I_1 (notice that $\alpha^p = x^p \otimes 1 - 1 \otimes x^p = 0$), and

$$\nabla(\alpha^\ell) = -\ell\alpha^{\ell-1} \otimes \mathrm{d}x. \tag{2.8}$$

Thus, as a free $\mathcal{O}_X$-module, I_ℓ has a basis $\{\alpha^\ell, \alpha^{\ell+1}, \dots, \alpha^{p-1}\}$, which means that $I_\ell/I_{\ell+1}$ has a basis α^ℓ, $(I_{\ell-1}/I_\ell)\otimes\Omega^1_X$ has a basis $\alpha^{\ell-1}\otimes\mathrm{d}x$ and $\nabla(\alpha^\ell) = -\ell\alpha^{\ell-1}\otimes\mathrm{d}x$. Therefore ∇ induces the isomorphism (2.5) since $(\ell, p) = 1$, which implies the isomorphism in (ii). $\square$

Theorem 2.2. *Let X be a smooth projective curve of genus $g \geq 1$. Then F_*W is semi-stable whenever W is semi-stable. If $g \geq 2$, then F_*W is stable whenever W is stable.*

Proof. Let $\mathcal{E} \subset F_*W$ be a nontrivial subbundle and

$$0 \subset V_m \cap F^*\mathcal{E} \subset \cdots \subset V_1 \cap F^*\mathcal{E} \subset V_0 \cap F^*\mathcal{E} = F^*\mathcal{E} \tag{2.9}$$

be the induced filtration. Let $r_\ell = \mathrm{rk}(\frac{V_\ell\cap F^*\mathcal{E}}{V_{\ell+1}\cap F^*\mathcal{E}})$ be the ranks of quotients. Then, by the filtration (2.9), we have

$$\mu(F^*\mathcal{E}) = \frac{1}{\mathrm{rk}(F^*\mathcal{E})}\sum_{\ell=0}^{m} r_\ell\cdot\mu(\frac{V_\ell\cap F^*\mathcal{E}}{V_{\ell+1}\cap F^*\mathcal{E}}). \tag{2.10}$$

By Lemma 2.1, $V_\ell/V_{\ell+1} \cong W\otimes(\Omega^1_X)^{\otimes\ell}$ is stable, we have

$$\mu(\frac{V_\ell\cap F^*\mathcal{E}}{V_{\ell+1}\cap F^*\mathcal{E}}) \leq \mu(W) + 2(g-1)\ell. \tag{2.11}$$

Then, notice that $\mu(V) = \mu(W) + (p-1)(g-1)$, we have

$$\mu(F_*W) - \mu(\mathcal{E}) \geq \frac{2g-2}{p\cdot\mathrm{rk}(\mathcal{E})}\cdot\sum_{\ell=0}^{m}(\frac{p-1}{2}-\ell)r_\ell \tag{2.12}$$

which becomes equality if and only if the inequalities in (2.11) become equalities. It is clear by (2.12) that $\mu(F_*W) - \mu(\mathcal{E}) > 0$ if $m \leq \frac{p-1}{2}$. Thus we can assume that $m > \frac{p-1}{2}$, then we can write

$$\begin{aligned}\sum_{\ell=0}^{m}(\frac{p-1}{2}-\ell)r_\ell &= \sum_{\ell=m+1}^{p-1}(\ell-\frac{p-1}{2})r_{p-1-\ell} \\ &+ \sum_{\ell>\frac{p-1}{2}}^{m}(\ell-\frac{p-1}{2})(r_{p-1-\ell}-r_\ell) \ \geq\ 0.\end{aligned} \tag{2.13}$$

On the other hand, since the isomorphisms $V_\ell/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_\ell)\otimes\Omega^1_X$ in Lemma 2.1 (ii) induce the injections

$$\frac{V_\ell\cap F^*\mathcal{E}}{V_{\ell+1}\cap F^*\mathcal{E}} \hookrightarrow \frac{V_{\ell-1}\cap F^*\mathcal{E}}{V_\ell\cap F^*\mathcal{E}}\otimes\Omega^1_X$$

we have $r_0\geq r_1\geq\cdots\geq r_{\ell-1}\geq r_\ell\geq\cdots\geq r_m$. Thus

$$\mu(F_*W)-\mu(\mathcal{E})\geq\frac{2g-2}{p\cdot\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^{m}(\frac{p-1}{2}-\ell)r_\ell\;\geq 0\,.$$

If $\mu(F_*W)-\mu(\mathcal{E})=0$, then (2.12) and (2.13) become equalities. That (2.12) becomes equality implies inequalities in (2.11) become equalities, which means $r_0=r_1=\cdots=r_m=\mathrm{rk}(W)$. Then that (2.13) become equalities implies $m=p-1$. Altogether imply $\mathcal{E}=F_*W$, we get contradiction. Hence F_*W is stable whenever W is stable. □

3. The filtration on higher dimension varieties

Let X be a smooth projective variety over k of dimension n and $F: X\to X_1$ be the relative k-linear Frobenius morphism, where $X_1 := X\times_k k$ is the base change of X/k under the Frobenius $\mathrm{Spec}\,(k)\to\mathrm{Spec}\,(k)$. Let W be a vector bundle on X and $V=F^*(F_*W)$. We have the straightforward generalization of the canonical filtration to higher dimensional varieties.

Definition 3.1. Let $V_0 := V = F^*(F_*W)$, $V_1=\ker(F^*(F_*W)\twoheadrightarrow W)$

$$V_{\ell+1} := \ker(V_\ell\xrightarrow{\nabla} V\otimes_{\mathcal{O}_X}\Omega^1_X\to(V/V_\ell)\otimes_{\mathcal{O}_X}\Omega^1_X) \tag{3.1}$$

where $\nabla: V\to V\otimes_{\mathcal{O}_X}\Omega^1_X$ is the canonical connection (cf. [5, Theorem 5.1]).

We first consider the special case $W=\mathcal{O}_X$ and give some local descriptions. Let $I_0=F^*(F_*\mathcal{O}_X)$, $I_1=\ker(F^*F_*\mathcal{O}_X\twoheadrightarrow\mathcal{O}_X)$ and

$$I_{\ell+1}=\ker(I_\ell\xrightarrow{\nabla} I_0\otimes_{\mathcal{O}_X}\Omega^1_X\to I_0/I_\ell\otimes_{\mathcal{O}_X}\Omega^1_X). \tag{3.2}$$

Locally, let $X=\mathrm{Spec}\,(A)$, $I_0=A\otimes_{A^p}A$, where $A=k[[x_1,\cdots,x_n]]$, $A^p=k[[x_1^p,\cdots,x_n^p]]$. Then the canonical connection $\nabla: I_0\to I_0\otimes\Omega^1_X$ is locally defined by

$$\nabla(g\otimes_{A^p}f)=\sum_{i=1}^{n}(g\otimes_{A^p}\frac{\partial f}{\partial x_i})\otimes_A \mathrm{d}x_i \tag{3.3}$$

Notice that I_0 has an A-algebra structure such that $I_0 = A\otimes_{A^p} A \twoheadrightarrow A$ is a homomorphism of A-algebras, its kernel I_1 contains elements

$$\alpha_1^{k_1}\alpha_2^{k_2}\cdots\alpha_n^{k_n},\ \text{ where } \alpha_i = x_i\otimes_{A^p} 1 - 1\otimes_{A^p} x_i,\ \sum_{i=1}^n k_i \geq 1. \tag{3.4}$$

Since $\alpha_i^p = x_i^p\otimes_{A^p}1-1\otimes_{A^p}x_i^p = 0$, the set $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}\,|\,k_1+\cdots+k_n\geq 1\}$ has p^n-1 elements. In fact, we have

Lemma 3.2. *Locally, as free A-modules, we have, for all $\ell\geq 1$,*

$$I_\ell = \bigoplus_{k_1+\cdots+k_n\geq\ell}(\alpha_1^{k_1}\cdots\alpha_n^{k_n})A. \tag{3.5}$$

Proof. We first prove for $\ell = 1$ that $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}\,|\,k_1+\cdots+k_n\geq 1\}$ is a basis of I_1 locally. By definition, I_1 is locally free of rank p^n-1, thus it is enough to show that as an A-module I_1 is generated locally by $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}\,|\,k_1+\cdots+k_n\geq 1\}$ since it has exactly p^n-1 elements.

It is easy to see that as an A-module I_1 is locally generated by $\{x_1^{k_1}\cdots x_n^{k_n}\otimes_{A^p}1-1\otimes_{A^p}x_1^{k_1}\cdots x_n^{k_n}\,|\,k_1+\cdots+k_n\geq 1,\ 0\leq k_i\leq p-1\}$. It is enough to show that any $x_1^{k_1}\cdots x_n^{k_n}\otimes_{A^p}1-1\otimes_{A^p}x_1^{k_1}\cdots x_n^{k_n}$ is a linear combination of $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}\,|\,k_1+\cdots+k_n\geq 1\}$. The claim is obvious when $k_1+\cdots+k_n = 1$, we consider the case $k_1+\cdots+k_n > 1$. Without loss generality, assume $k_n\geq 1$ and there are $f_{j_1,\ldots,j_n}\in A$ such that

$$x_1^{k_1}\cdots x_n^{k_n-1}\otimes_{A^p}1-1\otimes_{A^p}x_1^{k_1}\cdots x_n^{k_n-1} = \sum_{j_1+\cdots+j_n\geq 1}(\alpha_1^{j_1}\cdots\alpha_n^{j_n})\cdot f_{j_1,\ldots,j_n}.$$

Then we have

$$\begin{aligned} x_1^{k_1}\cdots x_n^{k_n}\otimes_{A^p}1-1\otimes_{A^p}x_1^{k_1}\cdots x_n^{k_n} &= \sum_{j_1+\cdots+j_n\geq 1}(\alpha_1^{j_1}\cdots\alpha_n^{j_n+1})\cdot f_{j_1,\ldots,j_n} \\ + \sum_{j_1+\cdots+j_n\geq 1}(\alpha_1^{j_1}\cdots\alpha_n^{j_n})\cdot f_{j_1,\ldots,j_n}x_n \quad &+ \quad \alpha_n\cdot(x_1^{k_1}\cdots x_n^{k_n-1}). \end{aligned}$$

For $\ell > 1$, to prove the lemma, we first show

$$\nabla(\alpha_1^{k_1}\cdots\alpha_n^{k_n}) = -\sum_{i=1}^n k_i(\alpha_1^{k_1}\cdots\alpha_i^{k_i-1}\cdots\alpha_n^{k_n})\otimes_A \mathrm{d}x_i \tag{3.6}$$

Indeed, (3.6) is true when $k_1+\cdots+k_n = 1$. If $k_1+\cdots+k_n > 1$, we assume $k_n\geq 1$ and $\alpha_1^{k_1}\cdots\alpha_n^{k_n-1} = \sum g_j\otimes_{A^p} f_j$. Then

$$\alpha_1^{k_1}\cdots\alpha_n^{k_n} = \sum_j x_n g_j\otimes_{A^p} f_j - \sum_j g_j\otimes_{A^p} f_j x_n\,.$$

Use (3.3), straightforward computations show

$$\nabla(\alpha_1^{k_1}\cdots\alpha_n^{k_n}) = \alpha_n\nabla(\alpha_1^{k_1}\cdots\alpha_n^{k_n-1}) - (\alpha_1^{k_1}\cdots\alpha_n^{k_n-1})\otimes_A \mathrm{d}x_n$$

which implies (3.6). Now we can assume the lemma is true for $I_{\ell-1}$ and recall that $I_\ell = \ker(I_{\ell-1}\xrightarrow{\nabla} I_0\otimes_A\Omega_X^1 \twoheadrightarrow (I_0/I_{\ell-1})\otimes_A\Omega_X^1)$. For any

$$\beta = \sum_{k_1+\cdots k_n\geq \ell-1}(\alpha_1^{k_1}\cdots\alpha_n^{k_n})\cdot f_{k_1,\ldots,k_n}\in I_{\ell-1},\quad f_{k_1,\ldots,k_n}\in A,$$

by using (3.6), we see that $\beta\in I_\ell$ if and only if

$$\sum_{k_1+\cdots+k_n=\ell-1}(\alpha_1^{k_1}\cdots\alpha_j^{k_j-1}\cdots\alpha_n^{k_n})\cdot k_jf_{k_1,\ldots,k_n}\ \in I_{\ell-1} \tag{3.7}$$

for all $1\leq j\leq n$. Since $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}\,|\,k_1+\cdots+k_n\geq 1\}$ is a basis of I_1 locally and the lemma is true for $I_{\ell-1}$, (3.7) is equivalent to

$$\begin{gathered}\text{For given } (k_1,\ldots,k_n) \text{ with } k_1+\cdots+k_n=\ell-1\\ k_jf_{k_1,\ldots,k_n}=0\ \text{ for all } j=1,\ldots,n\end{gathered} \tag{3.8}$$

which implies $f_{k_1,\ldots,k_n}=0$ whenever $k_1+\cdots+k_n=\ell-1$. Thus I_ℓ is generated by $\{\alpha_1^{k_1}\cdots\alpha_n^{k_n}\ |\,k_1+\cdots+k_n\geq\ell\,\}$. □

Lemma 3.3. (i) *$I_\ell=0$ when $\ell>n(p-1)$, and $\nabla(I_{\ell+1})\subset I_\ell\otimes\Omega_X^1$ for $\ell\geq 1$.*

(ii) *$I_\ell/I_{\ell+1}\xrightarrow{\nabla}(I_{\ell-1}/I_\ell)\otimes\Omega_X^1$ are injective in the category of vector bundles for $1\leq\ell\leq n(p-1)$. In particular, their composition*

$$\nabla^\ell : I_\ell/I_{\ell+1}\to(I_0/I_1)\otimes_{\mathcal{O}_X}(\Omega_X^1)^{\otimes\ell}=(\Omega_X^1)^{\otimes\ell} \tag{3.9}$$

is injective in the category of vector bundles.

Proof. (i) follows from Lemma 3.2 and Definition 3.1. (ii) follows from (3.6).

□

In order to describe the image of ∇^ℓ in (3.9), we recall a $\mathrm{GL}(n)$-representation $\mathrm{T}^\ell(V)\subset V^{\otimes\ell}$ where V is the standard representation of $\mathrm{GL}(n)$. Let S_ℓ be the symmetric group of ℓ elements with the action on $V^{\otimes\ell}$ by $(v_1\otimes\cdots\otimes v_\ell)\cdot\sigma = v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(\ell)}$ for $v_i\in V$ and $\sigma\in\mathrm{S}_\ell$. Let $e_1,\ldots,e_n$ be a basis of V, for $k_i\geq 0$ with $k_1+\cdots+k_n=\ell$ define

$$v(k_1,\ldots,k_n)=\sum_{\sigma\in\mathrm{S}_\ell}(e_1^{\otimes k_1}\otimes\cdots\otimes e_n^{\otimes k_n})\cdot\sigma \tag{3.10}$$

Definition 3.4. Let $\mathrm{T}^\ell(V)\subset V^{\otimes\ell}$ be the linear subspace generated by all vectors $v(k_1,\ldots,k_n)$ for all $k_i\geq 0$ satisfying $k_1+\cdots+k_n=\ell$. It is clearly a representation of $\mathrm{GL}(V)$. If $\mathcal{V}$ is a vector bundle of rank n, the subbundle $\mathrm{T}^\ell(\mathcal{V})\subset\mathcal{V}^{\otimes\ell}$ is defined to be the associated bundle of the frame bundle of $\mathcal{V}$ (which is a principal $\mathrm{GL}(n)$-bundle) through the representation $\mathrm{T}^{\ell}(V)$.

By sending any $e_1^{k_1}e_2^{k_2}\cdots e_n^{k_n} \in \mathrm{Sym}^{\ell}(V)$ to $v(k_1,\dots,k_n)$, we have

$$\mathrm{Sym}^{\ell}(V) \twoheadrightarrow \mathrm{T}^{\ell}(V) \tag{3.11}$$

which is an isomorphism in characteristic zero. When $\mathrm{char}(k)=p>0$, we have $v(k_1,\dots,k_n)=0$ if one of k_1, ..., k_n is bigger than $p-1$. Thus (3.11) is not injective when $\ell \geq p$, and $\mathrm{T}^{\ell}(V)$ is isomorphic to the quotient of $\mathrm{Sym}^{\ell}(V)$ by the relations $e_i^p=0$, $1\leq i\leq n$. In particular,

$$\mathrm{T}^{\ell}(V) \cong \mathrm{Sym}^{\ell}(V) \quad \text{when} \quad 0<\ell<p \tag{3.12}$$

and $\mathrm{T}^{\ell}(V)=0$ if $\ell > n(p-1)$. For any $0<\ell\leq n(p-1)$, $\mathrm{T}^{\ell}(V)$ is a simple representation of highest weight

$$(\overbrace{p-1,\cdots,p-1}^{a},\overbrace{b,0,\cdots,0}^{n-a}), \quad \text{where } \ell=(p-1)a+b,\ 0\leq b<p-1$$

and is called a 'Truncated symmetric power' (cf. [3]). In next proposition, we will describe $\mathrm{T}^{\ell}(V)$ using symmetric powers and exterior powers. The case of $\mathrm{GL}(2)$ is extremely simple, it is a tensor product of symmetric powers and exterior powers. In general, let F^*V denote the Frobenius twist of the standard representation V of $\mathrm{GL}(n)$ through the homomorphism $\mathrm{GL}(n)\to\mathrm{GL}(n)$ $((a_{ij})_{n\times n}\to(a_{ij}^p)_{n\times n})$, we have only a resolution of $\mathrm{T}^{\ell}(V)$ using symmetric powers of V and exterior powers of F^*V. Fix a basis e_1, ... , e_n of V, we define the k-linear maps

$$\mathrm{Sym}^{\ell-q\cdot p}(V)\otimes_k \bigwedge^{q}(V) \xrightarrow{\phi} \mathrm{Sym}^{\ell-(q-1)\cdot p}(V)\otimes_k \bigwedge^{q-1}(V) \tag{3.13}$$

such that for any $h=f_{\ell-q\cdot p}\otimes e_{k_1}\wedge\cdots\wedge e_{k_q}$ $(k_1<\cdots<k_q)$, we have

$$\phi(h)=\sum_{i=1}^{q}(-1)^{i-1}e_{k_i}^p f_{\ell-q\cdot p}\otimes e_{k_1}\wedge\cdots\wedge\hat{e}_{k_i}\wedge\cdots\wedge e_{k_q}. \tag{3.14}$$

Proposition 3.5. *(i) When $n=2$, as $\mathrm{GL}(2)$-representations, we have*

$$\mathrm{T}^{\ell}(V)=\begin{cases}\mathrm{Sym}^{\ell}(V) & \text{when } \ell<p;\\ \mathrm{Sym}^{2(p-1)-\ell}(V)\otimes\det(V)^{\ell-(p-1)} & \text{when } \ell\geq p\end{cases}$$

(ii) Let $\ell(p) \geq 0$ be the unique integer such that $0 \leq \ell - \ell(p) \cdot p < p$. Then, in the category of $\mathrm{GL}(n)$*-representations, we have exact sequence*

$$0 \to \mathrm{Sym}^{\ell-\ell(p)\cdot p}(V) \otimes_k \bigwedge^{\ell(p)}(F^*V) \xrightarrow{\phi} \mathrm{Sym}^{\ell-(\ell(p)-1)\cdot p}(V) \otimes_k \bigwedge^{\ell(p)-1}(F^*V)$$

$$\to \cdots \to \mathrm{Sym}^{\ell-q\cdot p}(V) \otimes_k \bigwedge^{q}(F^*V) \xrightarrow{\phi} \mathrm{Sym}^{\ell-(q-1)\cdot p}(V) \otimes_k \bigwedge^{q-1}(F^*V)$$

$$\to \cdots \to \mathrm{Sym}^{\ell-p}(V) \otimes_k F^*V \xrightarrow{\phi} \mathrm{Sym}^{\ell}(V) \to \mathrm{T}^{\ell}(V) \to 0.$$

Proof. (i) When $\ell < p$, $\mathrm{T}^\ell(V) = \mathrm{Sym}^\ell(V)$ follows the construction. When $\ell \geq p$, the simple representation $\mathrm{T}^\ell(V)$ has highest weight

$$(p-1, \ell-p+1) = (2p-2-\ell, 0) + (\ell-p+1)\cdot(1,1)$$

where $(2p-2-\ell, 0)$ and $(1,1)$ are the highest weights of the simple representations $\mathrm{Sym}^{2p-2-\ell}(V)$ and $\wedge^2(V) = \det(V)$ respectively. Thus

$$\mathrm{T}^\ell(V) = \mathrm{Sym}^{2(p-1)-\ell}(V) \otimes \det(V)^{\ell-(p-1)}.$$

(ii) The elements $e_1^p, e_2^p, \ldots, e_n^p \in \mathrm{Sym}^\bullet(V)$ form clearly a regular sequence for $\mathrm{Sym}^\bullet(V)$, thus the Koszul complex $K_\bullet(e_1^p, \ldots, e_n^p)$ of $\mathrm{Sym}^\bullet(V)$-modules is a resolution of

$$\frac{\mathrm{Sym}^\bullet(V)}{(e_1^p, e_2^p, \ldots, e_n^p)\mathrm{Sym}^\bullet(V)}$$

where $K_1 = \mathrm{Sym}^\bullet(V) \otimes_k V$ with basis $1 \otimes_k e_1, \ldots, 1 \otimes_k e_n$ and $K_i = \wedge^i K_1$. Notice $\wedge^i K_1 \cong \mathrm{Sym}^\bullet(V) \otimes_k \wedge^i V$ (as $\mathrm{Sym}^\bullet(V)$-modules), the sequence in the proposition is exact in the category of k-linear spaces (This was pointed out by Manfred Lehn).

We only need to show the k-linear maps ϕ in (3.13) are maps of $\mathrm{GL}(n)$-representations if $\wedge^\cdot V$ is twisted by Frobenius. It is enough to show, for any $A = (a_{ij})_{n\times n} \in \mathrm{GL}(n)$ and $h = 1 \otimes e_{k_1} \wedge \cdots \wedge e_{k_q}$, that

$$\phi(A \cdot h) = A \cdot \phi(h)$$

To simplify notation, we assume $h = 1 \otimes e_1 \wedge \cdots \wedge e_q$, then

$$A \cdot h = 1 \otimes \sum_{k_1 < \cdots < k_q} D\begin{pmatrix} k_1, k_2, \ldots, k_q \\ 1, 2, \ldots, q \end{pmatrix} e_{k_1} \wedge \cdots \wedge e_{k_q}, \quad \text{where}$$

$$D\begin{pmatrix} k_1, k_2, \ldots, k_q \\ 1, 2, \ldots, q \end{pmatrix} = \begin{vmatrix} a^p_{1k_1}, a^p_{1k_2}, \ldots, a^p_{1k_q} \\ a^p_{2k_1}, a^p_{2k_2}, \ldots, a^p_{2k_q} \\ \vdots \quad \vdots \quad \ldots \quad \vdots \\ a^p_{qk_1}, a^p_{qk_2}, \ldots, a^p_{qk_q} \end{vmatrix}$$

Then, by definition of ϕ, we have

$$\phi(A\cdot h) = \sum_{k_1<\cdots<k_q}\sum_{i=1}^{q}(-1)^{i-1}e_{k_i}^p\otimes_k D\begin{pmatrix} k_1,k_2,\ldots,k_q\\ 1,2,\ldots,q\end{pmatrix} e_{k_1}\wedge\cdots\wedge\hat{e}_{k_i}\wedge\cdots\wedge e_{k_q}$$
$$= \sum_{k_1<\cdots<k_q}\sum_{i=1}^{q}e_{k_i}^p\otimes_k\sum_{j=1}^{q}(-1)^{j-1}a_{jk_i}^p D\begin{pmatrix} k_1,\ldots,\hat{k}_i,\ldots,k_q\\ 1,\ldots,\hat{j},\ldots,q\end{pmatrix} e_{k_1}\wedge\cdots\wedge\hat{e}_{k_i}\wedge\cdots\wedge e_{k_q}$$
$$= \sum_{j=1}^{q}(-1)^{j-1}\sum_{k_1<\cdots<k_q}\sum_{i=1}^{q}a_{jk_i}^p e_{k_i}^p\otimes_k D\begin{pmatrix} k_1,\ldots,\hat{k}_i,\ldots,k_q\\ 1,\ldots,\hat{j},\ldots,q\end{pmatrix} e_{k_1}\wedge\cdots\wedge\hat{e}_{k_i}\wedge\cdots\wedge e_{k_q}.$$

On the other hand, we will show

$$\sum_{k_1<\cdots<k_q}\sum_{i=1}^{q}a_{jk_i}^p e_{k_i}^p\otimes_k D\begin{pmatrix} k_1,\ldots,\hat{k}_i,\ldots,k_q\\ 1,\ldots,\hat{j},\ldots,q\end{pmatrix} e_{k_1}\wedge\cdots\wedge\hat{e}_{k_i}\wedge\cdots\wedge e_{k_q}$$
$$= \left(\sum_{i=1}^{n}a_{ji}^p e_i^p\right)\otimes_k\left(\sum_{i=1}^{n}a_{1i}^p e_i\right)\wedge\cdots\widehat{\left(\sum_{i=1}^{n}a_{ji}^p e_i\right)}\cdots\wedge\left(\sum_{i=1}^{n}a_{qi}^p e_i\right)$$
$$-\sum_{i_1<\cdots<i_{q-1}}\left(\sum_{k=1}^{q-1}a_{ji_k}^p e_{i_k}^p\right)\otimes_k D\begin{pmatrix} i_1,\ldots,i_{q-1}\\ 1,\ldots\hat{j}\ldots,q\end{pmatrix} e_{i_1}\wedge\cdots\wedge e_{i_{q-1}}$$

and $\sum_{j=1}^{q}(-1)^{j-1}a_{ji_k}^p\cdot D\begin{pmatrix} i_1,\ldots,i_{q-1}\\ 1,\ldots\hat{j}\ldots,q\end{pmatrix}=0$ $(1\le k\le q-1)$. Thus

$$\phi(A\cdot h)=A\cdot\phi(h).$$

In fact, the second equality corresponds to developing a determinant having the i_k-th column repeated. To show the first equality, write

$$\left(\sum_{i=1}^{n}a_{ji}^p e_i^p\right)\otimes_k\left(\sum_{i=1}^{n}a_{1i}^p e_i\right)\wedge\cdots\wedge\widehat{\left(\sum_{i=1}^{n}a_{ji}^p e_i\right)}\wedge\cdots\wedge\left(\sum_{i=1}^{n}a_{qi}^p e_i\right).$$
$$= \sum_{i_1<\cdots<i_{q-1}}\left(\sum_{i=1}^{n}a_{ji}^p e_i^p\right)\otimes_k D\begin{pmatrix} i_1,\ldots,i_{q-1}\\ 1,\ldots\hat{j}\ldots,q\end{pmatrix} e_{i_1}\wedge\cdots\wedge e_{i_{q-1}}$$

For given $i_1 < \cdots < i_{q-1}$, let $S = \{i_1, \ldots, i_{q-1}\}$, write

$$\left(\sum_{i=1}^{n} a_{ji}^p e_i^p\right) \otimes_k D\begin{pmatrix} i_1, \ldots, i_{q-1} \\ 1, \ldots \hat{j} \ldots, q \end{pmatrix} e_{i_1} \wedge \cdots \wedge e_{i_{q-1}} =$$

$$\sum_{t \notin S} a_{jt}^p e_t^p \otimes_k D\begin{pmatrix} i_1, \ldots, i_{q-1} \\ 1, \ldots \hat{j} \ldots, q \end{pmatrix} e_{i_1} \wedge \cdots \wedge e_{i_{q-1}} +$$

$$\sum_{k=1}^{q-1} a_{ji_k}^p e_{i_k}^p \otimes_k D\begin{pmatrix} i_1, \ldots, i_{q-1} \\ 1, \ldots \hat{j} \ldots, q \end{pmatrix} e_{i_1} \wedge \cdots \wedge e_{i_{q-1}}$$

notice that for any $t \notin S$ there is a unique $k_1 < \cdots < k_q$ with $k_i = t$ such that $(k_1, ..., \hat{k}_i, ..., k_q) = (i_1, ..., i_{q-1})$, we have

$$\sum_{t \notin S} a_{jt}^p e_t^p \otimes_k D\begin{pmatrix} i_1, \ldots, i_{q-1} \\ 1, \ldots \hat{j} \ldots, q \end{pmatrix} e_{i_1} \wedge \cdots \wedge e_{i_{q-1}} =$$

$$\sum_{k_1 < \cdots < k_q} a_{jk_i}^p e_{k_i}^p \otimes_k D\begin{pmatrix} k_1, \ldots, \hat{k}_i, \ldots, k_q \\ 1, \ldots, \hat{j}, \ldots, q \end{pmatrix} e_{k_1} \wedge \cdots \wedge \hat{e}_{k_i} \wedge \cdots \wedge e_{k_q}$$

where the summation is taken for all $k_1 < \cdots k_q$ satisfying

$$(k_1, ..., \hat{k}_i, ..., k_q) = (i_1, ..., i_{q-1}).$$

Then, taking summation for all $i_1 < \cdots < i_{q-1}$ and exchange the order of two summations, we got the claimed equality. □

Lemma 3.6. *With the notation in Definition 3.4, the composition*

$$\nabla^\ell : I_\ell / I_{\ell+1} \to (\Omega_X^1)^{\otimes \ell} \tag{3.15}$$

of the $\mathcal{O}_X$-morphisms in Lemma 3.3 (ii) has image $\mathrm{T}^\ell(\Omega_X^1) \subset (\Omega_X^1)^{\otimes \ell}$.

Proof. It is enough to prove the lemma locally. By Lemma 3.2, $I_\ell / I_{\ell+1}$ is locally generated by

$$\{\alpha_1^{k_1} \cdots \alpha_n^{k_n} \mid k_1 + \cdots + k_n = \ell\}. \tag{3.16}$$

By using formula (3.6) and the formula of permutations with repeated objects, we have

$$\nabla^\ell(\alpha_1^{k_1} \cdots \alpha_n^{k_n}) = (-1)^\ell \sum_{\sigma \in \mathrm{S}_\ell} (\mathrm{d}x_1^{\otimes k_1} \otimes \cdots \mathrm{d}x_n^{\otimes k_n}) \cdot \sigma \tag{3.17}$$

which implies that $\nabla^\ell(I_\ell / I_{\ell+1}) = \mathrm{T}^\ell(\Omega_X^1) \subset (\Omega_X^1)^{\otimes \ell}$. □

Theorem 3.7. *The filtration defined in Definition 3.1 is*

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_* W) \tag{3.18}$$

which has the following properties

(i) $\nabla(V_{\ell+1})\subset V_\ell\otimes\Omega^1_X$ *for* $\ell\geq 1$, *and* $V_0/V_1\cong W$.

(ii) $V_\ell/V_{\ell+1}\xrightarrow{\nabla}(V_{\ell-1}/V_\ell)\otimes\Omega^1_X$ *are injective morphisms of vector bundles for* $1\leq\ell\leq n(p-1)$, *which induced isomorphisms*

$$\nabla^\ell: V_\ell/V_{\ell+1}\cong W\otimes_{\mathcal{O}_X}\mathrm{T}^\ell(\Omega^1_X),\quad 0\leq\ell\leq n(p-1).$$

The vector bundle $\mathrm{T}^\ell(\Omega^1_X)$ *is suited in the exact sequence*

$$0\to\mathrm{Sym}^{\ell-\ell(p)\cdot p}(\Omega^1_X)\otimes F^*\Omega^{\ell(p)}_X\xrightarrow{\phi}\mathrm{Sym}^{\ell-(\ell(p)-1)\cdot p}(\Omega^1_X)\otimes F^*\Omega^{\ell(p)-1}_X$$

$$\to\cdots\to\mathrm{Sym}^{\ell-q\cdot p}(\Omega^1_X)\otimes F^*\Omega^q_X\xrightarrow{\phi}\mathrm{Sym}^{\ell-(q-1)\cdot p}(\Omega^1_X)\otimes F^*\Omega^{q-1}_X$$

$$\to\cdots\to\mathrm{Sym}^{\ell-p}(\Omega^1_X)\otimes F^*\Omega^1_X\xrightarrow{\phi}\mathrm{Sym}^{\ell}(\Omega^1_X)\to\mathrm{T}^\ell(\Omega^1_X)\to 0$$

where $\ell(p)\geq 0$ *is the integer such that* $\ell-\ell(p)\cdot p<p$.

Proof. It is a local problem to prove the theorem. Thus $V_{n(p-1)+1}=0$ follows from Lemma 3.2. (i) is nothing but the definition. (ii) follows from Lemma 3.3, Proposition 3.5 and Lemma 3.6. □

Corollary 3.8. *When* $\dim(X)=2$, *we have*

$$V_\ell/V_{\ell+1}=\begin{cases} W\otimes\mathrm{Sym}^\ell(\Omega^1_X) & \textit{when } \ell<p\\ W\otimes\mathrm{Sym}^{2(p-1)-\ell}(\Omega^1_X)\otimes\omega_X^{\ell-(p-1)} & \textit{when } \ell\geq p\end{cases}$$

Proof. It follows from (i) of Proposition 3.5. □

4. STABILITY IN HIGHER DIMENSIONAL CASE

Let X be a smooth projective variety over k of dimension n and H a fixed ample divisor on X. For a torsion free sheaf $\mathcal{E}$ on X, we define

$$\mu(\mathcal{E})=\frac{c_1(\mathcal{E})\cdot\mathrm{H}^{n-1}}{\mathrm{rk}(\mathcal{E})}.$$

Definition 4.1. A torsion free sheaf $\mathcal{E}$ on X is called semistable (resp. stable) if, for any $0\neq\mathcal{E}'\subset\mathcal{E}$, we have

$$\mu(\mathcal{E}')\leq\mu(\mathcal{E})\quad(\text{resp. }\mu(\mathcal{E}')<\mu(\mathcal{E})).$$

For any torsion free sheaf E on X, there is a unique filtration, the so-called Harder-Narasimhan filtration

$$0=E_0\subset E_1\subset\cdots\subset E_k=E$$

such that E_i/E_{i-1} $(1\leq i\leq k)$ are semistable torsion free sheaves and

$$\mu_{\max}(E):=\mu(E_1)>\mu(E_2/E_1)>\cdots>\mu(E_k/E_{k-1}):=\mu_{\min}(E).$$

The instability of E was defined as

$$\mathrm{I}(E)=\mu_{\max}(E)-\mu_{\min}(E).$$

Then it is easy to see that for any subsheaf $F \subset E$ we have

$$\mu(F) - \mu(E) \le \mathrm{I}(E) = \mu_{\max}(E) - \mu_{\min}(E). \tag{4.1}$$

Let $F : X \to X_1$ be the relative k-linear Frobenius morphism and W a vector bundle of rank r on X.

Lemma 4.2. *Let $c_1(\Omega^1_X) = K_X$. Then, in the chow group $\mathrm{Ch}(X_1)_{\mathbb{Q}}$,*

$$c_1(F_*W) = \frac{r(p^n - p^{n-1})}{2} K_{X_1} + p^{n-1}c_1(W), \tag{4.2}$$
$$\mu(F^*F_*W) = p \cdot \mu(F_*W) = \frac{p-1}{2} K_X \cdot \mathrm{H}^{n-1} + \mu(W).$$

Proof. The proof is just an application of Riemann-Roch theorem. Indeed, by Grothendieck-Riemann-Roch theorem, we have

$$c_1(F_*W) = \frac{rp^n}{2} K_X + F_*(c_1(W) - \frac{r}{2}K_X). \tag{4.3}$$

We remark here that for any irreducible subvariety $Y \subset X$, its image $F_X(Y) \subset X$ (under the absolute Frobenius $F_X : X \to X$) equals to Y, and the induced morphism $F_X : Y \to F_X(Y) = Y$ is nothing but the absolute Frobenius morphism $F_Y : Y \to Y$ (which has degree $p^{\dim(Y)}$). In particular, $F_*(c_1(W) - \frac{r}{2}K_X) = p^{n-1}(c_1(W) - \frac{r}{2}K_{X_1})$ proves (4.2). That $\mu(F^*F_*W) = p \cdot \mu(F_*W)$ also follows from this remark. $\square$

Let $V = F^*F_*W$, recall Theorem 3.7, we have the canonical filtration

$$0 = V_{n(p-1)+1} \subset V_{n(p-1)} \subset \cdots \subset V_1 \subset V_0 = V = F^*(F_*W) \tag{4.4}$$

with $V_\ell/V_{\ell+1} \cong W \otimes_{\mathcal{O}_X} \mathrm{T}^\ell(\Omega^1_X)$.

Lemma 4.3. *With the same notation in Theorem 3.7, we have*

$$c_1(\mathrm{T}^\ell(\Omega^1_X)) = \frac{\ell}{n}\left(\sum_{q=0}^{\ell(p)} (-1)^q C^q_n \cdot C^{\ell-qp}_{n+\ell-q-1}\right) K_X \tag{4.5}$$
$$\mathrm{rk}(\mathrm{T}^\ell(\Omega^1_X)) = \sum_{q=0}^{\ell(p)} (-1)^q C^q_n \cdot C^{\ell-qp}_{n+\ell-q-1}.$$

In particular, we have $\mu(\mathrm{T}^\ell(\Omega^1_X)) = \frac{\ell}{n} K_X \cdot \mathrm{H}^{n-1}$.

Proof. The formula of $\mathrm{rk}(\mathrm{T}^\ell(\Omega^1_X))$ follows directly from the exact sequence in Theorem 3.7 (ii). To compute $c_1(\mathrm{T}^\ell(\Omega^1_X))$, we use the fact that for any vector bundle E of rank n, we have

$$c_1(\mathrm{Sym}^q(E)) = C^{q-1}_{n+q-1} \cdot c_1(E) \tag{4.6}$$

$$c_1(\wedge^q E) = C^{q-1}_{n-1} \cdot c_1(E). \tag{4.7}$$

Then, use the exact sequence in Theorem 3.7 (ii) and note that

$$c_1(F^*\Omega^q_X) = p\cdot c_1(\Omega^q_X),$$

we have the formula (4.5) of $c_1(\mathrm{T}^\ell(\Omega^1_X))$. □

Let $\mathcal{E}\subset F_*W$ be a nontrivial subsheaf, the canonical filtration (4.4) induces the filtration (we assume $V_m\cap F^*\mathcal{E}\neq 0$)

$$0\subset V_m\cap F^*\mathcal{E}\subset\ \cdots\ \subset V_1\cap F^*\mathcal{E}\subset V_0\cap F^*\mathcal{E}=F^*\mathcal{E}. \tag{4.8}$$

Lemma 4.4. *In the induced filtration* (4.8)*, let*

$$\mathcal{F}_\ell := \frac{V_\ell\cap F^*\mathcal{E}}{V_{\ell+1}\cap F^*\mathcal{E}}\subset\frac{V_\ell}{V_{\ell+1}},\qquad r_\ell=\mathrm{rk}(\mathcal{F}_\ell).$$

Then there is an injective morphism $\mathcal{F}_\ell\xrightarrow{\nabla}\mathcal{F}_{\ell-1}\otimes\Omega^1_X$ *and*

$$\begin{aligned}\mu(F_*W)-\mu(\mathcal{E})\geq&\frac{K_X\cdot\mathrm{H}^{n-1}}{np\cdot\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^{m}(\frac{p-1}{2}n-\ell)\cdot r_\ell\\&-\frac{1}{p}\sum_{\ell=0}^{m}\frac{r_\ell\cdot\mathrm{I}(W\otimes\mathrm{T}^\ell(\Omega^1_X))}{\mathrm{rk}(\mathcal{E})}\end{aligned} \tag{4.9}$$

the equality holds if and only if equalities hold in the inequalities

$$\mu(\mathcal{F}_\ell)-\mu(V_\ell/V_{\ell+1})\leq\mathrm{I}(W\otimes\mathrm{T}^\ell(\Omega^1_X))\quad(0\leq\ell\leq m). \tag{4.10}$$

Proof. The injective morphisms $V_\ell/V_{\ell+1}\xrightarrow{\nabla}(V_{\ell-1}/V_\ell)\otimes\Omega^1_X$ in Theorem 3.7 (ii) induces clearly the injective morphisms

$$\mathcal{F}_\ell\xrightarrow{\nabla}\mathcal{F}_{\ell-1}\otimes\Omega^1_X,\qquad\ell=1,\,\ldots,\,m.$$

To show (4.9), note $\mu(F_*W)-\mu(\mathcal{E})=\frac{1}{p}(\mu(F^*F_*W)-\mu(F^*\mathcal{E}))$ and

$$\mu(F^*\mathcal{E})=\frac{1}{\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^{m}r_\ell\cdot\mu(\mathcal{F}_\ell),$$

using Lemma 4.2, we have

$$\begin{gathered}\mu(F^*F_*W)-\mu(F^*\mathcal{E})=\\ \frac{1}{\mathrm{rk}(\mathcal{E})}\sum_{\ell=0}^{m}r_\ell\left(\frac{p-1}{2}K_X\cdot\mathrm{H}^{n-1}+\mu(W)-\mu(\mathcal{F}_\ell)\right).\end{gathered} \tag{4.11}$$

For $\mathcal{F}_\ell\subset V_\ell/V_{\ell+1}=W\otimes\mathrm{T}^\ell(\Omega^1_X)$ $(0\leq\ell\leq m)$, using Lemma 4.3,

$$\mu(\mathcal{F}_\ell)\leq\mu(W)+\frac{\ell}{n}K_X\cdot\mathrm{H}^{n-1}+\mathrm{I}(W\otimes\mathrm{T}^\ell(\Omega^1_X)). \tag{4.12}$$

Substitute (4.12) into (4.11), one get (4.9) and the equality holds if and only if all of inequalities (4.12) become equalities.

$\square$

Let K be a field of characteristic $p > 0$, consider the K-algebra

$$R = \frac{K[y_1, \cdots, y_n]}{(y_1^p, \cdots, y_n^p)} = \bigoplus_{\ell=0}^{n(p-1)} R^{\ell},$$

where R^{ℓ} is the K-linear space generated by

$$\{\, y_1^{k_1} \cdots y_n^{k_n} \,|\, k_1 + \cdots + k_n = \ell, \quad 0 \leq k_i \leq p-1 \,\}.$$

The polynomial ring $\mathrm{P} = K[\partial_{y_1}, \cdots, \partial_{y_n}]$ acts on R through partial derivations, which induces a P-module structure on R. Note that $\partial_{y_i}^p$ $(i = 1, 2, \ldots, n)$ act on R trivially, the P-module structure is in fact a D-module, where

$$\mathrm{D} = \frac{K[\partial_{y_1}, \cdots, \partial_{y_n}]}{(\partial_{y_1}^p, \cdots, \partial_{y_n}^p)} = K[t_1, t_2, \cdots, t_n] = \bigoplus_{\ell=0}^{n(p-1)} \mathrm{D}_{\ell},$$

where D_{ℓ} is the linear space of degree ℓ homogeneous elements and t_1, t_2, $\ldots$, t_n are the classes of ∂_{y_1}, ∂_{y_2}, $\ldots$, ∂_{y_n}.

Lemma 4.5. *Let $V \subset \mathrm{D}_{\ell}$ be a linear subspace. Then, when $\ell \leq \frac{n(p-1)}{2}$, there is a basis $\{d_i \in V\}$ of V and monomials $\{\delta_i \in \mathrm{D}_{n(p-1)-2\ell}\}$ such that $\{\delta_i d_i \in \mathrm{D}_{n(p-1)-\ell}\}$ are linearly independent.*

Proof. We reduce firstly the lemma to the case when V has a basis of monomials. Define the Lexicographic order on the set of monomials of D_{ℓ}, $\mathrm{D}_{n(p-1)-\ell}$ respectively. For any $v \in \mathrm{D}_{\ell}$, one can write uniquely

$$v = \lambda_v m_v + \sum_{m > m_v} \lambda_m m$$

where $0 \neq \lambda_v$, $\lambda_m \in K$, m_v and m are monomials of D_{ℓ}.

Let $\dim(V) = s$, then it is easy to see that there is a basis

$$d_i = \lambda_i m_i + \sum_{m > m_i} \lambda_{i,m} m\,,\ \lambda_i \neq 0, \quad (1 \leq i \leq s)$$

of V such that $\{m_1, \ldots, m_s\}$ are different monomials of D_{ℓ}. If there are monomials $\{\delta_i \in \mathrm{D}_{n(p-1)-2\ell}\}_{1 \leq i \leq s}$ such that $\{\delta_i m_i \in \mathrm{D}_{n(p-1)-\ell}\}_{1 \leq i \leq s}$ are different monomials, then we claim that

$$\{\delta_i d_i \in \mathrm{D}_{n(p-1)-\ell}\}_{1 \leq i \leq s}$$

are linearly independent. To prove the claim, we only remark that for any monomials m, $m' \in \mathrm{D}_{\ell}$ and monomial $\delta \in \mathrm{D}_{n(p-1)-2\ell}$, we have

$$m < m' \Rightarrow \delta m < \delta m' \quad \text{whenever } \delta m,\ \delta m' \text{ are nonzero.}$$

Thus we have

$$\delta_i d_i = \lambda_i \delta_i m_i + \sum_{\delta_i m > \delta m_i} \lambda_{i,m} \delta_i m \quad (1 \le i \le s),$$

which are linearly independent.

If we identify the set of monomials of D_ℓ with the set

$$M^\ell = \{\, v = (v_1, \ldots, v_n)\,|0 \le v_i \le p-1 \ (1 \le i \le n),\ \sum_{i=1}^n v_i = \ell \,\}.$$

Then the lemma is equivalent to the existence of an injective map

$$\varphi : M^\ell \to M^{n(p-1)-\ell}$$

such that for any $v \in M^\ell$, we have $v \le \varphi(v)$: $v_i \le \varphi(v)_i \quad (1 \le i \le n)$. The existence of φ is a special case of the following lemma. $\square$

For any $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\ge 0}$, let $M^\ell_n(a_1, \ldots, a_n)$ be the set

$$\{\, v = (v_1, \ldots, v_n)\,|0 \le v_i \le a_i \ (1 \le i \le n),\ \sum_{i=1}^n v_i = \ell \,\}.$$

For any $v \in M^\ell_n(a_1, \ldots, a_n)$ and $v' \in M^{\ell'}_n(a_1, \ldots, a_n)$, by $v \le v'$, we mean $v_i \le v'_i$ $(1 \le i \le n)$. Then we have the following lemma, its proof was suggested by Fusheng Leng

Lemma 4.6. *Let $\sigma = \sum_{i=1}^n a_i$. Then, when $\ell \le \frac{1}{2}\sigma$, there exists an injective map $\varphi : M^\ell_n(a_1, \ldots, a_n) \to M^{\sigma-\ell}_n(a_1, \ldots, a_n)$ such that*

$$v \le \varphi(v)\ , \quad \forall\ v \in M^\ell_n(a_1, \ldots, a_n).$$

Proof. The strategy of proof is to do induction for n and σ. The lemma is clearly true when $n = 1$. Assume the lemma is true for $n-1$. To show the lemma for n, we do induction for σ. The lemma is trivially true for any n when $\sigma = 1$. Thus we can assume $n \ge 2$ and $\sigma \ge 2$.

Without loss of generality, we assume $a_{n-1} > 0$ and $a_n > 0$. Let

$$S^\ell = \{\, v \in M^\ell_n(a_1, \ldots, a_n) \,|\, v_{n-1} = a_{n-1} \text{ or } v_n = 0 \,\},$$

$$S^{\sigma-\ell} = \{\, v \in M^{\sigma-\ell}_n(a_1, \ldots, a_n) \,|\, v_{n-1} = a_{n-1} \text{ or } v_n = 0 \,\},$$

$C^\ell = M^\ell_n(a_1, \ldots, a_n) \setminus S^\ell$ and $C^{\sigma-\ell} = M^{\sigma-\ell}_n(a_1, \ldots, a_n) \setminus S^{\sigma-\ell}$. We will show the existence of injective maps

$$\varphi_1 : S^\ell \to S^{\sigma-\ell}\ , \quad \varphi_2 : C^\ell \to C^{\sigma-\ell}$$

with $v \le \varphi_1(v)$, $v \le \varphi_2(v)$ ($\forall\ v \in S^\ell$, $\forall\ v \in C^\ell$) by induction of n, σ respectively. In order to use the induction, we identify S^ℓ (resp. $S^{\sigma-\ell}$) with $M^\ell_{n-1}(a_1,\ldots,a_{n-1}+a_n)$ (resp. $M^{\sigma-\ell}_{n-1}(a_1,\ldots,a_{n-1}+a_n)$) by

$$f_\ell : S^\ell \to M^\ell_{n-1}(a_1,\ldots,a_{n-1}+a_n), \quad f_\ell(v) = (v_1,\ldots,v_{n-2},v_{n-1}+v_n)$$

(resp. $f_{\sigma-\ell} : S^{\sigma-\ell} \to M^{\sigma-\ell}_{n-1}(a_1,\ldots,a_{n-1}+a_n)$). Indeed, f_ℓ (resp. $f_{\sigma-\ell}$) is a bijective map. To see the injectivity of f_ℓ, if $f_\ell(v) = f_\ell(v')$, then $v_i = v'_i$ ($1 \le i \le n-2$) and $v_{n-1} + v_n = v'_{n-1} + v'_n$. We claim that $v_{n-1}+v_n = v'_{n-1}+v'_n$ implies $v_n = v'_n$ (thus $v_{n-1} = v'_{n-1}$) since v, $v' \in S^\ell$. Indeed, if $v_n = 0$ then $v'_n = 0$, otherwise $v'_{n-1} = a_{n-1}$ (by definition of S^ℓ) and $v_{n-1} = a_{n-1} + v'_n > a_{n-1}$ (a contradiction to the definition of $M^\ell_n(a_1,\ldots,a_n)$). Similarly, $v'_n = 0$ implies $v_n = 0$. If both v_n and v'_n are not zero, by definition of S^ℓ, $v_{n-1} = a_{n-1} = v'_{n-1}$, thus $v_n = v'_n$. To see it being surjective, for any $w \in M^\ell_{n-1}(a_1,\ldots,a_{n-1}+a_n)$, notice that $w_i \le a_i$ ($1 \le i \le n-2$) and $w_{n-1} \le a_{n-1} + a_n$, we define

$$v = \begin{cases} (w_1,\ldots,w_{n-2},w_{n-1},0) & \text{if } w_{n-1} \le a_{n-1} \\ (w_1,\ldots,w_{n-2},a_{n-1},w_{n-1}-a_{n-1}) & \text{if } w_{n-1} > a_{n-1} \end{cases}$$

then $v \in S^\ell$ such that $f_\ell(v) = w$. Similarly, $f_{\sigma-\ell}$ is bijective.

By the inductive assumption for n, there exists an injective map

$$\psi_1 : M^\ell_{n-1}(a_1,\ldots,a_{n-1}+a_n) \to M^{\sigma-\ell}_{n-1}(a_1,\ldots,a_{n-1}+a_n)$$

such that $v \le \psi_1(v)$ ($\forall\ v \in M^\ell_{n-1}(a_1,\ldots,a_{n-1}+a_n)$). Then, we define

$$\varphi_1 = f^{-1}_{\sigma-\ell} \cdot \psi_1 \cdot f_\ell :\ S^\ell \to S^{\sigma-\ell}.$$

For any $v = (v_1,\ldots,v_n) \in S^\ell$, we need to show $v \le \varphi_1(v)$. Let

$$\psi_1(f_\ell(v)) = (w_1,\ldots,w_{n-2},w_{n-1}) \in M^{\sigma-\ell}_{n-1}(a_1,\ldots,a_{n-1}+a_n).$$

Then $v_i \le w_i$ ($1 \le i \le n-2$), $v_{n-1} + v_n \le w_{n-1}$ and

$$\varphi_1(v) = \begin{cases} (w_1,\ldots,w_{n-2},w_{n-1},0) & \text{if } w_{n-1} \le a_{n-1} \\ (w_1,\ldots,w_{n-2},a_{n-1},w_{n-1}-a_{n-1}) & \text{if } w_{n-1} > a_{n-1} \end{cases}$$

by the definition of f_ℓ, ψ_1 and $f_{\sigma-\ell}$. Thus $v_i \le \varphi_1(v)_i$ ($1 \le i \le n-2$). We still need to check $v_{n-1} \le \varphi_1(v)_{n-1}$ and $v_n \le \varphi_1(v)_n$. If $v_n = 0$ (thus $v_n \le \varphi_1(v)_n$), then $v_{n-1} \le w_{n-1}$ (since $v_{n-1} + v_n \le w_{n-1}$), thus

$$v_{n-1} \le \min\{w_{n-1}, a_{n-1}\} \le \varphi_1(v)_{n-1}.$$

If $v_n \ne 0$, by the definition of S^ℓ, $v_{n-1} = a_{n-1}$, which implies

$$a_{n-1} < a_{n-1} + v_n = v_{n-1} + v_n \le w_{n-1}.$$

Thus $\varphi_1(v)_{n-1} = a_{n-1}$ and $\varphi_1(v)_n = w_{n-1} - a_{n-1} = w_{n-1} - v_{n-1} \ge v_n$.

Next we construct the injective map $\varphi_2 : C^{\ell} \to C^{\sigma-\ell}$ by using induction for σ. By the definition of C^{ℓ} and $C^{\sigma-\ell}$, we have

$$C^{\ell} = \{v \in M_n^{\ell}(a_1, \dots, a_n) \,|\, v_{n-1} \leq a_{n-1} - 1,\ v_n \geq 1\,\}$$

$$C^{\sigma-\ell} = \{v' \in M_n^{\sigma-\ell}(a_1, \dots, a_n) \,|\, v'_{n-1} \leq a_{n-1} - 1,\ v'_n \geq 1\,\}.$$

Let $\bar{\sigma} = a_1 + \cdots + a_{n-2} + (a_{n-1} - 1) + (a_n - 1) = \sigma - 2$ and $\bar{\ell} = \ell - 1$, we have the following clear identifications

$$\pi_{\ell} :\, C^{\ell} \to M_n^{\bar{\ell}}(a_1, \dots, a_{n-2}, a_{n-1} - 1, a_n - 1)$$

$$\pi_{\sigma-\ell} :\, C^{\sigma-\ell} \to M_n^{\bar{\sigma}-\bar{\ell}}(a_1, \dots, a_{n-2}, a_{n-1} - 1, a_n - 1)$$

where $\pi_{\ell}(v) = (v_1, \dots, v_{n-1}, v_n - 1)$, $\pi_{\sigma-\ell}(v') = (v'_1, \dots, v'_{n-1}, v'_n - 1)$. Notice that $\bar{\ell} \leq \frac{1}{2}\bar{\sigma}$, by induction for σ, there exists an injective map

$$\psi_2 : M_n^{\bar{\ell}}(a_1, \dots, a_{n-2}, a_{n-1}-1, a_n-1) \to M_n^{\bar{\sigma}-\bar{\ell}}(a_1, \dots, a_{n-2}, a_{n-1}-1, a_n-1)$$

such that $v \leq \psi_2(v)$ for any $v \in M_n^{\bar{\ell}}(a_1, \dots, a_{n-2}, a_{n-1} - 1, a_n - 1)$. Let

$$\varphi_2 = \pi_{\sigma-\ell}^{-1} \cdot \psi_2 \cdot \pi_{\ell} :\ C^{\ell} \to C^{\sigma-\ell}.$$

For any $v = (v_1, \dots, v_n) \in C^{\ell}$, we have to check that $v \leq \varphi_2(v)$. Let

$$\psi_2(\pi_{\ell}(v)) = (w_1, \dots, w_n) \in M_n^{\bar{\sigma}-\bar{\ell}}(a_1, \dots, a_{n-2}, a_{n-1} - 1, a_n - 1),$$

then $v_i \leq w_i$ $(1 \leq i \leq n-1)$, $v_n - 1 \leq w_n$ and

$$\varphi_2(v) = (w_1, \dots, w_{n-1}, w_n + 1) \in C^{\sigma-\ell}$$

by the definition of π_{ℓ}, ψ_2 and $\pi_{\sigma-\ell}$. Thus $v_n \leq w_n + 1 = \varphi_2(v)_n$ and we have shown the lemma.

□

Proposition 4.7. *Let $V \subset R^{\ell}$ be a linear subspace, $\mathbb{L}(\mathrm{D}_{2\ell-n(p-1)} \cdot V)$ be the linear subspace generated by $\mathrm{D}_{2\ell-n(p-1)} \cdot V \subset R^{n(p-1)-\ell}$. Then,*

$$\dim(V) \leq \dim \mathbb{L}(\mathrm{D}_{2\ell-n(p-1)} \cdot V) \quad \textit{when} \quad \frac{n(p-1)}{2} \leq \ell \leq n(p-1).$$

Proof. Let $\omega = y_1^{p-1} y_2^{p-1} \cdots y_n^{p-1} \in R^{n(p-1)}$. Then the D-module structure on R induces surjective morphisms

$$\phi_{\ell} : \mathrm{D}_{\ell} \xrightarrow{\cdot\omega} R^{n(p-1)-\ell} \tag{4.13}$$

of linear spaces for any $0 \leq \ell \leq n(p-1)$. They must be isomorphisms since $\dim(\mathrm{D}_{\ell}) = \dim(R^{n(p-1)-\ell})$. To show the equality of dimensions, it is enough to show

$$\dim(\mathrm{D}_{\ell}) \geq \dim(R^{n(p-1)-\ell}) = \dim(\mathrm{D}_{n(p-1)-\ell}) \geq \dim(R^{\ell}) = \dim(\mathrm{D}_{\ell}).$$

The two inequalities hold because we have the surjective homomorphisms ϕ_ℓ and $\phi_{n(p-1)-\ell}$. The two equalities hold because

$$\bigoplus_{\ell=0}^{n(p-1)} R^\ell = R \cong \mathrm{D} = \bigoplus_{\ell=0}^{n(p-1)} \mathrm{D}_\ell$$

as (graded) K-algebras. In particular,

$$\phi_{n(p-1)-\ell} : \mathrm{D}_{n(p-1)-\ell} \to R^\ell\,, \quad \phi_\ell : \mathrm{D}_\ell \to R^{n(p-1)-\ell} \tag{4.14}$$

are isomorphisms. Since $0 \leq \bar{\ell} = n(p-1) - \ell \leq \frac{n(p-1)}{2}$, we can use Lemma 4.5 for $V' = \phi^{-1}_{n(p-1)-\ell}(V) \subset \mathrm{D}_{\bar{\ell}} = \mathrm{D}_{n(p-1)-\ell}$, thus there is a basis $\{d_i \in V'\}_{1\leq i\leq s}$ and monomials $\{\delta_i \in \mathrm{D}_{n(p-1)-2\bar{\ell}} = \mathrm{D}_{2\ell-n(p-1)}\}_{1\leq i\leq s}$ such that $\{\delta_i d_i \in \mathrm{D}_{n(p-1)-\bar{\ell}} = \mathrm{D}_\ell\}_{1\leq i\leq s}$ are linearly independent. Thus

$$\{\phi_\ell(\delta_i d_i) = \delta_i(d_i\omega) \in \mathrm{D}_{2\ell-n(p-1)} \cdot V \subset R^{n(p-1)-\ell}\}_{1\leq i\leq s}$$

are linearly independent, where $s = \dim(V') = \dim(V)$. We have proven the proposition. □

Let X be an irreducible smooth projective variety of dimension n over an algebraically closed field k with $\mathrm{char}(k) = p > 0$. For any vector bundle W on X, let

$$\mathrm{I}(W, X) = \max\{\mathrm{I}(W \otimes \mathrm{T}^\ell(\Omega^1_X)) \mid 0 \leq \ell \leq n(p-1)\}$$

be the maximal value of instabilities $\mathrm{I}(W \otimes \mathrm{T}^\ell(\Omega^1_X))$.

Theorem 4.8. *When $K_X \cdot \mathrm{H}^{n-1} \geq 0$, we have, for any $\mathcal{E} \subset F_*W$,*

$$\mu(F_*W) - \mu(\mathcal{E}) \geq -\frac{\mathrm{I}(W, X)}{p}. \tag{4.15}$$

In particular, if $W \otimes \mathrm{T}^\ell(\Omega^1_X)$, $0 \leq \ell \leq n(p-1)$, are semistable, then F_*W is semistable. Moreover, if $K_X \cdot \mathrm{H}^{n-1} > 0$, the stability of the bundles $W \otimes \mathrm{T}^\ell(\Omega^1_X)$, $0 \leq \ell \leq n(p-1)$, implies the stability of F_*W.

Proof. Since $K_X \cdot \mathrm{H}^{n-1} \geq 0$, by the inequality (4.9) in Lemma 4.4 (see also the notation in (4.8) and the lemma), it is enough to show

$$\sum_{\ell=0}^{m} (\frac{n(p-1)}{2} - \ell) r_\ell \geq 0.$$

If $m \le \frac{n(p-1)}{2}$, it is clear. If $m > \frac{n(p-1)}{2}$, then we have

$$\sum_{\ell=0}^{m}(\frac{n(p-1)}{2}-\ell)r_\ell = \sum_{\ell=m+1}^{n(p-1)}(\ell-\frac{n(p-1)}{2})r_{n(p-1)-\ell} + \sum_{\ell>\frac{n(p-1)}{2}}^{m}(\ell-\frac{n(p-1)}{2})(r_{n(p-1)-\ell}-r_\ell). \tag{4.16}$$

We will use Proposition 4.7 to show that

$$r_\ell \le r_{n(p-1)-\ell} \quad \text{when } \frac{n(p-1)}{2} \le \ell \le n(p-1).$$

It is clearly a local problem, we can consider all of the torsion free sheaves as vector spaces over the function field $K = k(X)$ of X. Without loss of generality, we assume $\mathrm{rk}(W) = 1$. Then, from the discussions in Section 3, we know that $V_\ell/V_{\ell+1} \cong \mathrm{T}^\ell(\Omega^1_X)$ $(0 \le \ell \le n(p-1))$ are precisely isomorphic to R^ℓ $(0 \le \ell \le n(p-1))$ in Proposition 4.7. Since the morphisms $V_\ell/V_{\ell+1} \xrightarrow{\nabla} (V_{\ell-1}/V_\ell)\otimes\Omega^1_X$ induce morphisms $\mathcal{F}_\ell \xrightarrow{\nabla} \mathcal{F}_{\ell-1}\otimes\Omega^1_X$, by the formula (3.6), we have

$$\mathrm{D}_{2\ell-n(p-1)}\cdot\mathcal{F}_\ell \subset \mathcal{F}_{n(p-1)-\ell}\,.$$

Then, by Proposition 4.7, $r_\ell = \dim(\mathcal{F}_\ell) \le \dim\mathbb{L}(\mathrm{D}_{2\ell-n(p-1)}\cdot\mathcal{F}_\ell)$, we have $r_\ell \le r_{n(p-1)-\ell}$, thus (4.15).

If the bundles $W\otimes\mathrm{T}^\ell(\Omega^1_X)$ $(0 \le \ell \le n(p-1))$ are stable, then

$$\mu(F_*W) - \mu(\mathcal{E}) \ge 0.$$

It becomes equality if and only if inequalities (4.10) become equalities and $\sum\limits_{\ell=0}^{m}(\frac{n(p-1)}{2}-\ell)r_\ell = 0$. Thus $m > \frac{n(p-1)}{2}$ and each term in (4.16) must be zero (since $K_X\cdot\mathrm{H}^{n-1} > 0$), which forces $m = n(p-1)$. Then the fact that inequalities (4.10) become equalities implies $\mathcal{E} = F_*W$. □

Corollary 4.9. *Let X be a smooth projective variety of $\dim(X) = n$, whose canonical divisor K_X satisfies $K_X\cdot\mathrm{H}^{n-1} \ge 0$. Then*

$$\mathrm{I}(F_*W) \le p^{n-1}\mathrm{rk}(W)\,\mathrm{I}(W,X).$$

Proof. It is just Theorem 4.8 plus the following trivial remark: For any vector bundle E, if there is a constant λ satisfying $\mu(E') - \mu(E) \le \lambda$ for any $E' \subset E$. Then $\mathrm{I}(E) \le \mathrm{rk}(E)\lambda$. □

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