Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric

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Let X be a simply connected complex manifold equipped with a real-analytic Kähler metric g. By the seminal work of Calabi's ([Ca], 1953), every germ of holomorphic isometry of (X, g) into the projective space $(\mathbb{P}^N, ds^2_{FS})$, $1 \leq N \leq \infty$, equipped with a Fubini-Study metric must extend to a holomorphic isometry on (X, g).

In the current article we study the extension problem for germs of holomorphic isometries $f : (D; x_0) \to (\Omega; f(x_0))$ up to normalizing constants between bounded domains in Euclidean spaces equipped with Bergman metrics ds^2_D on D and ds^2_Ω on Ω. Our basic extension results are of two types, viz., extension results of the germ Graph$(f) \subset D \times \Omega$ to a complex-analytic subvariety S of $D \times \Omega$, and extension results on S beyond the boundary of $D \times \Omega$ under certain assumptions. We call the former type interior extension results and the latter type boundary extension results. Interior extension follows from the work of Calabi [Ca] (cf. Remarks) after the proof of Theorem 2.1.1. Our main purpose will be on boundary extension for pairs of bounded domains (D, Ω) such that the Bergman kernel $K_D(z, w)$ extends meromorphically in $(z, \overline{w})$ to a neighborhood of $\overline{D} \times D$, and such that the analogous statement holds true for the Bergman kernel $K_\Omega(\zeta, \xi)$ on Ω. Examples include pairs (D, Ω) of bounded symmetric domains in their Harish-Chandra realizations. The special case where D is the unit disk Δ, Ω is a polydisk Δ^p, and where $f : \big(\Delta, \lambda ds^2_\Delta; 0\big) \to \big(\Omega, ds^2_\Omega; 0\big)$ is a germ of holomorphic isometry in which the normalizing constant λ is a positive integer q, was studied by Clozel-Ullmo ([CU], 2003) in connection to a problem in Arithmetic Dynamics. For such a germ of map they established a real-analytic functional identity arising from equating potential functions of Kähler metrics, and deduced as a consequence that the germ of subvariety Graph(f) in $\Delta \times \Delta^p$ extends algebraically to $\mathbb{C} \times \mathbb{C}^p$. In their case the germ of holomorphic map f arises from an algebraic correspondence on some finite-volume quotient of the unit disk, and, exploiting the action of the underlying lattice Γ on an extension of Graph(f) to $\Delta \times \Delta^p$, they proved that f must be totally geodesic, but conjectured ([CU, Conjecture 2.2, p.52]) that in fact any $f : \big(\Delta, q\, ds^2_\Delta; 0\big) \to \big(\Delta^p, ds^2_{\Delta^p}; 0\big)$ is totally geodesic.

To start with we consider the case of $f : \big(D, \lambda ds_D; 0\big) \to \big(\Omega, ds^2_\Omega; 0\big)$ between bounded complete circular domains with base points at 0. Generalizing the real-analytic functional identity expressed in terms of Bergman kernels, by polarization we obtain an infinite number of holomorphic identities, and the first question is to determine whether these identities are sufficiently non-degenerate to force analytic continuation. While examples show that in general this is not the case, we resolve the difficulty by studying deformations of simultaneous solutions of the holomorphic functional equations, and force analytic continuation by showing that, in the event that there are non-trivial deformations of simultaneous solutions to these equations, the germ of holomorphic isometry must take values in linear sections of the canonical image of the domain in the infinite-dimensional projective space $\mathbb{P}^\infty$, where the linear sections correspond to zeros of certain square-integrable holomorphic functions which are in some sense extremal with respect to the Bergman metric. For a bounded complete circular domain $G \Subset \mathbb{C}^m$ with Bergman kernel $K_G(z, w)$, the domains of definition of $K_{D,w} := K_G(z, w)$ grow to $\mathbb{C}^n$ as w shrinks to 0. Using this we prove the analytic continuation of Graph$(f) \subset D \times \Omega$ to a complex-analytic subvariety $S^\sharp$ in the Euclidean space. In the special case of bounded symmetric domains in their Harish-Chandra realizations, we prove the following stronger result.

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Theorem 1.3.1. *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded symmetric domains in their Harish-Chandra realizations. Let λ be any positive real number and $f : \big(D, \lambda ds_D^2; 0\big) \to \big(\Omega, ds_\Omega^2; 0\big)$ be a germ of holomorphic isometry at $0 \in D$, $f(0) = 0$. Then, the germ* $\mathrm{Graph}(f)$ *extends to an affine-algebraic subvariety $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ such that $S := S^\sharp \cap (D \times \Omega)$ is the graph of a proper holomorphic isometric embedding $F : D \to \Omega$ extending the germ of holomorphic map f.*

Bounded symmetric domains provide a first source of holomorphic isometries up to normalizing constants. A holomorphic totally geodesic embedding $F : D \to \Omega$ between bounded symmetric domains is a holomorphic isometry with respect to the Bergman metric up to a rational normalizing constant whenever D is irreducible. In terms of Borel embeddings, F extends algebraically to a holomorphic map between the dual Hermitian symmetric manifolds of the compact type, thus to rational maps on Euclidean spaces when $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are bounded symmetric domains in their Harish-Chandra realizations. At the same time, holomorphic totally geodesic embeddings of bounded symmetric domains into homogeneous disk bundles over them give examples of holomorphic isometries with any prescribed normalizing isometric *real* constant $\lambda > 1$. On the other hand we have now produced examples of holomorphic isometric embeddings of the Poincaré disk into certain bounded symmetric domains Ω which are *not* totally geodesic. More precisely, we have proved (cf. (3.2) for the meaning of 'congruence')

Theorem 3.2.1. *For every positive integer $p > 1$ there exists a holomorphic isometric embedding $F : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$, $F = (F_1, \cdots F_p)$, where each component $F_k, 1 \le k \le p$ is nonconstant, such that F is not totally geodesic. In particular, Conjecture 2.2 of Clozel-Ullmo* [CU] *is false. Furthermore, for $p \ge 3$ there exists a real-analytic 1-parameter family of mutually incongruent holomorphic isometric embeddings $F_t : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$, $t \in \mathbb{R}$.*

It is in general an interesting problem to construct non-standard holomorphic isometric embeddings of the Poincaré disk Δ into bounded domains Ω, including the case where Ω is a bounded symmetric domain. For the special case where Ω is the polydisk Δ^p, $p \ge 2$, the classification problem has been posed, but only very partial results are known (Ng [Ng]), including a complete classification for $p = 2, 3$. As a further example we give also an explicit construction of a non-trivial (proper) holomorphic isometric embedding $F : \Delta \to \mathcal{H}_3$ of the Poincaré disk into the Siegel upper half-plane $\mathcal{H}_3$ of genus 3. We will show that the latter is distinguishable from a holomorphic isometry into a polydisk by checking that the branch points of F do not lie on the Shilov boundary $Sh(\mathcal{H}_3)$ and invoking results of Ng [Ng]. It is also interesting to find domains D other than the Poincaré disk admitting non-standard holomorphic isometric embeddings into some bounded domain Ω. Restricted to the case where both D and Ω are assumed to be bounded symmetric domains, the main interest lies with $D = B^n$, $n \ge 2$. For a discussion on this and related problems cf. the survey article Mok [Mk5, §5].

Our study of extensions of germs of holomorphic isometries generalizes to those between arbitrary bounded domains. Interior extension holds true unconditionally, while boundary extension holds true under certain conditions on Bergman kernels, as given by

Theorem 2.1.2 (main part). *Let $D \Subset \mathbb{C}^n$* resp. *$\Omega \Subset \mathbb{C}^N$, be bounded domains. Let $x_0 \in D, y_0 \in \Omega, \lambda$ be a positive real number and $f : (D, \lambda ds_D^2; x_0) \to (\Omega, ds_\Omega^2; y_0)$ be a germ of holomorphic isometry. Suppose furthermore that the Bergman kernel $K_D(z, w)$ extends as a meromorphic function in $(z, \overline{w})$ to a neighborhood of $\overline{D} \times D$ and $K_\Omega(\zeta, \xi)$ extends as a meromorphic function in $(\zeta, \overline{\xi})$ to a neighborhood of $\overline{\Omega} \times \Omega$. Then, there exists a neighborhood $D^\sharp$ of $\overline{D}$ and a neighborhood $\Omega^\sharp$ of $\overline{\Omega}$ such that the germ of* $\mathrm{Graph}(f) \subset D \times \Omega$ *at (x_0, y_0) extends to an irreducible complex-analytic subvariety $S^\sharp$ of $D^\sharp \times \Omega^\sharp$.*

Theorem 2.1.2 further generalizes to relatively compact domains on complex manifolds provided that the domains admit Bergman metrics and the canonical maps on them are embeddings (cf. (2.2)).

Holomorphic isometries between bounded domains are meaningful for the study of holomorphic functions on such domains. As an illustration a *bona fide* holomorphic isometric embedding $F : (D, ds^2_D) \to (\Omega, ds^2_\Omega)$ between bounded circular domains star-shaped with respect to 0, $F(0) = 0$, is induced by a Hilbert space isomorphism $\mu : H^2(D) \to H^2(\Omega)$ onto the orthogonal complement of the Hilbert subspace $E \subset H^2(\Omega)$ consisting of functions vanishing on $Z := F(D)$, yielding for holomorphic functions square-integrable on Z (with respect to the measure induced from D) norm-preserving holomorphic extensions to Ω square-integrable with respect to the Lebesgue measure.

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§1 Extension of germs of holomorphic isometries with respect to the Bergman metric on bounded complete circular domains

(1.1) *Extension of germs of holomorphic isometries via holomorphic functional equations*

In connection to a problem in Arithmetic Dynamics, Clozel-Ullmo [CU] considered a germ of holomorphic isometry $f : \big(\Delta, q\, ds^2_\Delta; 0\big) \to \big(\Delta, ds^2_\Delta; 0\big)^p$ from the unit disk Δ into a polydisk, where q is a positive integer. (Here and in what follows, for a bounded domain D, ds^2_D stands for the Bergman metric.) They obtained a real-analytic functional identity arising from Kähler potentials, and proceeded from there to prove that Graph(f) extends as an affine-algebraic subvariety. In higher dimensions the method of [CU] is difficult to generalize directly. In Mok [Mk3] we considered the analogous problem for the complex unit ball B^n. There, by polarization we obtain instead a continuous family of holomorphic functional identities, and we solved the problem for B^n, $n \geq 2$, by forcing analytic continuation by means of these identities. Here we formulate the starting point of our argument more generally for germs of holomorphic isometries between bounded complete circular domains, allowing at the same time the normalizing constant λ to be any positive real number. Recall that a circular domain $D \subset \mathbb{C}^n$ is a domain invariant under the action of the circle group S^1 given by $\Phi : S^1 \times D \to D$; $\Phi(e^{i\theta}, z) = e^{i\theta} z$; $\theta \in \mathbb{R}$. D is complete if and only if $0 \in D$. For a bounded complete circular domain $D \Subset \mathbb{C}^n$ and for $\theta \in \mathbb{R}$, the Bergman kernel $K_D(\cdot, \cdot)$ satisfies $K_D(e^{i\theta} z, e^{i\theta} w) = K_D(z, w)$, so that $K_D(z, 0) = K_D(e^{i\theta} z, 0)$, implying that $K_D(z, 0)$ is a (positive) constant.

Proposition 1.1.1. *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded complete circular domains. Denote by ds^2_D,* resp. *ds^2_Ω, the Bergman metric on D,* resp. *Ω, and by K_D,* resp. *K_Ω, the Bergman kernel on D,* resp. *Ω. Let λ be any positive real number and $f : (D, \lambda ds^2_D; 0) \to (\Omega, ds^2_\Omega; 0)$ be a germ of holomorphic isometry at $0 \in D; f(0) = 0$. Then, there exists some real number $A > 0$ such that for $z, w \in D$ sufficiently close to 0 we have*

$$K_\Omega(f(z), f(z)) = A \cdot K_D(z, z)^\lambda; \quad \text{and hence}$$
$$K_\Omega(f(z), f(w)) = A \cdot K_D(z, w)^\lambda; \quad \text{where}$$
$$K_D(z, w)^\lambda = e^{\lambda \log K_D(z,w)} \ ,$$

in which log *denotes the principal branch of logarithm.*

Proof. Following the argument of Clozel-Ullmo [CU], we have from the hypothesis

$$\begin{aligned} \sqrt{-1}\partial\overline{\partial}\log K_\Omega(f(z),f(z)) &= \lambda\sqrt{-1}\partial\overline{\partial}\log K_D(z,z)\ ; \\ \log K_\Omega(f(z),f(z)) &= \lambda\log K_D(z,z) + \mathrm{Re}(\psi) \end{aligned} \tag{1}$$

for some holomorphic function ψ. Consider the Taylor expansion of $\log K_D(z,z)$ in $z_1,\cdots z_n$ and $\overline{z_1},\cdots\overline{z_n}$. For a multi-index $I = (i_1,\cdots,i_n)$; $i_1,\ldots,i_n \geq 0$; we write $z^I = z_1^{i_1}\cdots z_n^{i_n}$, and $|I| = i_1+\cdots+i_n$. By the invariance of the Bergman kernel under the circle group action $(e^{i\theta},z)\to e^{i\theta}z, \theta\in\mathbb{R}$, the coefficient of $z^I\overline{z^J}$ is zero whenever $|I|\neq|J|$. The analogue is true also for the complete circular domain Ω. Since $f(0)=0$, it follows by substitution that in the Taylor expansion of $\log K_\Omega\big(f(z),f(z)\big)$ at 0, the coefficients of terms of pure type z^I and $\overline{z^I}$ must vanish for any $I=(i_1,\ldots,i_n)$; $i_1,\ldots,i_n\geq 0$; such that at least one of the indices i_k, $1\leq k\leq n$, is nonzero. On the other hand, the Taylor expansion of $2\mathrm{Re}(\psi) = \psi+\overline{\psi}$ at 0 consists precisely of terms of pure type, and it follows by comparing the two sides of (1) that $\mathrm{Re}(\psi)$ must be a (real) constant.

We introduce now holomorphic functional identities by polarization, viz.,

$$\log K_\Omega(f(z),f(w)) = \lambda\log K_D(z,w) + a + H(z,w)\ , \tag{2}$$

where a is a real constant and

$$H(z,w) = \sum_{(I,J)\neq(0,0)} H_{I\overline{J}}z^I\overline{w^J}\ . \tag{3}$$

is holomorphic in z and anti-holomorphic in w. Recall that $K_D(0,0)$, resp. $K_\Omega(0,0)$, is real and 'log' stands for the principal branch of the logarithm. Restricting to the diagonal $\{z=w\}$ we have $H(z,z)=0$, i.e., $\sum_{(I,J)\neq(0,0)} H_{I\overline{J}}z^I\overline{z^J}=0$, so that $H_{I\overline{J}}=0$ for all $(I,J)\neq(0,0)$, hence $H(z,w)=0$ where defined, yielding

$$\log K_\Omega(f(z),f(w)) = \lambda\log K_D(z,w) + a\ ; \quad \text{hence} \tag{4}$$

$$K_\Omega(f(z),f(w)) = A\cdot K_D(z,w)^\lambda\ , \tag{5}$$

where $A := e^a$ and $K_D(z,w)^\lambda = e^{\lambda\log K_D(z,w)}$, as desired. ∎

For the application of Proposition 1.1.1 to extension problems, we recall first of all the following well-known fact about the Bergman kernel on a complete circular domain.

Lemma 1.1.1. *Let $D\Subset\mathbb{C}^n$ be a complete circular domain and denote by $K_D(z,w)$ the Bergman kernel on D. Suppose r is a real number, $0<r<1$, such that $rD\subset D$. Then, for $z\in D$ and $w\in rD$ we have $K_D(z,w) = K_D\big(rz,\frac{w}{r}\big)$. In particular, for every $w\in rD$ the holomorphic function $K_{D,w}(z) := K_D(z,w) = K_D\left(rz,\frac{w}{r}\right) := K_{D,\frac{w}{r}}(rz)$ in $z\in D$ extends holomorphically to $\frac{1}{r}D$ when we define $K_{D,w}(z) := K_{D,\frac{w}{r}}(rz)$ for $z\in\frac{1}{r}D$.*

Proof. From the invariance of D under the circle group action $(e^{i\theta},z)\mapsto e^{i\theta}z$ we have

$$K_D(z,w) = \sum_{|I|=|J|} a_{I\overline{J}}z^I\overline{w^J}\ .$$

Observing that $(rz)^I\overline{\big(\frac{w}{r}\big)^J} = z^I\overline{w^J}$ whenever $|I|=|J|$, we have $K_D(z,w) = K_D(rz,\frac{w}{r})$ for $z\in D$ and $w\in rD$. Fixing $w_0\in rD$, $K_D\big(rz,\frac{w_0}{r}\big)$ is defined for $z\in\frac{1}{r}D$. Hence, $K_{D,w_0}(z) = K_D(rz,\frac{w_0}{r})$ extends holomorphically from D to $\frac{1}{r}D$, as desired. ∎

For $r > 0$ we write $D_r := B^n(0;r)$. Choose $e > 0$ such that $f : \big(D, \lambda ds_D^2; 0\big) \to \big(\Omega, ds_\Omega^2; 0\big)$ is represented by a holomorphic embedding defined on D_e, and such that moreover $K_D(z,w)$ and $K_\Omega(f(z), f(w))$ are nonzero whenever $z, w \in D_e$. For notational convenience later on we will also require that $e < 1$. Similarly for $\rho > 0$ we write $\Omega_\rho := B^N(0;\rho)$. Choose δ_0 such that $0 < \delta_0 < 1$ and such that $\Omega_{\delta_0} \Subset \Omega$.

In Mok [Mk3] we studied germs of holomorphic isometries f from the unit ball $B^n, n \ge 2$, to its Cartesian products for the case where the normalizing constant is a positive integer q. There, making use of the explicit form of the holomorphic functional identities arising from equating potential functions, we extend Graph(f) to an affine-algebraic subvariety. To prove an analogue for the general case we encounter first of all the problem that the associated functional identities are in general not sufficiently 'non-degenerate' to force analytic continuation. We overcome difficulties arising from such degenerate situations by imposing additional constraints to cut down the set of simultaneous solutions to the functional equations. For its formulation recall that $K_D(z,0) = C > 0$. Let $D^\sharp \Subset \mathbb{C}^n$ be a neighborhood of $\overline{D}$. By Lemma 1.1.1, there exists ϵ_0 satisfying $0 < \epsilon_0 < e$ such that for any $w \in D_{\epsilon_0} = B^n(0;\epsilon_0)$, $K_D(z,w)$ is defined for $z \in D^\sharp$ (by analytic extension of $K_{D,w}(\cdot) = K_D(\cdot, w)$), and $\mathrm{Re}(K_D(z,w)) > 0$ for any $(z,w) \in D^\sharp \times D_{\epsilon_0}$. Then, $K_D(z,w)^\lambda = e^{\lambda \log K_D(z,w)}$ is defined for $(z,w) \in D^\sharp \times D_{\epsilon_0}$. We will further assume that $f(D_{\epsilon_0}) \Subset \Omega_{\delta_0}$. Suppose now $0 < \epsilon \le \epsilon_0$. In place of the germ of map, the symbol f will sometimes stand for the map $f : \big(D_\epsilon, \lambda ds_D^2\big|_{D_\epsilon}\big) \to (\Omega, ds_\Omega^2)$. Thus, $K_D(z,w)^\lambda$ is defined on $D^\sharp \times D_\epsilon$ as a function holomorphic in $(z, \overline{w})$. Writing $K_\Omega(0,0) := C'$ and $A := C'C^{-\lambda}$, we have

Proposition 1.1.2. *For each $w \in D_\epsilon$, let $V_w \subset D \times \Omega$ be the set of all $(z,\zeta) \in D \times \Omega$ such that*

$$(\mathbf{I}_w) \qquad K_\Omega\big(\zeta, f(w)\big) = A \cdot K_D\big(z,w\big)^\lambda \ .$$

Define $V_\epsilon = \bigcap_{w \in D_\epsilon} V_w$. Suppose for a general point $z \in D_\epsilon$, $\dim_{(z,f(z))}\big(V_\epsilon \cap (\{z\} \times \Omega)\big) \ge 1$. Then, there exists a family of holomorphic functions $h_\alpha \in H^2(\Omega)$, $\alpha \in \mathbf{A}$, such that

$$\mathrm{Graph}(f) \subset D_\epsilon \times E \ , \quad \textit{where} \quad E := \bigcap_{\alpha \in \mathbf{A}} \mathrm{Zero}(h_\alpha) \ ,$$

and such that $\dim_{(z,f(z))}\big(V_\epsilon \cap (\{z\} \times E)\big) = 0$ for a general point $z \in D_\epsilon$.

By a general point on a complex manifold we mean the complement of a nowhere dense complex-analytic subvariety. By the Identity Theorem on holomorphic functions, $V_\epsilon \subset D \times \Omega$ is independent of $\epsilon > 0$, and we will write V in place of V_ϵ. We say that the system of functional equations $(\mathbf{I}_w), w \in D_\epsilon$, is sufficiently non-degenerate whenever any irreducible branch of V containing Graph(f) must be of dimension $n = \dim(\mathrm{Graph}(f))$.

Proof of Proposition 1.1.2. Graph$(f) \subset D_\epsilon \times \Omega$ is by definition contained in V. By hypothesis, $\dim_{(z,f(z))}(V \cap (\{z\} \times \Omega)) := q \ge 1$ at a general point $z \in D_\epsilon$ (hence actually at any point $z \in D_\epsilon$ by upper semi-continuity of the fiber dimension). Fix a Stein neighborhood Ω_0 of 0 in Ω such that $f(D_{\epsilon_0}) \subset \Omega_0$. (We may take for instance $\Omega_0 = B^N(0;\delta_0)$.) Let $Z_\epsilon \subset V \cap (D_\epsilon \times \Omega_0)$ be an irreducible complex-analytic subvariety of $D_\epsilon \times \Omega_0$ containing Graph(f) such that $\dim_{(z,f(z))}(Z_\epsilon \cap (\{z\} \times \Omega_0)) = 1$ for a general point $z \in D_\epsilon$. $Z_\epsilon \subset V$ may be obtained by an inductive procedure, as follows. If $q = 1$, it suffices to take Z_ϵ to be an irreducible component of $V \cap (D_\epsilon \times \Omega_0)$ containing Graph(f). If $q > 1$, choose any $x_1 \in V \cap (D_\epsilon \times \Omega_0)$ lying outside the subvariety Graph$(f) \subset V \cap (D_\epsilon \times \Omega_0)$. Since $V \cap (D_\epsilon \times \Omega_0)$ is Stein there exists a holomorphic function g_1 on $V \cap (D_\epsilon \times \Omega_0)$ such that $g_1|_{\mathrm{Graph}(f)} \equiv 0$ and such that $g_1(x_1) \ne 0$. We define now $Z_\epsilon^{(n+q-1)} \subset V \cap (D_\epsilon \times \Omega_0)$

to be an irreducible component of the zero set $\mathrm{Zero}(g_1) \subset V \cap (D_\epsilon \times \Omega_0)$ of g_1 containing $\mathrm{Graph}(f)$. If $q = 2$ we take $Z_\epsilon := Z_\epsilon^{(n+1)}$. If $q > 2$ we proceed further with $V \cap (D_\epsilon \times \Omega_0)$ replaced by $Z_\epsilon^{(n+q-1)}$, x_1 replaced by $x_2 \in Z_\epsilon^{(n+q-1)} - \mathrm{Graph}(f)$ to find g_2 holomorphic on $Z_\epsilon^{(n+q-1)}$, $g_2|_{\mathrm{Graph}(f)} \equiv 0$ and $g_2(x_2) \neq 0$. Proceeding inductively we reach $Z_\epsilon := Z_\epsilon^{(n+1)}$ such that $Z_\epsilon \subset D_\epsilon \times \Omega$ is an irreducible subvariety containing $\mathrm{Graph}(f)$ and such that $\dim_{(z,f(z))}(Z_\epsilon \cap (\{z\} \times \Omega)) = 1$ at a general point $z \in D_\epsilon$.

Write $\nu : \widetilde{Z}_\epsilon \to Z_\epsilon$ for the normalization of Z_ϵ. Since the singular set of $\widetilde{Z}_\epsilon$ is of codimension ≥ 2, and $\nu^{-1}(\mathrm{Graph}(f)) \subset \widetilde{Z}_\epsilon$ is of pure codimension 1, a general point $\widetilde{p}$ of any irreducible branch B of $\nu^{-1}(\mathrm{Graph}(f))$ is a smooth point of $\widetilde{Z}_\epsilon$. We may choose $\widetilde{p}$ to be also a smooth point of B such that $\nu|_B : B \to \mathrm{Graph}(f)$ is a local biholomorphism at $\widetilde{p}$. Write $p := \nu(\widetilde{p})$, $p = (z_0, f(z_0)) \in D_\epsilon \times \Omega$, and denote by $\pi_D : D \times \Omega \to D$ the canonical projection. Choose a neighborhood W of $\widetilde{p}$ in $\widetilde{Z}_\epsilon$ and a neighborhood U of z_0 in D_ϵ such that $\pi_D \circ \nu|_{W\cap B} : W \cap B \to U$ is a biholomorphism which extends to a biholomorphism $\sigma : W \to U \times \Delta$ when U is identified with $U \times \{0\}$. (A neighborhood is always understood to be connected.) Write $\nu(\sigma^{-1}(z,t)) = (h(z,t), g(z,t))$. Since h is a holomorphic submersion at $(z_0, 0)$, h remains a holomorphic submersion at (z,t) sufficiently close to $(z_0, 0)$, and without loss of generality we may choose W, U and ν such that $h(z,t) = z$. For $t \in \Delta$, write $f_t(z) = g(z,t)$. We have

$$K_\Omega(f_t(z), f(w)) = A \cdot K_D(z,w)^\lambda \tag{1}$$

such that $f_0(z) = f(z)$. Assume that $\frac{\partial^k}{\partial t^k} f_t(z)\big|_{t=0} \equiv 0$ for $k < \ell$ and $\eta(f(z)) := \frac{\partial^\ell}{\partial t^\ell} f_t(z)\big|_{t=0} \not\equiv 0$. Let $\big(h_j\big)_{j=0}^{\infty}$ be an orthonormal basis of $H^2(\Omega)$. We have

$$K_\Omega(f_t(z), f(w)) = \sum_j h_j(f_t(z))\overline{h_j(f(w))} = A \cdot K_D(z,w)^\lambda \tag{2}$$

for every t. Hence, differentiating both sides of (2) ℓ times against t and noting that the right hand-side is independent of t, we have

$$\frac{\partial^\ell}{\partial t^\ell} K_\Omega(f_t(z), f(w))\Big|_{t=0} \equiv 0 \ ; \quad \text{i.e.}\ ,$$
$$\sum_{i,j} \frac{\partial h_j}{\partial \zeta_i}\, \frac{\partial^\ell f_t^i}{\partial t^\ell}(z)\overline{h_j(f(w))} \equiv 0\ , \quad \text{i.e.}\ , \sum_j dh_j\big(\eta(f(z))\big)\overline{h_j(f(w))} \equiv 0\ . \tag{3}$$

Denote by $\mathbb{H}$ the separable Hilbert space of square-integrable sequences of complex numbers. Let $\Phi : \Omega \to \mathbb{H}$ be defined by

$$\Phi(\zeta) = \big(h_0(\zeta), \dots, h_j(\zeta), \dots\big)\ . \tag{4}$$

By the choice of ϵ, f is injective on D_ϵ, hence $f\big|_U : U \to \Omega$ is a holomorphic embedding onto a locally closed complex submanifold Σ. In terms of the Hermitian inner product $\langle \cdot, \bar{\cdot} \rangle$ on the Hilbert space $\mathbb{H}$, the identity (3) is given by

$$\big\langle d\Phi(\eta(f(z)))\ ,\ \overline{\Phi(f(w))}\big\rangle = 0\,, \tag{5}$$

where $\eta(f(z))$ is interpreted as a vector field along Σ and $d\Phi(\eta)$ as a vector field along $\Xi := \Phi(\Sigma) \subset \mathbb{H}$. In other words, we have a non-trivial holomorphic vector field along Ξ which is orthogonal to the linear span of $\Phi(f(w))$ as w ranges over D_ϵ. We may assume

$\eta(f(z_0)) \neq 0$. Let h_0 be chosen so that $|h(f(z_0))|$ attains its maximum value at $h = h_0$ among all $h \in H^2(\Omega)$ of unit norm. Choose $h_1 \perp h_0$ such that $\big|dh\big(\eta(f(z_0))\big)\big|$ attains its maximum value at $h = h_1$ among all $h \in H^2(\Omega)$ of unit norm and orthogonal to h_0. Then, for any h such that $h \perp h_0$ and $h \perp h_1$ we have $h(f(z_0)) = 0$ and $dh\big(\eta(f(z_0))\big) = 0$. Thus, completing (h_0, h_1) to any orthonormal basis $(h_0, h_1, h_2, \cdots, h_j, \cdots)$ of $H^2(\Omega)$, for all w in D_ϵ we derive from (3) that

$$dh_0\big(\eta(f(z_0))\big)\overline{h_0(f(w))} + dh_1\big(\eta(f(z_0))\big)\overline{h_1(f(w))} = 0 \ . \tag{6}$$

Substituting at $w = z_0$, $h_1(f(z_0)) = 0$ and $h_0(f(z_0)) \neq 0$ imply that $dh_0\big(\eta(f(z_0))\big) = 0$. Since $\big|dh\big((\eta(f(z_0))\big)\big|$ attains its maximum among $h \perp h_0$ of unit norm at $h = h_1$, we must have $dh_1\big(\eta(f(z_0))\big) \neq 0$, and it follows from (6) that $h_1(f(w)) = 0$. Writing $(x_0, x_1, \cdots, x_j, \cdots)$ for a point in $\mathbb{H}$ we conclude that $\Phi(f(U))$ lies in a hyperplane section which is the zero set of a continuous linear functional on $\mathbb{H}$, given by

$$\Phi(f(U)) \subset \{x_1 = 0\} \subset \mathbb{H} \ . \tag{7}$$

Note that the function $h_\alpha = h_{1,z_0}$ is defined on all of Ω. Consider all deformations $f_t(z) = g(t,z)$ on some domain $U \subset D_\epsilon$ defined as in the above, and denote by $\mathbf{A}$ the set of indices α for all functions h_α thus obtained. Define $\mathcal{E} := \big\{h_\alpha \in H^2(\Omega) : \alpha \in \mathbf{A}\big\}$ and denote by $E \subset \Omega$ the common zero set of all $h_\alpha \in \mathcal{E}$. Thus $\Phi(E) \subset \mathbb{H}$ is a (closed) linear section of $\Phi(\Omega)$ containing Ξ. Now consider the functional equations $(\mathbf{I}_w)$, $w \in D_\epsilon$, together with a restriction on the indeterminate ζ, given by

$$K_\Omega(\zeta, f(w)) = A \cdot K_D(z,w)^\lambda \ ; \quad \zeta \in E \ . \tag{8}$$

For the proof of Proposition 1.1.2 it remains to prove that

$$(\dagger) \qquad \dim_{(z,f(z))}\big(V \cap (\{z\} \times E)\big) = 0$$

for a general point $z \in D_\epsilon$. Suppose otherwise. Repeating the same argument as in the above, we obtain a holomorphic 1-parameter family $\{f_t\}_{t\in\Delta}, f_0 = f$, defined on some domain $U \subset D_\epsilon$ such that f_t takes values in E, thereby deriving the existence of a holomorphic vector field η along $\Sigma = f(U)$ and $h_1 \in \mathcal{E}$ such that $dh_1\big(\eta(f(z_0))\big) \neq 0$ for a general point $z_0 \in U$. By definition h_1 must vanish identically on E, hence $h_1(f_t(z)) = 0$ for $z \in U$ and for $t \in \Delta$. Now, differentiating the latter identity ℓ times against t we conclude that $dh_1\big(\eta(f(z_0))\big) = 0$, contradicting the choice of h_1. Thus, we have established (†) by contradiction, proving Proposition 1.1.2. ■

An example where the functional equations are not sufficiently non-degenerate

The following example shows that the situation where the system of holomorphic functional equations are not sufficiently 'non-degenerate' does occur. In other words, the example is one for which $\dim_{(z,f(z))}\big(V \cap (\{z\} \times \Omega)\big) \geq 1$. Let $N > n \geq 1$ be integers and consider the totally geodesic holomorphic isometric embedding $f : (B^n, \frac{N+1}{n+1} ds^2_{B^n}) \to (B^N, ds^2_{B^N})$ given by $f(z) = (z, 0)$ for $z = (z_1, \cdots, z_n)$. In this case the holomorphic functional equations relating Bergman kernels are given by

$$K_{B^N}(\zeta, f(w)) = A \cdot K_{B^n}(z,w)^{\frac{N+1}{n+1}} \ , \tag{1}$$

for some $A > 0$. Denoting by $\langle \cdot, \bar{\cdot} \rangle$ the Euclidean Hermitian inner product, we have $K_{B^m}(z,w) = c_m \left(1 - \langle z, \overline{w} \rangle\right)^{-(m+1)}$ for some constant $c_m > 0$. We have thus

$$\frac{c_N}{\big(1 - \langle \zeta, \overline{(w,0)} \rangle\big)^{N+1}} = A\bigg(\frac{c_n}{\big(1 - \langle z, \overline{w} \rangle\big)^{n+1}}\bigg)^{\frac{N+1}{n+1}} \ . \tag{2}$$

Substituting at $(z,w)=(0,0)$ and $\zeta=f(0)=0$ we have $c_N = A c_n^{\frac{N+1}{n+1}}$. For w sufficiently small, $z\in B^n$, the functional equation (2) on ζ is equivalent to

$$1-\langle \zeta, \overline{(w,0)}\rangle = 1-\langle z,\overline{w}\rangle \tag{3}$$

for ζ sufficiently close to $(z,0)$. Clearly $\zeta = f(z) = (z,0)$, which describes the image of the holomorphic isometry, satisfies the functional equations (3). However, when (z,w) is fixed and we put $\zeta=(z,z')$, where $z'\in\mathbb{C}^{N-n}$ is arbitrary, (3) remains satisfied. In fact, they give all possible simultaneous solutions to (3), and we have

$$V=\left\{(z,\zeta)\in B^n\times B^N : \zeta=(z,z'), z'\in\mathbb{C}^{N-n}\right\}\ , \tag{4}$$

hence $\dim_{(z,f(z))}\big(V\cap(\{z\}\times B^N)\big)=N-n\geq 1$. Infinitesimal variations η of simultaneous solutions $f_t(z)=\big(z,g_t(z)\big)$, $g_0(z)\equiv 0$, to (3) are of the form

$$\eta(f(z))=\eta(z,0)=\sum_{\ell=n+1}^{N} a_\ell(z)\frac{\partial}{\partial\zeta_\ell}\ , \tag{5}$$

where $a_\ell(z)$ are holomorphic functions in z defined on some nonempty open subset $U\subset B^n$. Here the fiber of the canonical projection $\pi: V\to B^n$ over a general point $z\in B^n$ can be cut down to an isolated point when we impose the conditions $\zeta_{n+1}=\cdots=\zeta_N=0$, which in fact corresponds to cutting B^N by zero sets of extremal functions maximizing the derivatives in the direction $\frac{\partial}{\partial\zeta_\ell}$, $n+1\leq\ell\leq N$, at $(z,0)\in B^N$.

In the proof of Proposition 1.1.2, in the case where $(\mathbf{I}_w), w\in D_\epsilon$, are not sufficiently non-degenerate, we have to consider extremal functions $h\in\mathcal{E}$. Since these functions will play a crucial role in extension problems in the rest of the article, we will prove now a number of basic properties on such functions. Recall the initial choice of $\epsilon_0>0$ as specified in the paragraph preceding Proposition 1.1.2. The set of extremal functions $\mathcal{E}\subset H^2(\Omega)$ depends on the choice of $\epsilon>0$, $0<\epsilon\leq\epsilon_0$. We will write $\mathcal{E}(\epsilon)$, $\mathbf{A}(\epsilon)$, $E(\epsilon)$ to indicate this dependence. and from now on write $\mathcal{E}=\mathcal{E}(\epsilon_0)$, $E=E(\epsilon_0)$ and regard f as being defined on D_{ϵ_0}.

Note that each $h\in\mathcal{E}(\epsilon)$ is of the form h_1 in the notation of the proof of Proposition 1.1.2. More precisely, given a holomorphic 1-parameter family $\{f_t\}_{t\in\Delta}$ defined on a domain $U\subset D_\epsilon$ obtained as a deformation of $f_0=f|_U$ of simultaneous solutions of the holomorphic functional equations $(\mathbf{I}_w)$, by differentiation we obtain a holomorphic vector field η defined along $\Sigma=f(U)\subset\Omega$, and, for each $z_0\in U$ we have an h_1 which is determined by η and by the choice of z_0. We write $h_1=h_{\eta,z_0}$. We are going to relate h_1 to the Bergman kernel K_Ω on Ω, thereby extending its domain of definition by means of properties of Bergman kernels on complete circular domains as given in Lemma 1.1.1.

Recall that $h_0\in H^2(\Omega)$ has been chosen such that, among all $h\in H^2(\Omega)$ of unit norm, the maximum of $|h(f(z_0))|$ is attained at $h=h_0$. Moreover, $h_1\in H^2(\Omega)$ has been chosen such that, among all $h\in H^2(\Omega)$ of unit norm and orthogonal to h_0, the maximum of $\big|dh\big(\eta(f(z_0))\big)\big|$ is attained at $h=h_1$. Both h_0 and $h_1=h_{\eta,z_0}$ are uniquely determined only up to a scalar constant of modulus 1. We have

Lemma 1.1.2. *The extremal function $h_1=h_{\eta,z_0}\in\mathcal{E}$ can be expressed in terms of the Bergman kernel K_Ω as*

$$h_1(\zeta)=\frac{\overline{\partial_{\eta(f(z_0))}K_\Omega(f(z_0),\zeta)}-\big(\overline{\partial_{\eta(f(z_0))}h_0}\big)h_0(\zeta)}{\overline{\partial_{\eta(f(z_0))}h_1}}\ .$$

Furthermore, if we choose the unique $h_1 = h_{\eta,z_0}$ such that $dh_1(\eta) \neq 0$ is (real and) positive, then, with the vector field η along $\Sigma \subset \Omega$ being fixed and $h_1 = h_{\eta,z}$ depending on the base point $z \in U \subset D_{\epsilon_0}$, $h_{\eta,z}(\zeta)$ varies real-analytically in (z,ζ).

Here in $\partial_{\eta(f(z_0))}K_\Omega(f(z_0),\xi)$, the notation $\eta(f(z_0))$ signifies the (1,0)-tangent vector $(\eta(f(z_0)),0)$ at $(f(z_0),\xi) \in \Omega \times \Omega$, and $\partial_{\eta(f(z_0))}h_0$ means $\partial_{\eta(f(z_0))}h_0(f(z_0))$, etc. We will call $h_1 = h_{\eta,z_0}$ a normalized extremal function to mean that $dh_1(\eta)$ is positive.

Proof of Lemma 1.1.2. Complete (h_0, h_1) to an orthonormal basis $(h_0, h_1, \cdots, h_j, \cdots)$ of $H^2(\Omega)$. From the expansion of K_Ω in terms of the chosen orthonormal basis, for $\zeta, \xi \in \Omega$,

$$K_\Omega(\zeta,\xi) = h_0(\zeta)\overline{h_0(\xi)} + h_1(\zeta)\overline{h_1(\xi)} + h_2(\zeta)\overline{h_2(\xi)} + \cdots \,. \tag{1}$$

Note that $K_\Omega(\xi,\zeta) = \overline{K_\Omega(\zeta,\xi)}$. Substituting in (1) at $\zeta = f(z_0)$ and using the fact that $h_j(f(z_0)) = 0$ whenever $j \geq 1$, we deduce

$$K_\Omega(f(z_0),\xi) = h_0(f(z_0))\overline{h_0(\xi)}, \quad \text{so that} \tag{2}$$

$$h_0(\xi) = \frac{K_\Omega(\xi, f(z_0))}{\overline{h_0(f(z_0))}}\ , \tag{3}$$

expressing h_0 in terms of f and K_Ω. Furthermore, differentiating both sides of (1) against $\eta(f(z_0))$ and using the fact that $dh_j\big((\eta(f(z_0))\big) = 0$ whenever $j \geq 2$ we have

$$\partial_{\eta(f(z_0))}K_\Omega(f(z_0),\xi) = \big(\partial_{\eta(f(z_0))}h_0\big)\overline{h_0(\xi)} + \big(\partial_{\eta(f(z_0))}h_1\big)\overline{h_1(\xi)}\ , \quad \text{so that} \tag{4}$$

$$h_1(\zeta) = \frac{\overline{\partial_{\eta(f(z_0))}K_\Omega(f(z_0),\zeta)} - \big(\overline{\partial_{\eta(f(z_0))}h_0}\big)h_0(\zeta)}{\overline{\partial_{\eta(f(z_0))}h_1}}\ , \tag{5}$$

where we replace ξ in (4) by ζ in the formula (5), proving the first half of Lemma 1.1.2.

For the proof of the last statement of Lemma 1.1.2, we may also fix the choice of h_0 by requiring $h_0(z)$ to be (real and) positive. By the formulas (3) and (5) it suffices to check that $h_0(f(z))$ (with a hidden dependence of h_0 on z) and $dh_1(\eta) = \partial_{\eta(f(z))}h_1$ both depend real-analytically on z. Now from $K_\Omega(f(z),f(z)) = |h_0(f(z))|^2$ (by (2)) and the normalization that $h_0(f(z))$ is positive it follows that $h_0(f(z)) = \sqrt{K_\Omega(f(z),f(z))}$ depends real-analytically on z. On the other hand from (1) by differentiation against η in the ζ variable and then against $\overline{\eta}$ in the ξ variable and evaluating at $(f(z),f(z))$ it follows that $|\partial_{\eta(f(z))}h_1|^2$ can be expressed as a real-analytic function in z, noting that $h_j(f(z)) = dh_j(f(z)) = 0$ whenever $j \geq 2$ so that $h_2, h_3, \cdots$ do not enter into the formula for $|\partial_{\eta(f(z))}h_1|^2$, and $\partial_{\eta(f(z))}h_1$ varies real-analytically in z by our normalization that $\partial_{\eta(f(z))}h_1$ is real and positive, proving Lemma 1.1.2. ■

For the tangent bundle $\pi : T_\Omega \to \Omega$ we denote by $T'_\Omega \subset T_\Omega$ the subset of non-zero tangent vectors. In general, for $\tau \in T'_\Omega$ we have the notion of an extremal function adapted to τ, meaning an element $h_\tau \in H^2(\Omega)$ of unit norm such that $dh(\tau)$ attains maximal modulus at $h = h_\tau$ among all $h \in H^2(\Omega)$ of unit norm satisfying $h(\pi(\tau)) = 0$. h_τ is unique up to multiplication by a scalar of unit modulus. As in the above we can fix h_τ by requiring that $dh_\tau(\tau)$ is real and positive, and we call $h_\tau \in H^2(\Omega)$ the normalized extremal function adapted to $\tau \in T'_\Omega$. For a real-analytic manifold X, we will say that a mapping $\mathfrak{B} : X \to H^2(\Omega)$ is separately real-analytic to mean that $\mathfrak{B}(x)(\zeta_0)$ is a real-analytic function in $x \in X$ for any $\zeta_0 \in \Omega$. Obviously the Identity Theorem holds true for $\mathfrak{B}$ in the sense that $\mathfrak{B} \equiv 0$ whenever $\mathfrak{B}$ vanishes on a non-empty open subset $U \subset X$. Denote by $\mathfrak{H} : T'_\Omega \to H^2(\Omega)$ the mapping defined by $\mathfrak{H}(\tau) = h_\tau$ and denote its image by $\mathfrak{X}(\Omega) \subset H^2(\Omega)$. From the formula on h_τ implicit in Lemma 1.1.2, the mapping $\mathfrak{h} : \Omega \times T'_\Omega \to \mathbb{C}$ defined by $\mathfrak{h}(\zeta,\tau) = h_\tau(\zeta)$ is holomorphic in ζ and real-analytic in (ζ,τ), thus $\mathfrak{H} : T'_\Omega \to H^2(\Omega)$ is separately real-analytic.

For the further study of extremal functions $h_\alpha, \alpha \in \mathbf{A}(\epsilon)$, and extension problems on their common zero sets, it is convenient to give a variation on the description of $E(\epsilon) \subset \Omega$, $0 < \epsilon \le \epsilon_0$. Recall that $\Omega_0 \subset \Omega$ is a Stein neighborhood of 0, and $Z_\epsilon \subset V \cap (D_\epsilon \times \Omega_0)$ denotes an irreducible subvariety containing Graph(f) and consisting of solutions (z,ζ) of functional equations $(\mathbf{I}_w), w \in D_\epsilon$, such that $\dim_{(z,f(z))}(Z_\epsilon \cap (\{z\} \times \Omega)) = 1$ for a general point $z \in D_\epsilon$. For the normalization $\nu : \widetilde{Z}_\epsilon \to Z_\epsilon$, extremal functions $h_\alpha, \alpha \in \mathbf{A}(\epsilon)$, were constructed using $\pi_D \circ \nu : \widetilde{Z}_\epsilon \to D_\epsilon$, where $\pi_D : D \times \Omega \to D$ and (later on) $\pi_\Omega : D \times \Omega \to \Omega$ denote the canonical projections. Write $\gamma = \pi_D \circ \nu$. Denote by $\mathcal{E}(Z_\epsilon) \subset \mathcal{E}(\epsilon)$ the subset of extremal functions thus obtained through Z_ϵ and by $E(Z_\epsilon) \subset \Omega$ their common zero set. Write $\Gamma(\epsilon)$ for the set of all such $\gamma : \widetilde{Z}_\epsilon \to D_\epsilon$ and denote by $[Z_\epsilon]$ the member in $\Gamma(\epsilon)$ corresponding to the latter map. Then, $\mathcal{E}(\epsilon) = \bigcup \big\{\mathcal{E}(Z_\epsilon) : [Z_\epsilon] \in \Gamma(\epsilon)\big\}$, and $E(\epsilon) = \bigcap \big\{E(Z_\epsilon) : [Z_\epsilon] \in \Gamma(\epsilon)\big\}$. The extraction of extremal functions $h_\alpha \in \mathcal{E}(Z_\epsilon)$ depends on the choice of one of the finitely many irreducible components B_j of $\nu^{-1}(\mathrm{Graph}(f|_{D_\epsilon}))$. We denote by $\mathcal{E}(Z_\epsilon, B_j) \subset \mathcal{E}(Z_\epsilon)$ those arising from B_j, and by $E(Z_\epsilon, B_j) \subset \Omega$ the set of common zeros of $\mathcal{E}(Z_\epsilon, B_j)$. Clearly $E(Z_\epsilon) = \bigcap_j E(Z_\epsilon, B_j)$. We are ready to prove

Lemma 1.1.3. *For $0 < \epsilon_2 \le \epsilon_1 \le \epsilon_0$ we have $E(\epsilon_2) \subset E(\epsilon_1)$. Moreover, supposing that $\Omega' \supset \Omega$ is a domain such that every $h \in \mathcal{E}(\epsilon_1) \cup \mathcal{E}(\epsilon_2)$ extends holomorphically to Ω' and denoting by $E'(\epsilon_i) \subset \Omega'$; $i = 1, 2$; the common zero set of the extended functions h' on Ω' of $h \in \mathcal{E}(\epsilon_i)$, we have $E'(\epsilon_2) \subset E'(\epsilon_1)$.*

Proof. We continue with some generalities on $E(Z_\epsilon, B)$, where $0 < \epsilon \le \epsilon_0$, and $B = B_j$ is one of the irreducible branches of $\nu^{-1}(\mathrm{Graph}(f|_{D_\epsilon}))$. Since B is a hypersurface in the normal complex space $\widetilde{Z}_\epsilon$, for some hypersurface $H \subset \widetilde{Z}_\epsilon$ such that $\mathrm{Sing}(\widetilde{Z}_\epsilon) \subset H$ and $B \not\subset H$, any $p \in B - H$ is a nonsingular point of B and $\gamma = \pi_D \circ \nu$ is a submersion at p. Denote by $T_{\widetilde{Z}_\epsilon}$ the tangent sheaf of $\widetilde{Z}_\epsilon$ and by $\mathcal{F} \subset T_{\widetilde{Z}_\epsilon}$ the relative tangent sheaf of $\gamma : \widetilde{Z}_\epsilon \to D_\epsilon$. Since B is Stein, there is $\mu \in \Gamma(B, \mathcal{F})$, $\mu \not\equiv 0$. Write $\varphi = \pi_\Omega \circ \nu$. In particular, for $p \in B - H$, the fiber $\mathfrak{F}_{\gamma(p)} := \gamma^{-1}(\gamma(p))$ of $\gamma : \widetilde{Z}_\epsilon \to D_\epsilon$ is smooth at $p \in \mathfrak{F}_{\gamma(p)}$. Suppose the restriction of $\varphi - f(\gamma(p))$ to $\mathfrak{F}_{\gamma(p)}$ vanishes exactly to the order $\ell - 1$ at a general point of $B - H$. Let X be a holomorphic vector field defined on some non-empty open set $V \subset \widetilde{Z}_\epsilon - H$ tangent to fibers $\mathfrak{F}_{\gamma(p)}$ such that $X|_{B\cap V} \equiv \mu|_{B\cap V}$. Since $\varphi - f(\gamma(p))$ vanishes on $\mathfrak{F}_{\gamma(p)}$ to the order $\ell - 1$ at p, $X^\ell\varphi(p)$ is independent of the choice of $X \in \Gamma(V, \mathcal{F})$ extending $\mu|_{B\cap V}$. Thus, there exists $\sigma \in \Gamma(B - H, \mathcal{O}^N)$ such that $\sigma|_{B\cap V} = X^\ell\varphi|_{B\cap V}$ for any such choices of V and $X \in \Gamma(V, \mathcal{F})$. Since $\mathcal{F}$ is of rank 1, for $\sigma' \in \Gamma(B - H, \mathcal{O}^N)$ arising from any non-trivial section $\mu' \in \Gamma(B, \mathcal{F})$, we must have $\mu' = \lambda\mu$ for some non-trivial meromorphic function λ on B, hence $\sigma' = \lambda^\ell\sigma$ on $B - H$.

For $p \in B - H$, and $\sigma \in \Gamma(B - H, \mathcal{O}^N)$ as in the above, $\sigma(p)$ can be interpreted as an element $\tau(p) \in T_{\varphi(p)}\Omega \cong \mathbb{C}^N$. We have thus a holomorphic map $\tau : B - H - \mathrm{Zero}(\sigma) \to T'_\Omega$, and hence a separately real-analytic map $\mathfrak{A} : B - H - \mathrm{Zero}(\sigma) \to \mathfrak{X}(\Omega) \subset H^2(\Omega)$ given by $\mathfrak{A}(p) = h_{\tau(p)} = \mathfrak{h}(\cdot, \tau(p))$. Let $E^\sigma \subset \Omega$ be the common zero set of the extremal functions $\big\{\mathfrak{A}(p) : p \in B - H - \mathrm{Zero}(\sigma)\big\}$. For $\sigma' = \lambda^\ell\sigma$ as in the last paragraph, denoting by $\mathfrak{A}' : B - H - \mathrm{Zero}(\sigma') \to \mathfrak{X}(\Omega)$ the analogue of $\mathfrak{A}$, the two extremal functions $\mathfrak{A}'(p)$ and $\mathfrak{A}(p)$ are non-zero multiples of each other for p belonging to the dense open subset $B - H - \mathrm{Zero}(\sigma) - \mathrm{Zero}(\sigma') \subset B$, hence *a priori* the two closed subsets $E^\sigma, E^{\sigma'} \subset \Omega$ are the same. In other words, E^σ depends only on the rank-1 coherent subsheaf $\mathcal{F} \subset T_{\widetilde{Z}_\epsilon}$.

Consider any holomorphic deformation $\{f_t\}_{t\in\Delta}$ over $U \subset D_\epsilon$ constructed from (Z_ϵ, B) as in the proof of Proposition 1.1.2. There exists by construction $W \subset B$ such that $\nu|_W : W \to U$ is a biholomorphism, so that, writing $\gamma(p) = z$ for $p \in W$, at a general point $z \in U$ we have $\eta(f(z)) = \lambda(z)\tau(p)$ for some $\lambda(z) \in \mathbb{C}^*$. By Lemma 1.1.2 and by the Identity Theorem for real-analytic functions, the common zero set of the extremal functions $h_{\eta,z}$, $z \in U$, agrees with E^σ. Hence, $E^\sigma = E(Z_\epsilon, B)$.

We proceed now to prove $E(\epsilon_2) \subset E(\epsilon_1)$ whenever $0 < \epsilon_2 \le \epsilon_1 \le \epsilon_0$. For $[Z_{\epsilon_1}] \in \Gamma(\epsilon_1)$, write $\big[Z_{\epsilon_1}|_{D_{\epsilon_2}}\big] \in \Gamma(\epsilon_2)$ for the member obtained by restricting $\gamma : \widetilde{Z}_{\epsilon_1} \to D_{\epsilon_1}$ to D_{ϵ_2}, i.e., $\gamma|_{\gamma^{-1}(D_{\epsilon_2})} : \gamma^{-1}(D_{\epsilon_2}) \to D_{\epsilon_2}$. Let B' be an irreducible branch of $\nu^{-1}(\mathrm{Graph}(f|_{D_{\epsilon_2}}))$, and B be that of $\nu^{-1}(\mathrm{Graph}(f|_{D_{\epsilon_1}}))$ containing B'. Taking $\sigma_1 \in \Gamma(B - H, \mathcal{O}^N)$ as in the above (replacing ϵ by ϵ_1 and hence σ by σ_1), we have $E(Z_{\epsilon_1}, B) = E^{\sigma_1}$ and $E(Z_{\epsilon_1}|_{D_{\epsilon_2}}, B') = E^{\sigma_2}$, where σ_2 is the restriction of σ_1 to $B' - H$. By Lemma 1.1.2 and the Identity Theorem we have $E^{\sigma_2} = E^{\sigma_1}$, hence $E(Z_{\epsilon_1}|_{D_{\epsilon_2}}, B') = E(Z_{\epsilon_1}, B)$. Finally, $E(\epsilon_2) \subset E(\epsilon_1)$ follows from $E(\epsilon) = \bigcap \big\{E(Z_\epsilon) : [Z_\epsilon] \in \Gamma(\epsilon)\big\}$; $E(Z_\epsilon) = \bigcap_j E(Z_\epsilon, B_j)$. Exactly the same argument gives the other statement in Lemma 1.1.3 when any $h \in \mathcal{E}(\epsilon_1) \cup \mathcal{E}(\epsilon_2)$ extends to $\Omega' \supset \Omega$, as desired. ■

We are now ready to prove

Theorem 1.1.1. *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded complete circular domains. Denote by ds^2_D,* resp. *ds^2_Ω, the Bergman metric on D,* resp. *Ω. Let λ be any positive real number and $f : \big(D, \lambda ds^2_D; 0\big) \to (\Omega, ds^2_\Omega; 0)$ be a germ of holomorphic isometry. Then, there exists an irreducible complex-analytic subvariety $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ of dimension n which contains the germ of* Graph(f) *at* $(0,0)$.

Proof. Choose $\alpha \gg 1$ such that $D \Subset \alpha D_{\epsilon_0} = B^n(0; \alpha\epsilon_0)$. Let now $\epsilon' > 0$ be such that $\alpha\epsilon' < \epsilon_0$. By Lemma 1.1.1, $K_D|_{D\times D_{\epsilon'}}$ extends holomorphically as a function in $(z, \overline{w})$ to $\alpha D_{\epsilon_0} \times D_{\epsilon'}$ when we define

$$K_D\big(\alpha z, w\big) := K_D\big(z, \alpha w\big) \tag{1}$$

for $w \in D_{\epsilon'}$. In particular, for each $w \in D_{\epsilon'}$, the function $K_{D,w}(z) = K_D(z, w)$ extends holomorphically from D to $B^n(0; \alpha\epsilon_0)$. Recall for $w \in D_{\epsilon'}$ we have the functional equation

$$(\mathbf{I}_w) \quad K_\Omega\big(\zeta, f(w)\big) = A \cdot K_D\big(z, w\big)^\lambda \; . \tag{2}$$

To proceed we make use of the proof of Proposition 1.1.2 and the notation adopted there.

<u>The case where the functional equations are sufficiently non-degenerate</u>
Consider first of all the case where $(\mathbf{I}_w), w \in D_\epsilon$, are sufficiently non-degenerate. On $D_{\epsilon_0} \times D_{\epsilon_0}$ the function $\log K_D(z, w)$ is well-defined and on the right-hand side of (2) the expression $K_D\big(z, w\big)^\lambda := e^{\lambda \log K_D(z,w)}$ is holomorphic in $(z, \overline{w})$, hence by (1) the same holds true for $(z, w) \in \alpha D_{\epsilon_0} \times D_{\epsilon'}$, noting that $D \Subset \alpha D_{\epsilon_0} = B^n(0; \alpha\epsilon_0)$. Recall that $V_w \subset D \times \Omega$ is the set of all $(z, \zeta) \in D \times \Omega$ satisfying $(\mathbf{I}_w)$, $w \in D_\epsilon$, and that $V = \bigcap\{V_w : w \in D_{\epsilon'}\}$ (noting that $0 < \epsilon' < \epsilon_0$). Recall that $0 < \delta_0 < 1$ and $f(D_{\epsilon_0}) \Subset \Omega_{\delta_0} \Subset \Omega$ (cf. first and second paragraphs after Lemma 1.1.1). Choose now $\beta \gg 1$ such that $\Omega \Subset \beta\Omega_{\delta_0} = B^N(0; \beta\delta_0)$ and let $\delta > 0$ be such that $\beta\delta < \delta_0$. Then, by Lemma 1.1.1, $K_\Omega(\zeta, \xi)$ is defined by extension for $\zeta \in \beta\Omega_{\delta_0}$ and $\xi \in \Omega_\delta$. Hence, for $w \in D_\epsilon$ the functional equation (2) is defined for $(z, \zeta) \in \alpha D_{\epsilon_0} \times \beta\Omega_{\delta_0}$. The set of all solutions $(z, \zeta) \in \alpha D_{\epsilon_0} \times \beta\Omega_{\delta_0}$ gives a subvariety $V' \subset \alpha D_{\epsilon_0} \times \beta\Omega_{\delta_0}$ such that $V' \cap (D \times \Omega) = V$.

Let $k \ge 1$ be any positive integer. The function $K_{D,w}(z)$ can be extended holomorphically from D to $B^n(0; k) = D_k$ whenever $|w| < k^{-1}\epsilon_0^2$ $(< \epsilon_0)$. Likewise, letting $\ell \ge 1$ be any positive integer, the function $K_{\Omega,\xi}(\zeta)$ can be extended holomorphically from Ω to $B^N(0; \ell) = \Omega_\ell$ whenever $|\xi| < \delta_\ell := \ell^{-1}\delta_0^2$ $(< \delta_0)$. By the continuity of f at 0, for each $\ell \ge 1$ there exists $k(\ell)$ such that $f(D_{\epsilon_\ell}) \subset \Omega_{\delta_\ell}$ for $\epsilon_\ell := k(\ell)^{-1}\epsilon_0^2$. We will choose $k(\ell)$ to be strictly increasing as $\ell \to \infty$, and, from the argument in the last paragraph we have irreducible subvarieties $V_\ell^\sharp \subset D_{k(\ell)} \times \Omega_\ell$ such that $V_\ell^\sharp \cap (D \times \Omega) = V$ for ℓ sufficiently large and such that for $\ell' > \ell \ge 1$ we must have $V_{\ell'}^\sharp \cap (D_{k(\ell)} \times \Omega_\ell) = V_\ell^\sharp$, by the Identity

Theorem on holomorphic functions. Since $k(\ell) \to \infty$ as $\ell \to \infty$, writing $V^\sharp := \bigcup_{\ell \geq 1} V^\sharp_\ell$, we have obtained a subvariety $V^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ such that $V^\sharp \cap (D \times \Omega) = V$ and such that $V^\sharp \cap (D_{k(\ell)} \times \Omega_\ell) = V^\sharp_\ell$ for each positive integer ℓ. When the system of functional equations $(\mathbf{I}_w)$, $w \in D_\epsilon$, is sufficiently non-degenerate, it suffices to take $S^\sharp$ to be the irreducible component of $V^\sharp$ containing Graph(f), so that $\dim(S^\sharp) = n = \dim(\text{Graph}(f))$, and $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ extends Graph(f) as a subvariety.

<u>The case where the functional equations are not sufficiently non-degenerate</u>
For $0 < \epsilon \leq \epsilon_0$ we define $\widehat{\mathcal{E}}(\epsilon) := \bigcup \big\{\mathcal{E}(\beta) : 0 < \beta \leq \epsilon\big\}$, and write $\widehat{E}(\epsilon) \subset \Omega$ for the common zero set of $\widehat{\mathcal{E}}(\epsilon)$. Thus, $\widehat{E}(\epsilon) = \bigcap \big\{E(\beta) : 0 < \beta \leq \epsilon\big\}$. Obviously, $\widehat{E}(\epsilon) \supset \widehat{E}(\epsilon_0)$ whenever $0 < \epsilon \leq \epsilon_0$. By Lemma 1.1.3, we have $E(\epsilon_2) \subset E(\epsilon_1)$ whenever $0 < \epsilon_2 \leq \epsilon_1 \leq \epsilon_0$, hence $\widehat{E}(\epsilon) \subset \widehat{E}(\epsilon_0)$. Thus $\widehat{E}(\epsilon) = \widehat{E}(\epsilon_0) := \widehat{E}$ whenever $0 < \epsilon \leq \epsilon_0$.

From Proposition 1.1.2, for $0 < \epsilon \leq \epsilon_0$ we have Graph$(f) \subset V \cap (D \times E(\epsilon))$, hence Graph$(f) \subset V \cap (D \times \widehat{E})$. Recall that there exists an increasing sequence $k(\ell), 1 \leq \ell < \infty$, of positive integers such that $f(D_{\epsilon_\ell}) \subset \Omega_{\delta_\ell}$ for $\epsilon_\ell := k(\ell)^{-1}\epsilon_0^2$ and $\delta_\ell = \ell^{-1}\delta_0^2$. By Lemma 1.1.2, any $h_\alpha \in \widehat{\mathcal{E}}(\epsilon_\ell)$ is definable on Ω_ℓ, with common zero set on Ω_ℓ to be denoted by $\widehat{E}^\sharp_\ell \subset \Omega_\ell$. By Lemma 1.1.3 (cf. last paragraph), $\bigcup_\ell \widehat{E}^\sharp_\ell := \widehat{E}^\sharp \subset \mathbb{C}^N$ is a subvariety such that $\widehat{E}^\sharp \cap \Omega = \widehat{E}$. Define now $T^\sharp := V^\sharp \cap (\mathbb{C}^n \times \widehat{E}^\sharp) \supset$ Graph(f). Then, the unique irreducible component $S^\sharp$ of $T^\sharp$ containing Graph(f) extends the latter as a subvariety, as desired. ■

(1.2) *Holomorphic isometric embeddings defined by extensions of germs of graphs*
Let $f : (D, \lambda ds_D^2; 0) \to (\Omega, ds_\Omega^2; 0)$ be a germ of holomorphic isometry at 0 between bounded complete circular domains, $f(0) = 0$, and $S \subset D \times \Omega$ be the extension of Graph(f) to $D \times \Omega$ as a complex-analytic subvariety. For the study of properties of S we will need the following well-known lemma resulting from the Cauchy-Schwarz inequality.

Lemma 1.2.1. *Let G be a bounded domain and denote by $K_G(z,w)$ its Bergman kernel. Then, for any $z, w \in G$ we have $|K_G(z,w)|^2 \leq K_G(z,z)K_G(w,w)$. Moreover, equality holds if and only if $z = w$.*

Proof. Let $(g_j)_{j=0}^\infty$ be an orthonormal basis of the Hilbert space $H^2(G)$ of square-integrable holomorphic functions on G. Then, $K_G(z,w) = \sum_{j=0}^\infty g_j(z)\overline{g_j(w)}$, and the inequality $|K_G(z,w)|^2 \leq K_G(z,z)K_G(w,w)$ results from the Cauchy-Schwarz inequality for the Hilbert space $\mathbb{H}$ of square-integrable sequences of complex numbers. Writing $\Psi(z) = (g_0(z), \cdots, g_j(z), \cdots)$, equality holds if and only if $\Psi(z) = \alpha\Psi(w)$ for some complex number α. From the reproducing property of $K_G(z,w)$ this is the case if and only if $g(z) = \alpha g(w)$ for any $g \in H^2(G)$, which obviously holds true if and only if $z = w$. ■

Under some mild conditions we have a sharpened result on interior extension.

Theorem 1.2.1. *Let D and Ω be bounded complete circular domains, λ be any positive real number, and $f : (D, \lambda ds_D^2; 0) \to (\Omega, ds_\Omega^2; 0)$ be a germ of holomorphic isometry. Then,* Graph$(f) \subset D \times \Omega$ *extends to a complex-analytic subvariety $S \subset D \times \Omega$ which is the graph of a holomorphic isometry $F : (D', \lambda ds_D^2\big|_{D'}) \to (\Omega, ds_\Omega^2)$ for some connected open subset $D' \subset D$ containing D_ϵ. Suppose $\varphi_\Omega(\zeta) := K_\Omega(\zeta,\zeta)$ is an exhaustion function on Ω, then $D = D'$ and $F : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$ is a holomorphic isometry. Suppose furthermore $\varphi_D(z) := K_D(z,z)$ is an exhaustion function on D. Then, F is proper.*

Proof of Theorem 1.2.1. By Theorem 1.1.1, Graph(f) extends analytically to an irreducible subvariety $S \subset D \times \Omega$. Let $\rho_D : S \to D, \rho_\Omega : S \to \Omega$ be the canonical projections. By definition the real-analytic identity (†) $\lambda \rho_D^*(ds_D^2) = \rho_\Omega^*(ds_\Omega^2)$ holds true on Graph(f), hence on $\mathrm{Reg}(S)$ by analytic continuation. We claim that for any $p \in D$, the fiber $\Phi_p := \rho_D^{-1}(p)$ is 0-dimensional. Suppose otherwise. Let Φ_p be a positive-dimensional fiber and $(p, q) \in \Phi_p$ be a smooth point belonging to an irreducible branch of positive dimension. Let $\eta = (\eta', \eta'')$ be a non-zero real vector tangent to Φ_p at (p, q). Then, $\eta' = 0, \eta'' \neq 0$. Thus, $\rho_D^*(ds_D^2)(\eta, \eta) = 0$ while $\rho_\Omega^*(ds_\Omega^2)(\eta, \eta) = ds_\Omega^2(\eta'', \eta'') > 0$. If $(p, q) \in \mathrm{Reg}(S)$, then we have reached a contradiction since (†) holds true on $\mathrm{Reg}(S)$. In general, let $\mathcal{I} \subset \mathcal{O}_{D\times\Omega}$ be the ideal sheaf of $S \subset D\times\Omega$, and let $\mathcal{F} \subset \mathcal{O}(T_{D\times\Omega}|_S)$ be the coherent sheaf on S whose stalk at $s \in S$ consists of all $\xi \in \mathcal{O}_s(T_{D\times\Omega})$ such that $\xi f = 0$ for every $f \in \mathcal{I}_s$. Then, there exists $\xi \in \mathcal{F}_{(p,q)}$ such that $\mathrm{Re}\xi(p, q) = \eta$. Thus, writing $\xi = (\xi', \xi'')$, by analytic continuation the germ of function $\lambda\rho_D^* ds_D^2(\mathrm{Re}\xi', \mathrm{Re}\xi') - \rho_\Omega^* ds_\Omega^2(\mathrm{Re}\xi'', \mathrm{Re}\xi'')$ vanishes at (p, q), which is a contradiction at (p, q) since $\mathrm{Re}\xi'(p, q) = \eta' = 0$ and $\mathrm{Re}\xi''(p, q) = \eta'' \neq 0$.

Denote by $B \subsetneq S$ the subvariety over which ρ_D fails to be a local biholomorphism. Then $S - B$ is locally the graph of a holomorphic isometry between open subsets of D and Ω with respect to restrictions of the Kähler metrics λds_D^2 and ds_Ω^2. Since $\rho_D : S \to D$ is a local biholomorphism at a general point and its fibers are 0-dimensional, it is an open map. We claim that $\rho_D : S \to D$ is injective. Suppose otherwise. By the openness of ρ_D, there exists $x \in D$ and 2 distinct points $y_1, y_2 \in \Omega$ such that $(x, y_1), (x, y_2) \in S - B$. Thus, there exist some simply connected neighborhoods U of x and W_1, resp. W_2, of (x, y_1), resp. (x, y_2), such that $\rho_D|_{W_1} : W_1 \cong U$ and $\rho_D|_{W_2} : W_2 \cong U$ are biholomorphisms. For $z \in U$ and $i = 1, 2$ we describe W_i as the graph of $f_i : U \to \Omega$, which is a holomorphic isometry with respect to $\lambda ds_D^2\big|_U$ and ds_Ω^2. Recall that $K_D(z, 0)$ is a positive constant C. By Lemma 1.1.1, shrinking $\epsilon_0 > 0$ if necessary we may assume that $\mathrm{Re}(K_D(z, w)) > 0$ for any $(z, w) \in D \times D_{\epsilon_0}$, so that $K_D(z, w)^\lambda$ is defined as a function holomorphic in $(z, \overline{w})$ for $(z, w) \in D \times D_{\epsilon_0}$. By Proposition 1.1.2 we have $K_\Omega(f(z), f(w)) - A \cdot K_D(z, w)^\lambda = 0$ for $z, w \in D_{\epsilon_0}$. Thus, by analytic continuation $K_\Omega(y, f(w)) - A \cdot K_D(x, w)^\lambda = 0$ holds true for $w \in D_{\epsilon_0}$ and for any $(x, y) \in S$. In particular, we have

$$K_\Omega(y_1, f(w)) = A \cdot K_D(x, w)^\lambda = K_\Omega(y_2, f(w)) \ . \tag{1}$$

Since $x \in U$ is arbitrary, we conclude that

$$K_\Omega(f_1(z), f(w)) = K_\Omega(f_2(z), f(w)) \tag{2}$$

for any $(z, w) \in U \times D_{\epsilon_0}$. Fix an arbitrary point $z \in U$. Consider $\psi : \Omega \to \mathbb{C}$ defined by $\psi(\xi) = K_\Omega(\xi, f_1(z)) - K_\Omega(\xi, f_2(z))$. Define furthermore $s : S \to \mathbb{C}$ by $s(x, y) = \psi(y)$ for $(x, y) \in S$. By (2) we have $s(w, f(w)) = 0$ whenever $w \in D_{\epsilon_0}$. From the irreducibility of S, we deduce by analytic continuation that $s \equiv 0$ on S. In particular, substituting $(x, y) = (z, f_i(z)) \in S - B; i = 1, 2;$ we conclude from $s(z, f_1(z)) = s(z, f_2(z)) = 0$ that

$$K_\Omega(f_1(z), f_1(z)) = K_\Omega(f_1(z), f_2(z))\,;\ K_\Omega(f_2(z), f_1(z)) = K_\Omega(f_2(z), f_2(z)) \tag{3}$$

for any $z \in U$. Thus, $K(f_1(z), f_2(z))$ is real and we have

$$K_\Omega(f_1(z), f_2(z)) = K_\Omega(f_1(z), f_1(z)) = K_\Omega(f_2(z), f_2(z))\,. \tag{4}$$

From Lemma 1.1.2 we have

$$\big|K_\Omega(f_1(z), f_2(z))\big|^2 \leq K_\Omega(f_1(z), f_1(z))K_\Omega(f_2(z), f_2(z)) \ . \tag{5}$$

and equality holds if and only if $f_1(z) = f_2(z)$. Thus, (4) implies that $f_1(z) = f_2(z)$ for $z \in U$, proving that each fiber of $\rho_D : S \to D$ consists of at most one point. Hence, S is the graph of some holomorphic map $F : D' \to \Omega$ defined on some neighborhood

$D' \subset D$ of 0 containing D_{ϵ_0}. To prove that F is injective let $z_1, z_2 \in D'$ be such that $F(z_1) = F(z_2)$. For $w \in D_{\epsilon_0}$,

$$K_D(z_1,w)^\lambda = A^{-1}K_\Omega(F(z_1),f(w)) = A^{-1}K_\Omega(F(z_2),f(w)) = K_D(z_2,w)^\lambda\,. \qquad (6)$$

Since $K_D(z,0) = A$ is positive, (6) implies that for some ϵ sufficiently small, $0 < \epsilon \le \epsilon_0$,

$$K_D(z_1,w) = K_D(z_2,w) \qquad (7)$$

whenever $w \in D_{\epsilon_0}$, hence for any $w \in D$ by the Identity Theorem. By the reproducing property of $K_D(z,w)$, $h(z_1) = h(z_2)$ for any $h \in H^2(D)$, hence $z_1 = z_2$, i.e., F is injective.

Assume now $\varphi_\Omega(\zeta) := K_\Omega(\zeta,\zeta)$ to be an exhaustion function. Suppose $D' \subsetneq D$ and let $p \in \partial D' \cap D$. From the functional equations $(\mathbf{I}_w)$, $w \in D_{\epsilon_0}$, we have $K_\Omega(F(z),F(z)) = A \cdot K_D(z,z)^\lambda$. Since $K_\Omega(\zeta,\zeta)$ is an exhaustion function in ζ, any limit point (p,q) of points $(z,F(z))$ as z approaches p must lie in $D \times \Omega$, i.e., $q \in \Omega$. Since $S \subset D \times \Omega$ is a subvariety, in particular closed, it follows that $(p,q) \in S$, so that S is the graph of some holomorphic map in a neighborhood of $(p,q) \in S$, so that $p \in D'$, a plain contradiction. We conclude that $D' = D$, i.e., $F : D \to \Omega$ is a global holomorphic isometry.

Finally, assume $\varphi_D(z) = K_D(z,z)$ to be an exhaustion function. Then, for any discrete sequence of points $\big(z_m\big)_{m=0}^{\infty}$ on D, $K_D(z_m,z_m)$ must diverge to ∞ as $n \to \infty$. Hence, $K_\Omega(F(z_m),F(z_m)) = A \cdot K_D(z_m,z_m)^\lambda$ must also diverge to ∞, implying that $\big(F(z_m)\big)_{m=0}^{\infty}$ is discrete. As a consequence, $F : D \to \Omega$ must be proper, as desired. ■

Remarks For bounded complete circular domains D_1 and D_2, a biholomorphism $\Phi : (D_1;0) \to (D_2,0)$ must be linear, by a result of H. Cartan's (cf. Mok [Mk2, Chap. 4, §2, Thm. 1]). Thus, the exhaustive property of $\varphi_D(z)$ is a property of $(D;0)$ independent of its realization as a bounded complete circular domain marked at 0.

From the proof of Theorem 1.2.1 we deduce

Corollary 1.2.1. *In the notation of the proof of Theorem 1.1.1, let $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ be the irreducible component of $V^\sharp \cap (\mathbb{C}^n \times \widehat{E}^\sharp)$ containing* Graph(f)*. Suppose the function $\varphi_\Omega = K_\Omega(\zeta,\zeta)$ is an exhaustion function on Ω. Then, $S^\sharp \cap (D \times \Omega)$ is irreducible. In other words, denoting by S the irreducible component of $V \cap (D \times \widehat{E})$ containing* Graph(f)*, we have $S^\sharp \cap (D \times \Omega) = S$.*

Proof. By Theorem 1.2.1, S is the graph of $F : (D, \lambda ds^2_D) \to (\Omega, ds^2_\Omega)$. Suppose over some non-empty open subset $U \subset D \times \Omega$ there are two branches of $S^\sharp \cap (D \times \Omega)$ described by $(z, f_1(z))$ and $(z, f_2(z))$, where $f_i : U \to \Omega$, $i = 1, 2$, are holomorphic maps. The argument of analytic continuation leading to the identities $K_\Omega(f_1(z), f_2(z)) = K_\Omega(f_1(z), f_1(z)) = K_\Omega(f_2(z), f_2(z))$ remains valid. To conclude it suffices to note that the argument using the Cauchy-Schwarz inequality on $\Phi(\Omega) \subset \mathbb{H}$, which gives $f_1(z) = f_2(z)$ once the identities are established, remains applicable since both $f_1(z)$ and $f_2(z)$ lie on Ω. ■

As will be seen in (3.2), there exist non-standard holomorphic isometric embeddings of the Poincaré disk into polydisks. In such an example Graph(f) extends to an affine-algebraic variety $S^\sharp$, but $S^\sharp$ is no longer the graph of a 'univalent' map.

(1.3) *Holomorphic isometric embeddings between bounded symmetric domains* In 2003, Clozel-Ullmo proved an extension theorem for germs of holomorphic isometries up to integral normalizing constants from the unit disk into the polydisk equipped with the Bergman metric, showing that any such a germ of map extends to a holomorphic isometric immersion on the unit disk and that moreover its graph extends to an affine-algebraic variety. This was a crucial step in the proof of the total geodesy of such germs of holomorphic isometries arising from some special algebraic correspondences in [CU]. (For a discussion on methods of analytic continuation in relation to [CU], cf. Mok [Mk5, (2.2) and §4]). For germs of holomorphic isometries between bounded symmetric domains in general, applications of Theorem 1.1.1 and Theorem 1.2.1 and their proofs yield

Theorem 1.3.1. *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded symmetric domains in their Harish-Chandra realizations. Let λ be any positive real number and $f : (D, \lambda ds_D^2; 0) \to (\Omega, ds_\Omega^2; 0)$ be a germ of holomorphic isometry at $0 \in D$, $f(0) = 0$. Then, the germ* $\mathrm{Graph}(f)$ *extends to an affine-algebraic subvariety $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ such that $S := S^\sharp \cap (D \times \Omega)$ is the graph of a proper holomorphic isometric embedding $F : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$.*

For the proof of Theorem 1.3.1 we will make use of specific forms of Bergman kernels on bounded symmetric domains as given by the following well-known lemma.

Lemma 1.3.1. *Let $G \Subset \mathbb{C}^m$ be an irreducible bounded symmetric domain in its Harish-Chandra realization, and denote by $K_G(z, w)$ its Bergman kernel. Then $K_G(z, w) = \frac{1}{Q_G(z.w)}$, where Q_G is a polynomial in $(z_1, \cdots, z_m; \overline{w_1}, \cdots, \overline{w_m})$ such that $Q_G(z, z) > 0$ on G and $Q_G(z, z) = 0$ for $z \in \partial G$.*

We have more precisely $Q_G(z, w) = h_G(z, w)^{p_G}$, where $h_G(z, w)$ is some polynomial in $(z_1, \cdots, z_m; \overline{w_1}, \cdots, \overline{w_m})$ and p_G is a positive integer depending on G. The polynomial $h_G(z, w)$ in $(z, \overline{w})$ is characterized by the property (†) to be specified below (cf. Faraut-Korányi [FK, pp.76-77]). Denote by r the rank of G as a bounded symmetric domain. The isotropy subgroup K of $\mathrm{Aut}_0(G)$ acts as a group of G-preserving unitary transformations on the Euclidean space $\mathbb{C}^m$. Using Harish-Chandra coordinates, for each maximal polydisk $P \cong \Delta^r$ on G passing through 0 there exists $\gamma \in K$ such that $\gamma(P)$ is the unit polydisk $\Pi = \Delta^r \times \{0\}$. Each $z \in G$ is contained in a maximal polydisk $P \subset G$, hence there exists $\gamma \in K$ such that $\gamma(z) = (a_1, \cdots, a_r; 0) \in \Pi$. For some positive constant α_G the polynomial $h_G(z, w)$ in $(z, \overline{w})$ is characterized by the property (†) $h_G(z, z) = \alpha_G(1 - |a_1|^2) \times \cdots \times (1 - |a_r|^2)$. As examples, in the case of type-I domains $D^I_{p,q}$ in the complex Euclidean space $M(p, q)$ of p-by-q matrices with complex entries defined by $D^I_{p,q} := \{Z \in M(p.q) : I - \overline{Z}^t Z > 0\}$, the Bergman kernel is given by $K_{D^I_{p,q}}(Z, W) = \alpha_{p,q} \cdot \det(I - \overline{W}^t Z)^{-(p+q)}$ for some positive constant $\alpha_{p,q}$ (cf. Mok [Mk2, Chap. 4, p.80ff.] for this and other classical domains).

Proof of Theorem 1.3.1. Recall the functional equations $(\mathbf{I}_w), w \in D_\epsilon$, in Proposition 1.1.2, arising from a germ of holomorphic isometry $f : (D, \lambda ds_D^2; 0) \to (\Omega, ds_\Omega^2; 0)$, where f is assumed to be defined on $D_{\epsilon_0} = B^n(0; \epsilon_0) \Subset \mathbb{C}^n$. It may happen *a priori* that the the normalizing constant λ is irrational (cf. Proposition 3.1.2.) The functions $\varphi_D(z) = K_D(z, z)$ and $\varphi_\Omega(\zeta) = K_\Omega(\zeta, \zeta)$ are by Lemma 1.3.1 exhaustion functions. Thus, by Theorem 1.2.1, f extends to a proper holomorphic map $F : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$ such that $\mathrm{Graph}(F) \subset D \times \Omega$ extends to a complex-analytic subvariety $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$. By the fine structure of the boundary of bounded symmetric domains in their Harish-Chandra realizations (cf. Wolf [Wo]), there is a decomposition of ∂D into a finite union of orbits under $\mathrm{Aut}_0(D)$. The set of regular points $\mathrm{Reg}(\partial D)$ of ∂D is a locally closed real-analytic submanifold of $\mathbb{C}^n$ which is dense in ∂D. The preceding discussion holds analogously for the bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ in its Harish-Chandra realization.

We claim that λ must be a rational number. Since $\mathrm{Graph}(F)$ extends to a subvariety $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$, for a general point $b \in \mathrm{Reg}(\partial D)$, there is a neighborhood U_b of b in $\mathbb{C}^n$ and a holomorphic map $F^\flat : U_b \to \mathbb{C}^N$ such that $F^\flat\big|_{U_b \cap D}$ agrees with $F\big|_{U_b \cap D}$. We have

$$K_\Omega(F^\flat(z), F^\flat(z)) = A \cdot K_D(z, z)^\lambda \tag{1}$$

for $z, w \in U_b \cap D$. By Lemma 1.3.1 we have

$$A \cdot Q_\Omega(F^\flat(z), F^\flat(z)) = Q_D(z,z)^\lambda \qquad (2)$$

for $z \in U_b \cap D$. Write $\rho_D(z) = -h_D(z,z)$ on $\mathbb{C}^n$ and $\rho_\Omega(\zeta) = -h_\Omega(\zeta,\zeta)$ on $\mathbb{C}^N$. On U_b the function $\sigma(z) = \rho_\Omega(F^\flat(z))$ is real-analytic. We have $\sigma < 0$ on $U_b \cap D$ and $\sigma = 0$ on $U_b \cap \partial D$. ρ_D, resp. ρ_Ω vanishes to the order 1 along $\mathrm{Reg}(\partial D)$ resp. $\mathrm{Reg}(\partial\Omega)$. Letting $\ell \geq 1$ be the vanishing order of σ along $U_b \cap \partial\Omega$, by equating vanishing orders on both sides of (2) we conclude that $\ell p_\Omega = \lambda p_D$, hence $\lambda = \frac{\ell p_\Omega}{p_D}$ is a rational number, as claimed.

Write now $\lambda = \frac{p}{q}$, where p and q are positive integers. We adopt the notation in the proof of Theorem 1.1.1. There we have a subvariety $V^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$, a subvariety $\widehat{E}^\sharp \subset \mathbb{C}^N$ such that $T^\sharp = V^\sharp \cap (\mathbb{C}^n \times \widehat{E}^\sharp)$ contains $\mathrm{Graph}(f)$, $T^\sharp$ is irreducible and of dimension n at a general point of $\mathrm{Graph}(f)$, and $S^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ is the unique irreducible component of $T^\sharp$ containing $\mathrm{Graph}(f)$. In the current situation where $\lambda = \frac{p}{q}$ is rational, let $W^\sharp$ be the set of common solutions (z,ζ) on $\mathbb{C}^n \times \mathbb{C}^N$ to the equations $K_\Omega(\zeta, f(w))^q = A \cdot K_D(z,w)^p$ as w ranges over some $D_{\epsilon_0} = B^n(0;\epsilon_0)$. Then, $V^\sharp \subset W^\sharp$ and the germs of $V^\sharp$ and $W^\sharp$ at $(0,0)$ agree with each other. By Lemma 1.3.1, the functions $K_{D,w}(z) = K_D(z,w)$ and $K_{\Omega,\xi} = K_\Omega(\zeta,\xi)$ are rational functions, hence $W^\sharp \subset \mathbb{C}^n \times \mathbb{C}^N$ is affine-algebraic. The subvariety $\widehat{E} = \widehat{E}(\epsilon_0) \subset \Omega$ is defined by extremal functions $\{h_\alpha\}_{\alpha \in \mathbf{A}(\beta)}$, $0 < \beta \leq \epsilon_0$, and $\widehat{E} = \widehat{E}^\sharp \cap \Omega$. By the formula in Lemma 1.1.2 expressing $h_\alpha = h_{\eta,z_0}$ in terms of K_Ω, it follows that each h_α is a rational function. Thus $\widehat{E} = H \cap \Omega$ for some affine-algebraic variety $H \subset \mathbb{C}^N$. Finally, $S^\sharp$ is equivalently the irreducible component of $W^\sharp \cap (\mathbb{C}^n \times \widehat{H})$ containing $\mathrm{Graph}(f)$, hence also affine-algebraic, as desired. ■

When D is the unit disk Δ, and $F : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ is a holomorphic isometry, by Theorem 1.3.1, F is a proper holomorphic isometric embedding, and $S := \mathrm{Graph}(F)$ extends as a subvariety to an affine-algebraic subvariety $S^\sharp \subset \mathbb{C} \times \mathbb{C}^N$. It follows in particular that $F : \Delta \to \Omega$ extends to a continuous mapping $F^\flat : \overline{\Delta} \to \overline{\Omega}$. For a general point $b \in \partial\Delta$, there is a neighborhood U_b of b on $\mathbb{C}$ such that $F|_{U_b \cap \Delta}$ extends holomorphically to U_b. When the latter fails to be the case, b will be called a singular point of F, and we will say that b lies over the branched point $F^\flat(b) \in \partial\Omega$.

Germs of holomorphic isometries up to normalizing constants between bounded symmetric domains equipped with the Bergman metric may fail to be totally geodesic (cf. (3.2) and (3.3)). In view of such examples we pose the question of finding conditions under which germs of holomorphic isometries are necessarily totally geodesic. In the case where the domain is irreducible and of rank ≥ 2, as observed by Clozel-Ullmo [CU], total geodesy follows from the proof of Hermitian metric rigidity of Mok [Mk1,2]. Mok ([Mk3], 2002) proved an analogue on algebraic extension for germs of holomorphic isometries up to *integral* normalizing constants from an n-ball to a product of n-balls under a certain non-degeneracy assumption, showing in the case of $n \geq 2$ that any such map must necessarily be totally geodesic by applying Alexander's Theorem in [Al]. Using Theorem 1.3.1, the latter result can be improved by removing the non-degeneracy assumption and by allowing the normalizing constant λ to be *a priori* any positive real number. Regarding the characterization of totally geodesic maps among holomorphic isometries we have now

Theorem 1.3.2. *Let $D \Subset \mathbb{C}^n$, $\Omega \Subset \mathbb{C}^N$ be bounded symmetric domains, $\lambda > 0$, and $f : (D, \lambda ds^2_D; 0) \to (\Omega, ds^2_\Omega; 0)$ be a germ of holomorphic isometry. Then, f extends to a totally geodesic holomorphic embedding $F : (D, \lambda ds^2_D) \to (\Omega, ds^2_\Omega)$*

(a) *whenever each irreducible component of D is of rank ≥ 2;*

(b) *whenever D is of rank 1 and dimension ≥ 2, i.e., $D \cong B^n, n \geq 2$, and Ω is a Cartesian product of copies of B^n.*

Proof. (a) The zeros of holomorphic bisectional curvature are preserved by a holomorphic isometry. Thus, whenever $R^D_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0$, we have $R^\Omega_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0$ and $\|\sigma_{\alpha\zeta}\|^2 = R^\Omega_{\alpha\overline{\alpha}\zeta\overline{\zeta}} - R^D_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0$. When D is irreducible and of rank ≥ 2 the partial vanishing $\sigma_{\alpha\zeta} = 0$ is enough to imply $\sigma \equiv 0$, by Mok [Mk1, proof of Corollary to Theorem 3', p.138ff.], cf. also Clozel-Ullmo [CU, §3]. Assume now that D is reducible, $D = D_1 \times \cdots \times D_k$, $k \geq 2$, and each irreducible component D_i, $1 \leq i \leq k$, is of rank ≥ 2. Fix $x \in D$. For $\eta_i, \eta_i' \in T_x(D)$ tangent to the i-th direct factor we have $\sigma_{\eta_i\eta_i'} = 0$. On the other hand, if $\eta_j \in T_x(D)$ is tangent to the j-th direct factor and $i \neq j$, then $R^D_{\eta_i\overline{\eta_i}\eta_j\overline{\eta_j}} = 0$, and we conclude by $\|\sigma_{\eta_i\eta_j}\|^2 = R^\Omega_{\eta_i\overline{\eta_i}\eta_j\overline{\eta_j}} - R^D_{\eta_i\overline{\eta_i}\eta_j\overline{\eta_j}} = 0$ that $\sigma_{\eta_i\eta_j} = 0$. From $\sigma_{\eta_i\eta_i'} = \sigma_{\eta_i\eta_j} = 0$ we conclude that $\sigma \equiv 0$ on D, proving that $f : (D, \lambda ds_D^2; 0) \to (\Omega, ds_\Omega^2; 0)$ is totally geodesic.

(b) The statement for the germ of map $f : (B^n, \lambda ds^2_{B^n}; 0) \to \big((B^n)^p, ds^2_{(B^n)^p}; 0\big)$ was established in Mok [Mk3] under the assumptions that (i) the normalizing constant λ is a positive integer, and that (ii) writing $f = (f_1, \cdots f_p)$, $f_i : B^n \to B^n$, for $1 \leq i \leq p$, each f_i is of maximal rank at some point. When the normalizing constant $\lambda > 0$ is an arbitrary positive real number, results of the current article apply. In fact, by Theorem 1.3.1, Graph(f) extends as an affine-algebraic variety. The final argument in [Mk3] using Alexander's Theorem remains valid to show that f is totally geodesic, as follows. The functional identities as in Proposition 1.1.1 apply and we have especially the identity

$$\prod_{i=1}^{p} \big(1 - \|f_i\|^2\big) = \big(1 - \|z\|^2\big)^\lambda \tag{1}$$

analogous to Mok [Mk3, proof of Theorem (3.1)]. Pick $b \in \partial B^n$ where f extends holomorphically to a neighborhood U_b of b in $\mathbb{C}^n$. From (1) one of the factors $1 - \|f_i\|^2, 1 \leq i \leq p$, must vanish on ∂B^n. We may take $i = p$. Since $n \geq 2$ and f_p is obviously nonconstant, Alexander's Theorem (stated below) applies to force f_p to extend to a biholomorphism $F_p : B^n \to B^n$. Since $f_p(0) = 0$ we must have $\|f_p(z)\| = \|z\|$, hence by (1) we have $\prod_{i=1}^{p-1} \big(1 - \|f_i\|^2\big) = \big(1 - \|z\|^2\big)^{\lambda - 1}$, and (b) follows by induction, as desired. ∎

Theorem (Alexander [Al]). *Let $B^n \Subset \mathbb{C}^n$ be the complex unit ball of dimension $n \geq 2$. Let $b \in \partial B^n$, U_b be a connected open neighborhood of b in $\mathbb{C}^n$, and $f : U_b \to \mathbb{C}^n$ be a nonconstant holomorphic map such that $f(U_b \cap \partial B^n) \subset \partial B^n$. Then, there exists an automorphism $F : B^n \to B^n$ such that $F|_{U_b \cap B^n} \equiv f|_{U_b \cap B^n}$.*

§2 Generalizations of extension results for bounded domains and for complex manifolds

(2.1) *Extension of germs of holomorphic isometries for bounded domains* We have considered the extension problem for bounded complete circular domains on germs of holomorphic isometries f at 0, $f(0) = 0$. Here we generalize the results to holomorphic isometries $f : (D; \lambda ds_D^2; x_0) \to (\Omega, ds_\Omega^2; f(x_0))$ between arbitrary bounded domains.

Theorem 2.1.1. *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains. Let $x_0 \in D$, λ be a positive real number, and $f : (D, \lambda ds_D^2; x_0) \to (\Omega, ds_\Omega^2; f(x_0))$ be a germ of holomorphic isometry. Then, the germ of complex-analytic subvariety* Graph(f) *at $(x_0, f(x_0))$ extends to an irreducible complex-analytic subvariety $S \subset D \times \Omega$ which is the graph of a holomorphic isometric embedding $F : \big(D', \lambda ds_D^2\big|_{D'}\big) \to (\Omega, ds_\Omega^2)$ defined on some neighborhood D' of x_0 in D. If (Ω, ds_Ω^2) is complete as a Kähler manifold, then $D' = D$, so that the germ of holomorphic isometric immersion f extends to a holomorphic isometric embedding $F : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$.*

In what follows for $\epsilon, \delta > 0$ sufficiently small we will write $D_\epsilon := B^n(x_0;\epsilon) \Subset D$, $\Omega_\delta := B^N(x_0;\delta) \Subset \Omega$. The germ of holomorphic map $f : (D; x_0) \to (\Omega; f(x_0))$ will be taken to be defined on some $D_{\epsilon_0}, \epsilon_0 > 0$ being sufficiently small and fixed.

Proof of Theorem 2.1.1. With some minor differences Theorem 1.2.1 deals with the special case where $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are complete circular domains, $x_0 = 0$, and $f(x_0) = 0$. In the proof there we made use of the circle group action. With reference to the proof given there and in the notation used there, we examine what is needed on the coordinates (z_i) and (ζ_j) for the proof to work. We have

$$\log K_\Omega(f(z), f(z)) = \lambda \log K_D(z,z) + \mathrm{Re}(\psi) \ , \tag{1}$$

where ψ is a holomorphic function on D_{ϵ_0}. The pluriharmonic function $\mathrm{Re}(\psi)$ is shown to be a constant by the observations that for $|I| \neq |J|$, (a) the coefficient of $z^I\overline{z^J}$ in $\log K_D(z,z)$ is always 0; (b) the coefficient of $\zeta^I\overline{\zeta^J}$ in $\log K_\Omega(\zeta,\zeta)$ is always 0. By (b), substituting $\zeta = f(z)$, with $f(0) = 0$, we conclude that the coefficient of z^I (and hence of $\overline{z^I}$) in $\log K_\Omega(f(z), f(z))$ is always 0 whenever $I = (i_1, \cdots, i_n)$ is non-zero. Using (a) and (b) and comparing the two sides of (1) it follows that ψ must be a constant.

The observations (a) and (b) hold true because of the invariance of the Bergman kernels under the circle group action at 0. But, in order to conclude that ψ is a constant, it is sufficient that whenever $I = (i_1, \cdots, i_n)$ is non-zero, (a') the coefficient of $(z - x_0)^I$ in $\log K_D(z,z)$ is always 0; (b') the coefficient of $(\zeta - f(x_0))^I$ in $\log K_\Omega(\zeta,\zeta)$ is always 0. Such coordinates do not always exist. However, in place of using $\log K_D(z,z)$, resp. $\log K_\Omega(\zeta,\zeta)$, we can first remove pluriharmonic functions from the potential functions before comparing the two sides in the functional equations. For (a') and (b') to hold true it suffices that we choose a potential function at x_0 for the Bergman metric which is a convergent sum of $\pm|\theta|^2$ for a countable number of holomorphic functions θ on D vanishing at x_0, and an analogous potential function at $y_0 := f(x_0)$. For this purpose let $(s_0, s_1, \cdots, s_i, \cdots)$ be an orthonormal basis of $H^2(D)$ adapted to x_0 so that $s_i(x_0) = 0$ for $i \geq 1$. Then, the Bergman kernel K_D is given by $K_D(z,z) = |s_0|^2 K'_D(z,z)$, where $K'_D(z,z) = 1 + \sum_{i\geq 1} \left|\frac{s_i}{s_0}\right|^2$. Expanding in power series on some neighborhood of x_0, the function $\log K'_D(z,z)$ is the convergent sum of a countable number of functions of the form $\pm|\theta_k|^2$, where each θ_k is a holomorphic function vanishing at x_0. Choose now analogously an orthonormal basis $(r_0, r_1, \cdots, r_j, \cdots)$ of $H^2(\Omega)$ adapted to y_0 so that $r_j(y_0) = 0$ for every $j \geq 1$, and write in a similar way $K_\Omega(\zeta,\zeta) = |r_0|^2 K'_\Omega(\zeta,\zeta)$. Again, on some neighborhood of y_0 the function $\log K'_\Omega(\zeta,\zeta)$ is the convergent sum of a countable number of functions of the form $\pm|\chi_\ell|^2$, where each χ_ℓ is a holomorphic function on Ω vanishing at y_0. Noting that $\log|s_0|^2$, resp. $\log|r_0|^2$, is a pluriharmonic function on a neighborhood of x_0, resp. y_0, the hypothesis that $f : (D, x_0) \to (\Omega, y_0)$ is a holomorphic isometry up to a normalizing constant gives rise to

$$\begin{aligned} \sqrt{-1}\partial\overline{\partial} \log K_\Omega(f(z), f(z)) &= \lambda \sqrt{-1}\partial\overline{\partial} \log K_D(z,z) \ ; \\ \log K'_\Omega(f(z), f(z)) &= \lambda \log K'_D(z,z) + \mathrm{Re}(\psi') \ , \end{aligned} \tag{2}$$

where ψ' is a germ of holomorphic function at x_0. Thus, we have $\log K'_\Omega(f(z), f(z)) = \sum_\ell \pm|(\chi_\ell \circ f)(z)|^2$, where $(\chi_\ell \circ f)(x_0) = \chi_\ell(f(x_0)) = \chi_\ell(y_0) = 0$. Expanding in power series at x_0 and observing that $2\mathrm{Re}(\psi') = \psi' + \overline{\psi'}$ is a sum of terms of pure type, it follows that in fact the pluriharmonic function $\mathrm{Re}(\psi')$ vanishes identically, giving

$$\log K'_\Omega(f(z), f(z)) = \lambda \log K'_D(z,z) \ . \tag{3}$$

From $K'_D(z,z)$ we define the function $K'_D(z,w)$ holomorphic in z and anti-holomorphic in w such that one recovers the original definition by restricting to $z = w$. The same applies to $K'_\Omega(\zeta,\xi)$. Writing the extremal functions $s_0 \in H^2(D)$ and $r_0 \in H^2(\Omega)$ as

$$s_0(z) = \frac{K_D(z,x_0)}{\sqrt{K_D(x_0,x_0)}} \ , \quad r_0(\zeta) = \frac{K_\Omega(\zeta,y_0)}{\sqrt{K_\Omega(y_0,y_0)}} \ ; \tag{4}$$

from $K'_D(z,w) = \frac{K_D(z,w)}{s_0(z)\overline{s_0(w)}}$ and $K'_\Omega(\zeta,\xi) = \frac{K_\Omega(\zeta,\xi)}{r_0(\zeta)\overline{r_0(\xi)}}$ we have

$$K'_D(z,w) = \frac{K_D(z,w)K_D(x_0,x_0)}{K_D(z,x_0)K_D(x_0,w)} \ ; \quad K'_\Omega(\zeta,\xi) = \frac{K_\Omega(\zeta,\xi)K_\Omega(y_0,y_0)}{K_\Omega(\zeta,y_0)K_\Omega(y_0,\xi)} \ . \tag{5}$$

Observe from (5) that

$$K'_D(z,x_0) = 1 \ ; \ K'_\Omega(\zeta,y_0) = 1 \, . \tag{6}$$

Let $(h_j)_{j=0}^\infty$ be an orthonormal basis of $H^2(\Omega)$ and write $h'_j = \frac{h_j}{r_0}$. Define $\Phi : \Omega \to \mathbb{H}$ by

$$\Phi(\zeta) = \big(h_0(\zeta), \dots, h_j(\zeta), \dots\big) \ . \tag{7}$$

We also write

$$\Phi'(\zeta) = \big(h'_0(\zeta), \dots, h'_j(\zeta), \dots\big) = \frac{\Phi(\zeta)}{r_0(\zeta)} \ . \tag{8}$$

Each component h'_j of Φ' is meromorphic on Ω and may in general have poles. However, since $r_0(y_0) \neq 0$, without loss of generality we will assume that $f(D_{\epsilon_0}) \subset \Omega_{\delta_0}$ where r_0 has no zeros on Ω_{δ_0}, so that $\Phi' \circ f$ is holomorphic on D_{ϵ_0}. We are going to prove the extendibility of Graph(f) to $S \subset D \times \Omega$ as a complex-analytic subvariety by imposing first of all the following simplifying assumption on the Bergman kernel $K_D(z,w)$.

($\sharp$) The holomorphic function $K_D(z,x_0)$ in z does not have any zero on D.

Assuming ($\sharp$), the function $K'_D(z,w)$ is holomorphic in $(z,\overline{w})$ on $D \times D$. Let $G \Subset D$ be an open neighborhood of $\overline{D_{\epsilon_0}}$. Since $K'_D(z,x_0) \equiv 1$ by (6), from the continuity of $K'_D(z,w)$, for some ϵ satisfying $0 < \epsilon \le \epsilon_0$ we must have $\mathrm{Re}(K'_D(z,w)) > 0$ whenever $(z,w) \in G \times D_\epsilon$. Thus, for $(z,w) \in G \times D_\epsilon$, the function $\log K'_D(z,w)$ is well-defined and holomorphic in $(z,\overline{w})$ for the principal branch log of the natural logarithm, so that $\big(K'_D(z,w)\big)^\lambda = \exp(\lambda \log K'_D(z,w))$ is defined and holomorphic in $(z,\overline{w})$ over there. Consider

$$(\mathbf{I}_w) \quad K'_\Omega(\zeta, f(w)) = \big(K'_D(z,w)\big)^\lambda \ , \ w \in D_\epsilon \ ; \tag{9}$$

restricted to$(z,\zeta) \in G \times \Omega$ and denote by $V_G \subset G \times \Omega$ the set of common solutions to $(\mathbf{I}_w), w \in D_\epsilon$. By polarizing (3) and exponentiating, it follows that $(\mathbf{I}_w)$ is satisfied by $\zeta = f(z)$ for $w \in D_\epsilon$. Suppose connected open subsets G and G' are chosen such that $D_\epsilon \Subset G \Subset G' \Subset D$ and ϵ, ϵ' are chosen such that $0 < \epsilon' < \epsilon \le \epsilon_0$ and such that $\mathrm{Re}(K'_D(z,w)) > 0$ whenever $(z,w) \in G \times D_\epsilon$ or $(z,w) \in G' \times D_{\epsilon'}$. Then, $V_{G'} \cap (G \times \Omega) = V_G$ by the Identity Theorem for holomorphic functions. Choose a sequence $(G_k)_{k=1}^\infty$ of connected open subsets of D such that $D_\epsilon \Subset \cdots \Subset G_k \Subset G_{k+1} \Subset \cdots \Subset D$ and such that $\bigcup_{k\ge1} G_k = D$, and a corresponding strictly decreasing sequence of positive numbers $(\epsilon_k)_{k=1}^\infty$ converging to 0 such that $\mathrm{Re}(K'_D(z,w)) > 0$ whenever $(z,w) \in G_k \times D_{\epsilon_k}$ for some integer $k \ge 1$. Then, the union $V = \bigcup_{k\ge1} V_{G_k}$ gives a subvariety $V \subset D \times \Omega$.

Let $U \subset D_{\epsilon_0}$, and $\{f_t(z)\}$ for $t \in \Delta$ and $z \in U$, be a holomorphic 1-parameter family of solutions to the functional equations $(\mathbf{I}_w), w \in D_\epsilon$, given by

$$K'_\Omega(f_t(z), f(w)) = \big(K'_D(z,w)\big)^\lambda \ , \ w \in D_\epsilon \ . \qquad (10)$$

as w ranges over D_ϵ. Write $\Sigma := f(U) \subset \Omega$ and $\Xi' := \Phi'(\Sigma) \subset \mathbb{H}$. Again, let ℓ be the first positive integer such that $\frac{\partial^\ell}{\partial t^\ell} f_t(z)\big|_{t=0}$ is not identically zero on U. Then, as in the proof of Proposition 1.1.2, differentiating the identities (10) against t exactly ℓ times and evaluating at $t = 0$ we obtain a holomorphic vector field $\eta(f(z))$ on Σ, and corresponding a holomorphic vector field along $d\Phi'(\eta)$ along Ξ' satisfying

$$\big\langle d\Phi'\big(\eta(f(z))\big) \ , \ \overline{\Phi'(f(w))}\big\rangle = 0 \ . \qquad (11)$$

Write

$$\begin{aligned} K'_\Omega(\zeta,\xi) = \frac{1}{r_0(\zeta)\overline{r_0(\xi)}} K_\Omega(\zeta,\xi) &= \frac{1}{r_0(\zeta)\overline{r_0(\xi)}}\big(h_0(\zeta)\overline{h_0(\xi)} + h_1(\zeta)\overline{h_1(\xi)} + \cdots\big) \\ &= h'_0(\zeta)\overline{h'_0(\xi)} + h'_1(\zeta)\overline{h'_1(\xi)} + \cdots \ . \end{aligned} \qquad (12)$$

Choose now the orthonormal basis $(h_0, h_1 \cdots, h_j, \cdots)$ of $H^2(\Omega)$ to be adapted to a point z_0 on U and $\eta(f(z_0))$ as in the proof of Proposition 1.1.2, so that $h_j(z_0) = 0$ whenever $j \ge 1$, and $dh_j\big(\eta(f(z_0))\big) = 0$ whenever $j \ge 2$. Clearly, we have also $h'_j(z_0) = 0$ whenever $j \ge 1$, and $dh'_j\big((\eta(f(z_0))\big) = 0$ whenever $j \ge 2$. By the analogue of (3)-(5) in the proof of Lemma 1.1.2, applied instead to $\big(h'_j\big)_{j=0}^\infty$ we conclude that $h'_1(f(w)) = 0$ and hence $h_1(f(w)) = 0$ for any $w \in D_\epsilon$. Defining $\mathcal{E} \subset H^2(\Omega)$ to consist of $h_1 = h_{\eta,z_0}$ from infinitesimal variations of solutions to $(\mathbf{I}_w)$ and $E \subset \Omega$ to consist of common zeros of $h_\alpha \in \mathcal{E}$ (cf. the proof of Proposition 1.1.2), the irreducible component S of $V \cap (D \times E)$ containing Graph(f) gives an extension of Graph(f) to a subvariety of $D \times \Omega$.

In the absence of ($\sharp$) there is the problem of making sense out of the identity (3) and its polarization, formally written $\log K'_\Omega(f(z), f(w)) = \lambda \log K'_D(z,w)$, both sides of which can only be understood as multi-valued functions when the domain of definition of $f : D_{\epsilon_0} \to \Omega$ is enlarged. Recall that for $z, w \in D$ we write $K_{D,w}(z) = K_D(z,w)$ and likewise for $(\zeta,\xi) \in \Omega$ we write $K_{\Omega,\xi}(\zeta) = K_\Omega(\zeta,\xi)$. For each $w \in D_{\epsilon_0}$, denote by $\Theta_w \Subset D \times \Omega$ the complex-analytic subvariety given by

$$\Theta_w := \Big(\big(\mathrm{Zero}(K_{D,x_0}) \cup \mathrm{Zero}(K_{D,w})\big) \times \Omega\Big) \cup \Big(D \times \big(\mathrm{Zero}(K_{\Omega,f(x_0)}) \cup \mathrm{Zero}(K_{\Omega,f(w)})\big)\Big).$$

Given a relatively compact subdomain in $D \times \Omega - \Theta_{x_0}$ we will consider functional equations $(\mathbf{J}_w)$ which are well-defined on the subdomain provided that w is sufficiently close to x_0, where the requirement of proximity of w to x_0 depends on the subdomain chosen.

Let $G \Subset D - \mathrm{Zero}(K_{D,x_0})$ and $\mathcal{O} \Subset \Omega - \mathrm{Zero}(K_{\Omega,f(x_0)})$ be arbitrary relatively compact subdomains. Observe that for $\epsilon > 0$ sufficiently small, $K'_D(z,w)$ is holomorphic in $(z, \overline{w})$ for $(z,w) \in G \times D_\epsilon$, and we have $K'_D(z,x_0) \equiv 1$ for $z \in D - \mathrm{Zero}(K_{D,x_0})$. Likewise for $\delta > 0$ sufficiently small, $K'_\Omega(\zeta,\xi)$ is holomorphic in $(\zeta,\overline{\xi})$ for $(\zeta,\xi) \in \mathcal{O} \times \Omega_\delta$, and we have $K'_\Omega(\zeta, f(x_0)) \equiv 1$ for $\zeta \in \Omega - \mathrm{Zero}(K_{\Omega,f(x_0)})$. Hence, for some $\epsilon = \epsilon(G,\mathcal{O}) < \epsilon_0$ we have $\mathrm{Re}(K'_D(z,w)) > 0$ and $\mathrm{Re}(K'_\Omega(\zeta, f(w))) > 0$ whenever $w \in D_\epsilon$ and $(z,\zeta) \in G \times \mathcal{O}$. Let W_G be the set of common solutions $(z,\zeta) \in G \times \mathcal{O}$ to the functional equations

$$(\mathbf{I}'_w) \qquad \log K'_\Omega(\zeta, f(w)) = \lambda \log K'_D(z,w) \ , \ w \in D_\epsilon \ , \qquad (13)$$

where log stands for the principal branch of logarithm. W_G contains Graph(f) and the germs of W_G and V_G at a general point of Graph(f) agree with each other. Using (5) we have the following equivalent family of functional equations.

$$
\begin{aligned}
(\mathbf{J}_w) \qquad H(z,\zeta;w) := \log\left(\frac{K_\Omega(\zeta, f(w))}{K_\Omega(\zeta, f(x_0))K_\Omega(f(x_0), f(w))}\right) \\
-\lambda \log\left(\frac{K_D(z,w)}{K_D(z,x_0)K_D(x_0,w)}\right) + a = 0\ , \quad w \in D_\epsilon\ ,
\end{aligned}
\tag{14}
$$

where $a = \log K_\Omega(f(x_0), f(x_0)) - \lambda \log K_D(x_0, x_0)$. Thus $H_w(z,\zeta) := H(z,\zeta;w)$ is a holomorphic function on $G \times \mathcal{O}$. Note that $H(z,\zeta;w)$ depends anti-holomorphically on $w \in D_\epsilon$. For $1 \le i \le n$ and $w \in D_\epsilon$ consider now the new equations $(\mathbf{L}_w^i)$ defined by differentiating the equations $(\mathbf{J}_w)$, given by

$$
(\mathbf{L}_w^i) \quad L_w^i(z,\zeta) := \frac{\partial H_w}{\partial \overline{w_i}}(z,\zeta) = 0\ , \tag{15}
$$

where by definition $\frac{\partial H_w}{\partial \overline{w_i}}(z,\zeta) = \frac{\partial}{\partial \overline{w_i}} H(z,\zeta;w)$. More explicitly we have

$$
\begin{aligned}
(\mathbf{L}_w^i) \quad & \frac{\sum_{j=1}^N \frac{\partial}{\partial \xi_j} K_\Omega(\zeta,\xi)\Big|_{\xi=f(w)} \frac{\partial \overline{f^j}}{\partial \overline{w_i}}(w)}{K_\Omega(\zeta, f(w))} - \frac{\sum_{j=1}^N \frac{\partial}{\partial \xi_j} K_\Omega(f(x_0),\xi)\Big|_{\xi=f(w)} \frac{\partial \overline{f^j}}{\partial \overline{w_i}}(w)}{K_\Omega(f(x_0), f(w))} \\
& - \ \lambda\left(\frac{\frac{\partial}{\partial \overline{w_i}} K_D(z,w)}{K_D(z,w)} - \frac{\frac{\partial}{\partial \overline{w_i}} K_D(x_0,w)}{K_D(x_0,w)}\right) = 0\ ,
\end{aligned}
\tag{16}
$$

which shows that each $L_w^i(z,\zeta)$, *a priori* only defined on $G \times \mathcal{O}$, extends meromorphically to $D \times \Omega$, a crucial fact in the sequel. To proceed we need the following obvious lemma.

Lemma 2.1.1. *Let $U \subset \mathbb{C}^m$ be a domain and $E \subset U$ be the common zero set of a real-analytic family $\{\varphi_t : t = (t_1, \cdots, t_s) \in (-1,1)^s\}$ of holomorphic functions parametrized by an open cube $(-1,1)^s \subset \mathbb{R}^s$. Write $\psi(z,t) := \varphi_t(z)$, and define $\psi_{t,i}(z) := \frac{\partial \psi}{\partial t_i}(z,t)$ Then, E is the common zero set of φ_0 and of $\{\psi_{t,i} : t \in (-1,1)^s, 1 \le i \le s\}$.*

Returning to Theorem 2.1.1, for $w \in D_\epsilon$ consider the real-analytic family of holomorphic functions $H_w(z,\zeta) := H(z,\zeta;w)$ on $D \times \Omega$ as being parametrized by the real $2n$-dimensional parameter space D_ϵ in the variables $(\mathrm{Re}(w_i), \mathrm{Im}(w_i)), 1 \le i \le n$. Observe the crucial fact that $H_{x_0}(z,\zeta) = 0$ when $w = x_0$, so that in the application of Lemma 2.1.1 the function φ_0 there is the zero function, leaving us with only first derivatives of $H_w(z,\zeta)$ against w. Since H_w varies anti-holomorphically in w, to apply Lemma 2.1.1 above it suffices to take first derivatives against $\overline{w_i}, 1 \le i \le n$, i.e., to consider $L_w^i(z,\zeta) = \frac{\partial H_w}{\partial \overline{w_i}}(z,\zeta)$. Recall that for $(z,\zeta) \in G \times \mathcal{O}$ the functional equation $(\mathbf{J}_w)$ for $w \in D_\epsilon$ is well-defined. More generally, let $(G_k)_{k=1}^\infty$ be a sequence of subdomains of $D - \mathrm{Zero}(K_{D,x_0})$ such that $G_1 \Subset \cdots \Subset G_k \Subset G_{k+1} \Subset \cdots \Subset D$ and such that $\bigcup_{k \ge 1} G_k = D - \mathrm{Zero}(K_{D,x_0})$, and likewise let $(\mathcal{O}_k)_{k=1}^\infty$ be a sequence of subdomains of $\Omega - \mathrm{Zero}(K_{\Omega,f(x_0)})$ such that $\mathcal{O}_1 \Subset \cdots \Subset \mathcal{O}_k \Subset \mathcal{O}_{k+1} \Subset \cdots \Subset \Omega$ and such that $\bigcup_{k \ge 1} \mathcal{O}_k = \Omega - \mathrm{Zero}(K_{\Omega,f(x_0)})$. Then, there exists a strictly decreasing sequence $(\epsilon_k)_{k=1}^\infty$ of positive numbers converging to 0 such that $\mathrm{Re}(K'_D(z,w)) > 0$ whenever $(z,w) \in G_k \times D_{\epsilon_k}$ for some $k \ge 1$, and such that $\mathrm{Re}(K'_\Omega(\zeta, f(w))) > 0$ whenever $(w,\zeta) \in D_{\epsilon_k} \times \mathcal{O}_k$ for some $k \ge 1$. Thus, given $z \in G_k$, $H_w(z,\zeta)$ is defined whenever $(w,\zeta) \in D_{\epsilon_k} \times \mathcal{O}_k$. Define now the subvariety $W \subset (D \times \Omega) - \Theta_{x_0}$, resp. $V' \subset D \times \Omega$, by $(\mathbf{J}_w)$, resp. $(\mathbf{L}_w^i)$, as follows.

$$
\begin{aligned}
W &:= \Big\{(z,\zeta) \in (D \times \Omega) - \Theta_{x_0} : H_w(z,\zeta) = 0 \text{ for all } w \text{ sufficiently close to } x_0.\Big\} \\
V' &:= \Big\{(z,\zeta) \in D \times \Omega : L_w^i(z,\zeta) = 0 \text{ for all } w \in D_\epsilon,\ 1 \le i \le n.\Big\}
\end{aligned}
\tag{17}
$$

Thus, V' is the common solution set of $(\mathbf{L}^i_w), w \in D_\epsilon, 1 \le i \le n$, i.e., the intersection of the zero sets of the meromorphic functions $L^i_w(z,\zeta)$ on $D \times \Omega$. By Lemma 2.1.1, W agrees with $V' \cap ((D \times \Omega) - \Theta_{x_0})$, hence $\mathrm{Graph}(f) \subset V'$. In terms of exhaustion sequences as explained in the above, for $k \ge 1$ and for $(z,\zeta) \in G_k \times \mathcal{O}_k$ we consider only the functional equations $H_w(z,\zeta)$ for $w \in D_{\epsilon_k}$. If we denote by $W_k \subset G_k \times \mathcal{O}_k$ the intersection of the zero sets of $H_w(z,\zeta)$ as w ranges over D_{ϵ_k}, then $W_{k+1} \cap (G_k \times \mathcal{O}_k) = W_k$ for $k \ge 1$ by the Identity Theorem for (anti-)holomorphic functions, and we have $W = \bigcup_{k \ge 1} W_k$.

Using V' in place of V (as in the case satisfying the additional assumption ($\sharp$)) and the same extremal functions $h_\alpha \in \mathcal{E}$, with common zero set $E \subset \Omega$, the irreducible component S of $T := V' \cap (D \times E)$ containing $\mathrm{Graph}(f)$ gives the desired analytic continuation of $\mathrm{Graph}(f)$ to a subvariety of $D \times \Omega$. To prove that S is the graph of some holomorphic isometry $F : (D', \lambda ds^2_D\big|_{D'}) \to (\Omega, ds^2_\Omega)$, by the arguments of Theorem 1.2.1 and using the identities (3) in the above, for two branches $f_1(z)$, $f_2(z)$ of the analytic continuation of f over some subdomain of D, we have

$$K'_\Omega(f_1(z), f_1(z)) = K'_\Omega(f_1(z), f_2(z)) = K'_\Omega(f_2(z), f_2(z)) \ . \tag{18}$$

Since $K_\Omega(\zeta,\xi) = \left(\frac{K_\Omega(\zeta,y_0)K_\Omega(y_0,\xi))}{K_\Omega(y_0,y_0)}\right) K'_\Omega(\zeta,\xi)$, we conclude from (18) that

$$\left|K_\Omega(f_1(z), f_2(z))\right|^2 = K_\Omega(f_1(z), f_1(z)) K_\Omega(f_2(z), f_2(z)) \ . \tag{19}$$

Write $\Phi : \Omega \to \mathbb{H}$ for the canonical map defined in terms of any orthonormal basis of $H^2(\Omega)$. By the Cauchy-Schwarz inequality, it follows from (19) that for some non-zero complex number c we have

$$\Phi(f_1(z)) = c\Phi(f_2(z)), \quad \text{so that} \quad f_1(z) = f_2(z) \ . \tag{20}$$

Consequently, the argument $S = \mathrm{Graph}(F)$ for some $F : (D', \lambda ds^2_D\big|_{D'}) \to (\Omega, ds^2_\Omega)$ works *verbatim* as in the proof of Theorem 1.2.1. The proof of injectivity of F is also the same as for the latter theorem. Finally, supposing that (Ω, ds^2_Ω) is complete as a Kähler manifold, we have to prove that $D' = D$. Suppose otherwise, i.e., $D' \subsetneq D$. Let $r \in \partial D' \cap D$ and $\gamma : [0,1] \to D$ be a smooth curve such that $\gamma(0) = x_0$ and $\gamma(1) = r$. Among $t \in [0,1]$ let t_0 be the first element such that $\gamma(t_0) \notin D'$ and write $p = \gamma(t_0)$. Since $F : (D', \lambda ds^2_D) \to (\Omega, ds^2_\Omega)$ is a holomorphic isometry, restricting to $\gamma[0,t_0)$ we see that $F(\gamma(t))$ converges to some point $q \in \Omega$ as t increases to t_0. Since $S \subset D \times \Omega$ is closed we must have $(p,q) \in S$, contradicting with the statement that $p \notin D'$. ■

Remarks Theorem 2.1.1 can be deduced from Calabi [Ca]. Using the canonical embedding $\Psi_G : G \to \mathbb{P}(H^2(G)^*)$, by the existence and uniqueness theorems of [Ca] one can analytically continue holomorphic isometries along paths. Global extension can be deduced using the diastasis δ as defined and developed in [Ca], noting that $\delta_D(z,x_0) = \log K'_D(z,z)$. For a proof of interior extension using [Ca] we refer the reader to Mok [Mk5, (2.3)]. [Ca] does not however apply to boundary extension, since ∂G essentially disappears under Ψ_G. Here interior extension is presented as a natural intermediate outcome of our direct method which yields at the same time boundary extension.

For boundary extension results on bounded domains we have

Theorem 2.1.2. *Let* $D \Subset \mathbb{C}^n$ resp. $\Omega \Subset \mathbb{C}^N$*, be bounded domains. Let* $x_0 \in D$*,* λ *be a positive real number and* $f : (D, \lambda ds^2_D; x_0) \to (\Omega, ds^2_\Omega; f(x_0))$ *be a germ of holomorphic isometry. Suppose furthermore that the Bergman kernel* $K_D(z,w)$ *extends as a meromorphic function in* $(z,\overline{w})$ *to a neighborhood of* $\overline{D} \times D$ *and* $K_\Omega(\zeta,\xi)$ *extends as a* *meromorphic function in $(\zeta, \overline{\xi})$ to a neighborhood of $\overline{\Omega} \times \Omega$. Then, there exists a neighborhood $D^\sharp$ of $\overline{D}$ and a neighborhood $\Omega^\sharp$ of $\overline{\Omega}$ such that the germ of* $\mathrm{Graph}(f) \subset D \times \Omega$ *at $(x_0, f(x_0))$ extends to an irreducible complex-analytic subvariety $S^\sharp$ of $D^\sharp \times \Omega^\sharp$. If (Ω, ds^2_Ω) is complete as a Kähler manifold, then $S := S^\sharp \cap (D \times \Omega)$ is the graph of a holomorphic isometric embedding $F : (D, \lambda ds^2_D) \to (\Omega, ds^2_\Omega)$. If furthermore (D, ds^2_D) is complete, then $F : D \to \Omega$ is proper.*

Proof of Theorem 2.1.2. We refer to the proof of Theorem 2.1.1 and use the notation there. Under the hypothesis of Theorem 2.1.2 the domain of definition of the equations defining $V' \subset D \times \Omega$, viz., the functional equations $(\mathbf{L}^i_w)$, for $w \in D_\epsilon$ and for $1 \leq i \leq n$, can be extended from $D \times \Omega$ to $D^\sharp \times \Omega^\sharp$. Denote by $V'^\sharp$ the common solution set of the extension of the functional equations $(\mathbf{L}^i_w)$ thus defined. On the other hand, from the formula for $h_1 = h_\alpha = h_{\eta, z_0}, \alpha \in \mathbf{A}$ given in (3) and (5) in the proof of Lemma 1.1.2, under the assumption of Theorem 2.1.2, each h_α can be extended from Ω to $\Omega^\sharp$ as a *meromorphic* function $h^\sharp_\alpha$. Recall that $E \subset \Omega$, is the common zero set of $h_\alpha, \alpha \in \mathbf{A}$. Defining $E^\sharp$ to be the common zero set of the meromorphic functions $h^\sharp_\alpha, \alpha \in \mathbf{A}$, on $\Omega^\sharp$, and writing $S^\sharp \subset D^\sharp \times \Omega^\sharp$ for the irreducible component of $V'^\sharp \cap (D^\sharp \times E^\sharp)$ containing $\mathrm{Graph}(f)$, then $S^\sharp$ furnishes an extension of $\mathrm{Graph}(f)$ from $D \times \Omega$ to $D^\sharp \times \Omega^\sharp$ and $S^\sharp \cap (D \times \Omega) = S = \mathrm{Graph}(f)$, by the proof of Corollary 1.2.1. By Theorem 2.1.1, $S \subset D \times \Omega$ is the graph of a holomorphic isometric embedding $F : (D, \lambda\, ds^2_D) \to (\Omega, ds^2_\Omega)$.

It remains to prove that $F : D \to \Omega$ is proper whenever (D, ds^2_D) is complete. Suppose otherwise, then there exists $b := (p, y) \in S^\sharp - S$ such that $p \in \partial D$ and $y \in \Omega$. Let W be a neighborhood of (p, y) on $S^\sharp$ such that $W \subset D^\sharp \times \Omega$, and denote by $\rho : \widetilde{W} \to W$ a desingularization of W. Let $\eta = (x_1, y_1) \in W \cap \mathrm{Graph}(F)$ and denote by $\widetilde{\eta} \in \widetilde{W}$ the unique point lying over (x_1, y_1). Let $\widetilde{b} \in \widetilde{W}$ be any point such that $\rho(\widetilde{b}) = b$. Let $\gamma : [0, 1] \to \widetilde{W}$ be any smooth curve on $\widetilde{W}$ such that $\gamma(0) = \widetilde{\eta}$ and $\gamma(1) = \widetilde{b}$. Define $\gamma_1 : [0, 1] \to D^\sharp$, $\gamma_2 : [0, 1] \to \Omega$, by $\gamma_i(t) = \pi_i\big(\rho(\gamma(t)\big)$; $i = 1, 2$; where $\pi_1 : D^\sharp \times \Omega \to D^\sharp$ and $\pi_2 : D^\sharp \times \Omega \to \Omega$ are canonical projections. Let $0 < t^\flat \leq 1$ be the first point such that $\gamma_1(t^\flat) \in \partial D$ and write $x^\flat := \gamma_1(t^\flat) \in \partial D$, $y^\flat := \gamma_2(t^\flat) \in \Omega$. Then, $\gamma_1\big|_{[0,t^\flat]} : [0, t^\flat] \to D^\sharp$ joins $\gamma_1(0) = x_1$ to $x^\flat$ such that $\gamma_1(t) \in D$ for $0 \leq t < t^\flat$. On the other hand, $\gamma_2\big|_{[0,t^\flat]} : [0, t^\flat] \to \Omega$ joins $\gamma_2(0) = y_1$ to $\gamma_2(t^\flat) = y^\flat$. Since γ is smooth, $\gamma_2\big|_{[0,t^\flat]}$ is of finite length. Clearly $F\big(\gamma_1(t)\big) = \gamma_2(t)$ whenever $0 \leq t < t^\flat$. Since F is an isometry, $\gamma_1\big|_{[0,t^\flat)}$ must be of finite length with respect to the Bergman metric ds^2_D. However, $\gamma_1\big|_{[0,t^\flat]}$ is a smooth curve joining $x_1 \in D$ to $x^\flat \in \partial D$, and hence $\gamma_1\big|_{[0,t^\flat)}$ must be of infinite length on the complete Kähler manifold (D, ds^2_D). By contradiction we have proven that $F : D \to \Omega$ is proper, and the proof of Theorem 2.1.2 is complete. ■

Remarks In Theorem 1.1.1 we deal with boundary extension for germs of holomorphic isometries $f : (D, \lambda ds^2_D; 0) \to (\Omega, ds^2_\Omega; 0)$ between bounded complete circular domains with base points at 0. For arbitrary base points $x_0 \in D$ and $y_0 = f(x_0) \in \Omega$, Theorem 2.1.2 applies provided that $tD \subset D$ and $t\Omega \subset \Omega$ whenever $0 < t < 1$. To see this, by Lemma 1.1.1, $K_{D,w}(z) = K_D(z, w)$ extends holomorphically to some neighborhood $D^\sharp$ of $\overline{D}$ whenever w is sufficiently close to x_0, and the analogue holds true for $K_{\Omega,\xi}(\zeta) = K_\Omega(\zeta, \xi)$, whenever ξ is sufficiently close to y_0, so that Theorem 2.1.2 is applicable.

(2.2) *Generalizations to relatively compact subdomains of complex manifolds* We consider more generally extensions of germs of holomorphic isometries on complex manifolds equipped with Bergman metrics. First of all, we introduce some terminology, as follows.

Definition 2.2.1. *Let X be a complex manifold and denote by ω_X its canonical line bundle. Suppose the Hilbert space $H^2(X, \omega_X)$ of square-integrable holomorphic n-forms on*

X has no base points, and denote by $\mathcal{K}_X(z,w)$ the Bergman kernel form on X. Regarding $\mathcal{K}_X(z,z)$ as a Hermitian metric h on the anti-canonical line bundle ω_X^*, we denote by $\beta_X \geq 0$ the curvature form of the dual metric h^* on ω_X, and write ds_X^2 for the corresponding semi-Kähler metric on X. We say that (X, ds_X^2) is a Bergman manifold whenever ds_X^2 is positive definite. If furthermore the canonical map $\Psi_X : X \to \mathbb{P}((H^2(X,\omega_X)^*)$ is an embedding, we call (X, ds_X^2) a canonically embeddable Bergman manifold.

For a bounded domain $D \Subset \mathbb{C}^n$, we have $\mathcal{K}_D(z,w) = K_D(z,w)\big(\frac{i}{2}dz^1 \wedge \overline{dw^1}\big) \wedge \cdots \wedge \big(\frac{i}{2}dz^n \wedge \overline{dw^n}\big)$. Our extension results generalize to canonically embeddable Bergman manifolds, including bounded domains on Stein manifolds, as follows.

Theorem 2.2.1. *Let D (resp. Ω) be a canonically embeddable Bergman manifold. Let $D \Subset M$ (resp. $\Omega \Subset Q$) be a realization of D (resp. Ω) as a relatively compact domain on a complex manifold M (resp. Q) such that the Bergman kernel form $\mathcal{K}_D(z,w)$ (resp. $\mathcal{K}_\Omega(\zeta,\xi)$) extends meromorphically in $(z,\overline{w})$ to $M \times D$ (resp. in $(\zeta,\overline{\xi})$ to $Q \times \Omega$). Then, the analogue of Theorem 2.1.2 holds true with M replacing $D^\sharp$ and Q replacing $\Omega^\sharp$.*

Proof. Let μ be a square-integrable holomorphic n-form on D such that $\mu(x_0) \neq 0$, and ν be a square-integrable holomorphic N-form on Ω such that $\nu\big(f(x_0)\big) \neq 0$. For $m > 0$, write $\epsilon_m = \big(\sqrt{-1}\big)^{m^2}$ so that $\epsilon_m \alpha \wedge \overline{\alpha} \geq 0$ for any $(m,0)$-covector α on an m-dimensional complex manifold. Define $K_D^\flat(z,w)$ on $D \times D$, resp. $K_\Omega^\flat(\zeta,\xi)$ on $\Omega \times \Omega$ by

$$\mathcal{K}_D(z,w) = K_D^\flat(z,w)\big(\epsilon_n \mu(z) \wedge \overline{\mu(w)}\big) \ ; \quad \mathcal{K}_\Omega(\zeta,\xi) = K_\Omega^\flat(\zeta,\xi)\big(\epsilon_N \nu(\zeta) \wedge \overline{\nu(\xi)}\big) \ . \qquad (1)$$

Using $K_D^\flat(z,w)$, resp. $K_\Omega^\flat(\zeta,\xi)$, in place of $K_D'(z,w)$, resp. $K_\Omega'(\zeta,\xi)$, Theorem 2.1.1 and Theorem 2.1.2 generalize, as follows. Let $\sigma_0 \in H^2(D,\omega_D)$ be such that the (n,n)-vector $\epsilon_n \sigma(x_0) \wedge \overline{\sigma(x_0)}$ is maximized among square-integrable holomorphic n-forms of unit norm by $\sigma = \sigma_0$. Then, $\sigma(x_0) = 0$ for any $\sigma \perp \sigma_0$. Complete σ_0 to an orthonormal basis $\big(\sigma_i\big)_{i=0}^\infty$ of $H^2(D,\omega_D)$. Choosing $\mu = \sigma_0$, $K_D^\flat(z,z) = \sum_{i=0}^\infty \big|\frac{\sigma_i(z)}{\sigma_0(z)}\big|^2 = 1 + \sum_{i=1}^\infty \big|\frac{\sigma_i(z)}{\sigma_0(z)}\big|^2$. Similarly let $\big(\tau_i\big)_{i=0}^\infty$ be an orthonormal basis of $H^2(\Omega,\omega_\Omega)$ adapted to $y_0 = f(x_0)$ defined in exactly the same way. Choosing $\nu = \tau_0$, $K_\Omega^\flat(\zeta,\zeta) = \sum_{i=0}^\infty \big|\frac{\tau_i(\zeta)}{\tau_0(\zeta)}\big|^2 = 1 + \sum_{i=1}^\infty \big|\frac{\tau_i(\zeta)}{\tau_0(\zeta)}\big|^2$. Then $K_D^\flat(z,w)$, resp. $K_\Omega^\flat(\zeta,\xi)$, plays the role of $K_D'(z,w)$, resp. $K_\Omega'(\zeta,\xi)$, in Theorem 2.1.1, and by the analogues of (2) and (3) in the proof of Theorem 2.1.1 we have

$$\begin{aligned} \sqrt{-1}\partial\overline{\partial}\log K_\Omega^\flat(f(z),f(z)) &= \lambda\sqrt{-1}\partial\overline{\partial}\log K_D^\flat(z,z) \ ; \\ \log K_\Omega^\flat(f(z),f(z)) &= \lambda \log K_D^\flat(z,z) \ , \end{aligned} \qquad (2)$$

and the proofs there carry over with minor modifications to yield Theorem 2.2.1. ■

Remarks For a bounded symmetric domain $G \subset N$ embedded in its compact dual N by the Borel embedding, $\mathcal{K}_G(z,w)$ extends meromorphically in $(z,\overline{w})$ to N (cf. Lemma 1.3.1). Thus, Theorem 2.2.1 implies Theorem 1.3.1.

§3 Examples of holomorphic isometries with respect to the Bergman metric

(3.1) *Totally geodesic examples on bounded symmetric domains* The first examples of non-equidimensional holomorphic isometric embeddings $f : D \to \Omega$ up to normalizing constants with respect to the Bergman metric are given by holomorphic totally geodesic embeddings from an irreducible bounded symmetric domain into any bounded symmetric domain, such as the embedding of the Poincaré disk into the complex unit ball $B^n, n \geq 2$, given by $f(z) = (z,0)$, or the diagonal map into the polydisk Δ^n, $n \geq 2$, given by $f_n(z) = (z,\cdots,z)$. More generally, if Ω is a bounded symmetric domain of rank $r \geq 1$, then, up to automorphisms of Ω, there are exactly r such maps, obtained from a maximal polydisk $P \subset \Omega$, where $P \cong \Delta^r$, and $f : \Delta \to \Omega$ is given by composing the diagonal map $f_k : \Delta \to \Delta^k$ with the standard embedding $\Delta^k \times \{0\} \subset \Delta^r \cong P \subset \Omega$, $1 \leq k \leq r$.

Totally geodesic holomorphic embeddings $f : D \to \Omega$ from irreducible bounded symmetric domains into bounded symmetric domains have been classified by Satake [Sa] and Ihara [Ih]. As higher-dimensional examples write $M(p,q)$ for the complex vector space of p-by-q matrices with complex entries, and recall that the domain $D^I_{p,q} \subset M(p,q)$ consists of matrices Z satisfying $I - \overline{Z}^t Z > 0$. Let $M_a(n) \subset M(n,n)$, resp. $M_s(n) \subset M(n,n)$, be the complex vector subspace consisting of skew-symmetric, resp. symmetric, matrices. Define $D^{II}_n := D^I_{n,n} \cap M_a(n)$ and $D^{III}_n := D^I_{n,n} \cap M_s(n)$. Then, $D^I_{p,q} \Subset M(p,q)$, resp. $D^{II}_n \Subset M_a(n)$, resp. $D^{III}_n \Subset M_s(n)$ are classical symmetric domains of type I, resp. II, resp. III, in their Harish-Chandra realizations, and the inclusions $D^{II}_n \subset D^I_{n,n}$, $D^{III}_n \subset D^I_{n,n}$ are totally geodesic. They extend to holomorphic embeddings $M_a(n) \subset M(n,n)$, $M_s(n) \subset M(n,n)$. More generally, using the characterization of totally geodesic submanifolds on a Riemannian symmetric manifold in terms of Lie triple systems (cf. Helgason [He, §7, p.224ff.]), the Borel embedding between dual pairs of Hermitian symmetric spaces, and Harish-Chandra coordinates (cf. Wolf [Wo]), we have the following summary of basic facts for which the proof is omitted.

Proposition 3.1.1. *Let (D,h) and (Ω,g) be Hermitian symmetric manifolds of the noncompact type and denote by (M,h_c),* resp. *(Q,g_c), the compact dual of D,* resp. *Ω. Identify D and Ω as bounded symmetric domains $D \Subset \mathbb{C}^n$, $\Omega \Subset \mathbb{C}^N$ in their Harish-Chandra realizations, so that $D \Subset \mathbb{C}^n \subset M$ and $\Omega \Subset \mathbb{C}^N \subset Q$, where $D \subset M$ and $\Omega \subset Q$ are given by the Borel embedding. Let $F : D \to (\Omega, g)$ be a holomorphic totally geodesic embedding. Then, F extends to a holomorphic totally geodesic embedding $\Phi : M \to (Q, g_c)$. As a consequence,* Graph$(F) \subset D \times \Omega$ *extends to a complex submanifold $S \subset M \times Q$. When D is irreducible, F is a holomorphic isometry up to a normalizing constant. If $F(0) = 0$, then F is the restriction of a linear map $\Lambda : \mathbb{C}^n \to \mathbb{C}^N$.*

Let $D \Subset \mathbb{C}^n$ be an irreducible bounded symmetric domain in its Harish-Chandra realization. Denote by $\pi : L \to D$ the anti-canonical line bundle on D. Writing $(z_1, \cdots, z_n)$ for the Harish-Chandra coordinates on D, for $t \in \mathbb{C}$ the n-vector $t\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}$ at any $z \in D$ is identified with (z,t), giving a trivialization $L \cong D \times \mathbb{C}$. The action of $\mathrm{Aut}(D)$ on D induces an action on L, and $\pi : L \to D$ is equipped with an $\mathrm{Aut}(D)$-invariant Hermitian metric h. Thus, given any $z \in D$ and $\gamma \in \mathrm{Aut}(D)$ we have $f_*\big(\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}\big) = J_\gamma(z) \cdot \frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_n}$, where $J_\gamma(z) = \det(d\gamma(z))$ is the Jacobian determinant of γ, and the action of $\mathrm{Aut}(D)$ on L is given by $\Phi(\gamma)(z,t) = (\gamma(z), J_\gamma(z)\cdot t)$. On L we have the open subset $\Omega \subset L$ consisting of all n-vectors η of length < 1 with respect to h. By the Schwarz Lemma, the volume form of the Bergman metric ds^2_D is bounded from below by a constant multiple of the Euclidean volume form, so that $\Omega \subset D \times \Delta(R) \Subset \mathbb{C}^{n+1}$ for some $R > 0, \Delta(R)$ being the disk of radius R centered at 0. Let now α be a positive real number. We define $L^\alpha := D \times \mathbb{C}$ set-theoretically to be the same as L, but regard $\pi : L^\alpha \to D$ as being equipped with the Hermitian metric h^α, where, writing e for the basis of $L \cong D \times \mathbb{C}$ corresponding to $D \times \{1\}$, and writing e^α for the basis of $L^\alpha \cong D \times \mathbb{C}$ corresponding to $D \times \{1\}$, we have $\|e^\alpha\|_{h^\alpha} = \|e\|^\alpha_h$. We define $\Omega_\alpha \subset L^\alpha$ to consist of vectors η of length < 1 with respect to h_α, $\Omega_\alpha \Subset \mathbb{C}^{n+1}$. Thus, $\Omega_\alpha \subset L^\alpha$ is the unit disk bundle of $\pi : L^\alpha \to D$ with respect to a Hermitian metric of strictly negative curvature on $L^\alpha = D \times \mathbb{C}$, so that every boundary point $b \in \partial\Omega_\alpha - \partial D$ is strictly pseudoconvex (D being identified with $D \times \{0\}$). With this set-up we prove

Proposition 3.1.2. *Let $\alpha > 0$ and $f : D \to \Omega_\alpha$ be the embedding given by $f(z) = (z,0)$. Then, $f : (D, \lambda ds^2_D) \to (\Omega_\alpha, ds^2_{\Omega_\alpha})$ is a totally geodesic holomorphic isometric embedding for $\lambda = 1 + \alpha$. Furthermore, $(\Omega_\alpha, ds^2_{\Omega_\alpha})$ is a complete Kähler manifold.*

Proof. Since D is simply connected, for $\gamma \in \mathrm{Aut}(D)$ a holomorphic logarithm $\log J_\gamma(z)$ can be defined for the Jacobian determinant $J_\gamma(z) = \det(d\gamma(z))$, and the mapping

$\Psi_\gamma(z,\eta) = \big(\gamma(z), \exp(\alpha \log J_\gamma(z))\big)$ defines an automorphism of $\pi : L^\alpha \to D$ as a holomorphic line bundle which preserves the Hermitian metric h^α. Identify D as the zero section of $\pi : L^\alpha \to D$ and denote by $H \subset \mathrm{Aut}(\Omega_\alpha)$ the subgroup which leaves D invariant as a set. H acts transitively on $D \subset L^\alpha$ by means of Ψ_γ, $\gamma \in \mathrm{Aut}(D)$, hence the restriction of the Bergman kernel Ω_α to D can be computed from a single point, giving

$$K_{\Omega_\alpha}\left((z,0),(z,0)\right) = \left|\det(d\gamma(0))\right|^{-2(1+\alpha)} K_{\Omega_\alpha}(0,0)\ , \tag{1}$$

where γ is an automorphism of D such that $\gamma(0) = z$. On the other hand,

$$K_D(z,z) = \left|\det(d\gamma(0))\right|^{-2} K_D(0,0)\ . \tag{2}$$

Comparing (1) and (2) we conclude that

$$K_{\Omega_\alpha}\left((z,0),(z,0)\right) = c_\alpha \cdot K_D(z,z)^{1+\alpha} \tag{3}$$

for $c_\alpha > 0$. Writing $\varphi_D(z) := K_D(z,z)$ and $\varphi_{\Omega_\alpha}(\zeta) = K_{\Omega_\alpha}(\zeta,\zeta)$, from (3) we deduce

$$\sqrt{-1}\partial\overline{\partial}\ \log\varphi_{\Omega_\alpha}\big|_D = (1+\alpha)\sqrt{-1}\partial\overline{\partial}\ \log\varphi_D, \text{ i.e.,}$$

$$f^* ds^2_{\Omega_\alpha} = (1+\alpha) ds^2_D\ , \tag{4}$$

as desired. Since $D \subset \Omega_\alpha$ is the fixed point set of the circle group S^1 acting by $(e^{i\theta};(z,t)) \to (z, e^{i\theta}t)$, $D \subset \Omega_\alpha$ is totally geodesic with respect to $ds^2_{\Omega_\alpha}$. It remains to prove that $\big(\Omega_\alpha, ds^2_{\Omega_\alpha}\big)$ is complete. for which it suffices to show that, given any sequence of points $(x_j)_{1\le j<\infty}$ approaching $b \in \partial\Omega_\alpha$, $d(0,x_j)$ must diverges to ∞ as $j \to \infty$. Let $x \in \Omega_\alpha$ be any point and $\gamma : [0,1] \to \Omega_\alpha$ be a piecewise $\mathcal{C}^1$-curve joining 0 to x. Then, $\pi \circ \gamma : [0,1] \to D$ is a piecewise $\mathcal{C}^1$-curve joining 0 to $\pi(x) \in D$. Denote by $d_D(\cdot,\cdot)$, resp. $d_{\Omega_\alpha}(\cdot,\cdot)$, the distance function for the Kähler manifold (D, ds^2_D), resp. $\big(\Omega_\alpha, ds^2_{\Omega_\alpha}\big)$. For a complex manifold X we denote by κ_X its Carathédory pseudo-metric, which is an $\mathrm{Aut}(X)$-invariant continuous complex Finsler pseudo-metric, and by $\delta_X(\cdot,\cdot)$ the pseudo-distance function of (X,κ_X). When X is a bounded domain, κ_X is a metric, and $\delta_X(\cdot,\cdot)$ is a distance function. Since D is homogeneous, any two $\mathrm{Aut}(D)$-invariant continuous complex Finsler metrics are equivalent to each other, in particular $\delta_D(\cdot,\cdot) \ge c \cdot d_D(\cdot,\cdot)$ for some constant $c > 0$. By the distance-decreasing property of the Carathédory metric, $\delta_D(\pi(x),0) \le \delta_{\Omega_\alpha}(x,0)$. Since the Bergman metric on any bounded domain dominates the Carathéodory metric, $d_{\Omega_\alpha}(x,0) \ge \delta_{\Omega_\alpha}(x,0) \ge \delta_D(\pi(x),0) \ge c \cdot d_D(\pi(x),0)$. Let $(x_j)_{1\le j\le\infty}$ be a discrete sequence on Ω_α converging to $b \in \partial D \subset \partial\Omega_\alpha$. Then, $d_{\Omega_\alpha}(x_j,0) \ge c \cdot d_D(\pi(x_j),0) \to \infty$ since (D, ds^2_D) is complete. On the other hand, if $b \in \partial\Omega_\alpha - \partial D$, b is a smooth strictly pseudoconvex boundary point of Ω_α. By a standard localization argument $\delta_{\Omega_\alpha}(x_j,0) \to \infty$ as $j \to \infty$, and $d_{\Omega_\alpha}(x_j,0) \ge \delta_{\Omega_\alpha}(x_j,0) \to \infty$, proving that $\big(\Omega_\alpha, ds^2_{\Omega_\alpha}\big)$ is complete, as desired. ■

(3.2) *Examples of holomorphic isometric embeddings of the Poincaré disk into the polydisk*

Motivated by Clozel-Ullmo [CU], our first aim was to study germs of holomorphic isometries $f : (D;0) \to (\Omega;0)$ between bounded symmetric domains. In particular, in relation to the case where D is the unit disk Δ and Ω is the polydisk Δ^p, it was conjectured in [CU, Conjecture 2.2] that for any positive integer q, every germ of holomorphic isometry $f : (\Delta, q\, ds^2_\Delta; 0) \to (\Delta^p, ds^2_{\Delta^p}; 0)$ is necessarily totally geodesic. We can *a priori* allow the normalizing (positive) real constant λ to be arbitrary. By Theorem 1.3.1, f necessarily extends to a proper holomorphic embedding $F : \Delta \to \Delta^p$ whose graph extends to an irreducible affine-algebraic subvariety $S^\sharp \subset \mathbb{C} \times \mathbb{C}^p$. It follows readily that λ is necessarily a positive integer q. (This can be seen comparing Bergman kernels via a local holomorphic extension $F^\flat$ across a general boundary point $b \in \partial\Delta$.)

Let D and Ω be bounded symmetric domains, and $F, \widetilde{F} : D \to \Omega$ be holomorphic maps. We say that F and $\widetilde{F}$ are congruent whenever there exists $\varphi \in \mathrm{Aut}(D)$ and $\psi \in \mathrm{Aut}(\Omega)$ such that $\widetilde{F} = \psi \circ F \circ \varphi$, and incongruent otherwise. Concerning holomorphic isometric embeddings $F : (\Delta, q\, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, we have

Theorem 3.2.1. *For every positive integer $p > 1$ there exists a holomorphic isometric embedding $F : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, $F = (F_1, \cdots F_p)$, where each component $F_k, 1 \le k \le p$, is nonconstant, such that F is not totally geodesic. In particular, Conjecture 2.2 of Clozel-Ullmo* [CU] *is false. Furthermore, for $p \ge 3$ there exists a real-analytic 1-parameter family of mutually incongruent holomorphic isometric embeddings $F_t : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, $t \in \mathbb{R}$.*

We start with an example of a holomorphic isometric embedding of the Poincaré disk into the bi-disk. The unit disk is conformally equivalent to the upper half-plane $\mathcal{H}$. For $\tau \in \mathcal{H}$, $\tau = \rho e^{i\varphi}$, where $\rho > 0$, $0 < \varphi < \pi$, write $\sqrt{\tau} = \sqrt{\rho} e^{\frac{i\varphi}{2}}$. Then, we have

Lemma 3.2.1. *Equip $\mathcal{H}$ with the Poincaré metric $ds^2_{\mathcal{H}} = 2Re\frac{d\tau \otimes d\overline{\tau}}{2(\mathrm{Im}\tau)^2}$ of constant Gaussian curvature -1 and $\mathcal{H}^2$ with the product metric. Then, the proper holomorphic map $f : \mathcal{H} \to \mathcal{H}^2$ given by $f(\tau) = \big(\sqrt{\tau}, i\sqrt{\tau}\big)$ is a holomorphic isometric embedding.*

Proof. Let $\omega_{\mathcal{H}}$ resp. $\omega_{\mathcal{H}^2}$ be the Kähler forms of the chosen canonical Kähler metrics on $\mathcal{H}$ resp $\mathcal{H}^2$. Writing $\tau = s + it$, $\sqrt{\tau} = \alpha + i\beta$, where s, t, α and β are real, we have

$$\omega_{\mathcal{H}} = \sqrt{-1}\partial\overline{\partial}(-2\log t) = \sqrt{-1}\,\frac{d\tau \wedge d\overline{\tau}}{2t^2}\ ,$$
$$f^*\omega_{\mathcal{H}^2} = -2\sqrt{-1}\partial\overline{\partial}\big(\log(\mathrm{Im}(\sqrt{\tau})) + \log(\mathrm{Im}(i\sqrt{\tau}))\big) = -2\sqrt{-1}\partial\overline{\partial}\log\big(\mathrm{Im}(\sqrt{\tau})\cdot \mathrm{Im}(i\sqrt{\tau})\big),$$
$$\mathrm{Im}(\sqrt{\tau})\cdot\mathrm{Im}(i\sqrt{\tau}) = \beta\alpha\ = \frac{1}{2}\mathrm{Im}\big((\alpha^2-\beta^2)+2i\alpha\beta\big) = \frac{1}{2}\mathrm{Im}(\tau) = \frac{t}{2}\ ,$$
$$f^*\omega_{\mathcal{H}^2} = -2\sqrt{-1}\partial\overline{\partial}\log\Big(\frac{t}{2}\Big) = \sqrt{-1}\partial\overline{\partial}(-2\log t) = \omega_{\mathcal{H}}\ .$$

In other words, $f : (\mathcal{H}, ds^2_{\mathcal{H}}) \to (\mathcal{H}, ds^2_{\mathcal{H}}) \times (\mathcal{H}, ds^2_{\mathcal{H}})$ is a holomorphic isometry. It is an embedding since the function $\sqrt{\tau}$ is already injective on $\mathcal{H}$. ■

For $\tau \in \mathcal{H}$, $\tau = \rho e^{i\varphi}$, and an integer $p \ge 2$, write $\tau^{\frac{1}{p}} = \rho^{\frac{1}{p}} e^{\frac{i\varphi}{p}}$. Then, we have

Proposition 3.2.1. *Let $p \ge 2$ be a positive integer and $\gamma = e^{\frac{\pi i}{p}}$. Then, the proper holomorphic mapping $f : \big(\mathcal{H}, ds^2_{\mathcal{H}}\big) \to (\mathcal{H}, ds^2_{\mathcal{H}})^p$ defined by*

$$f(\tau) = \big(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}\big)$$

is a holomorphic isometric embedding.

Proof. Write $\tau^{\frac{1}{p}} = re^{i\theta}$, $0 < \theta < \frac{\pi}{p}$. Thus, $r^p = \rho$, $p\theta = \varphi$, and, for $0 \le k \le p-1$, $\mathrm{Im}\big(\gamma^k\tau^{\frac{1}{p}}\big) = r \cdot \mathrm{Im}\big(e^{i(\frac{k\pi}{p}+\theta)}\big)$. Let τ_k be the standard coordinate of the k-th direct factor of $\mathcal{H}^p$, and write $\tau_k = s_k + it_k$; s_k, t_k real. Then, to prove the proposition it suffices to check that $f^*(\log t_1 + \cdots + \log t_p) = a_p + \log t$ for some constant a_p. Now

$$f^*(\log t_1 + \cdots + \log t_p) = \log\Bigg(\prod_{k=0}^{p-1}\mathrm{Im}\big(e^{i(\frac{k\pi}{p}+\theta)}\big)\Bigg) + p\log r = \log\Bigg(\prod_{k=0}^{p-1}\sin\Big(\frac{k\pi}{p}+\theta\Big)\Bigg) + \log\rho\,.$$

Writing $t = \mathrm{Im}(\tau) = \rho\sin\varphi = \rho\sin(p\theta)$, it remains to verify the following identity.

Lemma 3.2.2. *Let $p \geq 2$ be a positive integer. Then, the trigonometric identity*

$$\sin\theta \sin\left(\frac{\pi}{p}+\theta\right)\cdots\sin\left(\frac{(p-1)\pi}{p}+\theta\right) = c_p \sin(p\theta)$$

holds true for some positive constant c_p.

Proof. Both sides of the displayed equation are trigonometric polynomials with exactly the same zero sets in θ consisting only of simple zeros. Hence, they must agree for some choice of nonzero constant c_p, which is positive by substitution at some $\theta \in \left(0, \frac{\pi}{p}\right)$. ■

Proof of Theorem 3.2.1. The p-th root map as in Proposition 3.2.1 gives via the Cayley transform a holomorphic isometry $f_p : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$. Here for the domain disk we use the Cayley transform $\iota : \mathcal{H} \to \Delta$ given by $z = \iota(\tau) = \frac{\tau-i}{\tau+i}$, and likewise the same map for each component of the target polydisk Δ^p. This gives examples proving the first half of Theorem 3.2.1. We have $f_p(0) = 0$, and f_p is singular exactly at two points $1, -1 \in \partial\Delta$ on the boundary circle, with images $f_p(1) = (1, \cdots, 1)$ and $f_p(-1) = (-1, \cdots, -1)$. An example of a real-analytic 1-parameter family of holomorphic isometries $F_t : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ which are mutually incongruent to each other can be constructed from f_{p-1} and f_2, as follows. Write $f_2(z) = \big(\alpha(z), \beta(z)\big)$, $f_{p-1}(z) = \big(\gamma_1(z), \cdots, \gamma_{p-1}(z)\big)$, and let $\varphi \in \mathrm{Aut}(\Delta)$ be an arbitrary automorphism. Define $h : \Delta \to \Delta^p$ by $h(z) := \big(\alpha(\varphi(\gamma_1(z))), \beta(\varphi(\gamma_1(z)), \gamma_2(z), \cdots, \gamma_{p-1}(z)\big)$. Then, $h = g \circ f_{p-1}$, where $g : \Delta^{p-1} \to \Delta^p$ is given by $g(z_1, \cdots z_{p-1}) = \big(f_2(\varphi(z_1)); z_2, \cdots z_{p-2}\big)$. Thus, g and hence h are holomorphic isometries with respect to Bergman metrics. Observe that $\gamma_1(z)$, which corresponds to taking the p-th in the coordinate $\tau = s + it$ of the upper half-plane $\mathcal{H}$ (cf. Proposition 3.2.1), maps the lower semi-circle $S^1_- := \big\{e^{i\theta} : -\pi < \theta < 0\big\}$ bijectively onto itself. (Note that the positive s-axis is mapped via $z = \iota(\tau) = \frac{\tau-i}{\tau+i}$ to S^1_- since $\iota(1) = -i$.) Given any two distinct points $a, b \in S^1_-$, we can choose $\varphi \in \mathrm{Aut}(\Delta)$ such that $\varphi(\gamma_1(a)) = 1$ and $\varphi(\gamma_1(b)) = -1$. Then, noting that in fact each component $\gamma_k, 1 \leq k \leq p-1$, of $f_{p-1} : \Delta \to \Delta^{p-1}$ can neither be analytically continued to a neighborhood of 1 nor of -1, h is singular precisely at the 4 distinct points $1, -1, a, b$. If we fix a and let b vary we get holomorphic isometries $h_b : (\Delta, ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ depending on b. For $b_1 \neq b_2$, h_{b_1} cannot be congruent to h_{b_2} since the two sets $\{1, -1, a, b_1\}$ and $\{1, -1, a, b_2\}$ cannot be transformed to each other by any automorphism of Δ. Letting b vary on a connected component of $S^1_- - \{a\}$, we have obtained a real-analytic one-parameter family of mutually incongruent holomorphic isometries $F_t : (\Delta, ds_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$, as desired. ■

(3.3) *An example of holomorphic isometric embedding of the unit disk into a Siegel upper half-plane* In this section we construct an example of a holomorphic isometric embedding from the Poincaré disk into some Siegel upper half-plane which does not arise from examples as given in (3.2). For a positive integer g, recall that $M_s(g)$ stands for the vector space of symmetric g-by-g matrices complex, and $\mathcal{H}_g \subset M_s(g)$ for the Siegel upper half-plane of genus g, $\mathcal{H}_g := \big\{\mathfrak{T} \in M_s(g) : \mathrm{Im}(\mathfrak{T}) > 0\big\}$. We have

Proposition 3.3.1. *For $\zeta = \rho e^{i\varphi}$, $\rho > 0$, $0 < \varphi < \pi$, n a positive integer, we write $\zeta^{\frac{1}{n}} := \rho^{\frac{1}{n}} e^{\frac{i\varphi}{n}}$. Then, the holomorphic mapping $G : \mathcal{H} \to M_s(3)$ defined by*

$$G(\tau) = \begin{bmatrix} e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}} & \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & 0 \\ \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}} \end{bmatrix}$$

maps $\mathcal{H}$ into $\mathcal{H}_3$, and $G : \big(\mathcal{H}, 2ds^2_{\mathcal{H}}\big) \to \big(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\big)$ is a holomorphic isometry.

Proof. Write $\tau^{\frac{1}{3}} = \alpha + i\beta$. We have $\tau = (\alpha + i\beta)^3 = (\alpha^3 - 3\alpha\beta^2) + i(3\alpha^2\beta - \beta^3)$. In particular, $\mathrm{Im}(\tau) = 3\alpha^2\beta - \beta^3 = \beta(3\alpha^2 - \beta^2)$. Note that $\tau \in \mathcal{H}$ if and only if $0 < \mathrm{Arg}(\tau^{\frac{1}{3}}) < \frac{\pi}{3}$, i.e., $0 < \beta < \sqrt{3}\alpha$. We compute

$$e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}} = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)\left((\alpha^2 - \beta^2) + 2i\alpha\beta\right), \text{ hence}$$
$$\mathrm{Im}\left(e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}}\right) = \frac{1}{2}\left(\alpha^2 - \beta^2\right) + \sqrt{3}\alpha\beta \ ;$$

$$\sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)(\alpha + i\beta), \text{ hence } \mathrm{Im}\left(\sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}}\right) = \frac{\sqrt{6}}{2}\beta - \frac{\sqrt{2}}{2}\alpha \ ;$$
$$e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(\alpha + i\beta), \text{ hence } \mathrm{Im}\left(e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}}\right) = \frac{\sqrt{3}}{2}\alpha + \frac{\beta}{2} \ .$$

Thus,

$$\det\left(\mathrm{Im}\,G\right) = \det\begin{bmatrix} \frac{1}{2}\left(\alpha^2 - \beta^2\right) + \sqrt{3}\alpha\beta & \frac{\sqrt{6}}{2}\beta - \frac{\sqrt{2}}{2}\alpha & 0 \\ \frac{\sqrt{6}}{2}\beta - \frac{\sqrt{2}}{2}\alpha & 1 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{2}\alpha + \frac{\beta}{2} \end{bmatrix}$$
$$= \left(-2\beta^2 + 2\sqrt{3}\alpha\beta\right)\left(\frac{\sqrt{3}}{2}\alpha + \frac{\beta}{2}\right) = \beta\left(\sqrt{3}\alpha - \beta\right)\left(\sqrt{3}\alpha + \beta\right) = \beta(3\alpha^2 - \beta^2) = \mathrm{Im}(\tau) \ .$$

Write $\lambda := \frac{\beta}{\alpha}$, $0 < \lambda < \sqrt{3}$. From the above, the determinant of the upper 2-by-2 matrix of $\mathrm{Im}G$ is positive. To check positivity of $\mathrm{Im}\,G$ it suffices to note that the entry $\frac{1}{2}\left(\alpha^2 - \beta^2\right) + \sqrt{3}\alpha\beta = \frac{\alpha^2}{2}\left(1 + \lambda\left(\sqrt{3} - \lambda\right)\right) > 0$ whenever $0 < \lambda < \sqrt{3}$. Noting that the Bergman kernel of $\mathcal{H}_3$ is of the form $c(\det(\mathrm{Im}(\mathfrak{T}))^{-4}$ we have

$$G^*\omega_{\mathcal{H}_3} = -4\sqrt{-1}\partial\overline{\partial}\log\left(\det\left(\mathrm{Im}\,G(\tau)\right)\right) = -4\sqrt{-1}\partial\overline{\partial}\log\left(\mathrm{Im}(\tau)\right) = 2\omega_{\mathcal{H}} \ ,$$

proving that $G : \left(\mathcal{H}, 2ds^2_{\mathcal{H}}\right) \to \left(\mathcal{H}_3, ds^2_{\mathcal{H}_3}\right)$ is a holomorphic isometry, as desired. ■

Recall the cube-root map $\rho_3 : \mathcal{H} \to \mathcal{H} \times \mathcal{H} \times \mathcal{H}$. Realizing the latter as a totally geodesic complex submanifold in $\mathcal{H}_3$ via a standard embedding $\iota : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}_3$ where the image consists precisely of all diagonal matrices in $\mathcal{H}_3$ we have a holomorphic isometry $F := \iota \circ \rho_3 : \left(\mathcal{H}, 2ds^2_{\mathcal{H}}\right) \to (\mathcal{H}_3, ds^2_{\mathcal{H}_3})$. Note that $\iota : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H}_3$ is a holomorphic isometric embedding with respect to the Bergman metric with normalizing constant $\lambda = 2$. For the holomorphic isometry $G : \mathcal{H} \to \mathcal{H}_3$, *a priori* it is not evident that F and G are incongruent to each other. They can however be distinguished by examining the nature of the branched points on $\partial\mathcal{H}_3$. More precisely, we have

Proposition 3.3.2. *The two holomorphic isometric embeddings $F, G : (\mathcal{H}, 2ds^2_{\mathcal{H}}) \to (\mathcal{H}_3, ds^2_{\mathcal{H}_3})$, $F := \iota \circ \rho_3$, are not congruent to each other. In fact, for any holomorphic isometric embedding $h : \mathcal{H} \to \mathcal{H} \times \mathcal{H} \times \mathcal{H}$, and for $H := \iota \circ h$, the two holomorphic embeddings $G, H : (\mathcal{H}, 2ds^2_{\mathcal{H}}) \to (\mathcal{H}_3, ds^2_{\mathcal{H}_3})$ are incongruent to each other.*

Proof. Regard $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$ as an open subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and likewise the Siegel upper half-plane $\mathcal{H}_3$ canonically (via the Borel embedding) as an open subset of the compact dual M of $\mathcal{H}_3$, the map $F : \mathcal{H} \to \mathcal{H}_3$ has two branched points on $\partial(\mathcal{H} \times \mathcal{H} \times \mathcal{H})$, viz. 0

and a point at infinity, both of which lie on the Shilov boundary of $\mathcal{H}\times\mathcal{H}\times\mathcal{H}$ and hence on the Shilov boundary $Sh(\mathcal{H}_3)$ of $\mathcal{H}_3$. The branched point at infinity corresponds to the point 0 on the boundary of the image of $\widehat{F} := -F(\tau)^{-1}$. Likewise the map $G : \mathcal{H} \to \mathcal{H}_3$ has two branched points on $\partial\mathcal{H}_3$, viz., the point $F(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the branched point at infinity correspond to the branched point $\begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{bmatrix}$ of the map $\widehat{G} := -G(\tau)^{-1}$. The finite part of $Sh(\mathcal{H}_3)$ consists precisely of the real (symmetric) matrices lying on $\partial\mathcal{H}_3$. Thus the two branched points of G on $\partial\mathcal{H}_3$ do not belong to the Shilov boundary, which implies that F and G are incongruent to each other.

For the general case of $H = \iota \circ h$ in place of F, according to [Ng, Theorem 8.1], the set of all holomorphic isometries $h : \mathcal{H} \to \mathcal{H}\times\mathcal{H}\times\mathcal{H}$ up to normalizing constants are completely determined. In particular, when the normalizing constant is $\lambda = 1$, h is either congruent to the cube-root map ρ_3, or it must be congruent to $(\rho(\sqrt{\tau}), i\sqrt{\tau})$, where $\rho : \mathcal{H} \to \mathcal{H}\times\mathcal{H}$ is a holomorphic map congruent the square-root map ρ_2. ρ is singular exactly at two distinct points $b_1, b_2 \in \partial\mathcal{H} \cup \{\infty\}$. If $H = \iota \circ h$ is congruent to G as maps from $\mathcal{H}$ to $\mathcal{H}_3$, then we must have $\{b_1, b_2\} = \{0, \infty\}$, and in this case we must have $\rho = \psi \circ \mu$, where $\mu(\tau) = (\sqrt{\tau}, i\sqrt{\tau})$, and $\psi \in \mathrm{Aut}(\mathcal{H}\times\mathcal{H})$. In this case H is congruent to the map $S(\tau) = \left(\tau^{\frac{1}{4}}, i\tau^{\frac{1}{4}}, i\tau^{\frac{1}{2}}\right)$. S has exactly two branched points, the point 0 and an infinite point corresponding to the branched point 0 of the map $\widehat{S} : \mathcal{H} \to \mathcal{H}_3$ defined by $\widehat{S}(\tau) = -S(\tau)^{-1}$. In particular, both branched points of H lie on $Sh(\mathcal{H}_3)$, implying that $G, H : \mathcal{H} \to \mathcal{H}_3$ are not congruent to each other. ■

§4 **Bona fide holomorphic isometries between complete circular domains**

(4.1) In this section we explore the meaning of holomorphic isometries in a special case, viz., *bona fide* holomorphic isometries between bounded complete circular domains. Here a holomorphic mapping between two Bergman manifolds is said to be a *bona fide* isometry if and only if it is an isometry with respect to the Bergman metric, i.e., the normalizing constant is $\lambda = 1$. We will show that they lead to norm-preserving extensions of square-integrable functions which can be expressed explicitly in terms of the Bergman kernel.

For a Hilbert space H we denote by H^* its dual space. For any vector subspace $S \subset H$ we denote by $S^\perp$ the orthogonal complement of S in H, and by $S^{\mathrm{Ann}} \subset H^*$ the annihilator of S consisting of continuous linear functionals on H vanishing on S.

For a bounded Euclidean domain G, we write $\Psi_G : G \hookrightarrow \mathbb{P}(H^2(G)^*)$ for the canonical embedding on G, $G^\natural \subset \mathbb{P}(H^2(G)^*)$ for its image $\Psi_G(G)$, to be called the canonical image. For $z \in G$, we denote by $\widehat{z} \in H^2(G)^*$ the continuous linear function on $H^2(G)$ given by $\widehat{z}(f) = f(z)$ for any $f \in H^2(G)$. Fixing an orthonormal basis $\left(h_i\right)_{i=0}^{\infty}$ and denoting by $\mathbb{H}$ the Hilbert space of square-integrable sequences of complex numbers, we also write $\Phi_G(z) = (h_0(z), \cdots, h_i(z), \cdots) \in \mathbb{H}$, and write $\Psi_G(z) = [\Phi_G(z)] \in \mathbb{P}(\mathbb{H})$.

Lemma 4.1.1. *$G^\natural \subset \mathbb{P}(H^2(G)^*)$ is topologically linearly non-degenerate, i.e., denoting by* $\mathrm{Span}(G^\natural) \subset \mathbb{P}(H^2(G)^*)$ *the projective linear span of $G^\natural$, we have* $\overline{\mathrm{Span}(G^\natural)} = \mathbb{P}(H^2(G)^*)$ *for its topological closure.*

Proof. Let $(a_0, \cdots, a_i, \cdots)$ be a square-integrable sequence of complex numbers orthogonal to the image of Φ_G. Then, writing $h := \overline{a_0}h_0 + \cdots + \overline{a_i}h_i + \cdots \in H^2(G)$ we have $h(z) = 0$ for every $z \in G$, which is absurd unless $a_i = 0$ for $0 \le i < \infty$, as desired. ■

Let now $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded complete circular domains. Suppose $F : D \to \Omega$ is a *bona fide* holomorphic isometric embedding with respect to the Bergman metric. Identifying D, resp. Ω, with its canonical image $D^\natural \subset \mathbb{P}(H^2(D)^*)$, resp. $\Omega^\natural \subset \mathbb{P}(H^2(\Omega)^*)$, $F : D \to \Omega$ corresponds to a holomorphic isometry $F^\natural : D^\natural \to \Omega^\natural$. Since $D^\natural \subset \mathbb{P}(H^2(D)^*)$ and $\Omega^\natural \subset \mathbb{P}(H^2(\Omega)^*)$ are topologically linearly non-degenerate, by Calabi [Ca], F^* is induced by some linear isometry $\Theta : H^2(D)^* \to H^2(\Omega)^*$. Identifying a Hilbert space with its dual by a conjugate linear map, Θ is equivalently given by a linear isometry $\mu : H^2(D) \to H^2(\Omega)$ onto a Hilbert subspace. In the case at hand, we determine μ in terms of the Bergman kernels, as follows.

Theorem 4.1.1. *Let $D \Subset \mathbb{C}^n$, resp. $\Omega \Subset \mathbb{C}^N$, be a complete circular domain, and assume that $tD \subset D$ and $t\Omega \subset \Omega$ for $0 < t < 1$. Let $F : (D, ds^2_D) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding, $F(0) = 0$. Write $Z := F(D) \subset \Omega$, and denote by $F^{-1} : Z \to D$ the inverse of $F : D \to Z$. Define $J := \{g \in H^2(\Omega) : g|_Z \equiv 0\}$. Then, for the canonical embedding $\Psi_\Omega : \Omega \hookrightarrow \mathbb{P}(H^2(\Omega)^*)$, we have $\overline{\mathrm{Span}(\Psi_\Omega(Z))} = \mathbb{P}(J^{\mathrm{Ann}})$. Moreover, the holomorphic isometry F is induced by a linear isometry $\mu : H^2(D) \to H^2(\Omega)$ such that $\mu(s)|_Z = s \circ F^{-1}$ for any $s \in H^2(D)$ and such that $E := Im(\mu) = J^\perp$.*

We write $K_{D,w}(z) := K_D(z, w)$; $K_{\Omega,\xi}(\zeta) := K_\Omega(\zeta, \xi)$. For $J \subset H^2(\Omega)$ we have

Lemma 4.1.2. *For the Hilbert subspace $J \subset H^2(\Omega)$ consisting of square-integrable holomorphic functions vanishing on Z, we have $J^\perp = \overline{\mathrm{Span}\left(\{K_{\Omega,\zeta} : \zeta \in Z\}\right)}$.*

Proof. By the reproducing property of K_Ω, we have $h(\zeta) = \int_\Omega K_\Omega(\zeta, \xi) h(\xi) dV(\xi)$, for any $h \in H^2(\Omega)$, where dV denotes the Euclidean volume form. Thus, for any $\zeta \in \Omega$, $h(\zeta) = 0$ whenever $h \perp K_{\Omega,\zeta}$, hence $h \in J$ whenever $h \perp K_{\Omega,\zeta}$ for every $\zeta \in Z$. It follows that $J^\perp$ is the minimal Hilbert subspace of $H^2(\Omega)$ containing $K_{\Omega,\zeta}$ for each $\zeta \in Z$, i.e., the topological closure of the linear span of $\{K_{\Omega,\zeta} : \zeta \in Z\}$, as desired. ■

Proof of Theorem 4.1.1. Without loss of generality we may assume that both $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are of Euclidean volume equal to 1, so that $K_D(z, 0) = 1$ for $z \in D$ and $K_\Omega(\zeta, 0) = 1$ for $\zeta \in \Omega$. By Proposition 1.1.1, $K_\Omega(F(z), F(w)) = K_D(z, w)$ for any $z, w \in D$. By the reproducing property of $K_D(z, w)$, for $s \in H^2(D)$ we have

$$s(z) = \int_D K_D(z, w) s(w) dV(w) \ , \tag{1}$$

where dV denotes the Euclidean volume form. For $0 < t < 1$ and $s \in H^2(D)$ define

$$\mu_t(s)(\zeta) = \int_D K_\Omega(\zeta, F(tw)) s(w) dV(w) \ , \tag{2}$$

noting that for $0 < t < 1$ the right-hand side is well-defined since in fact $tD \Subset D$, so that $K_\Omega(\zeta, F(tw))$ is bounded as a function in $w \in D$, and we have $\mu_t(s) \in H^2(\Omega)$ since $\left\|K_{\Omega,F(tw)}\right\|_{H^2(\Omega)}$ is uniformly bounded for $w \in D$. On the other hand, the right-hand side of (2) is *a priori* undefined when $t = 1$ since the holomorphic function $\varphi(w) := K_\Omega(F(w), \zeta)$ is not known to be in $H^2(D)$. We are going to show nonetheless that, as $t \to 1^-$, $\mu_t : H^2(D) \to H^2(\Omega)$ converges weakly to some linear isometry $\mu : H^2(D) \to H^2(\Omega)$. For $s \in H^2(D)$ and $0 < t < 1$, write $h_t = \mu_t(s)$. Then,

$$\begin{aligned} \left\|h_t\right\|^2_{H^2(\Omega)} &= \int_\Omega \Big(\int_D K_\Omega(\zeta, F(tw')) s(w') dV(w') \Big) \overline{\Big(\int_D K_\Omega(\zeta, F(tw)) s(w) dV(w) \Big)} dV(\zeta) \\ &= \int_D \left(\int_D \left(\int_\Omega K_\Omega(F(tw), \zeta) K_\Omega(\zeta, F(tw')) dV(\zeta) \right) s(w') dV(w') \right) \overline{s(w)} dV(w) \, . \end{aligned} \tag{3}$$

Since $t\Omega \subset \Omega$, by Lemma 1.1.1 it follows that $K_\Omega(\zeta, \xi)$ extends holomorphically in $(\zeta, \overline{\xi})$ to some neighborhood of $\overline{\Omega} \times t\Omega$. Hence $K_\Omega(\zeta, F(tw))$ is uniformly bounded on $\Omega \times D$, which justifies the change of order of integration by Fubini's theorem in (3). By the reproducing property of $K_\Omega(\zeta, \xi)$ applied to $\theta(\zeta) := K_\Omega(\zeta, F(tw')))$ on Ω, we have

$$\begin{aligned}\int_\Omega K_\Omega(F(tw), \zeta) K_\Omega(\zeta, F(tw')) dV(\zeta) &= \int_\Omega K_\Omega(F(tw), \zeta)\theta(\zeta) dV(\zeta) \\ &= \theta(F(tw)) = K_\Omega(F(tw), F(tw')) \ . \end{aligned} \tag{4}$$

Thus, we have

$$\begin{aligned}\int_\Omega \left|h_t(\zeta)\right|^2 dV(\zeta) &= \int_D \left(\int_D K_\Omega(F(tw), F(tw')) s(w') dV(w') \right) \overline{s(w)} dV(w) \\ &= \int_D \left(\int_D K_D(tw, tw') s(w') dV(w') \right) \overline{s(w)} dV(w) \\ &= \int_D \left(\int_D K_D(t^2 w, w') s(w') dV(w') \right) \overline{s(w)} dV(w) = \int_D s(t^2 w) \overline{s(w)} dV(w) \ . \end{aligned} \tag{5}$$

By exactly the same arguments, for $0 < t_1, t_2 < 1$ we have

$$\left\langle \mu_{t_1}(s), \overline{\mu_{t_2}(s)} \right\rangle_{H^2(\Omega)} = \int_\Omega h_{t_1}(\zeta) \overline{h_{t_2}(\zeta)} dV(\zeta) = \int_D s(t_1 t_2 w) \overline{s(w)} dV(w) \ ; \tag{6}$$

$$\left\| \mu_{t_1}(s) - \mu_{t_2}(s) \right\|^2_{H^2(\Omega)} = \int_D \left(s(t_1^2 w) + s(t_2^2 w) - 2s(t_1 t_2 w) \right) \overline{s(w)} dV(w) \ . \tag{7}$$

As $t_1, t_2 \to 1^-$, the function $\delta_{t_1,t_2}(s) : s(t_1^2 w) + s(t_2^2 w) - 2s(t_1 t_2 w)$ tends to 0 in $H^2(D)$, hence $\left\| \mu_{t_1}(s) - \mu_{t_2}(s) \right\|_{H^2(\Omega)}$ converges to 0. As a consequence, the weak limit μ of $\mu_t : H^2(D) \to H^2(\Omega)$ exists. By (5), $\|\mu(s)\|_{H^2(\Omega)} = \|s\|_{H^2(D)}$, i.e., $\mu : H^2(D) \to H^2(\Omega)$ is a Hilbert space isomorphism onto some Hilbert subspace $E \subset H^2(\Omega)$, which we proceed to identify. From the definition of μ_t for $0 < t < 1$ in (2), $\mu_t(s)$ is a limit in $H^2(\Omega)$ of linear combinations of $K_{\Omega, F(tw)}$ as w ranges over D. Now $f \in H^2(\Omega)$ is orthogonal to $\mathrm{Im}(\mu_t) := E_t \subset H^2(\Omega)$ whenever it vanishes at every point of $F(tw), w \in D$ (cf. proof of Lemma 4.1.2). Since $J = \left\{ f \in H^2(\Omega) : f|_Z \equiv 0 \right\}$, for $0 < t < 1$ we have $J \subset E_t^\perp$, and hence $J \subset E^\perp$ when one passes to the limit as $t \to 1^-$, i.e., $E \subset J^\perp$. From (2) and the reproducing property of $K_D(z, w)$, for $\zeta \in \Omega$ and $w \in D$ we have

$$\begin{aligned}\mu_t(K_{D,w})(\zeta) &= \overline{\int_D K_D(w, w') K_\Omega(F(tw'), \zeta)) dV(w')} \\ &= \overline{K_\Omega(F(tw), \zeta)} = K_\Omega(\zeta, F(tw)) = K_{\Omega, F(tw)}(\zeta) \, . \end{aligned} \tag{8}$$

Hence,

$$\mu(K_{D,w}) = \lim_{t \to 1^-} \mu_t(K_{D,w}) = \lim_{t \to 1^-} K_{\Omega, F(tw)} = K_{\Omega, F(w)} \ . \tag{9}$$

As a result, E contains $\overline{\mathrm{Span}\left\{ K_{\Omega, \zeta} : \zeta \in Z \right\}}$, which is precisely $J^\perp$. Thus, $E \supset J^\perp$ and hence $E = J^\perp$. Recall that for $z \in D$, $\widehat{z} \in H^2(D)^*$ is identified with $\Phi_D(z)$, and similarly for $\zeta \in \Omega$, $\widehat{\zeta} \in H^2(\Omega)^*$ is identified with $\Phi_\Omega(\zeta)$. Denoting by $\nu : E \to H^2(D)$ the inverse isomorphism of $\mu : H^2(D) \to E$, for the adjoint operator $\nu^* : H^2(D)^* \to E^*$ we have

$$\nu^*(\widehat{z})(K_{\Omega, F(w)}) = \widehat{z}(K_{D,w}) = K_D(z, w) = K_\Omega(F(z), F(w)) = \widehat{F(z)}(K_{\Omega, F(w)}) \, , \tag{10}$$

which gives $\Theta : H^2(D)^* \to H^2(\Omega)^*$ inducing the holomorphic isometry $F^\natural : D^\natural \to \Omega^\natural$ when we define $\Theta(\lambda)(f) = 0$ for any $\lambda \in H^2(D)^*$ and $f \in J$. $\overline{\Theta(H^2(D)^*)}$ is then precisely J^{Ann}, and $\overline{\mathrm{Span}(F^\natural(D^\natural)} = \mathbb{P}(\Theta(H^2(D)^*)) = \mathbb{P}(J^{\mathrm{Ann}})$. Finally, for $0 < t < 1$

$$\begin{aligned}\mu_t(s)(F(z)) = \int_D K_\Omega(F(z), F(tw))s(w)dV(w) &= \int_D K_D(z, tw)s(w)dV(w) \\ &= \int_D K_D(tz, w)s(w)dV(w) = s(tz)\ ; \end{aligned} \tag{11}$$

$$\mu(s)(F(z)) = \lim_{t\to 1^-} \mu_t(s)(F(z)) = \lim_{t\to 1^-} s(tz) = s(z); \quad \text{hence} \quad \mu(s)|_Z = s \circ F^{-1}\ , \tag{12}$$

completing the proof of Theorem 4.1.1. ■

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