Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products

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In their study on commutants of Hecke correspondences, Clozel-Ullmo [CU, 2003] considered germs of holomorphic maps arising from an algebraic correspondence Y which commutes with a given Hecke correspondence defined on the quotient $X := \Omega/\Gamma$ of an irreducible bounded symmetric domain by a torsion-free discrete group of automorphisms $\Gamma \subset \mathrm{Aut}(\Omega)$. Under certain conditions on the Hecke correspondence they asked the question whether the algebraic subvariety $Y \subset X \times X$ is modular in the sense that $Y \subset X \times X$ is a totally geodesic complex submanifold which descends from the graph of an automorphism of Ω. They reduced the problem first to a differential-geometric problem on the characterization of germs of measure-preserving holomorphic maps $f : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$. Specifically, given an algebraic correspondence $Y \subset X \times X$ such that the general fiber of the canonical projection $\mathrm{pr}_i : Y \to X_i$ of Y to the i-th factor $X_i = X$ consists of precisely d_i points; $i = 1, 2$; then at a general point $x \in X$, $\mathrm{pr}_2^{-1}(x) = \{y_1, \cdots, y_{d_2}\}$, taking inverse images of pr_2 we obtain a germ of holomorphic map $f_0 : (X; x) \to (X; y_1) \times \cdots \times (X; y_{d_2})$. Lifting X locally to its universal cover Ω and lifting each base point to $0 \in \Omega$ we have equivalently $f : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$. For $1 \leq \alpha \leq d_2$ we write Ω_α for the α-th direct factor of Ω^{d_2}, and $\pi_\alpha : \Omega^{d_2} \to \Omega_\alpha$ for the canonical projection onto $\Omega_\alpha = \Omega$. Let $d\mu_\Omega$ stand for the volume form of the Bergman metric on Ω. Then, an algebraic correspondence $Y \subset X \times X$ is measure-preserving if and only if for a general point $x \in X$ and for the germ of holomorphic map $f : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$ defined as in the above, we have $f^*(\pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega) = d_1\, d\mu_\Omega$, Ω_α being identified with Ω.

When Ω is the unit disk $\Delta \subset \mathbb{C}$, a germ of measure-preserving holomorphic map $f : (\Delta; 0) \to (\Delta; 0) \times \cdots \times (\Delta; 0)$ is equivalently a holomorphic isometry $f : (\Delta, d_1\, ds^2_\Delta; 0) \to (\Delta, ds^2_\Delta; 0)^{d_2}$, where ds^2_Δ denotes the Bergman metric on Δ, i.e., the Poincaré metric on Δ of constant Gaussian curvature -1. In this case Clozel-Ullmo [CU] showed that $\mathrm{Graph}(f) \subset \mathbb{C}\times\mathbb{C}^{d_2}$ extends to an affine-algebraic subvariety in $\mathbb{C}\times\mathbb{C}^{d_2}$, and deduced as a consequence that f is totally geodesic whenever it arises from an algebraic correspondence $Y \subset X \times X$ on some finite-volume quotient $X = \Delta/\Gamma$. For the general problem of characterizing commutants of (a certain type of) Hecke correspondences on finite-volume quotients $X = \Omega/\Gamma$ of irreducible bounded symmetric domains Ω, Clozel-Ullmo [CU] did not solve the problem on germs of measure-preserving holomorphic maps. In its place they further reduced the characterization problem for commutants to another differential-geometric problem of characterizing germs of holomorphic isometries $f : (\Omega, \lambda\, ds^2_\Omega; 0) \to (\Omega, ds^2_\Omega; 0) \times \cdots \times (\Omega, ds^2_\Omega; 0)$, where ds^2_Ω stands for the Bergman metric on Ω, and where in the case of $\dim(\Omega) > 1$ the normalizing constant λ is *a priori*

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only known to be a positive real number. They observed that in the case where Ω is of rank ≥ 2, any germ of holomorphic isometry f as in the above is necessarily totally geodesic as a consequence of the arguments on Hermitian metric rigidity in Mok (cf. [Mk1, 1987] and [Mk2, 1989]), and the total geodesy of f holds true without assuming that it arises from an algebraic correspondence on some finite-volume quotient $X = \Omega/\Gamma$. In the remaining case of the complex unit ball $\Omega = B^n$, $n \geq 2$, Mok [Mk4] proved that f is necessarily totally geodesic under the assumption that λ is a positive integer. A slight modification of the arguments in [Mk4] yields the result also for an arbitrary normalizing constant $\lambda > 0$, and this generalization has been incorporated in Mok [Mk5, §3]. We note that even in the case of $\Omega = B^n$, $n \geq 2$, the germ of map f is totally geodesic whenever it is a germ of holomorphic isometry, without further assuming that it arises from algebraic correspondences. Here as in [CU] the proof proceeds first with proving algebraic extension of Graph(f), but for the proof of total geodesy of the map we made use of the functional identity on potential functions and the result of Alexander [Al] characterizing automorphisms of B^n of complex dimension ≥ 2.

In this article we solve the problem of Clozel-Ullmo on the characterization of germs of measure-preserving holomorphic maps $f = (f_1, \cdots, f_{d_2}) : (\Omega, d_1\, d\mu_\Omega; 0) \to (\Omega^{d_2}, \pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega; 0)$ for irreducible bounded symmetric domains Ω, where each component map $f_\alpha : \Omega \to \Omega_\alpha$, $1 \leq \alpha \leq d_2$, is of maximal rank at some point. When $\Omega = \Delta$, by Clozel-Ullmo [CU], f is totally geodesic provided that it arises from an algebraic correspondence $Y \subset X \times X$ on some finite-volume quotient $X = \Delta/\Gamma$. Moreover, without the latter assumption, in general f need not be totally geodesic as shown by the nonstandard examples of Mok [Mk4] given by p-th root maps and their composites. In the current article, we prove on the other hand that for irreducible bounded symmetric domains Ω of dimension ≥ 2, the germ of measure-preserving holomorphic map $f : (\Omega; 0) \to (\Omega^{d_2}; 0)$ is totally geodesic without further assumptions.

For the proof of our main results we make use of extension theorems in Several Complex Variables. With respect to the Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$ as a bounded symmetric domain, it is known that the Bergman kernel $K(z, w)$ on Ω is a rational function in $(z, \overline{w})$. It follows that by passing to unit sphere bundles the germ of measure-preserving holomorphic map $f : (\Omega; 0) \to (\Omega^{d_2}; 0)$ induces a germ of CR-mapping $\widetilde{f}$ between certain algebraic hypersurfaces. By curvature considerations the target algebraic hypersurface is pseudoconvex and strongly pseudoconvex at a general point. We check that, modifying the base point of the germ of map f if necessary, $\widetilde{f}$ maps its base point to a strongly pseudoconvex point of the target algebraic hypersurface. As a consequence, we can apply the result of Huang [Hu] to obtain an extension of Graph$(f) \subset \Omega \times \Omega^{d_2}$ to an affine-algebraic variety $V \subset \mathbb{C} \times \mathbb{C}^{d_2}$, which may be regarded as the 'graph' of a multivalent holomorphic map F from $\mathbb{C}^n$ into $(\mathbb{C}^n)^{d_2}$. When Ω is the complex unit ball B^n of dimension $n \geq 2$, after the step of algebraic extension we conclude our argument again by using Alexander's Theorem [Al], according to which a nonconstant holomorphic map $h : U_b \to \mathbb{C}^n$, $n \geq 2$, defined on a neighborhood U_b of a boundary point $b \in \partial B^n$ must necessarily agree with an automorphism of B^n whenever $h(U_b \cap \partial B^n) \subset \partial B^n$. The latter condition is checked for component maps F_α, $1 \leq \alpha \leq d_2$ of any local branch at a general boundary point $b \in \partial B^n$

of the extended multivalent map F by means of the functional identity arising from the measure-preserving property, and we conclude from Alexander's result [Al] that F restricts to a totally geodesic holomorphic embedding $F\big|_{B^n} : B^n \to (B^n)^{d_2}$. When Ω is of rank ≥ 2 we make use of an analogous result due to Henkin-Tumanov [TK1], in which automorphisms of an irreducible bounded symmetric domain Ω of rank ≥ 2 are given a local characterization in terms of boundary points lying on the Shilov boundary. In order to apply the result of Henkin-Tumanov [TK1], we show first of all that the lifting of the bad set of the multivalent holomorphic map F on $\mathbb{C}^n$ lies on an affine-algebraic variety which necessarily avoids general points on the Shilov boundary. Furthermore, by means of the fine structure of $\partial\Omega$ (Wolf [Wo]), which decomposes $\partial\Omega$ into a disjoint union of finitely many $\mathrm{Aut}_0(\Omega)$-orbits, we show that a general point $b \in Sh(\Omega)$ of the Shilov boundary $Sh(\Omega)$ is mapped into $Sh(\Omega)$ by each component map of a local branch of the multivalent extension F, thereby allowing us to apply [TK1] and to conclude the total geodesy of $f : (\Omega; 0) \to (\Omega^{d_2}; 0)$.

In the last section we give a new Alexander-type characterization theorem for automorphisms on irreducible bounded symmetric domains Ω of rank ≥ 2, where in place of the Shilov boundary as considered by Henkin-Tumanov we consider a boundary-preserving biholomorphism defined on a neighborhood of a point on the smooth locus $\mathrm{Reg}(\partial\Omega)$ of the boundary. We deem it natural to present the latter Alexander-type theorem as it gives an alternative argument to complete the proof of our Main Theorem in a way parallel to the rank-1 case, and the new statement and its proof could provide a useful tool in the future for the study of rigidity phenomena on irreducible bounded symmetric domains of rank ≥ 2 related to the theme of the current article.

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§1 Background materials and statement of results

(1.1) *Motivation and statement of results.* Let Ω be an irreducible bounded symmetric domain, of complex dimension n, and Γ be a torsion-free discrete subgroup of $\mathrm{Aut}(\Omega)$. Write $X := \Omega/\Gamma$. In the case where X is compact, by an algebraic correspondence on X we will simply mean an irreducible subvariety $Y \subset X \times X$ such that the restriction to Y of the canonical projection to each of the two Cartesian factors is a surjective finite map. When X is of finite volume with respect to the canonical measure but non-compact, we consider a non-singular projective-algebraic model $\overline{X}$ of the minimal compactification $\overline{X}_{\min}$, and regard $X \subset \overline{X}$ naturally as a quasi-projective manifold. By an algebraic correspondence we will mean an irreducible quasi-projective subvariety $Y \subset X \times X$ such that the restriction to Y of the canonical projection to each of the two Cartesian factors is a surjective finite proper map. The assumption that $Y \subset X \times X$ is a quasi-projective subvariety means equivalently that the topological closure $\overline{Y} \subset \overline{X} \times \overline{X}$ (called the closure of Y in the sequel) is an irreducible projective-algebraic subvariety (of complex dimension n). We are going to recall the notion of measure-preserving algebraic correspondences on $X = \Omega/\Gamma$ taken from Clozel-Ullmo [CU]. For the basic definitions we refer the reader to [CU, §1] and the references given there.

Denote by $\mathrm{pr}_i : Y \to X_i$ the restriction to Y of the canonical projection of $X \times X \to X_i$ to the i-th factor $X_i = X$; $i = 1, 2$; and by $\overline{\mathrm{pr}}_i : \overline{Y} \to \overline{X_i}$ the analogue on $\overline{Y}$. Write d_i for the degree of $\overline{\mathrm{pr}}_i$; $i = 1, 2$. At a general point $x \in X$, $\mathrm{pr}_2^{-1}(x) = \{y_1, \cdots, y_{d_2}\}$, taking inverse images of pr_2 we obtain a germ of holomorphic map $f_0 : (X; x) \to (X; y_1) \times \cdots \times (X; y_{d_2})$. By locally lifting X to its universal cover Ω with the base points identified with $0 \in \Omega$ we obtain a germ of holomorphic map $f : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$. For $1 \le \alpha \le d_2$, we write Ω_α for the α-th direct factor of Ω^{d_2}, and $\pi_\alpha : \Omega^{d_2} \to \Omega_\alpha$ for the canonical projection onto $\Omega_\alpha = \Omega$.

By the canonical measure $d\mu_\Omega$ on an irreducible bounded symmetric domain Ω we will mean the volume form of the Bergman metric ds^2_Ω on Ω. For a surjective finite proper holomorphic map $\varphi : M \to Z$ from an irreducible complex-analytic space M onto a complex manifold Z, there is the notion of the order (multiplicity) of φ at $a \in M$, written $\mu_\varphi(a)$, such that $\mu_\varphi(a) = 1$ whenever φ is unramified at a, and such that $\sum_{a \in \varphi^{-1}(z)} \mu_\varphi(a) = s(\varphi)$, the sheeting number of $\varphi : M \to Z$. Coming back to the algebraic correspondence $Y \subset X \times X$, for each $x \in X$ we have the 0-cycle $T_Y.x := \mathrm{pr}_{2*}\mathrm{pr}_1^* x$, and for a function α on X we define $T_Y^*\alpha(x) := \sum_{z \in T_Y.x} \alpha(z)$. By [CU, Lemma 1.1], T_Y^* transforms continuous functions α on X to continuous functions. Denoting by $d\mu_X$ the volume form on X induced from $d\mu_\Omega$, the algebraic correspondence $Y \subset X \times X$ is said to be measure-preserving ([CU, immediately after the proof of Lemma 1.1]) if and only if $\frac{1}{d_1}\int_X T_Y^*\alpha \, d\mu_X = \int_X \alpha \, d\mu_X$, noting that the normalizing factor $\frac{1}{d_1}$ is imposed by the special case of the constant function $\alpha \equiv 1$. By Clozel-Ullmo [CU] (cf. (2.1)) the algebraic correspondence $Y \subset X \times X$ is measure-preserving if and only if the germ of holomorphic map $f = (\Omega; 0) \to (\Omega; 0)^{d_2}$ defined as in the above by taking inverse images under pr_2 on a neighborhood of a general point $x \in X_2$ satisfies the identity $f^*(\pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega) = d_1 d\mu_\Omega$. In other words, $f = (f_1, \cdots, f_{d_2})$ satisfies

$$(\natural) \qquad \frac{1}{d_1}\sum_{\alpha=1}^{d_2} f_\alpha^* d\mu_\Omega = d\mu_\Omega.$$

In relation to a problem of characterizing modular correspondences of X among algebraic correspondences, Clozel and Ullmo [CU] raised the following question: *If an algebraic correspondence preserves the canonical measure, is the correspondence necessarily modular?* To give an affirmative answer to the question, it suffices to prove the total geodesy of a measure-preserving algebraic correspondence $Y \subset X \times X$. Let the Kähler form of (Ω, ds^2_Ω) be $\omega = \sqrt{-1}\sum(g_{i\bar{j}} dz^i \wedge d\bar{z}^j)$ and $g = \det(g_{i\bar{j}})$. Then

$d\mu_\Omega = \left(\sqrt{-1}\right)^{n^2} g dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$. Now ($\natural$) can be rewritten as

$$(\dagger) \qquad \sum_{\alpha=1}^{d_2} (g \circ f_\alpha) \big| \det(Jf_\alpha) \big|^2 = d_1 g,$$

where Jf_α is the Jacobian matrix of f_α. Clozel-Ullmo [CU] considered the special case where Ω is the unit disk, and they proved

Theorem 1.1.1. (Clozel-Ullmo[CU]) *Let $\Gamma \subset \mathrm{Aut}(\Delta)$ be a torsion-free lattice, and $X = \Delta/\Gamma$ be the quotient Riemann surface, $\overline{X}$ be its uniquely determined compactification to an algebraic curve. Let $Y \subset X \times X$ be an algebraic correspondence on X such that the canonical projection $\mathrm{pr}_i : \overline{Y} \to \overline{X_i}$; $i = 1, 2$; is of degree d_i, where X_i denotes the i-th direct factor of $X \times X = X_1 \times X_2$. Suppose the algebraic correspondence $Y \subset X \times X$ is measure-preserving. Let $f = (f_1, \cdots, f_{d_2}) : (\Delta, d_1\, ds^2_\Delta; 0) \to (\Delta, ds^2_\Delta; 0) \times \cdots \times (\Delta, ds^2_\Delta; 0)$ be a germ of measure-preserving holomorphic map arising from taking inverse images under pr_2 at a general point $x \in X$. Then, $\mathrm{Graph}(f) \subset \Delta \times \Delta^{d_2} \subset \mathbb{C} \times \mathbb{C}^{d_2}$ extends to an affine-algebraic variety $V \subset \mathbb{C} \times \mathbb{C}^{d_2}$ which is the graph of a totally geodesic holomorphic embedding $F : (\Delta, ds^2_\Delta) \to (\Delta, ds^2_\Delta) \times \cdots \times (\Delta, ds^2_\Delta)$. In particular, $Y \subset X \times X$ is a modular correspondence.*

Our main result in the current article is

Main Theorem. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of complex dimension ≥ 2, and $d\mu_\Omega$ be the volume form of the Bergman metric on Ω. Suppose d_1 and d_2 are positive integers and $f = (f_1, \cdots, f_{d_2}) : (\Omega, d_1\, d\mu_\Omega; 0) \to (\Omega^{d_2}, \pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega; 0)$ is a measure-preserving holomorphic map such that each $f_\alpha, 1 \leq \alpha \leq d_2$, is of maximal rank at some point. Then, $d_1 = d_2$ and f extends to a totally geodesic holomorphic embedding $f : \Omega \to \Omega^{d_2}$.*

Here in the statement of the the Main Theorem we do not assume that f arises from an algebraic correspondence, but, following the question posed in Clozel-Ullmo, we assume that the implicit normalizing constant λ, given by d_1, is a positive integer, and that each of the component map f_α is of maximal rank at some point, so that $\det(Jf_\alpha)$ does not vanish identically. A holomorphic isometry between bounded symmetric domains up to normalizing constants with respect to the Bergman metric will be called nonstandard if and only if it is not totally geodesic. In Mok [Mk5] one of the authors has constructed nonstandard examples of holomorphic isometric (proper) embeddings of the unit disk into polydisks. For this case, holomorphic isometries are the same as measure-preserving holomorphic maps. Our Main Theorem says that, unlike the case of the unit disk Δ, there is no nonstandard measure-preserving holomorphic map $F : \Omega \to \Omega^{d_2}$ whenever the irreducible bounded symmetric domain Ω is the complex unit ball B^n of dimension $n \geq 2$ or it is of rank ≥ 2. For holomorphic isometries arising from algebraic correspondences in the case of the unit disk, the result of Clozel-Ullmo (Theorem 1.1.1) says that the algebraic extension is forced to be totally geodesic because the algebraic extension $V \subset \Delta \times \Delta^{d_2}$ is equivariant with respect to Γ. Combining Theorem 1.1.1 and our Main Theorem we have resolved the problem of Clozel-Ullmo [CU] on measure-preserving algebraic correspondences, as follows.

Theorem 1.1.2. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain, and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free lattice. Write $X := \Omega/\Gamma$ and let $Y \subset X \times X$ be a measure-preserving algebraic correspondence with respect to the canonical measure $d\mu_\Omega$ on Ω. Then, Y is necessarily a modular correspondence.*

From Clozel-Ullmo [CU, Sections 2, 3], and in the terminology used there, Theorem 1.1.2 implies a characterization of algebraic correspondences commuting with certain modular correspondences. In the notation of Theorem 1.1.2, an irreducible modular correspondence on X is defined by an element $g \subset \mathrm{Aut}(\Omega)$ such that g and $g^{-1}\Gamma g$ are commensurable. Defining $i_g : \Omega \to \Omega \times \Omega$ by $i_g(z) = (z, g(z))$, the modular correspondence associated to g is given by $S_g = \pi(i_g(\Omega))$, where $\pi : \Omega \times \Omega \to X \times X$ is the canonical projection. Following [CU], S_g will be called an interior modular correspondence (correspondance intérieure) if the subgroup generated by Γ and g in $\mathrm{Aut}(\Omega)$ is discrete, and called an exterior modular correspondence (correspondance extérieure) otherwise. From Theorem 1.1.2 here and the proofs of [CU, Theorems 2.10 and 3.8] (for the case of the unit disk resp. the case of rank ≥ 2) it follows that the latter results hold also in the case where Ω is of rank 1 and of dimension ≥ 2. In other words, we have

Corollary 1.1.1. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain and identify $\mathrm{Aut}(\Omega)$ as a linear algebraic group $\mathcal{G}$ defined over $\mathbb{Q}$. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free lattice which is a congruence subgroup of $\mathcal{G}$ and write $X := \Omega/\Gamma$. Let $S_g \subset X \times X$ be an exterior modular correspondence defined by g, where g is a rational point in $\mathcal{G}$. Suppose $Y \subset X \times X$ is an algebraic correspondence which commutes with S_g. Then, Y is necessarily a modular correspondence.*

(1.2) *Algebraic extension of germs of measure-preserving holomorphic maps.* Denote by d_i; $i = 1, 2$; the degree of the canonical projection of Y onto the i-th factor. To prove the Main Theorem, we first establish the algebraic extension of the holomorphic map $f : U \to \Omega^{d_2}$ induced by Y, where U is some open neighborhood of $0 \in \Omega$. Equivalently, we consider f as a germ of holomorphic map at 0, written as $f : (\Omega; 0) \to (\Omega; 0)^{d_2}$. In the sequel we will make no distinction between a germ of map and a representative of the germ of map, thus in the latter interpretation for f it is understood that f refers to a map on some open neighborhood U of 0, and $\mathrm{Graph}(f)$ both refers to the germ of the graph and its representative, viz., the graph of f over U.

Consider the anti-canonical line bundle L of Ω equipped with the Hermitian metric $g = \det\left(g_{i\bar{j}}\right)$, then (L, g) is a negative line bundle because $-\sqrt{-1}\partial\bar{\partial}\log g = \mathrm{Ric}(\Omega, \omega)$ which is negative definite. Let $\pi_\alpha : \Omega^{d_2} \longrightarrow \Omega$ be the canonical projection onto the α-th factor and write $(L_\alpha, g_\alpha) = (\pi_\alpha^* L, g \circ \pi_\alpha)$ for the pull-back of (L, g) by π_α to Ω^{d_2}. Let $(\mathcal{L}, \mathbf{g}) = (\oplus_{\alpha=1}^{d_2} L_\alpha, \oplus_{\alpha=1}^{d_2} g_\alpha)$ be the direct sum of L_α equipped with the product metric. It then follows that $(\mathcal{L}, \mathbf{g})$ is a seminegative Hermitian holomorphic vector bundle in the sense of Griffiths.

Write the canonical coordinates in L and $\mathcal{L}$ as $(z_1, \ldots, z_n, u)$ and $(z_1^1, \ldots, z_n^1, \ldots, z_1^{d_2}, \ldots, z_n^{d_2}, u_1 \ldots, u_{d_2})$ respectively. Then the unit sphere bundles S_L

and $S_{\mathcal{L}}$ of $(L, d_1 g)$ and $(\mathcal{L}, \mathbf{g})$ are respectively defined by the equations

$$d_1 g(z_1, \ldots, z_n)|u|^2 = 1 \tag{1}$$

and

$$\sum_{\alpha=1}^{d_2} g(z_1^\alpha, \ldots, z_n^\alpha)|u_\alpha|^2 = 1. \tag{2}$$

Now by considering the derivative of f on the tangent bundle of Ω, we obtain an induced locally defined map $\widetilde{f} : V \subset (L, d_1 g) \longrightarrow (\mathcal{L}, \mathbf{g})$ given by

$$(\sharp) \qquad \widetilde{f} = (f_1, \ldots, f_{d_2}, \det(Jf_1)u, \ldots, \det(Jf_{d_2})u).$$

By (†), $\widetilde{f}$ maps some non-empty connected open subset of S_L into $S_{\mathcal{L}}$. We will make use of the following theorem of Huang.

Theorem 1.2.1. (Huang [Hu]). *Let $M_1 \subset \mathbb{C}^m$ and $M_2 \subset \mathbb{C}^{m+k}$ be real algebraic hypersurfaces with $m > 1$ and $k \geq 0$. Let $p \in M_1$ be a strongly pseudoconvex point. Suppose that h is a holomorphic mapping from a neighborhood U_p of p to $\mathbb{C}^{m+k}$ so that $h(U_p \cap M_1) \subset M_2$ and $h(p)$ is also a strongly pseudoconvex point, then h is algebraic.*

In Huang [Hu], h is said to be algebraic if each of its component function satisfies a non-trivial algebraic equation. For our purpose, we will take another equivalent definition, viz., h is algebraic if and only if Graph(h) is contained in an irreducible affine-algebraic variety of the same dimension, i.e., of dimension m. Theorem 1.2.1 is the Main Theorem of Huang [Hu]. In the original version of the theorem, M_1 and M_2 are assumed to be strongly pseudoconvex real-algebraic hypersurfaces and f is defined on a neighborhood of M_1 such that $h(M_1) \subset M_2$. However, we note that the proof is local in nature, and the assumptions can be slightly relaxed as stated here in the theorem.

To prove the algebraicity of $f : U \to \Omega^{d_2}$ in our situation, we first need two lemmas. In what follows on $\mathbb{C}^n$ we write $dV = (\sqrt{-1})^{n^2} dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$, which is 2^n times the Euclidean volume form.

Lemma 1.2.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization. Write $K_\Omega(z, w)$ for the Bergman kernel of Ω. Then, there exists a polynomial $Q_\Omega(z, w)$ which is holomorphic in z and anti-holomorphic in w such that $K_\Omega(z, w) = \frac{1}{Q_\Omega(z,w)}$. As a consequence, denoting by $d\mu_\Omega$ the volume form of the Bergman metric on Ω and writing $d\mu_\Omega = gdV$, $g(z)$ is the restriction of a rational function in $(z, \overline{z})$ to Ω.*

Proof. The formula for the Bergman kernel $K_\Omega(z, w) = \frac{1}{Q_\Omega(z,w)}$ on the bounded symmetric domain Ω can be found in Faraut-Korányi [FK, pp.76-77, especially Eqns.(3.4) and (3.9)]. Since the automorphism group Aut(Ω) acts transitively on Ω, and both $K_\Omega(z, z)dV$ and $d\mu_\Omega = gdV$ are invariant under Aut(Ω), we must have $g(z) = c_\Omega K_\Omega(z, z)$ for some positive constant c_Ω, and the lemma follows. □

Next, in order to apply the result of Huang (Theorem 1.2.1) on extension of CR-maps between real-algebraic hypersurfaces, we have to check the condition of strong pseudoconvexity. For a Hermitian holomorphic vector bundle (E, h) on a domain D, we denote by $\Theta = \Theta_{E,h}$ the associated End(E)-valued curvature (1,1) form. At each point $z \in D$, the tensor $\Theta_z = \Theta(z)$ can equivalently be considered as a Hermitian bilinear pairing Q_z on $E_z \otimes T_z^{0,1}$. Then, (E, h) is of strictly negative curvature in the dual sense of Nakano if and only if $Q_z < 0$ at each point $z \in D$. For the purpose of checking the condition of strong pseudoconvexity we have

Lemma 1.2.2. *Let (E, h) be a Hermitian vector bundle holomorphic of rank r on a domain $D \subset \mathbb{C}^n$ with the Hermitian metric h. Let S_E be the unit sphere bundle of E. If (E, h) is seminegative in the sense of Griffiths, then S_E is a pseudoconvex real hypersurface in $\mathbb{C}^{r+n}$. Furthermore, if $(z, v) \in S_E$ is a weakly pseudoconvex point, then there exists some non-zero vector $\xi \in T_z^{1,0}D$ such that $Q_z(v\otimes\overline{\xi}, v\otimes\overline{\xi}) = 0$. In particular, if (E, h) is strictly negative in the sense of Griffiths, then S_E is strongly pseudoconvex.*

Proof. Write $\pi : E \to D$ for the canonical projection. At $z \in D$, let U be a sufficiently small open neighborhood of z such that $\pi\big|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is holomorphically trivial. Let (w^i, v^α), $1 \le i \le n, 1 \le \alpha \le r$, be local holomorphic coordinates over $E|_U := \pi^{-1}(U)$, where w^i are holomorphic coordinates on U, $w^i(z) = 0$ for $1 \le i \le n$, and v^α are holomorphic fiber coordinates for the vector bundle $E\big|_U$. We choose the fiber coordinates adapted to the point z, i.e. $h_{\alpha\bar\beta}(0) = \delta_{\alpha\beta}$ and $dh_{\alpha\bar\beta}(0) = 0$, where $\delta_{\alpha\beta}$ is the Kronecker delta. S_E is defined by $\varphi(w^i; v^\alpha) = \sum_{\alpha,\beta} h_{\alpha\bar\beta}(w^1, \ldots, w^n)u^\alpha\bar{u}^\beta = 1$. The complex Hessian of φ at $(0, v^\alpha)$ is

$$\begin{bmatrix} \sum_{\mu,\nu} \frac{\partial^2 h_{\mu\bar\nu}}{\partial w_i \partial \overline{w_j}}(0)v^\mu \bar{v}^\nu & 0 \\ 0 & \delta_{\alpha\beta} \end{bmatrix} . \tag{1}$$

On the other hand, the curvature tensor at z is given by

$$\Theta_{\mu\bar\nu i\bar j}(0) = -\frac{\partial^2 h_{\mu\bar\nu}}{\partial w_i \partial \overline{w_j}}(0) . \tag{2}$$

Thus, given a $(1, 0)$ tangent vector $\eta = \sum\limits_{\alpha=1}^{n} \xi^i \dfrac{\partial}{\partial w_i} + \sum\limits_{\alpha=1}^{r} \lambda^\alpha \dfrac{\partial}{\partial u_\alpha}$ at $(0, v)$ we have

$$\sqrt{-1}\partial\overline{\partial}\varphi\Big(\frac{1}{\sqrt{-1}}\eta \wedge \overline{\eta}\Big)$$

$$= -\sum_{i,j=1}^{n} \sum_{\mu,\nu=1}^{r} \Theta_{\mu\bar\nu i\bar j}(0)v^\mu\overline{v^\nu}\xi^i\overline{\xi^j} + \sum_{\alpha=1}^{r} \left|\lambda^\alpha\right|^2 . \tag{3}$$

By definition, (E, h) is seminegative in the sense of Griffiths at 0 if and only if

$$\sum_{\mu,\nu,i,j} \Theta_{\mu\bar\nu i\bar j}(0)v^\mu\overline{v^\nu}\xi^i\overline{\xi^j} \le 0 , \tag{4}$$

and it follows from (3) that the Levi form $\sqrt{-1}\partial\overline{\partial}\varphi$ is semipositive on S_E if and only if $Q_z(v \otimes \overline{\xi}, v \otimes \overline{\xi}) \le 0$ for any $v \in E_z, \xi \in T_z^{1,0}$. Moreover, it is strictly positive at

$(z,v), v \neq 0$, unless there exists some nonzero $\xi \in T_z^{1,0}$ such that $Q_z(v \otimes \overline{\xi}, v \otimes \overline{\xi}) = \sum \Theta_{\mu\bar{\nu}i\bar{j}}(0) v^\mu \overline{v^\nu} \xi^i \overline{\xi^j} = 0$, as desired. $\square$

Since $(L, d_1 g)$ is a (strictly) negative line bundle, S_L is a strongly pseudoconvex real-algebraic hypersurface in $\mathbb{C}^{n+1}$ as g is rational for the Bergman metric. On the other hand, $(\mathcal{L}, \mathbf{g})$ is seminegative in the sense of Griffiths, so we only know that $S_{\mathcal{L}}$ is a pseudoconvex real-algebraic hypersurface in $\mathbb{C}^{d_2(n+1)}$. In order to apply Theorem 1.2.1, we need to show that $\widetilde{f}$ maps some point in S_L to a strongly pseudoconvex point in $S_{\mathcal{L}}$.

From the definition of $(\mathcal{L}, \mathbf{g})$, we see that the weakly pseudoconvex points on $S_{\mathcal{L}}$ are those where at least one of the components u_α vanishes, where $(u_1, \ldots, u_{d_2})$ are the canonical fiber coordinates on $\mathcal{L}$. Since none of the component maps of f is degenerate, the set $\{\det(Jf_1) = 0\} \cup \cdots \cup \{\det(Jf_{d_2}) = 0\}$ is a proper subvariety in $U \subset \Omega$. By the definition of $\widetilde{f}$ as in ($\sharp$) in the first paragraphs of (1.2), it follows that that $\widetilde{f}$ maps some point in S_L to a strongly pseudoconvex point in $S_{\mathcal{L}}$. Therefore, by Theorem 1.2.1., $\mathrm{Graph}(\widetilde{f}) \subset (\Omega \times \mathbb{C}) \times (\Omega^{d_2} \times \mathbb{C}^{d_2})$ extends to an irreducible affine-algebraic variety $W \subset (\mathbb{C}^n \times \mathbb{C}) \times \big((\mathbb{C}^n)^{d_2} \times \mathbb{C}^{d_2}\big)$ of complex dimension $n+1$. Restricting $\widetilde{f}$ to $\Omega \times \{0\}$ we recover f, and as a consequence $\mathrm{Graph}(f) \subset \Omega \times \Omega^{d_2}$ extends to an affine-algebraic variety $V \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$ of complex dimension n. To summarize, we have established the following intermediate result toward the proofs of Main Theorem and Theorem 1.1.2, noting that the preceding arguments apply to any irreducible bounded symmetric domain including the unit disk Δ.

Proposition 1.2.1. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain, and $d\mu_\Omega$ be the volume form of the Bergman metric on Ω. Suppose d_1 and d_2 are positive integers and $f = (f_1, \cdots, f_{d_2}) : (\Omega, d_1\, d\mu_\Omega; 0) \to (\Omega^{d_2}, \pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega; 0)$ is a measure-preserving holomorphic map. Then,* $\mathrm{Graph}(f) \subset \Omega \times \Omega^{d_2} \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$ *extends to an affine-algebraic variety $V \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$.*

From Proposition 1.2.1 we deduce readily

Proposition 1.2.2. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. Denote by $d\mu_\Omega$ the canonical measure on Ω given by the volume form of its Bergman metric. Let $f = (f_1, \cdots, f_{d_2}) : (\Omega; 0) \to (\Omega^{d_2}, \pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega; 0)$ be a germ of measure-preserving holomorphic map. Then, there exists an affine-algebraic subvariety $R \subset \mathbb{C}^n$ such that for any point $b \in \partial\Omega - R$, the germ of holomorphic map f at $0 \in \Omega$ can be analytically continued along some continuous path $\gamma : [0,1] \to \overline{\Omega} - R$ satisfying $\gamma([0,1)) \subset \Omega - R$, $\gamma(0) = 0$ and $\gamma(1) = b$ to a holomorphic map into $(\mathbb{C}^n)^{d_2}$ defined on a neighborhood U_b of b in $\mathbb{C}^n$.*

Proof. By Proposition 1.2.1, $\mathrm{Graph}(f) \subset \Omega \times \Omega^{d_2} \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$ can be extended to an affine-algebraic variety $V \subset \mathbb{C}^n \times (\mathbb{C}^n)^{d_2}$. Denote by M the compact dual of Ω, so that $\Omega \subset \mathbb{C}^n \subset M$ gives at the same time the Harish-Chandra embedding $\Omega \subset \mathbb{C}^n$ and the Borel embedding $\Omega \subset M$. The compactification $\mathbb{C}^n \subset M$ is birational to the standard compactification $\mathbb{C}^n \subset \mathbb{P}^n$. As a consequence, the topological closure $\overline{V} \subset M \times M^{d_2}$ is a projective subvariety of complex dimension n. Denote by $\pi_0 : \overline{V} \to M$ the canonical projection onto the factor M of $M \times M^{d_2}$. Let $S \subset \overline{V}$ be the union of the singular locus of $\overline{V}$, the subset of $\mathrm{Reg}(\overline{V})$ consisting of points where π_0 fails to be a local biholomorphism, and the set of points $w \in \overline{V}$ such that $\pi_\alpha(w) \in M - \mathbb{C}^n$ for one of the canonical projections $\pi_\alpha : \overline{V} \to M, 1 \le \alpha \le d_2$, onto the α-th direct factor of M^{d_2}. Then $S \subset \overline{V}$ is a projective subvariety such that each irreducible component is of complex dimension at most $n-1$. By the Proper Mapping Theorem, $E := \pi_0(S) \subsetneqq M$ is a subvariety of M. $R := E \cap \mathbb{C}^n \subsetneqq \mathbb{C}^n$ is an affine-algebraic subvariety. Then $\pi_0\big|_{V-\pi_0^{-1}(R)} : V - \pi_0^{-1}(R) \to \mathbb{C}^n - R$ is a topological covering map. The rest of Proposition 1.2.2 on analytic continuation follows readily. $\square$

§2 Proof of the Main Theorem and Theorem 1.1.2

(2.1) *Proof of the Main Theorem in the rank-1 case.* From Proposition 1.2.1 we have established the algebraic extension of the graph of the germ of measure-preserving holomorphic map $f : (\Omega; 0) \to (\Omega; 0)^{d_2}$. To proceed we will make use of the real-analytic functional identity satisfied by f and study boundary behavior of component maps f_α, $1 \le \alpha \le d_2$, of the holomorphic map, still denoted by f, obtained by analytic continuation along continuous paths on $\Omega - R$. By means of algebraic extension and the functional identity, we will obtain holomorphic maps defined on open neighborhoods of a general boundary point which preserve the boundary, and we will need to make use of the extension results due to Alexander [Al] in the rank 1 case, and due to Henkin-Tumanov [TK1] in the case where Ω is of rank ≥ 2. We start with the rank-1 case.

Theorem 2.1.1. (Alexander [Al]) *Let $B^n \Subset \mathbb{C}^n$ be the complex unit ball of dimension $n \ge 2$. Let $b \in \partial B^n$, U_b be a connected open neighborhood of b in $\mathbb{C}^n$, and $f : U_b \to \mathbb{C}^n$ be a nonconstant holomorphic map such that $f(U_b \cap \partial B^n) \subset \partial B^n$. Then, there exists an automorphism $F : B^n \to B^n$ such that $F|_{U_b \cap B^n} \equiv f|_{U_b \cap B^n}$.*

Using the result on the algebraic extension of the germ of graph of a measure-preserving map as given in Proposition 1.2.1 and Theorem 2.1.1 (Alexander's Theorem) we are now ready to prove the Main Theorem in the rank-1 case, i.e., for the complex unit ball B^n, $n \ge 2$.

Proof of the Main Theorem in the case of B^n, $n \ge 2$. Recall that for an irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^n$ in its Harish-Chandra realization, $K_\Omega(z, w)$ stands for the Bergman kernel on Ω and ds^2_Ω stands for the Bergman metric on Ω. Denote by dV the Euclidean volume form on $\mathbb{C}^n$. Both the (n, n)-form $K_\Omega(z, z)dV$ and the volume form $d\mu_\Omega$ of (Ω, ds^2_Ω) are invariant under the action of the group $\mathrm{Aut}(\Omega)$ of holomorphic automorphisms. Since $\mathrm{Aut}(\Omega)$ acts transitively on Ω, $d\mu_\Omega = c_\Omega K_\Omega(z, z)dV$ for some constant $c_\Omega > 0$. From the functional identity (†) in (1.1) we deduce that

$$\sum_{\alpha=1}^{d_2} K_\Omega(f_\alpha(z), f_\alpha(z)) |\det(Jf_\alpha(z))|^2 = d_1 K_\Omega(z, z)\,. \tag{1}$$

For $\Omega = B^n$, the Bergman kernel on B^n is given by $K_{B^n}(z, w) = \dfrac{c_n}{(1 - < z, \bar{w} >)^{n+1}}$ for some constant $c_n > 0$. Hence, by (1)

$$\sum_{\alpha=1}^{d_2} \frac{|\det(Jf_\alpha(z))|^2}{(1 - |f_\alpha(z)|^2)^{n+1}} = \frac{d_1}{(1 - |z|^2)^{n+1}}\,. \tag{2}$$

Let $b \in \partial B^n - R$ where $R \subsetneq \mathbb{C}^n$ is the affine-algebraic subvariety as in the statement of Proposition 1.2.2. Then $f : U \to \Omega^{d_2}$ can be analytically continued along some continuous path on $\Omega - R$ reaching b to give a holomorphic mapping on a neighborhood U_b of b, still to be denoted $f = (f_1, \cdots, f_{d_2})$. Noting that $\det(Jf_\alpha(z))$ is bounded on $U_b' \cap B^n$ for any neighborhood U_b' of b in $\mathbb{C}^n$ relatively compact in U_b, applying the functional equation (2) to $f|_{U_b \cap B^n}$ and comparing the two sides near points on $U_b \cap \partial B^n$, we conclude that for some f_α, say f_1, we must have $|f_1(b')| = 1$ for any $b' \in U_b \cap \partial B^n$, i.e., $f_1(U_b \cap \partial B^n) \subset \partial B^n$. When $n \geq 2$, by Alexander's Theorem [Al] as stated here in Theorem 2.1.1, $f_1|_{U_b \cap B^n}$ extends to an automorphism of B^n. Since an automorphism preserves the volume form of the Bergman metric, we have

$$\frac{|\det(Jf_1(z))|^2}{(1 - |f_1(z)|^2)^{n+1}} = \frac{1}{(1 - |z|^2)^{n+1}} . \tag{3}$$

and it follows from Eqn.(2) that

$$\sum_{\alpha=2}^{d_2} \frac{|\det(Jf_\alpha(z))|^2}{(1 - |f_\alpha(z)|^2)^{n+1}} = \frac{d_1 - 1}{(1 - |z|^2)^{n+1}} . \tag{4}$$

If $d_1 - 1 > 0$ the same argument can be repeated, and we conclude by induction that there are exactly d_1 of the components f_α such that $f_\alpha(U_b \cap B^n) \subset \partial B^n$, say those f_α for which $1 \leq \alpha \leq d_1$. What remains gives

$$\sum_{\alpha=d_1+1}^{d_2} \frac{1}{(1 - |f_\alpha(z)|^2)^{n+1}} \left|\det(Jf_\alpha(z))\right|^2 = 0 . \tag{5}$$

The possibility $d_1 < d_2$ plainly cannot occur because each of the component maps f_α is assumed to be of maximal rank at some point, and the same property must be propagated by analytic continuation to U_b, showing that each of the Jacobian determinants $\det(Jf_\alpha(z))$, $1 \leq \alpha \leq d_2$, is not identically 0 on U_b. We have thus established in the rank-1 case, where $\Omega = B^n$, $n \geq 2$, that in fact $d_1 = d_2$ and that $f : (B^n; 0) \to (B^n; 0)^{d_2}$ extends to a totally geodesic holomorphic embedding $F : B^n \to (B^n)^{d_2}$, where each component $F_i : B^n \to B^n$ is a biholomorphism. As a consequence, we have completed the proof of the Main Theorem in the special case where $\Omega = B^n, n \geq 2$. $\square$

(2.2) *Boundary behavior of the algebraic extension along the Shilov boundary.* For the proof of the Main Theorem in the case of rank ≥ 2, to start with we need the following special case of a result of Henkin-Tumanov [TK1, Theorem 1] analogous to Alexander's Theorem.

Theorem 2.2.1. (Henkin-Tumanov [TK1]) *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization, and denote by $Sh(\Omega) \subset \partial\Omega$ its Shilov boundary. Suppose $b \in Sh(\Omega)$. Let $U_b \subset \mathbb{C}^n$ be a connected open neighborhood of b in $\mathbb{C}^n$, and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap \Omega) = f(U_b) \cap \Omega$ and $f(U_b \cap Sh(\Omega)) = f(U_b) \cap Sh(\Omega)$. Then, there exists an automorphism $F : \Omega \to \Omega$ such that $F|_{U_b \cap \Omega} \equiv f|_{U_b \cap \Omega}$.*

Remarks The result of Henkin-Tumanov [TK1] is stated in the general form for Cartesian products of irreducible bounded symmetric domains of complex dimension ≥ 2, and a complete proof is given there for irreducible classical domains of Type-I. A simplification of the proof in the latter case is given in Henkin-Tumanov [TK2, §4] basing on the use of geometric structures defined by irreducible Hermitian symmetric spaces of the compact type of rank ≥ 2. (The work of Goncharov [Go] was cited in [TK2], but the result needed was first due to Ochiai [Oc]). The scheme of proof in [TK1] together with the simplification as given in [TK2] applies to yield Theorem 2.2.1.

Imitating the proof of the Main Theorem in the rank-1 case, we need to show that there exists some point b on $Sh(\Omega)$ such that the germ of holomorphic mapping $f = (f_1, \cdots, f_{d_2}) : (\Omega; 0) \to (\Omega; 0) \times \cdots \times (\Omega; 0)$ can be analytically continued along a continuous path in Ω to a neighborhood of $b \in \mathbb{C}^n$, and such that, with respect to any choice of analytic continuation of f to U_b, one of the components of the mapping, say $f_1 : U_b \to \mathbb{C}^n$, satisfies $f_1(U_b \cap Sh(\Omega)) \subset Sh(\Omega)$. To start with, we have

Lemma 2.2.1. *With reference to Proposition 1.2.2 and in the notation there, the subvariety $R \subsetneq \mathbb{C}^n$ does not contain the Shilov boundary $Sh(\Omega)$.*

Proof. In the notation of the proof of Proposition 1.2.2, the affine-algebraic variety $R \subsetneq \mathbb{C}^n$ is exactly the common zero set of a finite number of polynomials $\{h_1, \cdots, h_\ell\}$. By the property of the Shilov boundary, given any continuous function $s : \overline{\Omega} \to \mathbb{C}$ such that $s|_\Omega$ is holomorphic, the maximum of the moduli $\{|s(x)| : x \in \overline{\Omega}\}$ is precisely attained on the Shilov boundary. If $Sh(\Omega)$ were contained in R, then the maximum modulus of each of the defining functions h_i, $1 \leq i \leq \ell$, would have to be 0, and hence $h_i \equiv 0$ on $\mathbb{C}^n$, a plain contradiction. Thus, $Sh(\Omega) - R \neq \emptyset$, as desired. □

For the proof of the Main Theorem we need some structure theory about bounded symmetric domains regarding maximal polydisks and Harish-Chandra realizations. Let Ω be an irreducible bounded symmetric domain. Write G for the identity component of the group $\mathrm{Aut}(\Omega)$ of biholomorphic automorphisms of Ω, and $K \subset G$ for the isotropy subgroup at the origin $0 \in \Omega$. Denote by $\mathfrak{g}$ the Lie algebra of G, and by $\mathfrak{k}$ the Lie algebra of K. With respect to the the involution at $0 = eK$ of Ω we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is canonically identified with the real tangent space $T_0^{\mathbb{R}}(\Omega)$ at $0 = eK$. Equipping Ω with the Bergman metric, Ω can be identified with G/K as a Riemannian symmetric manifold. Let $G^{\mathbb{C}}$ be the complexification of G, $K^{\mathbb{C}} \subset G$ be the complexification of K in $G^{\mathbb{C}}$, and $P \subset G^{\mathbb{C}}$ be the maximal parabolic subgroup containing $K^{\mathbb{C}}$ (as a Levi factor). Then $M := G/P$ is the rational homogeneous manifold which is the underlying complex manifold of the Hermitian symmetric manifold of the compact type dual to Ω. As a complex manifold Ω can be identified with an open subset of M by means of the Borel Embedding Theorem, given by the natural map $G/K \hookrightarrow G^{\mathbb{C}}/P := M$. Write $\mathfrak{g}^{\mathbb{C}}$ for the (complex) Lie algebra of $G^{\mathbb{C}}$. The real Lie algebra $\mathfrak{g}$ is a real form of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$, i.e., $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We have the Harish-Chandra decomposition $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$ in standard notations (cf. Wolf [Wo] and Mok [Mk2]), where $\mathfrak{m}^+ \oplus \mathfrak{m}^- = \mathfrak{m}^{\mathbb{C}} := \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{m}^+$ is canonically identified with $T_0(\Omega) = T_0^{1,0}(\Omega)$, $\mathfrak{m}^-$ is canonically identified with $\overline{T_0(\Omega)} = T_0^{0,1}(\Omega)$, $\mathfrak{k}^{\mathbb{C}}$ (being the complex Lie algebra $K^{\mathbb{C}}$) is the complexification of $\mathfrak{k}$, and $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$ is the Lie algebra of $P \subset G^{\mathbb{C}}$. If we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$, and denote by Δ the set of roots with respect to $\mathfrak{h}$, then we have a decomposition of $\mathfrak{g}^{\mathbb{C}}$ into the direct sum of $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ and the (1-dimensional) eigenspaces $\mathfrak{g}^{\varphi}$, and $T_0(\Omega) = \mathfrak{m}^+$ can be identified with the direct sum of the eigenspaces $\mathfrak{g}^{\varphi}$ as φ ranges over the set Δ_0^+ of positive noncompact roots.

For each $\varphi \in \Delta$ we write $\mathfrak{g}^{\varphi} = \mathbb{C}e_{\varphi}$. We choose e_{φ} as in Wolf [Wo, §3], as follows. Denote by $(\cdot,\cdot)$ the restriction of the Killing form B of $\mathfrak{g}^{\mathbb{C}}$ to the complexified Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$, and by the same symbol the induced bilinear form on the dual space $(\mathfrak{h}^{\star})^{\mathbb{C}}$. For $\varphi \in \Delta$ we define $h_{\varphi} \in i\mathfrak{h}$ by the relation $2\varphi(h) = (\varphi,\varphi)(h_{\varphi},h)$ for every $h \in \mathfrak{h}$. We choose now root vectors $e_{\varphi} \in \mathfrak{g}^{\varphi}$ subject to the normalization $e_{-\varphi} = \overline{e_{\varphi}}$, $[e_{\varphi}, e_{-\varphi}] = h_{\varphi}$, where conjugation in $\mathfrak{g}^{\mathbb{C}}$ is taken with respect to the real structure given by $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$.

Regarding G/K as an open subset of M by the Borel embedding, the mapping $\xi : \mathfrak{m}^+ \to M = G^{\mathbb{C}}/P$ given by $\xi(z) = \exp(m)P$ is a biholomorphism onto a Zariski open subset of M containing G/K. The inverse map $\eta = \xi^{-1} : G/K \xrightarrow{\cong} \Omega \Subset \mathfrak{m}^+ \cong \mathbb{C}^n$ is the Harish-Chandra embedding. Enumerating the positive noncompact roots as $\Delta_0^+ = \{\varphi_1, \cdots, \varphi_n\}$, and identifying a point $z = z_1 e_{\varphi_1} + \cdots z_n e_{\varphi_n}$ with $(z_1, \cdots, z_n)$, we have obtained the Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$, and we will refer to $(z_1, \cdots, z_n)$ as the Harish-Chandra coordinates.

Maximal polydisks $\Pi \subset \Omega$ can be constructed as follows. Two roots $\varphi_1, \varphi_2 \in \Delta$ are said to be strongly orthogonal if and only if neither $\varphi_1 + \varphi_2$ nor $\varphi_1 - \varphi_2$ is a root. When φ_1 and φ_2 are positive roots, $\varphi_1 + \varphi_2$ is never a root. Let $\Psi \subset \Delta_0^+$ be a maximal set of mutually strongly orthogonal positive noncompact roots. Ψ consists of precisely r elements, $\Psi = \{\psi_1, \cdots, \psi_r\}$, where r denotes the rank of Ω as a Hermitian symmetric manifold. For each $\psi \in \Psi$, the real 3-dimensional vector space $\mathfrak{q}_{\psi} := \mathfrak{g}^{\psi} + \mathfrak{g}^{-\psi} + [\mathfrak{g}^{\psi}, \mathfrak{g}^{-\psi}]$ gives a Lie algebra isomorphic to $\mathfrak{su}(1,1)$, and $Q_{\psi} := \exp(\mathfrak{q}_{\psi}) \subset G$ gives a Lie group isomorphic to $\mathrm{SU}(1,1)/\{\pm I\}$ such that the orbit of $0 \in \Omega$ under Q_{ψ} is a minimal disk on Ω. $T_0(P) \subset T_0(\Omega) \cong \mathfrak{m}^+ \subset \mathfrak{g}$ is spanned by root vectors belonging to a maximal set of strongly orthogonal noncompact positive roots. Furthermore, from the strong orthogonality condition $Q_{\Psi} := Q_{\psi_1} \times \cdots \times Q_{\psi_r}$ acts as a group of automorphisms on Ω and the orbit of $0 \in \Omega$ under Q_{Ψ} is a maximal polydisk $\Pi \subset \Omega$ passing through the origin $0 \in \Omega$. We have

Theorem 2.2.2. (Polydisk Theorem, cf. Wolf [Wo, p.280]) *Let Ω be a bounded symmetric domain of rank r, equipped with an $\mathrm{Aut}(\Omega)$-invariant Kähler metric g. Then, there exists an r-dimensional totally geodesic complex submanifold Π biholomorphic to the polydisk Δ^r. Moreover, the identity component G of the group of automorphisms $\mathrm{Aut}(\Omega)$ acts transitively on the space of all such polydisks.*

A maximal strongly orthogonal set $\Psi \subset \Delta_0^+$ can be constructed inductively, as follows. Choose a lexicographic ordering on the set Δ of roots with respect to $\mathfrak{h}$ and let $\psi_1 := \mu \in \Delta_0^+$ be the dominant root thus defined. If a set $\{\psi_1, \cdots, \psi_k\}$ of mutually strongly orthogonal positive noncompact roots has been defined, $1 \le k < r$, we pick $\psi_{k+1} \in \Delta_0^+$ to be the highest root with respect to the chosen lexicographic ordering among all positive noncompact roots φ strongly orthogonal to each ψ_i, $1 \le i \le k$. This way we end up with a maximal strongly orthogonal set $\Psi \subset \Delta_0^+$ of cardinality equal to $r = \mathrm{rank}(\Omega)$ and a corresponding maximal polydisk $\Pi \subset \Omega$. In our choice of Harish-Chandra coordinates we will take $\varphi_i = \psi_i$ for $1 \le i \le r$, where $\psi_1 = \mu$ is the dominant root. By Wolf [Wo, §3, Eqn.(3.22)] in terms of Harish-Chandra coordinates, the maximal polydisk Π as constructed in the above is precisely the unit polydisk $\Delta^r \times \{0\}$.

Denote by $B(\cdot,\cdot)$ the Killing form on $\mathfrak{g}^{\mathbb{C}}$. With respect to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, the restriction $B|_{\mathfrak{k}}$ on the compact semisimple Lie algebra $\mathfrak{k}$ is negative definite, while the restriction $B|_{\mathfrak{m}}$ is positive. Write $\mathfrak{g}_c := \mathfrak{k} + i\mathfrak{m} \subset \mathfrak{g}^{\mathbb{C}}$ for the compact real form of $\mathfrak{g}^{\mathbb{C}}$. Let $\lambda > 0$ be any positive constant and define $\langle \cdot,\cdot\rangle$ by $\langle \alpha, \beta\rangle = -\lambda B(\alpha, \tau_c(\beta))$. Since $B|_{\mathfrak{g}_c}$ is negative definite, $\langle \cdot,\cdot\rangle$ is a Hermitian inner product with respect to the real structure defined by $\mathfrak{g}_c \subset \mathfrak{g}^{\mathbb{C}}$, i.e., with respect to conjugation given by τ_c, which is invariant under K. The Harish-Chandra decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$ is an orthogonal decomposition with respect to $\langle \cdot,\cdot\rangle$. In terms of the conjugation $\tau_0(g) = \overline{g}$ on $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$, we have $\langle m_1, m_2\rangle = \lambda B(m_1, \overline{m_2})$ for $m_1, m_2 \in \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, while $\langle k_1, k_2\rangle = -\lambda B(k_1, \overline{k_2})$ for $k_1, k_2 \in \mathfrak{k}^{\mathbb{C}}$.

To study the boundary of the irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^n$ in its Harish-Chandra realization we will make use of the Hermann Convexity Theorem, as follows.

Theorem 2.2.3 (cf. Wolf [Wo, p.286]). *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. Let $B(\cdot,\cdot)$ be the Killing form on $\mathfrak{g}^{\mathbb{C}}$, $\lambda > 0$ be any positive number, $\langle \cdot,\cdot\rangle$ be the Hermitian inner product on $\mathfrak{g}^{\mathbb{C}}$ defined by $\langle g, h\rangle = -\lambda B(g, \tau_c(h))$, and $|g| = \langle g, g\rangle^{\frac{1}{2}}$. Then, the Harish-Chandra realization $\Omega \Subset \mathfrak{m}^+ \cong \mathbb{C}^n$ is given by $\Omega = \{\xi \in \mathfrak{m}^+ : \|ad(\mathrm{Re}\,\xi)\| < 1\}$, where $\|\cdot\|$ is the Banach norm on $ad(\mathfrak{g})$ defined by $\|ad(u)\| := \sup\left\{\|ad(u)(g)\| : g \in \mathfrak{g}^{\mathbb{C}}, |g| = 1\right\}$. In particular, $\Omega \Subset \mathbb{C}^n$ is a bounded convex domain.*

In the definition $\|ad(u)\|$ is in fact the operator norm of $ad(u) : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$, and is thus independent of the choice of $\lambda > 0$ in the definition of $\langle \cdot,\cdot\rangle$. We will not make use of the convexity but rather the more precise description of Ω as the unit ball with respect to a Banach norm. We have

Lemma 2.2.2. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. Let $\Pi \subset \Omega$ be a maximal polydisk passing through 0 and suppose the Harish-Chandra coordinates $(z_1, \cdots, z_n)$ have been chosen so that the basis vectors are root vectors with respect to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$, and $(z_1, \cdots, z_r)$ are Euclidean coordinates on the maximal polydisk $\Pi \subset \Omega$. Then, for $1 \le k \le r$, the function $\pi : \Omega \to \mathbb{C}$ defined by $\pi(z) = z_k$ maps Ω onto the unit disk Δ.*

Proof. In the notation of Theorem 2.2.3, the restriction of $\langle \cdot,\cdot\rangle$ to $\mathfrak{m}^+$ defines a Hermitian inner product on $\mathfrak{m}^+$. In terms of the Harish-Chandra coordinates $(z_1, \cdots, z_n)$ as described we have $\Pi = \Delta^r \times \{0\} \subset \Omega$ for the maximal polydisk Π. Suppose now $z \in \Omega$ and $z = ae_\mu + \sum_{\varphi \in \Delta_0^+, \varphi \neq \mu} b_\varphi e_\varphi$. To prove the lemma it suffices to show that $|a| < 1$, We normalize $\langle \cdot,\cdot\rangle$ by choosing the constant $\lambda > 0$ such that $|[e_\mu, \overline{e_\mu}]| = 1$. With this normalization from standard calculations on $\mathfrak{sl}(2, \mathbb{C})$, we have $|e_\mu| = \frac{1}{\sqrt{2}}$ and

$|\mathrm{Re}\,(e_\mu)| = \frac{1}{2}$. We have

$$[z+\overline{z}, e_\mu + \overline{e_\mu}] = 2\mathrm{Re}\Big(a[e_\mu,\overline{e_\mu}] + \textstyle\sum_{\varphi\in\Delta_0^+,\varphi\neq\mu} b_\varphi[e_\varphi,\overline{e_\mu}]\Big)\ . \tag{1}$$

Note that $[e_\mu,\overline{e_\mu}] \in i\mathfrak{h}$ is purely imaginary. Replacing z by $e^{i\theta}z$, we may assume that a is purely imaginary. For $\varphi\in\Delta_0^+$ distinct from 0, either $[e_\varphi,\overline{e_\mu}]$ is 0 or $\varphi-\mu\in\Delta$, in which case $[e_\varphi,\overline{e_\mu}]$ is a generator of the root space $\mathfrak{g}^{\varphi-\mu}$. The root spaces are mutually orthogonal to each other and they are orthogonal to the complexified Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$. Taking real parts it remains the case that the non-zero summands of the right-hand side of (1) are mutually orthogonal. It follows that

$$2|a| \le |[2\mathrm{Re}\,z, 2\mathrm{Re}\,(e_\mu)]| = 4\,|[\mathrm{Re}\,z, \mathrm{Re}\,(e_\mu)]| = 4\ |ad(\mathrm{Re}\,z)(\mathrm{Re}\,(e_\mu))| < 4|\mathrm{Re}\,(e_\mu)| = 2\ , \tag{2}$$

hence $|a|<1$, as desired. □

Remarks

(a) From (2), the statement that $z = ae_\mu$ lies on Ω if and only if $|a|<1$ is equivalent to the fact that $\|ad(\mathrm{Re}(e_\mu)\|=1$, which results from the Restricted Root Theorem and is used in the proof of Hermann Convexity Theorem (cf Wolf [Wo, §3]).

(b) In terms of the Harish-Chandra embedding $\eta: G/K \xrightarrow{\cong} \Omega \Subset \mathbb{C}^n$, $d\eta(e_{\varphi_k} \mathrm{mod}\, P) = \frac{\partial}{\partial z_k}$. The normalization on $\langle\cdot,\cdot\rangle$ chosen in the proof is precisely the one with respect to which $\left|\frac{\partial}{\partial z_k}\right| = \frac{1}{\sqrt{2}}$, i.e., the one for which, writing $z_k = x_k + iy_k$, $x_{n+k} := y_k$, the set $\left\{\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{2n}}\right\}$ constitutes an orthonormal basis of the underlying real vector space of $\mathbb{C}^n$. In other words, $\langle\cdot,\cdot\rangle$ induces the standard Euclidean metric on $\mathbb{C}^n$. The latter interpretation is however irrelevant to the proof.

In order to apply the result of Henkin-Tumanov [TK1] (Theorem 2.2.1 here) to our situation of measure-preserving holomorphic maps in the higher rank case, we prove the following general result on biholomorphisms defined on a neighborhood of a point on the Shilov boundary.

Lemma 2.2.3. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization, and denote by $Sh(\Omega)$ its Shilov boundary. Let $U\subset\mathbb{C}^n$ be a connected open set such that $U\cap Sh(\Omega)\neq\emptyset$. Let $h: U\to\mathbb{C}^n$ be a biholomorphism onto an open subset of $\mathbb{C}^n$ (regarded as a Euclidean space containing a copy of Ω in its Harish-Chandra realization) such that $h(U\cap\Omega)\subset\Omega$ and such that $h(U\cap\partial\Omega)\subset\partial\Omega$. Then, $h(U\cap Sh(\Omega))\subset Sh(\Omega)$. As a consequence, there exists $H\in\mathrm{Aut}(\Omega)$ such that $H\big|_{U_b\cap\Omega}\equiv h$.*

Proof. Each $\gamma\in G_0$ extends to an automorphism of the compact dual M, and as such it restricts to a homeomorphism of $\overline{\Omega}$ mapping $\partial\Omega$ homeomorphically onto $\partial\Omega$. By the structure of boundary components of bounded symmetric domains (cf. Wolf [Wo]), $\partial\Omega$ decomposes into the union of exactly r orbits under the action of G_0. This can be deduced from the Polydisk Theorem, as follows. Denote by $\Pi\subset\Omega$, $\Pi\cong\Delta^r$, a maximal polydisk passing through 0 defined by a maximal strongly orthogonal subset $\Psi\subset\Delta_0^+$ of noncompact positive roots, $\Psi=\{\psi_1,\cdots,\psi_r\}$. Denote by $(z_1,\cdots,z_r)$ the

Euclidean coordinates on Π, so that $\frac{\partial}{\partial z_i} \in \mathfrak{g}^{\psi_i}$ and $\Pi = \Delta^r$ in terms of these coordinates. Extend the Euclidean coordinates $(z_1, \cdots, z_r)$ to Harish-Chandra coordinates $(z_1, \cdots, z_r; z_{r+1}, \cdots z_n)$ on Ω so that each $\frac{\partial}{\partial z_k}, 1 \leq k \leq n$, is a root vector belonging to a positive noncompact root. By the Polydisk Theorem each point $x \in \overline{\Omega}$ is equivalent under the action of K to a point $y \in \overline{P}$. We have a decomposition $\partial P = A_1 \cup A_2 \cup \cdots A_r$, where A_k consists of boundary points $b = (b_1, \cdots, b_r)$ in which exactly k of the coordinates b_i are of norm 1, and exactly $r - k$ of the coordinates b_i are of norm strictly less than 1. Denote by ϵ_k the point $(1, \cdots, 1; 0, \cdots, 0)$ on ∂P with the first k coordinates equal to 1 and the other $r - k$ coordinates equal to 0. Thus, if we write e_i for the i-th unit vector, $1 \leq i \leq n$, then $\epsilon_k = e_1 + \cdots + e_k$. Now a point $b \in \partial P$ lies on A_k if and only if it is of the form $\gamma(\epsilon_k)$ for some $\gamma \in \mathrm{Aut}(P)$, noting that $\mathrm{Aut}(P)$ is a semi-direct product of $\mathrm{Aut}_0(P) = (\mathrm{Aut}(\Delta))^r$ with the permuting group S_r on a set of r elements, where $\sigma \in S_r$ acts by $\sigma(z_1, \cdots, z_r) = (z_{\sigma(1)}, \cdots, z_{\sigma(r)})$. Now the full group $\mathrm{Aut}(P)$ of automorphisms extends to automorphisms of Ω (cf. Wolf [Wo]). Hence, given any $b \in \partial\Omega$, there exists $\gamma \in G$ such that $\gamma(b) = \epsilon_k$ for some k, $1 \leq k \leq r$. Furthermore, for $1 \leq k < \ell \leq r$, ϵ_k and ϵ_ℓ are inequivalent under the action of G. As a consequence, we have a decomposition $\partial\Omega = E_1 \cup E_2 \cup \cdots \cup E_r$ into the disjoint union of orbits $E_k := G\epsilon_k$. We claim

($\flat$) Let $1 \leq \ell \leq r$ and write $K_\ell := E_\ell \cup E_{\ell+1} \cup \cdots \cup E_r$. Then, $b \in K_\ell$ is a smooth point of K_ℓ if and only if $b \in E_\ell$.

We observe first of all that for $1 \leq k \leq r-1$, E_{k+1} is always in the topological closure of E_k, as can be seen from the action of $\mathrm{Aut}_0(P)$ on ∂P. To prove ($\flat$) we may assume that $\ell < r$ and it suffices to show that any point $b \in E_{\ell+1}$ cannot be a smooth point of K_ℓ. Since G acts transitively on each E_k, it suffices to show that $\epsilon_{\ell+1} \notin \mathrm{Reg}(K_\ell)$. Suppose $\epsilon_{\ell+1}$ were a smooth point of K_ℓ. Then, the real tangent space $T^{\mathbb{R}}_{\epsilon_{\ell+1}}(K_\ell)$ must contain limits of real vectors v_j tangent to p_j, where (p_j) is any sequence of points lying on $E_\ell \subset \mathrm{Reg}(K_\ell)$ and converging to $\epsilon_{\ell+1}$. In particular, writing $z_k = x_k + \sqrt{-1}y_k$ as usual for $1 \leq k \leq n$, the point $p = \epsilon_\ell + z_\ell e_{\ell+1}$ lies on E_ℓ whenever $|z_\ell| < 1$, and the vector $v = \frac{\partial}{\partial x_{\ell+1}}$ lies on $T^{\mathbb{R}}_p(E_\ell)$, and thus $v \in T^{\mathbb{R}}_{\epsilon_{\ell+1}}(K_\ell)$. Since $z_{\ell+1}(\epsilon_{\ell+1}) = 1$, it follows that there exists some point $b \in K_\ell$ such that $|z_{\ell+1}(b)| > 1$, contradicting Lemma 2.2.2.

We proceed to prove $h(U \cap Sh(\Omega)) \subset Sh(\Omega)$ by induction. It suffices to consider the case where $\mathrm{rank}(\Omega) := r \geq 2$. By ($\flat$), $\mathrm{Reg}(\partial\Omega) = E_1$. Since $h : U \to \mathbb{C}^n$ is an open embedding such that $h(U \cap \partial\Omega) \subset \partial\Omega$, a singular point of $U \cap \partial\Omega$ must be mapped by h to a singular point of $\partial\Omega$. In other words $h(U \cap K_2) \subset K_2$. If Ω is of rank 2 then $K_2 = E_2 = Sh(\Omega)$ and we are done. If $r \geq 3$, let ℓ be any integer such that $2 \leq \ell < r$. Suppose by induction hypothesis we have $h(U \cap K_\ell) \subset K_\ell$. By ($\flat$), $\mathrm{Reg}(K_\ell) = E_\ell$, i.e., $\mathrm{Sing}(K_\ell) = K_{\ell+1}$, and exactly the same argument as in the above shows that $h(U \cap K_{\ell+1}) \subset K_{\ell+1}$. Thus, by induction we reach $\ell = r$, showing that $h(U \cap K_r) \subset K_r$. But K_r is nothing other than the Shilov boundary $Sh(\Omega)$ and we have shown that $h(U \cap Sh(\Omega)) \subset Sh(\Omega)$. By the Theorem of Henkin-Tumanov (Theorem 2.2.1 here), there exists $H \in \mathrm{Aut}(\Omega)$ such that $h|_{U\cap\Omega} = H|_{U\cap\Omega}$, as desired. $\square$

(2.3) *Proof of the Main Theorem and its consequences in the case of rank ≥ 2.* We are ready to complete the proof of the Main Theorem and Theorem 1.1.2. To start with we need the following standard fact about the Bergman kernel on bounded symmetric domains.

Lemma 2.3.1. *On an irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^n$ in its Harish-Chandra realization, denote by $K_\Omega(z,w)$ the Bergman kernel. Write $\varphi_\Omega(z) := K_\Omega(z,z)$. Then, $\varphi_\Omega(z)$ is an unbounded exhaustion function on Ω.*

Proof. Write $n := \dim_{\mathbb{C}}(\Omega)$ and $r := \text{rank}(\Omega)$. In the statement of Lemma 1.2.1, we have $K_\Omega(z,w) = \frac{1}{Q(z.w)}$, where Q is a polynomial in $(z_1,\cdots,z_n;\overline{w_1},\cdots,\overline{w_n})$ such that $Q(z,z) \neq 0$ whenever $z \in \Omega$. More precisely, $Q(z,w) = h(z,w)^p$, where $h(z,w)$ is some polynomial in $(z_1,\cdots,z_n;\overline{w_1},\cdots,\overline{w_n})$ and p is a positive integer, with the following property (cf. Faraut-Korányi [FK, pp.76-77]). Let $\Pi \cong \Delta^r$ be a maximal polydisk on Ω passing through 0. We may choose Harish-Chandra coordinates such that Π is exactly the unit polydisk $\Delta^r \times \{0\}$. For $z \in \Omega$, there exists $\gamma \in K = \text{Aut}_0(\Omega)$ such that $\gamma(z) = (a_1,\cdots,a_r) \in P$ and we have

$$h(z,z) = (1-|a_1|^2)\times\cdots\times(1-|a_r|^2)\ .$$

We may normalize $a_1,\cdots,a_r$; $a_i = a_i(z)$; so that each a_i is nonnegative and we have $a_1 \geq \cdots \geq a_r \geq 0$, and refer to $(a_1(z),\cdots,a_r(z))$ as the normal form of z modulo K. Then, a sequence of points $(z_k)_{k=0}^\infty$ is discrete if and only if $a_1(z_k) \to 1$ as $k \to \infty$ so that $h(z_k,z_k) \to 0$ as $k \to \infty$. It follows that $\varphi_\Omega(z) = K_\Omega(z,z)$ is an exhaustion function, as desired. $\square$

In order to apply the result of Henkin-Tumanov (Theorem 2.2.1) in the case of rank ≥ 2 in analogy to using Alexander's Theorem (Theorem 2.1.1) in the rank-1 case, we need to consider topological properties concerning the structure of $\partial\Omega$ of the Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$. More precisely, we will need the following connectedness statement.

Lemma 2.3.2. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization, and denote by $Sh(\Omega) \subset \partial\Omega$ its Shilov boundary. Let $p \in Sh(\Omega)$ be any point on the Shilov boundary, $\text{Reg}(\partial\Omega)$ be the smooth locus of $\partial\Omega$, and $Q'_p \subset \partial\Omega$ be any connected open neighborhood of p on $\partial\Omega$. Then, there exists a connected open neighborhood Q_p of p in $\partial\Omega$ such that $Q_p \subset Q'_p$ and such that $Q_p \cap \text{Reg}(\partial\Omega)$ is connected.*

Proof. We will make use of a canonical unbounded realization of the bounded symmetric domain Ω. By Korányi-Wolf [KW] there is a biholomorphism $\Phi : \Omega \to \mathcal{D}$ of Ω onto a Siegel domain $\mathcal{D} \subset \mathbb{C}^n$ of the first or second kind such that $\Phi(p) = 0$, where Φ^{-1} is a Cayley transform in the terminology of [KW]. Siegel domains were defined in Pyatetskii-Shapiro [Py]. A Siegel domain $\mathcal{D} \subset \mathbb{C}^n$ of the first kind is a tube domain over a cone $V \subset \mathbb{R}^n$ where V does not contain any affine line. A Siegel domain of the second kind $\mathcal{D} \subset \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ defined by $\mathcal{D} = \{(z_1,z_2) : \text{Im}(z_1) - F(z_2,z_2) \in V\}$, where $V \subset \mathbb{R}^{n_1}$ is an open convex cone not containing any affine line, and where $F : \mathbb{C}^{n_2} \times \mathbb{C}^{n_2} \to \mathbb{C}^{n_1}$ is a $\mathbb{C}^{n_1}$-valued Hermitian form such that $F(z_2,z_2) \in V$ for any non-zero $z_2 \in \mathbb{C}^{n_2}$. In both cases $\mathcal{D} \subset \mathbb{C}^n$ will be referred to as a Siegel domain in the sequel. When the Siegel domain $\mathcal{D}$ is of the first kind, for any positive real number t the mapping $\alpha_t(z) = tz$ is an automorphism of $\mathcal{D}$. When $\mathcal{D}$ is of the second kind, the mapping $\alpha_t(z_1,z_2) = (tz_1,\sqrt{t}z_2)$

is an automorphism of $\mathcal{D}$. In either case α_t is a complex linear map, and it extends therefore to a homeomorphism of $\overline{\mathcal{D}} = \mathcal{D} \cup \partial\mathcal{D}$ such that $\alpha_t(\mathcal{D}) = \mathcal{D}$, $\alpha_t(\partial\mathcal{D}) = \partial\mathcal{D}$, $\alpha_t(0) = 0$. Write $\Omega \Subset \mathbb{C}^n \subset M$ to incorporate both the Harish-Chandra realization and the Borel embedding $\Omega \subset M$. The inverse Cayley transform $\Phi : \Omega \to \mathcal{D}$ is the restriction to Ω of an automorphism of M, still to be denoted by Φ, where we have $\mathcal{D} \subset \mathbb{C}^n \subset M$ canonically. The affine part $Sh^\flat(\mathcal{D}) \subset \mathbb{C}^n$ of the Shilov boundary $Sh(\mathcal{D}) \subset M$ is given by $Sh^\flat(\mathcal{D}) = \{(z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} : \mathrm{Im}(z_1) = F(z_2, z_2)\}$. In particular, $0 \in \partial\mathcal{D}$ is a point on the Shilov boundary.

In terms of an unbounded realization $\Phi : \Omega \to \mathcal{D}$ as a Siegel domain, $\Phi(p) = 0$ we have equivalently to prove that, given any connected open neighborhood $P_0' \subset \partial\mathcal{D}$ of 0 in $\partial\mathcal{D}$, there exists a connected open neighborhood P_0 of 0 in $\partial\mathcal{D}$, such that $P_0 \subset P_0'$ and such that $P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$ is connected. Without loss of generality we may assume that $P_0' \Subset \partial\mathcal{D}$. Define $P_0^\sharp := \bigcup_{0<t<1} \alpha_t(P_0')$, $P_0^\sharp \Subset \partial\mathcal{D}$. Let now s be a sufficiently small positive number such that $\alpha_s(P_0^\sharp) \subset P_0'$ and we define $P_0 := \alpha_s(P_0^\sharp) \subset P_0'$. By construction $\alpha_t(P_0) \subset P_0$ for $0 < t \le 1$.

We proceed to prove that $P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$ is path-connected, i.e., given any two points $q_1, q_2 \in P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$, there exists some continuous path μ on $P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$ joining q_1 to q_2. Write $\Phi^{-1}(q_i) := b_i \in \partial\Omega; i = 1, 2$. From ($\flat$) in Lemma 2.2.3 the smooth locus $\mathrm{Reg}(\partial\Omega) \subset \partial\Omega$ is an orbit under the identity component G_0 of $\mathrm{Aut}(\Omega)$ and it is hence connected. Thus there exists a continuous path γ on $\partial\Omega$ joining b_1 to b_2. The hypersurface $H := M - \mathbb{C}^n$ corresponds to a hypersurface $L := \Phi^{-1}(H) \subset M$ such that $\Phi^{-1}(\mathrm{Reg}(\partial\mathcal{D})) = \mathrm{Reg}(\partial\Omega) - L$. (Here $\partial\mathcal{D}$ stands for the boundary of $\mathcal{D}$ in $\mathbb{C}^n$, not in M.) For the proof of Lemma 2.2.2 in what follows without loss of generality we will assume that the irreducible bounded symmetric domain Ω is of rank ≥ 2. Now $H \subset M$ is of complex codimension 1, and hence $L \cap \mathrm{Reg}(\partial\Omega)$ is at least of real codimension 1 in $\mathrm{Reg}(\partial\Omega)$. If the codimension is 1, then $\mathrm{Reg}(\partial\Omega)$ must contain some open subset of L, which is impossible since any locally closed complex submanifold lying on $\mathrm{Reg}(\partial\Omega)$ must be contained in a boundary component of maximal dimension on $\partial\mathcal{D}$, and the latter are necessarily of real codimension ≥ 3 whenever Ω is an irreducible bounded symmetric domain of rank ≥ 2. Thus, $L \cap \mathrm{Reg}(\partial\Omega)$ is at least of real codimension 2, and it follows that we can choose a continuous path on $\mathrm{Reg}(\partial\Omega) - L$ joining b_1 to b_2. Equivalently, we can find a continuous path $\nu : [0, 1] \to \mathrm{Reg}(\partial\mathcal{D})$ joining q_1 to q_2. Choose now $\epsilon > 0$ sufficiently small so that $\alpha_\epsilon(\nu([0, 1])) \subset P_0$. Since $\mathrm{Reg}(\partial\mathcal{D})$ is invariant under the automorphism α_t, $t > 0$, we have $\alpha_\epsilon(\nu([0, 1])) \subset P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$. Thus $\alpha_\epsilon(q_1)$ is joined to $\alpha_\epsilon(q_2)$ by $\alpha_\epsilon \circ \nu$ on $P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$. On the other hand, $\alpha_t(q_i) \in P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$ for $0 < t \le 1$; $i = 1, 2$. Thus, for $i = 1, 2$ the point q_i is joined to $\alpha_\epsilon(q_i) \in P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$ through $\alpha_t(q_i)$ as t decreases from 1 to ϵ. It follows that for an arbitrary pair of points $q_1, q_2 \in P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$, q_1 is joined to q_2 by a continuous path on $P_0 \cap \mathrm{Reg}(\partial\mathcal{D})$, and the latter is path-connected, hence connected, as desired. □

Remarks $\partial\Omega \subset \mathbb{C}^n$ is a semi-analytic subset, and as such it is locally connected [Ło]. Hence, given any open neighborhood $U_p \subset \mathbb{C}^n$ of p in $\mathbb{C}^n$, there exists some connected open neighborhood $Q_p' \subset \partial\Omega \cap U_p$ of p in $\partial\Omega$. Lemma 2.3.2 is a topological statement, and it can also be derived from the structure of $\partial\Omega$ as a semi-analytic set and the fact that the singular locus $\mathrm{Sing}(\partial\Omega) := \partial\Omega - \mathrm{Reg}(\partial\Omega) \subset \partial\Omega$ is of real codimension ≥ 2. In fact, $Q'_p \cap \mathrm{Reg}(\partial\Omega)$ is already connected.

We are now ready to give a proof of the Main Theorem when rank ≥ 2.

Proof of the Main Theorem for rank ≥ 2. In the statement of the Main Theorem recall that $\Omega \Subset \mathbb{C}^n$ is an irreducible bounded symmetric domain of complex dimension at least 2 in its Harish-Chandra realization, and $f = (f_1, \cdots, f_{d_2}) : (\Omega, d_1\, d\mu_\Omega; 0) \to (\Omega^{d_2}, \pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega; 0)$ is a measure-preserving holomorphic map. Furthermore, in the notation of the statement of Proposition 1.2.2, for some affine-algebraic variety $R \subsetneqq \mathbb{C}^n$, and for every $b \in \partial\Omega - R$ the germ of holomorphic map f can be analytically continued along some continuous path in $\Omega - R$ to a holomorphic map into $(\mathbb{C}^n)^{d_2}$ defined on a neighborhood U_b of b in $\mathbb{C}^n$. We still denote by $f = (f_1, \cdots, f_{d_2})$ such an analytic continuation on U_b. By the structural equation (†) in (1.1) for measure-preserving holomorphic maps we have

$$\sum_{\alpha=1}^{d_2} K_\Omega(f_\alpha(z), f_\alpha(z)) |\det(Jf_\alpha(z))|^2 = d_1 K_\Omega(z, z)\,, \tag{1}$$

By Lemma 2.3.1, $K_\Omega(z,z) = \frac{1}{h(z,z)^p}$ is an exhaustion function. Here $h(z,w)$ is a polynomial in z and $\overline{w}$ such that $h(z,z) > 0$ whenever $z \in \Omega$ and $h(z,z) = 0$ whenever $z \in \partial\Omega$. Let $b \in Sh(\Omega) - R$, which is non-empty by Lemma 2.2.1. Define $\varphi(z) := -h(z,z)$. Then $\varphi(z) < 0$ for $z \in \Omega$ and $\varphi(z) = 0$ for $z \in \partial\Omega$, so that φ is an algebraic (in particular real-analytic) defining function of $\Omega \Subset \mathbb{C}^n$. Imitating the proof of the rank-1 case in (2.1) for B^n, $n \geq 2$, we assert that one of the components $f_\alpha : \Omega \to \Omega$, say f_1, must satisfy $f_1(U_b \cap \partial\Omega) \subset \partial\Omega$, which is not altogether obvious when $\mathrm{rank}(\Omega) \geq 2$. By Lemma 2.3.2, without loss of generality we may assume that for $Q_b := U_b \cap \partial\Omega$, $Q_b \cap \mathrm{Reg}(\partial\Omega)$ is connected. Write $\varphi(z) := h(z,z)$. From the definition of f by analytic continuation and from the functional equation (1) it follows that $f(U_b \cap \Omega) \subset \Omega^{d_2}$. In particular $f_i(U_b \cap \Omega) \subset \Omega$ for any $i, 1 \leq i \leq d_2$. Again from (1) we can choose a component, say f_1 after renumbering the components if necessary, such that $f_1(N_b) \subset \partial\Omega$ for some non-empty open subset $N_b \subset Q_b$. In order to apply the extension result of Henkin-Tumanov stated here in [(2.2), Theorem 2.2.1], we have to check that $f_1(U_b \cap \partial\Omega) \subset \partial\Omega$. Now $\psi := \varphi \circ f_1 : U_b \to \mathbb{R}$ is a real-analytic function which vanishes on the non-empty open subset $N_b \cap \mathrm{Reg}(\partial\Omega) \subset Q_b \cap \mathrm{Reg}(\partial\Omega)$, which is connected. From the real-analyticity of ψ and the Identity Theorem for real-analytic functions it follows that ψ must vanish identically on the dense open subset $Q_b \cap \mathrm{Reg}(\partial\Omega) \subset Q_b$, hence identically on $Q_b = U_b \cap \partial\Omega$. Since $f_1(U_b \cap \overline{\Omega}) \subset \overline{\Omega}$ and $\varphi|_\Omega < 0$, it follows from $\psi\big|_{Q_b} \equiv 0$ that in fact $f_1(U_b \cap \partial\Omega) \subset \partial\Omega$. By Lemma 2.2.3, we have furthermore $f_1(U_b \cap Sh(\Omega)) \subset Sh(\Omega)$, and there exists an automorphism $F_1 : \Omega \to \Omega$ such that F_1 agrees with f_1 on $U_b \cap \Omega$. The proof of the Main Theorem for the general case then follows exactly as in the rank-1 case. □

Finally, combining the Main Theorem and the result of Clozel-Ullmo (Theorem 1.1.1) we deduce Theorem 1.1.2.

Proof of Theorem 1.1.2. Recall that $\Omega \Subset \mathbb{C}^n$ is an irreducible bounded symmetric domain in its Harish-Chandra realization and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a torsion-free discrete group of automorphisms such that $X := \Omega/\Gamma$ is of finite volume with respect to the canonical measure induced by the Bergman metric ds^2_Ω on Ω. Let $Y \subset X \times X$ be a measure-preserving algebraic correspondence. In the case where $\Omega = \Delta$, by the result of Clozel-Ullmo (Theorem 1.1.1) the algebraic correspondence Y is necessarily modular. When $\Omega \Subset \mathbb{C}^n$ is of complex dimension greater than 1, by the Main Theorem any germ of measure-preserving holomorphic map $f : (\Omega, d_1 d\mu_\Omega; 0) \to (\Omega^{d_2}, \pi_1^* d\mu_\Omega + \cdots + \pi_{d_2}^* d\mu_\Omega; 0)$ is already totally geodesic, and Theorem 1.1.2 follows. $\square$

Proof of Corollary 1.1.1. Corollary 1.1.1 follows immediately from Theorem 1.1.2 and from the same argument as in Clozel-Ullmo [CU, Theorems 2.10 and 3.8]. $\square$

(2.4) *From algebraic extension to total geodesy owing to* Γ*-equivariance: a differential-geometric proof in the case of the Poincaré disk.* To make the article more self-contained, for the proof of Theorem 1.1.2 in the case of $\Omega = \Delta$ we will provide an alternative argument deducing the total geodesy of f from the algebraicity of Graph(f) and from Γ-equivariance. We use a differential-geometric argument by studying the boundary behavior of f. In the case of the Poincaré disk the Bergman metric is given by $ds^2_\Delta = 2\mathrm{Re}\Big(\frac{2dw\otimes d\overline{w}}{(1-|w|^2)^2}\Big)$, where w is the Euclidean coordinate on Δ.

Proof of Theorem 1.1.2 from the algebraic extension by differential-geometric means. By the measure-preserving property of $f = (f_1, \cdots, f_{d_2})$ we deduce

$$\sum_{\alpha=1}^{d_2} \frac{2|f'_\alpha(w)|^2}{\big(1-|f_\alpha(w)|^2\big)^2} = \frac{2d_1}{\big(1-|w|^2\big)^2}, \tag{1}$$

on a neighborhood of $0 \in \Delta$. By Proposition 1.2.2., for a general point $b \in \partial\Delta$, there exists an open neighborhood U_b of b in $\mathbb{C}$ such that $f = (f_1, \cdots, f_{d_2})$ admits an analytic continuation along some continuous path in Δ to a holomorphic map, still denoted as $f = (f_1, \ldots, f_{d_2})$, such that $f_\alpha(U_b \cap \Delta) \subset \Delta$ for any i, $1 \le \alpha \le d_2$. The functional identity (1) then holds true for this branch of the holomorphic map f on U_b. Suppose $f_\alpha(U_b \cap \partial\Delta) \subset \partial\Delta$. Clearly $f'_\alpha \not\equiv 0$ on $U_b \cap \partial\Delta$. Choosing $b \in \partial\Delta$ sufficiently general and shrinking U_b if necessary we may assume that $f'_\alpha(p) \neq 0$ for $p \in U_b \cap \partial\Delta$. Then, there exists a smooth function φ_α on U_b such that $1 - |f_\alpha(w)|^2 = (1-|w|^2)e^{\varphi_\alpha(w)}$ on U_b and we have

$$\begin{aligned} \frac{|f'_\alpha(w)|^2}{\big(1-|f_\alpha(w)|^2\big)^2} &= -\frac{\partial^2}{\partial w \partial\overline{w}} \log(1-|f_\alpha|^2) \\ = -\frac{\partial^2}{\partial w \partial\overline{w}} \log(1-|w|^2) - \frac{\partial^2 \varphi_\alpha}{\partial w \partial\overline{w}} &= \frac{1}{(1-|w|^2)^2} - \frac{\partial^2 \varphi_\alpha}{\partial w \partial\overline{w}}. \end{aligned} \tag{2}$$

If we choose the point $b \in \partial\Delta$ to be sufficiently general and the open neighborhood U_b to be sufficiently small, by comparing the boundary behavior of both sides of (2) as $w \in U_b$ approaches b, we conclude readily that exactly d_1 of the functions f_α; say $i = 1, \cdots, d_1$; map boundary points to boundary points, i.e., $f_\alpha(U_b \cap \partial\Delta) \subset \partial\Delta$ for $1 \le \alpha \le d_1$, and $f_\beta(U_b) \Subset \Delta$ for $d_1 + 1 \le \beta \le d_2$. From (1), $(U_b \cap \Delta, f^* ds^2_{\Delta^{d_2}})$ is of constant Gaussian curvature $-\frac{1}{d_1}$. We may assume that f is an embedding on U_b. Write $Z = f(U_b \cap \Delta)$. For $w \in U_b \cap \Delta$ denote by $\eta(w) \in T_{f(w)}(Z)$ a vector of unit length with respect to $ds^2_{\Delta^{d_2}}$. Denote by σ the second fundamental form of Z as a (locally closed) complex submanifold of Δ^{d_2}. By the Gauss equation we have

$$R_{\eta(w)\overline{\eta(w)}\eta(w)\overline{\eta(w)}} - \|\sigma(f(w))\|^2 = -\frac{1}{d_1}\ . \tag{3}$$

For $w \in \Delta$, write $\delta(w) = 1 - |w|$ for the Euclidean distance to the boundary $\partial\Delta$. From (2), writing $f'(w) = (f_1'(w), \cdots, f_{d_2}'(w))$, it follows that the tangent vector $f_\alpha'(w)\frac{\partial}{\partial z_i}$ is of length $\frac{\sqrt{2}}{1-|w|^2} + O(\delta(w))$ for $1 \le \alpha \le d_1$, and of length $O(1)$ for $d_1 + 1 \le \alpha \le d_2$. It follows readily that $\eta(w)$ is equivalent under the action of $\mathrm{Aut}(\Delta^{d_2})$ to the unit vector $\eta_w \in T_0(\Delta^{d_2})$ given by

$$\eta_w = \frac{1}{\sqrt{2d_1}}\Big(1 + O\big(\delta(w)^2\big), \cdots, 1 + O\big(\delta(w)^2\big); O(\delta(w)), \cdots, O(\delta(w))\Big) \tag{4}$$

where precisely the first d_1 components are of the form $1 + O\big(\delta(w)^2\big)$. Comparing (3) and (4) we conclude that

$$R_{\eta(w)\overline{\eta(w)}\eta(w)\overline{\eta(w)}} = -\frac{1}{d_1} + O\big(\delta(w)^2\big); \quad \text{hence} \quad \|\sigma(f(w))\| = O(\delta(w))\ . \tag{5}$$

As a consequence, $f : U_b \cap \Delta \to Z \subset \Delta^{d_2}$ is asymptotically totally geodesic as w approaches $U_b \cap \partial\Delta$. On the other hand, since $\Gamma \subset \mathrm{Aut}(\Delta)$ is a lattice, for almost every point $b' \in U_b \cap \partial\Delta$, there exists a sequence of elements $\gamma_j \in \Gamma$ such that $\gamma_j(x)$ approaches the boundary point b' for any $x \in \Delta$. If we pick $x \in U_b \cap \Delta$, for j sufficiently large $\gamma_j(x) \in U_b \cap \Delta$, and we have $\|\sigma(f(x))\| = \|\sigma(f(\gamma_j(x)))\|$ in view of the way that f is defined from $Y \subset X \times X$, $X = \Delta/\Gamma$. Taking the limit as j tends to ∞ we conclude from the asymptotic total geodesy of f on $U_b \cap \Delta$ that in fact $\sigma(f(x)) = 0$ for any $x \in U_b \cap \Delta$. As a consequence, $f : U_b \cap \Delta \to \Delta^{d_2}$ is in fact a totally geodesic embedding such that f_α extends to an automorphism $F_\alpha \in \mathrm{Aut}(\Delta)$ for $1 \le \alpha \le d_1$ and f_α is a constant function for $d_1 + 1 \le \alpha \le d_2$. However from the way that f is defined from an algebraic correspondence it follows that each component map f_k must be of maximal rank at some point, hence $d_1 = d_2$, and $f : (\Delta; 0) \to (\Delta; 0)^{d_2}$ extends to a totally geodesic embedding F congruent to the diagonal map $\Phi(w) = (w, \cdots, w)$ in the sense that $\psi \circ f \circ \varphi = \Phi$ for some $\varphi \in \mathrm{Aut}(\Delta)$, $\psi \in \mathrm{Aut}(\Delta^{d_2})$. In particular, $Y \subset X \times X$ is a modular correspondence, as desired. $\square$

§3 An Alexander-type theorem for automorphisms of irreducible bounded symmetric domains of rank ≥ 2 in terms of smooth boundary points

(3.1) *Alternative proof of Main Theorem in the case of rank ≥ 2 by a new Alexander-type characterization theorem* For an alternative way to complete the proof of Main Theorem we give here another Alexander-type characterization theorem for automorphisms of irreducible bounded symmetric domains Ω of rank ≥ 2 in their Harish-Chandra realization, where in place of the Shilov boundary $Sh(\Omega)$ we consider holomorphic maps defined on a neighborhood of a smooth point $b \in \partial\Omega$. We have

Theorem 3.1.1. *Let $\Omega \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. Suppose b be a smooth point on $\partial\Omega$. Let $U_b \subset \mathbb{C}^n$ be an open neighborhood of b in $\mathbb{C}^n$ and $f : U_b \to \mathbb{C}^n$ be an open holomorphic embedding such that $f(U_b \cap \Omega) \subset \Omega$ and $f(U_b \cap \partial\Omega) \subset \partial\Omega$. Then, there exists an automorphism $F : \Omega \to \Omega$ such that $F|_{U_b\cap\Omega} \equiv f|_{U_b\cap\Omega}$.*

Theorem 3.1.1 allows us to give an alternative proof of our Main Theorem in the case of rank ≥ 2 without the need to examine the behavior near the Shilov boundary $Sh(\Omega)$ of the multivalent map given by the algebraic extension of Graph(f) of the germ of measure-preserving holomorphic mapping $f : (\Omega; 0) \to (\Omega; 0)^{d_2}$.

Alternative proof of Main Theorem in the case of rank ≥ 2. The Main Theorem in the case of rank ≥ 2 follows immediately from the extension result [(1.2), Proposition 1.2.2]. the functional identity for measure-preserving holomorphic maps as in the structural equation (†) in (1.1) for such maps and reformulated in [(2.3), Eqn.(1)] in the proof there of the Main Theorem, and from Theorem 3.1.1, exactly as in the case of the unit ball B^n, $n \geq 2$, given in (2.1). □

We have chosen to give in (2.2) a proof of the Main Theorem in the case where rank$(\Omega) \geq 2$ by resorting to the result of Henkin-Tumanov [TK1], stated here as Theorem 2.2.1, since the latter is the well-known form of Alexander-type theorem in the rank ≥ 2 case. Here we present a proof of Theorem 3.1.1 for two reasons. First of all, as explained it completes a proof of the Main Theorem in the case of rank ≥ 2 in a way parallel to the rank-1 case. Secondly, Theorem 3.1.1 is of independent interest in the function theory of bounded symmetric domains and may serve other purposes for rigidity phenomena in the case of rank ≥ 2.

Remarks We note furthermore that Theorem 3.1.1 implies Theorem 2.2.1. In fact, given any point $b_0 \in Sh(\Omega)$ and an open holomorphic embedding $f : U_{b_0} \to \mathbb{C}^n$ satisfying $f(U_{b_0} \cap \Omega) = f(U_{b_0}) \cap \Omega$ and $f(U_{b_0} \cap Sh(\Omega)) = f(U_{b_0}) \cap Sh(\Omega)$, for a smooth point $b \in U_{b_0} \cap \partial\Omega$ and any connected open neighborhood U_b of b such that $U_b \subset U_{b_0}$, the open holomorphic embedding $f|_{U_b} : U_b \to \mathbb{C}^n$ satisfies the hypothesis of Theorem 3.1.1.

(3.2) G-*structures modeled on irreducible Hermitian symmetric manifolds of rank ≥ 2* The proof of Theorem 3.1.1 will be based on Ochiai's result [Oc] from the theory of G-structures modeled on irreducible Hermitian symmetric manifolds M of the compact type and of rank ≥ 2. The reader is referred to Mok [Mk3] for an introduction to such G-structures. We adopt the notations in (2.2), writing $\Omega \Subset \mathbb{C}^n \subset M$ for the Harish-Chandra and Borel embeddings of Ω and representing the compact dual $M = G^{\mathbb{C}}/P$ of Ω as a rational homogeneous manifold. At each $x \in M$ denote by $P_x \subset G^{\mathbb{C}}$ the isotropy subgroup of x. There is a natural homomorphism $\varphi_x : P_x \to \mathrm{GL}(T_x(M))$ given by $\varphi_x(\gamma)(\eta) = d\gamma(\eta)$ for $\eta \in T_x(M)$, and we denote its image by Γ_x. At $0 \in \Omega$, $\varphi_0|_K : K \to \mathrm{GL}(T_0(\Omega))$ is an injective homomorphism on the isotropy subgroup $K \subset G$, and K will be naturally identified with its image in $\mathrm{GL}(T_0(\Omega)) \cong \mathrm{GL}(n; \mathbb{C})$. With respect to the trivialization of the holomorphic tangent bundle over $\mathbb{C}^n \subset M$ given by the Harish-Chandra coordinates, the image of φ_x is identified with $K^{\mathbb{C}} \subset \mathrm{GL}(T_x(M)) \cong \mathrm{GL}(n, \mathbb{C})$, where $K^{\mathbb{C}}$ is the complexification of $K \subset \mathrm{GL}(n, \mathbb{C})$. Covering M by charts admitting Harish-Chandra coordinates we have equipped M with a flat (or integrable)

$K^{\mathbb{C}}$-structure, i.e., a holomorphic reduction of T_M from $\mathrm{GL}(n,\mathbb{C})$ to $K^{\mathbb{C}}$ by means of holomorphic coordinates on the base manifold. There is a notion of preservation of G-structures, which in our case can be equivalently formulated in terms of minimal rational tangents, as follows (cf. Mok [Mk3]). A rational curve $C \subset M$ is said to be a minimal rational curve if and only if its homology class is a generator of $H_2(M,\mathbb{Z}) \cong Z$. At $x \in \mathbb{C}^n$, the reductive complex Lie group $\Gamma_x \cong K^{\mathbb{C}}$ acts on $T_x(M)$, and the highest weight orbit of the semisimple part of Γ_x defines a highest weight variety $\mathcal{W}_x \subset \mathbb{P}T_x(M)$. The latter agrees with the variety of minimal rational tangents at $x \in M$, i.e., the variety of tangents to minimal rational curves passing through x, and, for $x \in \mathbb{C}^n$, such a curve is precisely the topological closure of an affine line $\ell \subset \mathbb{C}^n$ passing through x such that $[T_x(\ell)] \in \mathbb{P}\mathcal{W}_x$. We have the following equivalent formulation of the main result of Ochiai [Oc] (cf. Goncharov [Go]).

Theorem 3.2.1 (Ochiai [Oc]). *Let M be an irreducible compact Hermitian symmetric manifold of the compact type and of rank ≥ 2; $U, V \subset M$ be connected open subsets, and $f : U \to V$ be a biholomorphism. Suppose for every $x \in U$ the projectivization $[df(x)]$ of $df(x) : T_x(M) \to T_{f(x)}(M)$ satisfies $[df(x)](\mathcal{W}_x) = \mathcal{W}_{f(x)}$. Then, there exists an automorphism $F \in \mathrm{Aut}(M)$ such that $F\big|_U \equiv f$.*

(3.3) *Proof of Theorem 3.1.1* In order to check that the given holomorphic map in Theorem 3.1.1 preserves the $K^{\mathbb{C}}$-structure modeled on M, we make use of the fine structure of bounded symmetric domains Ω, especially the foliation of the smooth locus of $\partial\Omega$ by boundary components of maximal dimension. To pass from boundary values to the mapping on $U_b \cap \Omega$ we resort to the method of Mok-Tsai [MT] for the study of boundary values of holomorphic functions on irreducible bounded symmetric domains of rank ≥ 2 by restriction to certain complex submanifolds which are product domains

To streamline the presentation, we recall the notion of invariantly geodesic submanifolds introduced in Tsai [Ts, §4]. Equip Ω with the canonical Kähler-Einstein metric g, and M with the K-invariant Kähler-Einstein metric g_c on M, so that (Ω, g) and (M, g_c) form a dual pair of Hermitian symmetric spaces. In the terminology of [Ts], a complex submanifold $S \subset M$ is called an invariantly geodesic submanifold if and only if $\gamma(S) \subset M$ is totally geodesic in (M, g_c) for any $\gamma \in G^{\mathbb{C}}$. (Such submanifolds are completely classified in [Ts, Proposition 4.6].) A complex submanifold $S_0 \subset \Omega$ will be called an invariantly geodesic submanifold if and only if $\gamma(S_0) \cap \Omega \subset \Omega$ is totally geodesic in (Ω, g) for any $\gamma \in G^{\mathbb{C}}$ such that $\gamma(S_0) \cap \Omega \neq \emptyset$. If $0 \in S_0$, then it follows from the total geodesy of $S_0 \subset \Omega$ and the definition of the Harish-Chandra embedding that $S_0 = W \cap \Omega$ for some complex vector subspace $W \subset \mathbb{C}^n$. From [Ts, Lemma 4.3] it follows readily that $S_0 \subset \Omega$ is invariantly geodesic if and only if $\gamma(\overline{W}) = \overline{W}$ for any $\gamma \in P$, where $\overline{E}$ denotes the topological closure of E in M for any subset $E \subset M$. Hence, $S_0 \subset \Omega$ is invariantly geodesic if and only if $S_0 = S \cap \Omega$ for some invariantly geodesic submanifold $S \subset M$. An affine-linear subspace $A \subset \mathbb{C}^n$ will be called invariantly geodesic if and only if $\overline{A} \subset M$ is totally geodesic. From [Ts, Lemma 4.3] any invariantly geodesic submanifold $S \subset M$ such that $S \cap \mathbb{C}^n \neq \emptyset$ must be of the form $\overline{A}$ for some invariantly geodesic affine-linear subspace $A \subset \mathbb{C}^n$. Regarding invariantly geodesic submanifolds we have the following obvious lemma.

Lemma 3.3.1. *Let $\Omega \subset M$ be an irreducible bounded symmetric domain Ω realized as an open subset of its compact dual M by the Borel embedding. Let $\{S_\alpha\}_{\alpha\in A}$ be any family of invariantly geodesic submanifolds $S_\alpha \subset M$ such that $N := \bigcap_{\alpha\in A} S_\alpha$ is non-empty. Then, $N \subset M$ is an invariantly geodesic submanifold. Consequently, if $\{D_\alpha\}_{\alpha\in A}$ is any family of invariantly geodesic submanifolds $D_\alpha \subset \Omega$ such that $\Psi := \bigcap_{\alpha\in A} D_\alpha$ is non-empty, then $\Psi \subset \Omega$ is an invariant geodesic submanifold.*

We are now ready to give a proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Write $r := \mathrm{rank}(\Omega) \geq 2$. Without loss of generality we may assume that U_b is convex and that both $U_b \cap \partial\Omega$ and its image $f(U_b \cap \partial\Omega)$ consist entirely of smooth points of $\partial\Omega$. By the fine structure of bounded symmetric domains (cf. Wolf [Wo]) in their Harish-Chandra realization, the smooth locus $\mathrm{Reg}(\partial\Omega)$ of $\partial\Omega$ admits a smooth foliation $\mathcal{F}$ by boundary components. For $p \in \mathrm{Reg}(\partial\Omega)$, the leaf Φ_p of $\mathcal{F}$ passing through p is a maximal boundary component of $\partial\Omega$, i.e., a boundary component of maximal complex dimension (and of rank $r-1$), and the group $G = \mathrm{Aut}_0(\Omega)$ acts transitively on the set of such boundary components Φ_p. Let Π be a maximal polydisk on Ω such that $b \in \partial P$. Replacing b by $\gamma(b)$ for some $\gamma \in G$, we may assume that Π is a Euclidean polydisk $\Delta^r \subset \mathbb{C}^r \times \{0\} \subset \mathbb{C}^n$ in terms of Harish-Chandra coordinates, and that $b = (1, 0, \cdots, 0) \in \partial P \subset \partial\Omega$. We have $\Pi = \Delta \times \Delta^{r-1}$, where Δ^{r-1} is a maximal polydisk of an irreducible bounded symmetric domain Ω' of rank $r-1$ lying on some complex vector subspace $V \subset \mathbb{C}^n$ such that $\Omega' \Subset V$ is the Harish-Chandra embedding.

$\Delta \times \Omega' \subset \Omega$ is a totally geodesic complex submanifold. For each point $p = (a, q) \in \partial\Delta \times \Omega' \subset \partial\Omega$, the boundary component Φ_p passing through p is given by $\{a\} \times \Omega' \subset \partial\Omega$. In the terminology of Mok-Tsai [MT, Definition 1.5.2], for any $t \in \Delta$, the complex submanifold $\Omega_t \subset \{t\} \times \Omega' \subset \Omega$ is a characteristic symmetric subspace. By [MT, Proposition 1.12] and Tsai [Ts, Lemma 4.4], $\Omega_t \subset \Omega$ is an invariantly geodesic submanifold. Denoting by $\Omega_0 \Subset \mathbb{C}^{n_0} \subset M_0$ the Harish-Chandra and Borel embeddings, $M_0 \subset M$ is an invariantly geodesic submanifold. Consider the complex submanifold $\Delta \times \Omega' \subset \Omega$. Since the point $b = (1, 0, \cdots, 0) \in \partial P$ lies on U_b, there exists an open neighborhood W of 1 in $\mathbb{C}$, and a connected open neighborhood D of 0 in Ω' such that $W \times D \subset U_b$. For any $\zeta \in \partial\Delta \cap W$, $f\big|_{\{\zeta\}\times D}$ is a biholomorphism of $\{\zeta\} \times D$ onto its image $f(\{\zeta\} \times D)$, which is an open subset of the maximal boundary component $\Phi_{f(\zeta;0)}$, which is a bounded domain on some characteristic affine-linear subspace A_ζ. As in the proof of [MT, Proposition 2.3], by taking higher-order partial derivatives in the directions of Ω' along the zero-section $\{0\}\times D \subset U_b$ and verifying their linear dependence on first-order derivatives (owing to holomorphicity and linear dependence on $W \cap \partial\Delta$), it follows that for each $t \in U_b$, $f\big|_{\{t\}\times D}$ is a biholomorphism of $\{t\} \times D$ onto an open subset of some affine-linear subspace A_t. We have thus an induced holomorphic map $f^\sharp : W \to \mathcal{G}$, where $\mathcal{G}$ is the Grassmannian of affine-linear subspaces of $\mathbb{C}^n$ of dimension n_0. Let $\mathcal{H}$ be the set of all affine-linear subspaces $A \subset \mathcal{G}$ such that $\overline{A} = \gamma(M_0)$ for some $\gamma \in G^{\mathbb{C}}$. Then $\mathcal{H} \subset \mathcal{G}$ is a complex submanifold. For each maximal boundary component $\Phi_p \subset \mathrm{Reg}(\partial\Omega)$, Φ_p is an open subset of an n_0-dimensional affine-linear subspace belonging to $\mathcal{H}$. Hence the map $f^\sharp : W \to \mathcal{G}$ is such that $f^\sharp(W \cap \partial\Delta) \subset \mathcal{H}$, and it follows that $f^\sharp(W) \subset \mathcal{H}$. Fix now a maximal characteristic symmetric subspace of the form $\Theta = \{t_0\} \times \Omega'$ for some $t_0 \in W \cap \Delta$ so that $\Theta \cap U_b \neq \emptyset$, and $f\big|_{\Theta \cap U_b}$ is a biholomorphism onto an open subset of a maximal characteristic symmetric subspace Ξ.

In order to apply Ochiai's result as in Theorem 3.2.1 we need to check that $[df]$ preserves varieties of minimal rational tangents. In the case of rank equal to 2 this follows readily from the last paragraph since in that case tangents to maximal characteristic symmetric subspaces are minimal rational tangents. For arbitrary rank $r \geq 2$ we need to have a procedure of recovering minimal rational tangents from maximal characteristic symmetric subspaces, which are of rank $r - 1 \geq 1$.

There is an open neighborhood N of the identity element e in $G^{\mathbb{C}}$ with the following property. For any $\gamma \in N$, $\gamma(W \times D) \cap (U_b \cap \partial\Omega) \neq \emptyset$. Then $\gamma(\Theta) \cap U_b \neq \emptyset$ and the same argument as in the above then shows that $f\big|_{\gamma(\Theta)\cap U_b}$ is a biholomorphism of $\gamma(\Theta) \cap U_b$ onto an open subset of a maximal characteristic symmetric subspace Ξ_γ. Write $x_0 := (t; 0) \in \Delta \times \Omega'$. For $\mu \in N$ write $x := \mu(x_0)$. N contains an open neighborhood of μ in the right coset $K_x\mu$. Then, for $\gamma \in K_x\mu$ we have $\gamma(x_0) = x$, $\gamma(\Theta)$ is a maximal characteristic symmetric subspace passing through x and there is a maximal characteristic symmetric Ξ_γ passing through $f(x)$ such that $f(\gamma(\Theta)\cap U_b)$ is an open subset of Ξ_γ. Fix a non-zero minimal rational tangent $\alpha \in T_x(\Omega)$ and denote by Δ_α the unit minimal disk passing through x and tangent to α. By [MT, Proposition 1.9], Δ_α is the intersection of all maximal characteristic symmetric subspaces Θ containing it. By the Identity Theorem for holomorphic functions the same remains true if in place of all such Θ we take a non-empty open subset of such Θ. Then $f(\Delta_\alpha \cap U_b)$ lies on $\Psi := \bigcap \big\{\Xi_\gamma : \gamma(x_0) = x, \gamma \in N\big\}$, and $f(\Delta_\alpha \cap U_b) \subset \Psi$ is an open subset. On the other hand, Ψ is the intersection of a family of maximal characteristic symmetric subspaces, and it follows that Ψ is an open subset of an affine line. By Lemma 3.3.1, $\Psi \subset \Omega$ is an invariantly geodesic submanifold. Thus, for some $\nu \in G$ we have $\nu(\Psi) \subset \mathcal{P}$, a maximal polydisk passing through 0. As can be easily checked using the action of $\mathrm{Aut}_0(\mathcal{P}) \subset G$, such a geodesic submanifold can be invariantly geodesic only if it is a minimal disk, and we conclude that $\Psi \subset \Omega$ is a minimal disk. As a conclusion, we have shown that for some non-empty connected open subset $\mathcal{O} \subset U_b \cap \Omega$, we have $[df(x)](\mathcal{W}_x) \subset \mathcal{W}_{f(x)}$ for $x \in \mathcal{O}$, and by Ochiai's result as given in Theorem 3.2.1 we conclude that there exists $F_0 \in \mathrm{Aut}(M)$ such that $F_0\big|_{U_b} \equiv f$.

It remains to check that $F := F_0\big|_\Omega$ is an automorphism of Ω. For that purpose it suffices to check that the germ of F at some point $x_0 \in \Omega$ is a germ of holomorphic isometry of (Ω, g). Choose $x_0 \in \Omega$ such that for some minimal rational tangent $\alpha_0 \neq 0$ at x_0, we have $\Delta_{\alpha_0} \cap U_b \neq \emptyset$. Δ_{α_0} is an open set of a minimal rational curve C_{α_0} on M such that $F\big|_{C_{\alpha_0}}$ maps C_{α_0} biholomorphically onto a minimal rational curve $C' \subset M$. The image of $\partial\Delta_{\alpha_0}$ under F must be a circle on the affine part of $C' \cong \mathbb{P}^1, C' \cap \mathbb{C}^n \cong \mathbb{C}$. Since $F(\partial\Delta_{\alpha_0} \cap U_b) \subset C' \cap \partial\Omega$, the restriction $F\big|_{\Delta_{\alpha_0}}$ must map Δ_{α_0} isometrically onto the minimal disk $C' \cap \Omega$. The analogous statement holds true for x sufficiently close to x_0 and for a minimal rational tangent α at x sufficiently close to α_0 in the tangent bundle T_Ω. It follows that for x sufficiently close to x, F is an isometry when restricted to a non-empty open set of minimal disks passing through x, hence for all minimal disks passing through x by the Identity Theorem for real-analytic functions. Writing $s = g - F^*g$ on a neighborhood $\mathcal{O}$ of x_0, for $x \in \mathcal{O}$ and for any minimal rational tangent α at x we have $s_{\alpha\overline{\alpha}} = 0$. Since the set of (non-zero) minimal rational tangents at x is complex-analytic, expanding α in Taylor series at some point α_1 and polarizing the identity we conclude that $s_{\xi\overline{\eta}} = 0$ for any $\xi, \eta \in \mathcal{W}_x$, noting that $\mathcal{W}_x \subset \mathbb{P}T_x(\Omega)$ is linearly non-degenerate. Thus the germ of F at x_0 is a germ of holomorphic isometry of (Ω, g) at x_0 and we have $F \in \text{Aut}(\Omega)$, as desired. The proof of Theorem 3.1.1 is complete. □

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