

Cycle spaces of flag domains on Grassmannians and rigidity of holomorphic mappings

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1 Introduction

1.1 Background and statement of results

In this article we study certain cycle spaces on a particular type of flag domains on Grassmannians and their relations with the rigidities of holomorphic mappings between Grassmannians. There is an extensive literature on the studies of cycle spaces of flag domains, see [1] for a comprehensive account on the subject. The structural results on these cycle spaces have found many applications in other areas, for instance, on moduli problems on marked K3 surfaces and variation of Hodge structure. The main purpose of the current article is to illustrate how the structure of cycle spaces may lead to rigidities of holomorphic mappings between flag domains and their ambient rational homogeneous spaces.

Denote by $G_{k+1,n-k}$ the Grassmannian of $(k+1)$ -planes in \mathbb{C}^{n+1} . Let $\ell \in \{0, \dots, n-1\}$, we equip \mathbb{C}^{n+1} with the standard non-degenerate Hermitian form $H_{\ell+1}$ of signature $(\ell+1, n-\ell)$, where $\ell+1$ eigenvalues are 1 and the other $n-\ell$ eigenvalues are -1 . The flag domains we are looking at are the domains in $G_{k+1,n-k}$ corresponding to the $(k+1)$ -planes in \mathbb{C}^{n+1} on which the restriction of $H_{\ell+1}$ is positive definite. In particular, we require that $\ell \geq k$. They are open orbits on $G_{k+1,n-k}$ under the actions of the real forms $SU(\ell+1, n-\ell)$ of $SL(n+1, \mathbb{C})$ which acts on $G_{k+1,n-k}$ as biholomorphisms. We denote these domains by $\mathbb{D}_{k,n}^{\ell}$. In the later section we will also give a definition of $\mathbb{D}_{k,n}^{\ell}$ using the homogeneous coordinates in $G_{k+1,n-k}$. When $\ell = k$, $\mathbb{D}_{k,n}^{\ell} = \mathbb{D}_{k,n}^k$ can be recognized as the type-I irreducible bounded symmetric domain $\Omega_{k+1,n-k}$ embedded in $G_{k+1,n-k}$ by the Borel embedding. We will exclude this case because the domain (being a domain in a Euclidean space) does not contain any non-trivial compact subvarieties. In addition, we actually need the cycle spaces to *extend across the boundary*, more precisely, we will look at those flag domains whose boundaries also contain similar compact subvarieties as the domains. For this reason we will also exclude the cases with $\ell \geq n-k-1$ as there the boundaries of the flag domains do not contain any compact subvarieties. Thus, in summary, we will be dealing with the domains $\mathbb{D}_{k,n}^{\ell} \subset G_{k+1,n-k}$, where $k+1 \leq \ell \leq n-k-2$.

Since the boundaries of the flag domains play a crucial role in the study, we will first make some general remarks on their structures. For each fixed $\mathbb{D}_{k,n}^{\ell}$, its boundary stratifies into $k+1$ orbits under the action of $SU(\ell+1, n-\ell)$. We denote these strata by $\partial_r \mathbb{D}_{k,n}^{\ell}$,

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$1 \leq r \leq k + 1$, where each $\partial_r \mathbb{D}_{k,n}^\ell$ is the orbit of $SU(\ell + 1, n - \ell)$ corresponding to the set of $(k + 1)$ -planes in \mathbb{C}^{n+1} on which the restriction of $H_{\ell+1}$ is semi-positive definite with the dimension of the null space being r . The most important stratum for us is $\partial_{k+1} \mathbb{D}_{k,n}^\ell$ and we will also denote it by $\Sigma(\mathbb{D}_{k,n}^\ell)$. Note that for $\mathbb{D}_{k,n}^k$, which is a bounded symmetric domain, $\Sigma(\mathbb{D}_{k,n}^k)$ is just the Shilov boundary. We can now state our main result.

Theorem 1.1. *Let $\ell \in \{k + 1, \dots, n - k - 2\}$. Let $U \subset G_{k+1, n-k}$ be a connected open set and $U \cap \Sigma(\mathbb{D}_{k,n}^\ell) \neq \emptyset$. Suppose $f : U \rightarrow G_{k+1, m-k}$ is a holomorphic embedding in one of the following cases:*

- (i) $k + 1 \leq \ell < \frac{n}{2}$, $f(U \cap \Sigma(\mathbb{D}_{k,n}^\ell)) \subset \Sigma(\mathbb{D}_{k,m}^\ell)$;
- (ii) $\frac{n}{2} \leq \ell \leq n - k - 2$, $f(U \cap \Sigma(\mathbb{D}_{k,n}^\ell)) \subset \Sigma(\mathbb{D}_{k,m}^{\ell+m-n})$,

then f extends to a standard embedding of $G_{k+1, n-k}$ into $G_{k+1, m-k}$. If furthermore, we have either

- (i) $k + 1 \leq \ell < \frac{n}{2}$, $f(U \cap \mathbb{D}_{k,n}^\ell) \subset \mathbb{D}_{k,m}^\ell$; or
- (ii) $\frac{n}{2} \leq \ell \leq n - k - 2$, $f(U \cap \mathbb{D}_{k,n}^\ell) \subset \mathbb{D}_{k,m}^{\ell+m-n}$,

then the extension restricting on $\mathbb{D}_{k,n}^\ell$ is a proper holomorphic embedding into $\mathbb{D}_{k,m}^\ell$ in case (i) and $\mathbb{D}_{k,m}^{\ell+m-n}$ in case (ii).

In the above theorem, by a standard embedding between two Grassmannians, we mean an embedding induced by a linear embedding of the corresponding Euclidean spaces.

For $k = 0$, Theorem 1.1 covers half of the cases in the following theorem of Baouendi-Huang [2] on the rigidity of Cauchy-Riemann mappings between real hyperquadrics with positive signatures in projective spaces. To simplify the notations, we will write \mathbb{D}_n^ℓ for the flag domains $\mathbb{D}_{0,n}^\ell$ on \mathbb{P}^n (In Baouendi-Huang [2], the notation is \mathbb{B}_ℓ^n).

Theorem 1.2 (Baouendi-Huang). *Let $1 \leq \ell \leq n - 2$ and $U \subset \mathbb{P}^n$ be a connected open set such that $U \cap \partial \mathbb{D}_n^\ell \neq \emptyset$. Suppose $f : U \rightarrow \mathbb{P}^m$ is a holomorphic mapping satisfying either (i) $f(U \cap \partial \mathbb{D}_n^\ell) \subset \partial \mathbb{D}_m^\ell$; or (ii) $f(U \cap \partial \mathbb{D}_n^\ell) \subset \partial \mathbb{D}_m^{\ell+m-n}$, then f extends to a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^m . If furthermore, we have either in case (i) $f(U \cap \mathbb{D}_n^\ell) \subset \mathbb{D}_m^\ell$; or in case (ii) $f(U \cap \mathbb{D}_n^\ell) \subset \mathbb{D}_m^{\ell+m-n}$, then the extension restricting on \mathbb{D}_n^ℓ is a proper holomorphic embedding into \mathbb{D}_m^ℓ in case (i) and into $\mathbb{D}_m^{\ell+m-n}$ in case (ii).*

In the first half of the article we single out the case $k = 0$, i.e. the flag domains \mathbb{D}_n^ℓ in the projective space \mathbb{P}^n . The reason for this is twofold. On one hand, the cycles being considered in this case are linear subvarieties in \mathbb{P}^n whose structures and description are easier and hence better for illustrating the basic principles employed in the article. On the other hand, there are some major differences between the extension theorems on the projective spaces and on the Grassmannians of higher rank that we are going to apply. These differences will make themselves transparent later in the article.

We now give the scheme of proof of Theorem 1.1 for $k = 0$ in case (i). Case (ii) is similar. We first determine the set of maximal linear subvarieties contained in \mathbb{D}_n^ℓ and its boundary. They will be shown to be of dimension ℓ . The corresponding cycle space turns out to be the closure $\overline{\mathbb{D}}_{\ell,n}^\ell$ in $G_{\ell+1, n-\ell}$ which is just the closure of the type-I bounded symmetric domain $\Omega_{\ell+1, n-\ell}$ in $\mathbb{C}^{(\ell+1)(n-\ell)}$. This can be obtained by recalling how the domains \mathbb{D}_n^ℓ are defined using the non-degenerate Hermitian forms in \mathbb{C}^{n+1} . However, in the article we use explicit

matrix computations to obtain the result. This first of all provides us with an easy way to specify a particular linear subvariety and is at the same time important for our analysis on the boundary. After determining the cycle space of maximal linear subvarieties, we show that any maximal linear subvariety in the boundary must be mapped into some maximal linear subvariety in the boundary of the target domain by the holomorphic maps satisfying the stated assumptions in the theorem. Then the main idea is to apply a function theoretic lemma to the cycle space and deduce that the maximal linear subvarieties in the domain must also be mapped into the maximal linear subvarieties in the target domain. This implies that lines in the domain are mapped to lines in the target and then we can apply a standard extension theorem in algebraic geometry to conclude the proof.

When $k \geq 1$, we basically follow the same principle and we need to study the structures of certain totally geodesic subgrassmannians in $\mathbb{D}_{k,n}^\ell$. By demonstrating the preservation of these subgrassmannians under a mapping satisfying the hypotheses, we deduce that such a mapping preserves a distinguished set of tangent directions at each point on the flag domain. Then the last step is to invoke a Cartan-Fubini type extension theorem of Mok [3] on the characterization of standard embeddings between Grassmannians.

It could be interesting to put Theorem 1.1 in another perspective. As has been mentioned the boundary stratum $\Sigma(\mathbb{D}_{k,n}^\ell)$ is a generalization of the Shilov boundary of bounded symmetric domains to the flag domains $\mathbb{D}_{k,n}^\ell$. We recall first of all the following theorem of Alexander [4] (for rank = 1) and Henkin-Tumanov [5] (for rank ≥ 2) on local characterizations of the automorphisms of irreducible bounded symmetric domains of dimension at least two in relation to the Shilov boundary.

Theorem 1.3 (Alexander, Henkin-Tumanov). *Let $N \geq 2$, $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain and $S(\Omega)$ be its Shilov boundary. Let $U \subset \mathbb{C}^N$ be a connected open set such that $U \cap S(\Omega) \neq \emptyset$. Suppose $f : U \rightarrow \mathbb{C}^N$ is a local biholomorphism satisfying $f(U \cap \Omega) \subset \Omega$ and $f(U \cap S(\Omega)) \subset S(\Omega)$, then f extends to an automorphism of Ω .*

Now the equidimensional case of Theorem 1.1 can be regarded as an analogue theorem of Alexander-Henkin-Tumanov for the flag domains $\mathbb{D}_{k,n}^\ell$.

Corollary 1.4 (of Theorem 1.1). *Let $\ell \in \{k+1, \dots, n-k-2\}$. Let $U \subset G_{k+1, n-k}$ be a connected open set and $U \cap \Sigma(\mathbb{D}_{k,n}^\ell) \neq \emptyset$. Suppose $f : U \rightarrow G_{k+1, n-k}$ is a local biholomorphism satisfying $f(U \cap \mathbb{D}_{k,n}^\ell) \subset \mathbb{D}_{k,n}^\ell$ and $f(U \cap \Sigma(\mathbb{D}_{k,n}^\ell)) \subset \Sigma(\mathbb{D}_{k,n}^\ell)$, then f extends to an automorphism of $\mathbb{D}_{k,n}^\ell$.*

Remarks. The special case (for $\Omega = \Omega_{k+1, n-k}$, $n \geq 2$) of the theorem of Alexander-Henkin-Tumanov can be incorporated to the above corollary for the case $\ell = k$. However, our method is not directly applicable to bounded symmetric domains.

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1.2 Notations and Definitions

Matrix notations

We use $M(p, q; \mathbb{C})$ to denote be the set of p by q complex matrices and $M(n; \mathbb{C}) := M(n, n; \mathbb{C})$. For a matrix A , we will denote its Hermitian transpose by A^H . If A is a square matrix, we will write $A > 0$ (resp. $A \geq 0$) to signify that A is positive definite (resp. positive semi-definite). We use the notations $A < 0$ and $A \leq 0$ similarly.

Grassmannians

Denote by $G_{p,q}$ the Grassmannian of p -planes in \mathbb{C}^{p+q} . As a Riemannian symmetric space, the rank of $G_{p,q}$ is $\min\{p, q\}$. For good notation consistency in the article, we will almost always look at $G_{k+1, n-k}$, $0 \leq k < n$, i.e. set of $(k+1)$ -planes in \mathbb{C}^{n+1} . A homogeneous coordinates matrix for a point $Z \in G_{k+1, n-k}$ is a matrix $[Z] \in M(k+1, n+1; \mathbb{C})$,

$$[Z] = \begin{bmatrix} z_{1,0} & \cdots & z_{1,n} \\ \vdots & \ddots & \vdots \\ z_{k+1,0} & \cdots & z_{k+1,n} \end{bmatrix},$$

where the row vectors of $[Z]$ constitute a basis for the $(k+1)$ -plane Z . Homogeneous coordinates matrices are only uniquely determined up to left multiplications by non-singular matrices in $M(k+1; \mathbb{C})$.

Flag domains

Let $Z \in G_{k+1, n-k}$ and $[Z] \in M(k+1, n+1; \mathbb{C})$ be a homogeneous coordinates matrix for Z . For $j \in \{0, \dots, n\}$, we use the notation $[Z] = [Z', Z'']_j$ to split $[Z]$ into two block matrices with $Z' \in M(k+1, j+1; \mathbb{C})$ and $Z'' \in M(k+1, n-j; \mathbb{C})$. For $\ell \in \{k, \dots, n\}$, define

$$\mathbb{D}_{k,n}^\ell = \{[Z', Z'']_\ell \in G_{k+1, n-k} : Z' Z'^H - Z'' Z''^H > 0\}.$$

It is easy to see that the validity of the above matrix inequality is independent of the choice of the homogeneous coordinates matrix. Moreover, it is also clear that the above definition coincide with the one appears at the beginning of the introduction. That is, $\mathbb{D}_{k,n}^\ell$ is the set of $(k+1)$ -planes in \mathbb{C}^{n+1} on which the restriction of the non-degenerate Hermitian form $H_{\ell+1}$ is positive definite.

When $k = 0$, we will write $\mathbb{D}_{0,n}^\ell$ as \mathbb{D}_n^ℓ . In addition, we split the homogeneous coordinates of \mathbb{P}^n as $[\mathbf{z}', \mathbf{z}'']_\ell$, where $\mathbf{z}' = (z_0, \dots, z_\ell)$ and $\mathbf{z}'' = (z_{\ell+1}, \dots, z_n)$. Let $\|\mathbf{z}'\|^2 = \sum_{i=0}^{\ell} |z_i|^2$ and

$$\|\mathbf{z}''\|^2 = \sum_{j=\ell+1}^n |z_j|^2. \text{ Then we have } \mathbb{D}_n^\ell = \{[\mathbf{z}', \mathbf{z}'']_\ell \in \mathbb{P}^n : \|\mathbf{z}'\|^2 > \|\mathbf{z}''\|^2\}.$$

Particular cases:

(1) $\mathbb{D}_{k,n}^n = G_{k+1, n-k}$.

(2) $\mathbb{D}_{k,n}^k \subset G_{k+1, n-k}$ is the non-compact dual of $G_{k+1, n-k}$ as a Hermitian symmetric space. That is, $\mathbb{D}_{k,n}^k \cong \Omega_{k+1, n-k}$, the type-I irreducible bounded symmetric domain defined by $\{A \in M(k+1, n-k; \mathbb{C}) : AA^H - I < 0\}$.

2 On projective spaces

2.1 Projective linear subspaces in $\overline{\mathbb{D}}_n^\ell$

In this section, we will determine all linear subspaces contained in \mathbb{D}_n^ℓ and its boundary. We will first consider the maximal linear subspaces.

2.1.1 Maximal linear subspaces

Write the homogeneous coordinates in \mathbb{P}^n as $[\mathbf{z}', \mathbf{z}'']_\ell$. Let $L' \subset \mathbb{P}^n$ (resp. $L'' \subset \mathbb{P}^n$) be the linear subspace defined by the equation $\mathbf{z}' = 0$ (resp. $\mathbf{z}'' = 0$). Let also $U' = \mathbb{P}^n - L'$. Consider the projection map $\pi : U' \rightarrow L''$ given by $[\mathbf{z}', \mathbf{z}'']_\ell \mapsto [\mathbf{z}', 0]_\ell$.

Lemma 2.1. *Let $X \subset U'$ be a compact subvariety. Then the projection $\pi|_X : X \rightarrow L''$ is a finite map.*

Proof. Fix a point $[\mathbf{a}] \in \mathbb{P}^\ell$ and consider the preimage $\pi^{-1}([\mathbf{a}, \mathbf{0}]_\ell)$. Its closure in \mathbb{P}^n is the set $\{[z\mathbf{a}, \mathbf{z}'']_\ell \in \mathbb{P}^n : [z, \mathbf{z}'''] \in \mathbb{P}^{n-\ell}\} \cong \mathbb{P}^{n-\ell}$, we denote it by $\mathbb{P}_{\mathbf{a}}^{n-\ell}$.

Let $X \subset U'$ be a compact subvariety, in particular, $X \cap L' = \emptyset$. Consider the intersection $X_{\mathbf{a}} := X \cap \mathbb{P}_{\mathbf{a}}^{n-\ell}$. If $X_{\mathbf{a}}$ is at least one dimensional, then $X_{\mathbf{a}} \cap L' \neq \emptyset$ as L' is a hyperplane in $\mathbb{P}_{\mathbf{a}}^{n-\ell}$. This contradicts $X \cap L' = \emptyset$.

Thus, $X_{\mathbf{a}}$ is a finite set. By the construction of $\mathbb{P}_{\mathbf{a}}^{n-\ell}$, this precisely means that the preimage $\pi|_X^{-1}([\mathbf{a}, \mathbf{0}]_\ell)$ is a finite set in X . The proof is complete. \square

Proposition 2.2. *Let $1 \leq \ell < n$. The domain $\mathbb{D}_n^\ell \subset \mathbb{P}^n$ contains a family of ℓ -dimensional projective linear subspaces. They are compact complex analytic subvarieties in \mathbb{D}_n^ℓ of the maximal dimension. Moreover, the set of all such \mathbb{P}^ℓ is parametrized by the points in the irreducible bounded symmetric domain $\Omega_{\ell+1, n-\ell}$.*

Proof. We first give a family of projective linear subspaces \mathbb{P}^ℓ in \mathbb{D}_n^ℓ parametrized by $\Omega_{\ell+1, n-\ell}$, as follows. Let $A \in M(\ell+1, n-\ell; \mathbb{C})$. Consider the ℓ -dimensional linear subspace

$$\{[\mathbf{z}', \mathbf{z}'']_\ell \in \mathbb{P}^n : \mathbf{z}'' = \mathbf{z}'A\} \cong \mathbb{P}^\ell \subset \mathbb{P}^n.$$

Then as $\mathbf{z}'\mathbf{z}'^H > \mathbf{z}'AA^H\mathbf{z}'^H$ for all \mathbf{z}' if and only if $I - AA^H > 0$, we see that such \mathbb{P}^ℓ is contained in \mathbb{D}_n^ℓ if and only if $A \in \Omega_{\ell+1, n-\ell}$. In addition, it is clear that a different choice of A will give different linear subspace.

Now suppose X is an irreducible compact subvariety contained in \mathbb{D}_n^ℓ , in particular, $X \subset U'$. By Lemma 2.1, the projection $\pi : X \rightarrow L''$ is a finite map and hence $\dim(X) \leq \dim(L'') = \ell$. This shows that the ℓ -dimensional linear subspaces we constructed above are of the maximal dimension among compact subvarieties of \mathbb{D}_n^ℓ . Now if X is an ℓ -dimensional linear subspace, then Lemma 2.1 implies that $\pi : X \rightarrow L''$ is biholomorphic and therefore we have proved that any ℓ -dimensional linear subspace contained in \mathbb{D}_n^ℓ is of the form $\{[\mathbf{z}', \mathbf{z}'']_\ell : \mathbf{z}'' = \mathbf{z}'A\}$, $A \in \Omega_{\ell+1, n-\ell}$. \square

From the previous proposition, we see that the points in $\Omega_{\ell+1, n-\ell}$ and the maximal linear subspaces contained in \mathbb{D}_n^ℓ are in one to one correspondence. It is natural to extend this correspondence to the closure $\overline{\Omega}_{\ell+1, n-\ell}$. In fact, by replacing the strict inequalities by non-strict inequalities, the same proof gives the following

Proposition 2.3. *Let $1 \leq \ell < n$. The closure $\overline{\mathbb{D}}_n^\ell \subset \mathbb{P}^n$ contains a family of ℓ -dimensional projective linear subspaces. They are maximal compact complex analytic subvarieties in $\overline{\mathbb{D}}_n^\ell$. Moreover, the set of all such \mathbb{P}^ℓ is parametrized by the points in $\overline{\Omega}_{\ell+1, n-\ell}$.*

We can say more about the boundary $\partial\mathbb{D}_n^\ell$ and the boundary $\partial\Omega_{\ell+1, n-\ell}$ of the cycle space.

Lemma 2.4. *The maximal compact complex analytic subvarieties in $\partial\mathbb{D}_n^\ell$ are ℓ -dimensional projective linear subspaces when $\ell < n/2$ and $(n-\ell-1)$ -dimensional projective linear subspaces when $\ell \geq n/2$. Moreover, any germ of complex submanifold lying inside $\partial\mathbb{D}_n^\ell$ must be contained in one of these projective linear subspaces.*

Proof. It suffices to give the proof for the case $\ell < n/2$ because the complement of $\overline{\mathbb{D}}_n^\ell$ in \mathbb{P}^n is $\mathbb{D}_n^{n-\ell-1}$ by a change of coordinates. In homogeneous coordinates $[z_0, \dots, z_n]$, $\partial\mathbb{D}_n^\ell \subset \mathbb{P}^n$ is defined by the equation

$$\sum_{i=0}^{\ell} |z_i|^2 = \sum_{j=\ell+1}^n |z_j|^2.$$

When $\ell < n/2$, let $A \in M(\ell+1, n-\ell; \mathbb{C})$ and $AA^H = I$. It is clear that the ℓ -dimensional linear subspace $\{[\mathbf{z}', \mathbf{z}'']_\ell \in \mathbb{P}^n : \mathbf{z}'' = \mathbf{z}'A\}$ is contained in $\partial\mathbb{D}_n^\ell$.

Suppose we are given a germ of holomorphic curve $g : \Delta \rightarrow \partial\mathbb{D}_n^\ell$, where Δ is the unit disk. Without loss of generality, we may assume that $g(\Delta) \subset \partial\mathbb{D}_n^\ell \cap U$, where $U = \mathbb{P}^n - \{z_0 = 0\} \cong \mathbb{C}^n$. Expressing in inhomogeneous coordinates, $g = (g_1, \dots, g_n)$ will satisfy the equation

$$1 + \sum_{i=1}^{\ell} |g_i|^2 = \sum_{j=\ell+1}^n |g_j|^2.$$

This implies $(g_{\ell+1}, \dots, g_n) = (1, g_1, \dots, g_\ell)A$ for some $A \in M(\ell+1, n-\ell; \mathbb{C})$ and $AA^H = I$. (For an elementary proof of this fact, see [6].) We therefore see that such a germ must be contained in one of the ℓ -dimensional linear subspaces described above. By considering instead a holomorphic map from a polydisk into $\partial\mathbb{D}_n^\ell$, the same argument works and thus the same conclusion holds for any germ of complex submanifold contained in $\partial\mathbb{D}_n^\ell$. This in particular has also proved that those linear subspaces are maximal subvarieties in $\partial\mathbb{D}_n^\ell$. \square

Proposition 2.5. *In Proposition 2.3, the boundary points of $\overline{\Omega}_{\ell+1, n-\ell}$ correspond to those projective linear subspaces which intersect $\partial\mathbb{D}_n^\ell$. Furthermore, the points on the Shilov boundary of $\overline{\Omega}_{\ell+1, n-\ell}$ correspond to those whose intersection with $\partial\mathbb{D}_n^\ell$ contains one of the maximal projective linear subspaces described in Lemma 2.4.*

Proof. By comparing Proposition 2.2 and Proposition 2.3, we can already deduce the first part of the proposition. For the second part, note that the Shilov boundary of $\Omega_{\ell+1, n-\ell}$ is precisely the matrices $A \in \overline{\Omega}_{\ell+1, n-\ell}$ such that $AA^H - I$ attains the lowest rank. Fix one such A , either we have $AA^H = I$ (when $\ell < n/2$) or there exist unitary matrices U and V such that $UAV^H = \begin{bmatrix} I \\ 0 \end{bmatrix}$ (when $\ell \geq n/2$).

For $\ell < n/2$, the proposition follows as the ℓ -dimensional projective linear subspace $\{[\mathbf{z}', \mathbf{z}'A]_\ell \in \mathbb{P}^n : \mathbf{z}' \in \mathbb{P}^\ell\}$ completely lies in $\partial\mathbb{D}_n^\ell$ if and only if $\|\mathbf{z}'\|^2 = \|\mathbf{z}'A\|^2$ for all $\mathbf{z}' \in \mathbb{P}^\ell$ if and only if $AA^H = I$.

When $\ell \geq n/2$, let $A \in \overline{\Omega}_{\ell+1, n-\ell}$ be a point on the Shilov boundary of $\Omega_{\ell+1, n-\ell}$, the ℓ -dimensional projective linear subspace $\{[\mathbf{z}', \mathbf{z}'A]_\ell \in \mathbb{P}^n : \mathbf{z}' \in \mathbb{P}^\ell\}$ cannot lie completely inside $\partial\mathbb{D}_n^\ell$ (see Lemma 2.4). On the other hand, if we write $UAV^H = I^\sharp$, where $I^\sharp = \begin{bmatrix} I \\ 0 \end{bmatrix}$, after the $SU(\ell+1, n-\ell)$ transformation $[\mathbf{z}', \mathbf{z}'']_\ell \mapsto [\mathbf{z}'U^H, \mathbf{z}''V^H]_\ell$, the above ℓ -dimensional linear subspace becomes $\{[\mathbf{z}'U^H, \mathbf{z}''U^HI^\sharp]_\ell : \mathbf{z}' \in \mathbb{P}^\ell\}$. From this form we immediately see that it contains an $(n-\ell-1)$ -linear subspace which lies completely inside $\partial\mathbb{D}_n^\ell$. Since both \mathbb{D}_n^ℓ and $\partial\mathbb{D}_n^\ell$ are $SU(\ell+1, n-\ell)$ invariant, we have proved that the intersection of the above ℓ -dimensional linear subspace and $\partial\mathbb{D}_n^\ell$ contains an $(n-\ell-1)$ -linear subspace. Conversely, suppose the intersection of the ℓ -dimensional linear subspace $\{[\mathbf{z}', \mathbf{z}'A]_\ell : \mathbf{z}' \in \mathbb{P}^\ell\}$ and $\partial\mathbb{D}_n^\ell$ contains an $(n-\ell-1)$ -linear subspace. We can then find $(n-\ell)$ linearly independent vectors $\{\mathbf{z}'_1, \dots, \mathbf{z}'_{n-\ell}\}$ in $\mathbb{C}^{\ell+1}$ such that $\|\mathbf{z}'_i\|^2 = \|\mathbf{z}'_iA\|^2$, for every i . This precisely means $AA^H - I$ has a maximal null space and hence is of the lowest rank among $A \in \overline{\Omega}_{\ell+1, n-\ell}$. \square

2.1.2 Linear subspaces of lower dimensions

In the above, we have determined the set of maximal linear subspaces contained in $\overline{\mathbb{D}}_n^\ell$. For the study holomorphic mappings among \mathbb{D}_n^ℓ , we will also determine the set of lower dimensional linear subspaces contained in $\overline{\mathbb{D}}_n^\ell$. The cycle spaces turn out to be $\overline{\mathbb{D}}_{e,n}^\ell \subset G_{e+1, n-e}$, $e \geq 1$.

Proposition 2.6. *Let $1 \leq e \leq \ell$. The set of e -dimensional linear subspaces contained in \mathbb{D}_n^ℓ (resp. $\overline{\mathbb{D}}_n^\ell$) is parametrized by $\mathbb{D}_{e,n}^\ell$ (resp. $\overline{\mathbb{D}}_{e,n}^\ell$).*

Proof. Take a point $A \in \overline{\mathbb{D}}_{e,n}^\ell$ and choose a homogeneous coordinates matrix $[A', A'']_\ell$ for A . We have $A'A'^H - A''A''^H \geq 0$. Define the e -dimensional linear subspace

$$\{[\mathbf{z}A', \mathbf{z}A'']_\ell \in \mathbb{P}^n : [\mathbf{z}] \in \mathbb{P}^e\} \cong \mathbb{P}^e.$$

Note that it is independent of the choice of the homogeneous coordinates matrix for A . Moreover this linear subspace is contained in $\overline{\mathbb{D}}_n^\ell$ because $\|\mathbf{z}A'\|^2 \geq \|\mathbf{z}A''\|^2$ for all $[\mathbf{z}] \in \mathbb{P}^e$ if and only if $\mathbf{z}A'A'^H\mathbf{z}^H - \mathbf{z}A''A''^H\mathbf{z}^H \geq 0$ for all $[\mathbf{z}] \in \mathbb{P}^e$ if and only if $A'A'^H - A''A''^H \geq 0$.

On the other hand, we can deduce from Lemma 2.1 that any e -dimensional linear subspace in $\overline{\mathbb{D}}_n^\ell$ is of the form $\{[\mathbf{z}B', \mathbf{z}B'']_\ell \in \mathbb{P}^n : [\mathbf{z}] \in \mathbb{P}^e\}$ for some matrices B' and B'' . And $\|\mathbf{z}B'\|^2 > \|\mathbf{z}B''\|^2$ for all $[\mathbf{z}]$ implies that $B'B'^H - B''B''^H \geq 0$ or $[B', B'']_\ell \in \overline{\mathbb{D}}_{e,n}^\ell$.

Replacing non-strict inequalities by strict inequalities, the other half of the proposition follows. The proof is complete. \square

2.1.3 Foliations

We have seen that the domain \mathbb{D}_n^ℓ is filled up by many linear subspaces. In general, given an integer $e \in \{1, \dots, \ell\}$, there are many e -dimensional linear subspaces passing through a given point and therefore these linear subspaces do not give us a foliation of \mathbb{D}_n^ℓ . However, if we lift everything to some Grassmann bundle over \mathbb{D}_n^ℓ (see the definition below), there is indeed a foliation. For illustration, we will first look at \mathbb{D}_n^1 .

Proposition 2.7. *Let $\pi : \mathbb{P}T\mathbb{D}_n^1 \rightarrow \mathbb{D}_n^1$ be the projectivized tangent bundle of \mathbb{D}_n^1 . There is an open set $V_n \subset \mathbb{P}T\mathbb{D}_n^1$, $\pi(V_n) = \mathbb{D}_n^1$ such that V_n has a holomorphic \mathbb{P}^1 -foliation and $V_n \cong \Omega_{2,n-1} \times \mathbb{P}^1$.*

Proof. Fix a point $p \in \mathbb{D}_n^1$. Since \mathbb{P}^1 is compact and \mathbb{D}_n^1 is a domain, we deduce that there is an open set $U_p \subset \mathbb{P}T_p\mathbb{D}_n^1$ consisting of precisely all the projectivized tangent vectors which are tangent to some line contained in \mathbb{D}_n^1 . Since the tangent direction at a point uniquely determine the line, combining with Proposition 2.2 when $\ell = 1$, the statements in the proposition are immediate. \square

When $\ell \geq 2$, to obtain a similar foliation picture as in Proposition 2.7, we need to consider the more general Grassmann bundles in place of the projectivized tangent bundle. Take a point $p \in \mathbb{D}_n^\ell$ and let $T_p\mathbb{D}_n^\ell$ be its tangent space. The set of all e -planes in $T_p\mathbb{D}_n^\ell$ is just the Grassmannian $G_{e,n-e}$, and in the present situation we will denote it by $G_eT_p\mathbb{D}_n^\ell$. We define the e -Grassmann bundle $G_eT\mathbb{D}_n^\ell := \bigcup_{p \in \mathbb{D}_n^\ell} G_eT_p\mathbb{D}_n^\ell$. Note that $\mathbb{P}T\mathbb{D}_n^\ell = G_1T\mathbb{D}_n^\ell$.

The argument in Proposition 2.7 can be directly carried over to the cases $\ell \geq 2$ and $e \in \{1, \dots, \ell\}$ and we then have

Proposition 2.8. *Let $1 \leq e \leq \ell$ and $\pi_e : G_eT\mathbb{D}_n^\ell \rightarrow \mathbb{D}_n^\ell$ be the e -Grassmann bundle of \mathbb{D}_n^ℓ . There is an open set $V_{e,n}^\ell \subset G_eT\mathbb{D}_n^\ell$, $\pi_e(V_{e,n}^\ell) = \mathbb{D}_n^\ell$ such that $V_{e,n}^\ell$ is a holomorphic \mathbb{P}^e -bundle over $\mathbb{D}_{e,n}^\ell$.*

We will now restrict ourselves to the case $e = \ell$. On \mathbb{P}^n , there is the tautological \mathbb{P}^ℓ -foliation on $G_\ell T\mathbb{P}^n$ and $G_\ell T\mathbb{P}^n \rightarrow G_{\ell+1,n-\ell}$ is the universal family (note that $G_{\ell+1,n-\ell}$ is the space of ℓ -planes in \mathbb{P}^n). The foliation in Proposition 2.8 is certainly the restriction of the tautological foliation to the open subset $V_{\ell,n}^\ell \subset G_\ell T\mathbb{P}^n$. We can still add one more between these two. Let $U_n^\ell = \{[\mathbf{z}', \mathbf{z}'']_\ell \in \mathbb{P}^n : \mathbf{z}' \neq 0\}$. The ℓ -plane $\{[\mathbf{z}', \mathbf{z}'A]_\ell \in \mathbb{P}^n : [\mathbf{z}'] \in \mathbb{P}^\ell\}$ is contained in U_n^ℓ for any $A \in M(\ell+1, n-\ell; \mathbb{C})$. And we can deduce from Lemma 2.1 that they are all the ℓ -planes in U_n^ℓ . Hence, as in the case of \mathbb{D}_n^ℓ , we can find an open set $W_{\ell,n} \subset G_\ell T U_n^\ell$, $\pi_\ell(W_{\ell,n}) = U_n^\ell$ such that $W_{\ell,n} \rightarrow \mathbb{C}^{(\ell+1)(n-\ell)}$ is a holomorphic \mathbb{P}^ℓ -bundle. In fact, we have $W_{\ell,n} \cong \mathbb{C}^{(\ell+1)(n-\ell)} \times \mathbb{P}^\ell$ by our explicit parametrization. In summary, we have the following inclusion relations:

$$\begin{array}{ccccc} \mathbb{D}_n^\ell & \subset & U_n^\ell & \subset & \mathbb{P}^n \\ \uparrow & & \uparrow & & \uparrow \\ V_{\ell,n}^\ell & \subset & W_{\ell,n} & \subset & G_\ell T\mathbb{P}^n \\ \downarrow & & \downarrow & & \downarrow \\ \Omega_{\ell+1,n-\ell} & \Subset & \mathbb{C}^{(\ell+1)(n-\ell)} & \subset & G_{\ell+1,n-\ell} \end{array}$$

Remark. It is clear from our explicit parametrization that the inclusion relation among the cycle spaces $\Omega_{\ell+1,n-\ell} \Subset \mathbb{C}^{(\ell+1)(n-\ell)} \subset G_{\ell+1,n-\ell}$ is just the classical Harish-Chandra embedding.

2.2 Rigidity of global proper holomorphic maps

To illustrate better the basic strategy for obtaining rigidity, we will consider global proper holomorphic maps in this section and return to local maps in the next section. We first state two lemmas.

Lemma 2.9. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain and $S(\Omega) \subset \partial\Omega$ be its Shilov boundary. Let $U \subset \mathbb{C}^n$ be a connected open set such that $U \cap S(\Omega) \neq \emptyset$. Suppose u is a holomorphic function on U and $u|_{U \cap S(\Omega)} \equiv 0$, then $u \equiv 0$ on U .*

Proof. We will first prove the lemma by induction for the polydisks Δ^m , $m \geq 1$. For the unit disk Δ , since $S(\Delta) = \partial\Delta$, the result immediately follows from the discreteness of zeros of holomorphic functions in one variable. Assume the lemma has been proved for Δ^k , $1 \leq k \leq p-1$. For $\Delta^p \subset \mathbb{C}^p$, we have $S(\Delta^p) = (\partial\Delta)^p$. Without loss of generality, we may assume that U is a connected open neighborhood of the point $(1, \dots, 1) \in S(\Delta^p)$ in \mathbb{C}^p and we can find an open arc $\gamma \subset \partial\Delta$, $1 \in \gamma$ such that $\gamma^p \subset U \cap S(\Delta^p)$. Let $\mathbf{b}_0 \in \gamma^{p-1}$ and $L_{\mathbf{b}_0} = \{(z, \mathbf{b}_0) \in \mathbb{C}^p : z \in \mathbb{C}\}$. Since u vanishes on γ^p , we have $u(z, \mathbf{b}_0) = 0$ for $z \in \gamma$ and hence $u|_{U \cap L_{\mathbf{b}_0}} \equiv 0$. In particular, u vanishes on $U \cap \Delta_{\mathbf{b}_0}$, where $\Delta_{\mathbf{b}_0} \subset L_{\mathbf{b}_0} \cong \mathbb{C}$ is the unit disk. Now if we choose a small enough open neighborhood $B \subset \gamma^{p-1}$ of \mathbf{b}_0 , then we can find an open set $A \subset \Delta$ such that $(z, \mathbf{b}) \in U \cap \Delta_{\mathbf{b}}$ for all $z \in A$ and $\mathbf{b} \in B$. Following the previous reasoning, we have $u(z, \mathbf{b}) = 0$ for all $z \in A$ and $\mathbf{b} \in B$. On the other hand, for a fix $z_0 \in A$, consider $E_{z_0} = \{(z_0, \mathbf{z}) \in \mathbb{C}^p : \mathbf{z} \in \mathbb{C}^{p-1}\}$ and the $(p-1)$ -disk $\Delta_{z_0}^{p-1} = \{(z_0, \mathbf{z}) \in \mathbb{C}^p : \mathbf{z} \in \Delta^{p-1}\}$. Now since $u(z_0, \mathbf{b}) = 0$ for $\mathbf{b} \in B \subset \gamma^{p-1} \subset S(\Delta^{p-1})$, it follows from the induction hypothesis that $u|_{U \cap E_{z_0}} \equiv 0$. By varying $z_0 \in A$, we see that u vanishes on an open set in U and hence $u \equiv 0$ on U . We have therefore proved the lemma for the polydisks.

When $\Omega \Subset \mathbb{C}^n$ is a bounded symmetric domain with $\text{rank}(\Omega) = r$, by the Polydisk Theorem [7], there is a totally geodesic embedding $0 \in \Delta^r \subset \Omega$ of the r -disk into Ω . Moreover, if $K \subset \text{Aut}(\Omega)$ is the isotropy group at $0 \in \Omega$, then $\bigcup_{k \in K} k \cdot \Delta^r = \Omega$. The elements in K extend to unitary transformations in \mathbb{C}^n . The Shilov boundary of Ω is precisely the K -orbit of $S(\Delta^r)$, i.e. $S(\Omega) = \bigcup_{k \in K} k \cdot S(\Delta^r) = \bigcup_{k \in K} k \cdot (\partial\Delta)^r$. We can choose Harish-Chandra coordinates (z_1, \dots, z_n) of Ω such that $\Delta^r = \{(z_1, \dots, z_r, 0, \dots, 0) \in \Omega\}$. Without loss of generality, we may assume that $U \cap S(\Delta^r) \neq \emptyset$. This implies in particular $U \cap \Delta^r \neq \emptyset$. Now take a sufficiently small open set $\mathcal{K} \subset K$ containing the identity such that $U \cap (k \cdot S(\Delta^r)) \neq \emptyset$ and hence also $U \cap (k \cdot \Delta^r) \neq \emptyset$ for all $k \in \mathcal{K}$. For each $k \in \mathcal{K}$, since $k \cdot S(\Delta^r) \subset S(\Omega)$, we have by assumption $u|_{U \cap (k \cdot S(\Delta^r))} \equiv 0$ and hence $u|_{U \cap (k \cdot \Delta^r)} \equiv 0$ by the present lemma in the polydisk case. This implies that u is identically zero when restricting on $U \cap (\bigcup_{k \in \mathcal{K}} k \cdot \Delta^r)$. Finally, it suffices to note that $U \cap (\bigcup_{k \in \mathcal{K}} k \cdot \Delta^r)$ contains an open subset of U and hence $u \equiv 0$ on U . The proof of the lemma is complete. \square

Lemma 2.10. *Let $U \subset \mathbb{P}^n$ be a connected open set. Let $h : U \rightarrow \mathbb{P}^m$ be a holomorphic embedding such that for every line L in \mathbb{P}^n with $L \cap U \neq \emptyset$, we have $h(U \cap L)$ being an open subset of a line in \mathbb{P}^m . Then h extends to a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^m .*

Proof. See [8], for example. \square

Theorem 2.11. *Let $1 \leq \ell < n/2$. If $f : \mathbb{D}_n^\ell \rightarrow \mathbb{D}_m^\ell$ is a proper holomorphic map which is non-degenerate at some point, then f extends to a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^m .*

Proof. The map $f : \mathbb{D}_n^\ell \rightarrow \mathbb{D}_m^\ell$ induces a meromorphic map $[df] : G_\ell T\mathbb{D}_n^\ell \rightarrow G_\ell T\mathbb{D}_m^\ell$ by differentiation. Recalling the notations in Proposition 2.8, we restrict $[df]$ to $V_{\ell, n}^\ell \subset G_\ell T\mathbb{D}_n^\ell$ and regard it as a map into $G_\ell T\mathbb{P}^m$. Since $\mathbb{D}_{\ell, n}^\ell \cong \Omega_{\ell+1, n-\ell}$, we have $V_{\ell, n}^\ell \cong \Omega_{\ell+1, n-\ell} \times \mathbb{P}^\ell$. Thus, we are now considering the meromorphic map

$$[df] : \Omega_{\ell+1, n-\ell} \times \mathbb{P}^\ell \rightarrow G_\ell T\mathbb{P}^m.$$

Let $\Pi : G_\ell T\mathbb{P}^m \rightarrow G_{\ell+1, m-\ell}$ be the universal family of ℓ -planes in \mathbb{P}^m and let $f^\sharp = \Pi \circ [df]$. We are going to prove that $f^\sharp : \Omega_{\ell+1, n-\ell} \times \mathbb{P}^\ell \rightarrow G_{\ell+1, m-\ell}$ is constant on each fibre $\{z\} \times \mathbb{P}^\ell$, $z \in \Omega_{\ell+1, n-\ell}$.

Since $\mathbb{D}_n^\ell \subset \mathbb{P}^n$ is pseudoconcave, f can be extended to an open neighborhood of $\overline{\mathbb{D}_n^\ell}$. In particular, f^\sharp can be extended to an open neighborhood W of $\overline{\Omega_{\ell+1, n-\ell}} \times \mathbb{P}^\ell$ in $\mathbb{C}^{(\ell+1)(n-\ell)} \times \mathbb{P}^\ell$

(c.f. the paragraph after Proposition 2.8). We will still denote the extension by f^\sharp . Take a point b on the Shilov boundary of $\Omega_{\ell+1, n-\ell}$. The fibre $\{b\} \times \mathbb{P}^\ell \subset \Omega_{\ell+1, n-\ell} \times \mathbb{P}^\ell$ is the tautological lifting of an ℓ -plane L_b lying inside the boundary $\partial\mathbb{D}_n^\ell$ (see Proposition 2.5 and we need $1 \leq \ell < n/2$ here). Being a proper holomorphic map, f (more precisely, its extension) will map L_b into the boundary $\partial\mathbb{D}_m^\ell$. By Lemma 2.4, $f(L_b)$ must be contained in an ℓ -plane $\tilde{L}_b \subset \partial\mathbb{D}_m^\ell$. From the definition of f^\sharp , it follows that f^\sharp is constant on the fibre $\{b\} \times \mathbb{P}^\ell$. We have therefore proved that f^\sharp is constant along every fibre over the Shilov boundary of $\Omega_{\ell+1, n-\ell}$.

Now pick again an arbitrary point b on the Shilov boundary of $\Omega_{\ell+1, n-\ell}$ and let $U \subset \mathbb{C}^{(\ell+1)(n-\ell)}$ be a connected open set containing b such that $U \times \mathbb{P}^\ell \subset W$. By shrinking U if necessary, we can choose an open set $V \subsetneq \mathbb{P}^\ell$ such that $f^\sharp(U \times V)$ is contained in some Euclidean chart of $G_{\ell+1, m-\ell}$. Then on $U \times V$, the map f^\sharp can be expressed in terms of holomorphic functions over some open set in a Euclidean space. We identify U with a section $U \times \{v\} \subset U \times V$ and consider the derivatives of f^\sharp in the vertical direction (i.e. along V). From the previous paragraph, we know that these vertical derivatives are zero on $U \cap S(\Omega_{\ell+1, n-\ell})$, where $S(\Omega_{\ell+1, n-\ell})$ is the Shilov boundary of $\Omega_{\ell+1, n-\ell}$. By Lemma 2.9, they are identically zero on U . This precisely means that f^\sharp is constant along every fibre over U . Now by the Identity Theorem of meromorphic functions, we can conclude that $f^\sharp : \Omega_{\ell+1, n-\ell} \times \mathbb{P}^\ell \rightarrow G_{\ell+1, m-\ell}$ is constant on each fibre $\{z\} \times \mathbb{P}^\ell$, $z \in \Omega_{\ell+1, n-\ell}$. This is equivalent to the map

$$\begin{array}{ccc} \Omega_{\ell+1, n-\ell} \times \mathbb{P}^\ell & \xrightarrow{[df]} & G_\ell T\mathbb{P}^m \\ \downarrow & & \downarrow \\ \Omega_{\ell+1, n-\ell} & & G_{\ell+1, m-\ell} \end{array}$$

between two universal families of ℓ -planes being fibre-preserving which is in turn equivalent to $f : \mathbb{D}_n^\ell \rightarrow \mathbb{D}_m^\ell$ mapping ℓ -planes into ℓ -planes. Since f is non-degenerate at a general point, by considering the intersections of ℓ -planes, it follows that f also maps lines into lines. Finally, as the lines contained in \mathbb{D}_n^ℓ constitute an open set in the space of lines in \mathbb{P}^n , we deduce further that for any line $L \subset \mathbb{P}^n$ intersecting \mathbb{D}_n^ℓ , $f(\mathbb{D}_n^\ell \cap L)$ is an open subset of some line in \mathbb{P}^m . The theorem now follows from Lemma 2.10. \square

Remark: It can actually be checked, like in [2], that any proper holomorphic map $f : \mathbb{D}_n^\ell \rightarrow \mathbb{D}_m^\ell$ is not totally degenerate.

Theorem 2.12. *Let $\frac{n}{2} \leq \ell < n - 1$ and $k \in \mathbb{N}$. If $f : \mathbb{D}_n^\ell \rightarrow \mathbb{D}_{n+k}^{\ell+k}$ is a proper holomorphic map, then f extends to a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^{n+k} .*

Proof. The proof is similar to that of Theorem 2.11 except that we now look at the holomorphic $\mathbb{P}^{n-\ell-1}$ -foliation of $G_{n-\ell-1}T\mathbb{D}_n^\ell$. By differentiation, f induces a meromorphic map $[df] : G_{n-\ell-1}T\mathbb{D}_n^\ell \rightarrow G_{n-\ell-1}T\mathbb{D}_{n+k}^{\ell+k}$. Now we restrict $[df]$ to $V_{n-\ell-1, n}^\ell \subset G_{n-\ell-1}T\mathbb{D}_n^\ell$, where $V_{n-\ell-1, n}^\ell \rightarrow \mathbb{D}_{n-\ell-1, n}^\ell$ is a holomorphic $\mathbb{P}^{n-\ell-1}$ -bundle. This time $\mathbb{D}_{n-\ell-1, n}^\ell \subset G_{n-\ell, \ell+1}$ is no longer a bounded symmetric domain. Instead, we will see that when expressing in homogeneous coordinates in $G_{n-\ell, \ell+1}$, the domain $\mathbb{D}_{n-\ell-1, n}^\ell$ is disjoint with another flag domain $\Omega_{n-\ell, \ell+1} \subset G_{n-\ell, \ell+1}$ and at the same time, the Shilov boundary of $\Omega_{n-\ell, \ell+1}$ is part of the boundary of $\mathbb{D}_{n-\ell-1, n}^\ell$. This is crucial and it makes Lemma 2.9 applicable in the present situation as in the proof of Theorem 2.11. Recall first of all the definition $\mathbb{D}_{n-\ell-1, n}^\ell = \{[A', A'']_\ell \in G_{n-\ell, \ell+1} : A'A'^H - A''A''^H > 0\}$. On the other hand, $\Omega_{n-\ell, \ell+1}$ can be realized, using the same set of homogeneous coordinates, as the domain: $\Omega_{n-\ell, \ell+1} = \{[A', A'']_\ell \in G_{n-\ell, \ell+1} : A'A'^H - A''A''^H < 0\}$. In this realization, when $\ell \geq n/2$, the Shilov boundary

is $S(\Omega_{n-\ell, \ell+1}) = \{[A', A'']_\ell \in G_{n-\ell, \ell+1} : A'A'^H = A''A''^H\}$ and we immediately see that it is contained in the boundary of $\mathbb{D}_{n-\ell-1, n}^\ell$. From this point, the proof in Theorem 2.11 can be carried over here, *verbatim*. \square

2.3 Local characterization

In this section, we will give the local versions of Theorem 2.11 and Theorem 2.12 in the previous section. We will deal with the situation where a local embedding f of a connected open set U of \mathbb{P}^n into \mathbb{P}^m is given and f preserves the boundary of \mathbb{D}_n^ℓ . This is clearly relevant to the study of *local* proper holomorphic maps among the flag domains \mathbb{D}_n^ℓ as such a local proper map automatically extends across the boundary by pseudoconcavity and the extension will preserve the boundary. On the other hand, in the equidimensional case, we may also regard the theorem as a counterpart of Alexander's Theorem [4] on the local characterization of the automorphisms of the unit ball in the complex Euclidean space of dimension at least two. We start with a lemma in linear algebra.

Lemma 2.13. *Let $2 \leq n \leq m$ and V, W be complex vector spaces of dimensions n, m respectively. Suppose V and W are equipped with non-degenerate Hermitian forms $H_V(\cdot, \cdot)$ and $H_W(\cdot, \cdot)$ of signatures $(p, n-p)$ and $(p, m-p)$ respectively, where $1 \leq p \leq n-1$. Let $A : V \rightarrow W$ be a linear map which preserves the null vectors and there exists $v_+ \in V$ with positive length such that $A(v_+)$ is of positive length. Then $A^*H_W = \lambda H_V$ for some $\lambda > 0$.*

Proof. Let $\|\cdot\|_V, \|\cdot\|_W$ be the norms induced by H_V and H_W respectively. We fix an orthonormal basis $\{v_1, \dots, v_n\}$ for V such that $\|v_i\|_V = 1$ for $1 \leq i \leq p$ and $\|v_j\|_V = -1$ for $p+1 \leq j \leq n$. Now fix $j \in \{p+1, \dots, n\}$. The vector $v_1 + e^{i\theta}v_j$ is null and hence $\|A(v_1 + e^{i\theta}v_j)\|_W^2 = 0$ for any $\theta \in \mathbb{R}$. Then we have

$$H_W(Av_1, Av_1) + e^{i\theta}H_W(Av_j, Av_1) + e^{-i\theta}H_W(Av_1, Av_j) + H_W(Av_j, Av_j) = 0$$

for any $\theta \in \mathbb{R}$. From this we readily deduce that $H_W(Av_1, Av_1) + H_W(Av_j, Av_j) = 0$ and $H_W(Av_1, Av_j) = 0$. Similarly, we have $H_W(Av_i, Av_i) + H_W(Av_j, Av_j) = 0$ and $H_W(Av_i, Av_j) = 0$ for any $i \in \{1, \dots, p\}$ and $j \in \{p+1, \dots, n\}$. Therefore if we let $\|A(v_1)\|_W^2 = \lambda$ for some $\lambda \in \mathbb{R}$, we will have $A^*H_W = \lambda H_V$. Finally, $\lambda > 0$ because $\|A(v_+)\|_W > 0$ for some $v_+ \in V$ with $\|v_+\|_V > 0$. \square

We are now ready to give the proof for Theorem 1.1 in the case of $k = 0$.

Theorem 2.14. *Let $1 \leq \ell \leq n-2$ and $U \subset \mathbb{P}^n$ be a connected open set such that $U \cap \partial\mathbb{D}_n^\ell \neq \emptyset$. Suppose $f : U \rightarrow \mathbb{P}^m$ is a holomorphic embedding in one of the following cases:*

- (i) $1 \leq \ell < \frac{n}{2}$, $f(U \cap \partial\mathbb{D}_n^\ell) \subset \partial\mathbb{D}_m^\ell$;
- (ii) $\frac{n}{2} \leq \ell \leq n-2$, $f(U \cap \partial\mathbb{D}_n^\ell) \subset \partial\mathbb{D}_m^{\ell+m-n}$,

then f extends to a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^m . If furthermore, we have either in case (i) $f(U \cap \mathbb{D}_n^\ell) \subset \mathbb{D}_m^\ell$; or in case (ii) $f(U \cap \mathbb{D}_n^\ell) \subset \mathbb{D}_m^{\ell+m-n}$, then the extension restricting on \mathbb{D}_n^ℓ is a proper holomorphic embedding into \mathbb{D}_m^ℓ in case (i) and into $\mathbb{D}_m^{\ell+m-n}$ in case (ii).

Proof. We will again make use of Lemma 2.10. To apply the lemma, we need to first prove that for any line $L \subset \mathbb{P}^n$ intersecting U , $f(L \cap U)$ is an open subset of some line in \mathbb{P}^m .

We will deal with case (i) first. Consider the ℓ -Grassmann bundle of U , i.e. $G_\ell TU$ which is a connected open subset of $G_\ell T\mathbb{P}^n$. Recall that the holomorphic \mathbb{P}^ℓ -bundle $\pi^{uni} : G_\ell T\mathbb{P}^n \rightarrow G_{\ell+1, n-\ell}$ is the universal family of ℓ -planes in \mathbb{P}^n and $\pi^{uni}(G_\ell TU)$ is the set of ℓ -planes whose intersection with U is non-empty. Also recall that by Proposition 2.3 the fibres over $\overline{\mathbb{D}}_{\ell, n}^\ell \cong \overline{\Omega}_{\ell+1, n-\ell} \subset G_{\ell+1, n-\ell}$ in the universal family $\pi^{uni} : G_\ell T\mathbb{P}^n \rightarrow G_{\ell+1, n-\ell}$ are those ℓ -planes contained in $\overline{\mathbb{D}}_n^\ell$. Furthermore, the fibres over the Shilov boundary $S(\Omega_{\ell+1, n-\ell})$ correspond to those lying inside $\partial\mathbb{D}_n^\ell$ by Proposition 2.5. Therefore, $U \cap \partial\mathbb{D}_n^\ell \neq \emptyset$ implies that $\pi^{uni}(G_\ell TU) \cap S(\Omega_{\ell+1, n-\ell}) \neq \emptyset$.

Now for each point $b \in \pi^{uni}(G_\ell TU) \cap S(\Omega_{\ell+1, n-\ell})$, denote by L_b the ℓ -plane in $\partial\mathbb{D}_n^\ell$ corresponding to b . By the assumption that $f(U \cap \partial\mathbb{D}_n^\ell) \subset \partial\mathbb{D}_m^\ell$ and Lemma 2.4, we see that $f(U \cap L_b)$ is an open subset of some ℓ -plane $L_{f(b)}$ lying inside $\partial\mathbb{D}_m^\ell$. Then using exactly the same argument given in the proof of Theorem 2.11 involving taking derivatives along the fibre direction and applying Lemma 2.9, we can conclude that for any ℓ -plane L_z corresponding to a point $z \in \pi^{uni}(G_\ell TU)$, $f(U \cap L_z)$ is an open subset of some ℓ -plane $L_{f(z)}$ in \mathbb{P}^m . In other words, f maps the intersection of any ℓ -plane with U to an open subset of an ℓ -plane in \mathbb{P}^m . As f is an embedding, f also maps the intersection of any line with U to an open subset of a line in \mathbb{P}^m . By Lemma 2.10, f extends to a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^m .

Note that a totally geodesic embedding of \mathbb{P}^n into \mathbb{P}^m is induced by a linear embedding $A : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$. If we equip \mathbb{C}^{n+1} with the standard non-degenerate Hermitian form of signature $(\ell+1, n-\ell)$, the points on \mathbb{D}_n^ℓ and on $\partial\mathbb{D}_n^\ell$ are then lines containing vectors of positive norm and zero norm in \mathbb{C}^{n+1} respectively. On the other hand, by the Identity Theorem for real analytic functions, the extension of f maps the whole $\partial\mathbb{D}_n^\ell$ into $\partial\mathbb{D}_m^\ell$ and therefore the corresponding linear map A maps null vectors to null vectors. Now if furthermore we have $f(U \cap \mathbb{D}_n^\ell) \subset \mathbb{D}_m^\ell$, by applying Lemma 2.13 to A it follows immediately that the extension of f restricting on \mathbb{D}_n^ℓ is a proper holomorphic embedding into \mathbb{D}_m^ℓ . This completes the proof for case (i).

For case (ii), let $\ell' = n - \ell - 1$. Then we have $1 \leq \ell' < n/2$. Observe that the complement of $\overline{\mathbb{D}}_n^\ell$ in \mathbb{P}^n is just $\mathbb{D}_n^{\ell'}$ by a change of coordinates. Thus, $\partial\mathbb{D}_n^\ell \cong \partial\mathbb{D}_n^{\ell'}$ and similarly $\partial\mathbb{D}_m^{\ell+m-n} \cong \partial\mathbb{D}_m^{\ell'}$ and it reduces to case (i). \square

3 On Grassmannians of higher rank

3.1 Confined subgrassmannians in $\overline{\mathbb{D}}_{k, n}^\ell$

We are going to generalize the results obtained in the previous sections to the flag domains $\mathbb{D}_{k, n}^\ell \subset G_{k+1, n-k}$, $k \geq 1$ and $\ell \in \{k+2, \dots, n-k-2\}$. This time we will look at the subgrassmannians in $G_{k+1, n-k}$ corresponding to linear subspaces in \mathbb{C}^{n+1} . More explicitly, for a linear subspace $E \subset \mathbb{C}^{n+1}$ with $\dim(E) = e+1 \geq k+1$, we consider the Grassmannian $G_{k+1, e-k}^E := \{(k+1)\text{-planes contained in } E\}$, which is a totally geodesic submanifold of $G_{k+1, n-k}$. We will call it an $(e+1)$ -confined subgrassmannian or simply *confined subgrassmannians* in this article. It is clearly a generalization of the projective linear subspaces in \mathbb{P}^n .

Similar to the case $\overline{\mathbb{D}}_n^\ell \subset \mathbb{P}^n$, we will first determine the set of confined subgrassmannians of $G_{k+1, n-k}$ which are contained in $\overline{\mathbb{D}}_{k, n}^\ell$. The following proposition is a generalization of Proposition 2.6.

Proposition 3.1. *Let $k \leq e \leq \ell$. The set of $(e+1)$ -confined subgrassmannians contained in $\mathbb{D}_{k,n}^\ell$ (resp. $\overline{\mathbb{D}}_{k,n}^\ell$) is parametrized by $\mathbb{D}_{e,n}^\ell$ (resp. $\overline{\mathbb{D}}_{e,n}^\ell$). Furthermore, among them the $(\ell+1)$ -confined subgrassmannians are maximal compact complex analytic subvarieties in $\mathbb{D}_{k,n}^\ell$.*

Proof. The proof is parallel to those of Proposition 2.2 and Proposition 2.6 and we will just simply write down the explicit parametrization. Take a point $E \in \mathbb{D}_{e,n}^\ell$ (resp. $\overline{\mathbb{D}}_{e,n}^\ell$) and choose a homogeneous coordinates matrix $[E', E'']_\ell$ for E . The corresponding $(e+1)$ -confined subgrassmannian is $G_{k+1,e-k}^E = \{[ZE', ZE'']_\ell \in G_{k+1,n-k} : [Z] \in G_{k+1,e-k}\}$ and it is contained in $\mathbb{D}_{k,n}^\ell$ (resp. $\overline{\mathbb{D}}_{k,n}^\ell$) because $E \in \mathbb{D}_{e,n}^\ell$ (resp. $\overline{\mathbb{D}}_{e,n}^\ell$). \square

Recall from the introduction section we will always consider the cases where $k+1 \leq \ell \leq n-k-2$. We now analyze the boundary $\partial\mathbb{D}_{k,n}^\ell$ as in Lemma 2.4 and Proposition 2.5. The difference here is that $\partial\mathbb{D}_{k,n}^\ell$ is no longer homogeneous for $k \geq 1$ and it actually consists of $k+1$ strata. They can be described as follows. For each $r \in \{1, \dots, k+1\}$, the r th-stratum of $\partial\mathbb{D}_{k,n}^\ell$, denoted by $\partial_r\mathbb{D}_{k,n}^\ell$, is defined to be the set of $[Z', Z'']_\ell \in G_{k+1,n-k}$ such that $Z'Z'^H - Z''Z''^H \geq 0$ with the null space being of dimension r . The most important stratum for us is $\Sigma(\mathbb{D}_{k,n}^\ell) := \partial_{k+1}\mathbb{D}_{k,n}^\ell$ which can be defined equivalently as the set $\{[Z', Z'']_\ell \in G_{k+1,n-k} : Z'Z'^H = Z''Z''^H\}$.

Proposition 3.2. *Let $k+1 \leq \ell \leq n-k-2$. The maximal compact complex analytic subvarieties in $\Sigma(\mathbb{D}_{k,n}^\ell)$ are $(\ell+1)$ -confined subgrassmannians when $\ell < n/2$ and $(n-\ell)$ -confined subgrassmannians when $\ell \geq n/2$. Moreover, any germ of complex submanifold lying inside $\Sigma(\mathbb{D}_{k,n}^\ell)$ must be contained in one of these subgrassmannians.*

Proof. We first note that $\Sigma(\mathbb{D}_{k,n}^\ell)$ and $\Sigma(\mathbb{D}_{k,n}^{n-\ell-1})$ are equivalent under an automorphism of $G_{k+1,n-k}$. To see this, consider the domain defined by $D = \{[Z', Z'']_\ell \in G_{k+1,n-k} : Z'Z'^H - Z''Z''^H < 0\}$. It is clearly biholomorphic to $\mathbb{D}_{k,n}^{n-\ell-1}$ under the automorphism of $G_{k+1,n-k}$ given by $[Z', Z'']_\ell \mapsto [Z'', Z']_{n-\ell-1}$. Now it suffices to know that both $\Sigma(D)$ and $\Sigma(\mathbb{D}_{k,n}^\ell)$ are defined by the equation $Z'Z'^H = Z''Z''^H$. Thus, we just need to prove the proposition for $\ell < n/2$.

Suppose we are given a germ of holomorphic curve $g : \Delta \rightarrow \Sigma(\mathbb{D}_{k,n}^\ell)$, where Δ is the unit disk. Without loss of generality, we may assume that $g(\Delta)$ lies in the Euclidean cell $U := \{[Z', Z'']_k \in G_{k+1,n-k} : \det(Z') \neq 0\} \cong \mathbb{C}^{(k+1)(n-k)}$ and in the standard inhomogeneous coordinates of U , we can write $g(z) = (G'(z), G''(z))$, $G'(z) \in M(k+1, \ell-k; \mathbb{C})$, $G''(z) \in M(k+1, n-\ell; \mathbb{C})$ and so that $[(I, G'), G'']_\ell$ is a homogeneous coordinates matrix for $g(z)$, where $I \in M(k+1; \mathbb{C})$ is the identity matrix. Then $g(\Delta) \subset \Sigma(\mathbb{D}_{k,n}^\ell)$ implies that $(I, G')(I, G')^H = G''G''^H$ and hence there exists a matrix $A \in M(\ell+1, n-\ell; \mathbb{C})$, $AA^H = I$, such that $G'' = (I, G')A$. Therefore, $g(\Delta)$ is contained in the $(\ell+1)$ -confined subgrassmannian $G_{k+1,\ell-k}^{[I,A]} = \{[Z, ZA]_\ell \in G_{k+1,n-k} : [Z] \in G_{k+1,\ell-k}\}$. Note that $G_{k+1,\ell-k}^{[I,A]} \subset \Sigma(\mathbb{D}_{k,n}^\ell)$. Replacing a germ of curve in the above argument by a germ of holomorphic map from some polydisk into $\Sigma(\mathbb{D}_{k,n}^\ell)$, we conclude similarly that any germ of complex submanifold inside $\Sigma(\mathbb{D}_{k,n}^\ell)$ must be contained in some $(\ell+1)$ -confined subgrassmannian in $\Sigma(\mathbb{D}_{k,n}^\ell)$. In particular, for $A \in M(\ell+1, n-\ell; \mathbb{C})$ with $AA^H = I$, the $(\ell+1)$ -confined subgrassmannians $G_{k+1,\ell-k}^{[I,A]}$ are maximal compact complex analytic subvarieties of $\Sigma(\mathbb{D}_{k,n}^\ell)$. \square

3.1.1 Foliation

Fix one $G_{k+1,n-k}$ and take a point $p \in G_{k+1,n-k}$. Let $e \in \{k, \dots, n\}$. An $(e+1)$ -confined subgrassmannian passing through p will give one $(k+1)(e-k)$ -dimensional tangent subspace in $T_p G_{k+1,n-k}$, or equivalently, a point in the Grassmannian of $(k+1)(e-k)$ planes in $T_p G_{k+1,n-k}$, i.e. $G_{(k+1)(e-k)} T_p G_{k+1,n-k}$ (see the definitions before Proposition 2.8). The collection of all such $(k+1)(e-k)$ -dimensional tangent subspaces at p gives a locally closed complex analytic subvariety \mathfrak{F}_p^e of $G_{(k+1)(e-k)} T_p G_{k+1,n-k}$. This can be seen by noting that \mathfrak{F}_p^e is an orbit of the isotropy action of some parabolic subgroup of $SL(n+1, \mathbb{C})$ on $G_{(k+1)(e-k)} T_p G_{k+1,n-k}$. Within a Euclidean cell of $G_{k+1,n-k}$, any Euclidean translation extends to a biholomorphism of $G_{k+1,n-k}$ and hence \mathfrak{F}_p^e at any point is simply the Euclidean translation of \mathfrak{F}_0^e at the origin. Let us write $\mathfrak{F}^e := \bigcup_{p \in G_{k+1,n-k}} \mathfrak{F}_p^e$. Since a confined subgrassmannian is uniquely determined by its tangent space at one point, we have therefore a holomorphic $G_{k+1,e-k}$ -foliation of \mathfrak{F}^e . Moreover, as the set of $(e+1)$ -confined subgrassmannians in $G_{k+1,n-k}$ is in one to one correspondence to the set of $(e+1)$ -planes in \mathbb{C}^{n+1} , we have the universal family $\mathfrak{F}^e \rightarrow G_{e+1,n-e}$ as a holomorphic $G_{k+1,e-k}$ -bundle.

When $p \in \mathbb{D}_{k,n}^\ell$, for each $e \in \{k, \dots, \ell\}$, if we restrict to those $(e+1)$ -confined subgrassmannians contained in $\mathbb{D}_{k,n}^\ell$, then their tangent spaces constitute an open subset $\mathfrak{D}_p^e \subset \mathfrak{F}_p^e$ and we let $\mathfrak{D}^e = \bigcup_{p \in \mathbb{D}_{k,n}^\ell} \mathfrak{D}_p^e$. The foliation of \mathfrak{F}^e restricts to a foliation of \mathfrak{D}^e and we have the universal family $\mathfrak{D}^e \rightarrow \mathbb{D}_{e,n}^\ell$ as a holomorphic $G_{k+1,e-k}$ -bundle.

We now restrict to the case $e = \ell$. As in the case of \mathbb{D}_n^ℓ , we add one more between the foliations of \mathfrak{D}^ℓ and \mathfrak{F}^ℓ . Let $U^\ell = \{[Z', Z'']_\ell \in G_{k+1,n-k} : \det(Z') \neq 0\}$. Then the $(\ell+1)$ -confined subgrassmannian $\{[ZA', ZA'']_\ell \in G_{k+1,n-k} : [Z] \in G_{k+1,\ell-k}\}$ is contained in U^ℓ for any $[A', A'']_\ell \in G_{\ell+1,n-\ell}$ with $\det(A') \neq 0$. Note that the set $\{[A', A'']_\ell \in G_{\ell+1,n-\ell} : \det(A') \neq 0\} \cong \mathbb{C}^{(\ell+1)(n-\ell)}$. Then similar to $\mathbb{D}_{k,n}^\ell$, we can find an open subset $\mathfrak{W}^\ell \subset \mathfrak{F}^\ell$, $\pi_\ell(\mathfrak{W}^\ell) = U^\ell$ such that $\mathfrak{W}^\ell \rightarrow \mathbb{C}^{(\ell+1)(n-\ell)}$ is a holomorphic $G_{k+1,\ell-k}$ -bundle. In summary, we have the following inclusion relations:

$$\begin{array}{ccccc}
 \mathbb{D}_{k,n}^\ell & \subset & U^\ell & \subset & G_{k+1,n-k} \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathfrak{D}^\ell & \subset & \mathfrak{W}^\ell & \subset & \mathfrak{F}^\ell \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega_{\ell+1,n-\ell} & \Subset & \mathbb{C}^{(\ell+1)(n-\ell)} & \subset & G_{\ell+1,n-\ell}
 \end{array}$$

3.2 Local characterization

We will make use of the following Cartan-Fubini type extension theorem of Mok [3] on holomorphic maps between Grassmannians. To state and apply the theorem, we need to recall a few basic facts about Grassmannians. Since $G_{m,n}$ is biholomorphic to $G_{n,m}$ we will assume that $m \leq n$. Each $G_{m,n}$ can be embedded into the projective space $\mathbb{P}\mathbb{A}^m(\mathbb{C}^{m+n})$ by the Plücker embedding. The image of $G_{m,n}$ in $\mathbb{P}\mathbb{A}^m(\mathbb{C}^{m+n})$ contains certain m -dimensional projective linear subspaces in $\mathbb{P}\mathbb{A}^m(\mathbb{C}^{m+n})$ corresponding to the $(m+1)$ -confined subgrassmannians $G_{m,m+1} \cong \mathbb{P}^m$. In addition, there is a notion of *minimal rational curves* for $G_{m,n}$ (see [9]). In the present context, they are the lines in $\mathbb{P}\mathbb{A}^m(\mathbb{C}^{m+n})$ which lie inside the image of $G_{m,n}$ under the Plücker embedding. In fact, they are just the lines in those m -dimensional projective linear subspaces we have just described. At each point $p \in G_{m,n}$, a tangent vector is called a *characteristic vector* if it is the tangent of some minimal rational curve in $G_{m,n}$. The set of projectivized characteristic vectors at p is called the *characteristic variety* at p ,

denoted by \mathcal{C}_p . At each point $p \in G_{m,n}$, the characteristic variety $\mathcal{C}_p \subset \mathbb{P}T_p G_{m,n} \cong \mathbb{P}^{mn-1}$ can be identified as the image of the Segre embedding $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{mn-1}$.

Theorem 3.3 (Mok). *Let G and G' be Grassmannians of rank at least 2. Let $h : U \subset G \rightarrow G'$ be a local holomorphic map defined on some connected open set U and let dh be its differential. If dh maps characteristic vectors to characteristic vectors and $h(U)$ is not contained in any projective linear subspace in G' , then h extends to a standard embedding of G into G' .*

In the present context, a standard embedding of $G_{k+1,n-k}$ into $G_{k+1,m-k}$ means a totally geodesic embedding of $G_{k+1,n-k}$ into $G_{k+1,m-k}$ induced by a linear embedding $A : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$.

The proof of the following proposition is along a similar approach as in the case of \mathbb{D}_n^ℓ .

Proposition 3.4. *Let $k+1 \leq \ell < n/2$. Let $U \subset G_{k+1,n-k}$ be a connected open set and $U \cap \Sigma(\mathbb{D}_{k,n}^\ell) \neq \emptyset$. Let $f : U \rightarrow G_{k+1,m-k}$ be a holomorphic embedding and $f(U \cap \Sigma(\mathbb{D}_{k,n}^\ell)) \subset \Sigma(\mathbb{D}_{k,m}^\ell)$. Then the differential df maps characteristic vectors to characteristic vectors.*

Proof. Recall the notations in Section 3.1.1. The holomorphic $G_{k+1,\ell-k}$ -bundle $\pi^{uni} : \mathfrak{F}^\ell \rightarrow G_{\ell+1,n-\ell}$ is the universal family of $(\ell+1)$ -confined subgrassmannians in $G_{k+1,n-k}$. Let $\mathfrak{F}_U^\ell := \bigcup_{p \in U} \mathfrak{F}_p^\ell$. It is an open subset of \mathfrak{F}^ℓ and $\pi^{uni}(\mathfrak{F}_U^\ell)$ are precisely those subgrassmannians intersecting U . Moreover, the fibres over $\overline{\mathbb{D}}_{\ell,n}^\ell \cong \overline{\Omega}_{\ell+1,n-\ell} \subset G_{\ell+1,n-\ell}$ are those contained in $\overline{\mathbb{D}}_{k,n}^\ell$ and the fibres over the Shilov boundary $S(\Omega_{\ell+1,n-\ell})$ are those contained in $\Sigma(\mathbb{D}_{k,n}^\ell)$. Since $U \cap \Sigma(\mathbb{D}_{k,n}^\ell) \neq \emptyset$, we have $\pi^{uni}(\mathfrak{F}_U^\ell) \cap S(\Omega_{\ell+1,n-\ell}) \neq \emptyset$.

Now take a point $b \in U \cap \Sigma(\mathbb{D}_{k,n}^\ell)$ and $v_b \in \mathfrak{F}_b^\ell$ such that $\pi^{uni}(v_b) \in S(\Omega_{\ell+1,n-\ell})$.

In the target space, by assumption, we have $f(b) \in \Sigma(\mathbb{D}_{k,m}^\ell)$ and by Proposition 3.2, $df(v_b) \in \mathfrak{F}_{f(b)}^\ell$. We now choose a connected open neighborhood $Y \subset G_{k+1,m-k}$ of $f(b)$ and a connected open neighborhood $\Upsilon \subset G_{(k+1)(\ell-k)}T_{f(b)}G_{k+1,m-k}$ of $df(v_b)$. We may choose Y and Υ small enough so that Y (resp. Υ) lies in some Euclidean cell of $G_{k+1,m-k}$ (resp. $G_{(k+1)(\ell-k)}T_{f(b)}G_{k+1,m-k}$) and $\Upsilon \cap \mathfrak{F}_{f(b)}^\ell$ is precisely the zero set of a family of holomorphic functions $\{h_i\}_{i \in I}$ on Υ . Recall that each fibre in $\mathfrak{F}_Y^\ell := \bigcup_{y \in Y} \mathfrak{F}_y^\ell$ is just the Euclidean parallel transport of one another. Thus, $\mathfrak{F}_Y^\ell \cap (Y \times \Upsilon)$ is also precisely the zero set of $\{h_i\}_{i \in I}$ when these functions are trivially extended from Υ to $Y \times \Upsilon$.

Back to the domain, we choose small enough connected open sets $\mathcal{W} \subset \pi^{uni}(\mathfrak{F}_U^\ell)$ and $\mathbf{W} \subset G_{k+1,\ell-k}$ such that $\mathcal{W} \cap S(\Omega_{\ell+1,n-\ell}) \neq \emptyset$, \mathcal{W} (resp. \mathbf{W}) lies in some Euclidean cell of $G_{\ell+1,n-\ell}$ (resp. $G_{k+1,\ell-k}$) and $v_b \in \mathcal{W} \times \mathbf{W}$, $df(\mathcal{W} \times \mathbf{W}) \subset Y \times \Upsilon$. We pull back the functions $\{h_i\}_{i \in I}$ from $Y \times \Upsilon$ to $\mathcal{W} \times \mathbf{W}$ by df and denote the pulled-back functions by $\{h_i^*\}_{i \in I}$.

Take a section $\mathcal{W}_{\mathbf{w}} := \mathcal{W} \times \{\mathbf{w}\} \subset \mathcal{W} \times \mathbf{W}$ and $\mathcal{W}_{\mathbf{w}}$ can be identified as a connect open set in $\mathbb{C}^{(\ell+1)(n-\ell)}$ whose intersection with $S(\Omega_{\ell+1,n-\ell})$ is non-empty. Now we claim that each h_i^* is zero on $\mathcal{W}_{\mathbf{w}} \cap S(\Omega_{\ell+1,n-\ell})$ and hence identically zero on $\mathcal{W}_{\mathbf{w}}$ by Lemma 2.9. To see this, note that by construction, each point $w \in \mathcal{W}_{\mathbf{w}} \cap S(\Omega_{\ell+1,n-\ell})$ is the tangent plane at a point $b_w \in \Sigma(\mathbb{D}_{k,n}^\ell)$ of some $(\ell+1)$ -confined subgrassmannian in $G_{k+1,n-k}$ which completely lies inside $\Sigma(\mathbb{D}_{k,n}^\ell)$. By Proposition 3.2, $df(w) \in \mathfrak{F}_{f(b_w)}^\ell$ and hence we have $h_i^*(w) = 0$ for all $i \in I$ and thereby we have the claim. Since $\mathbf{w} \in \mathbf{W}$ is arbitrary, we then have $h_i^* \equiv 0$ on $\mathcal{W} \times \mathbf{W}$ for every $i \in I$. It is equivalent to saying that for every $w \in \mathcal{W} \times \mathbf{W}$, $df(w) \in \mathfrak{F}^\ell$, where $\mathfrak{F}^\ell \rightarrow G_{k+1,m-k}$ is the bundle in the target. As $\mathcal{W} \times \mathbf{W} \subset \mathfrak{F}_U^\ell$ is open and \mathfrak{F}_U^ℓ is connected, by cooperating with the standard connectedness argument we conclude that $df(\mathfrak{F}_U^\ell) \subset \mathfrak{F}^\ell$ over $G_{k+1,m-k}$.

Note that any non-empty intersection of two confined-subgrassmannian is again a confined subgrassmannian and any $(k + 2)$ -confined subgrassmannian is the intersection of a set of $(\ell + 1)$ -confined subgrassmannians (recall that $\ell \geq k + 1$). Since f is an embedding, we therefore see that $df(\mathfrak{F}_U^{k+1}) \subset \mathfrak{F}^{k+1}$, where $\mathfrak{F}^{k+1} \rightarrow G_{k+1, m-k}$ is the bundle in the target. Finally, as have been mentioned in the beginning of this section, the characteristic vectors on $G_{k+1, n-k}$ or $G_{k+1, m-k}$ are just those tangent vectors tangent to any of the $(k + 2)$ -confined subgrassmannians. Hence, df maps characteristic vectors to characteristic vectors. The proof is complete. \square

Corollary 3.5. *Let $\frac{n}{2} \leq \ell \leq n - k - 2$ and $U \subset G_{k+1, n-k}$ be a connected open set, $U \cap \Sigma(\mathbb{D}_{k, n}^\ell) \neq \emptyset$. Let $f : U \rightarrow G_{k+1, m-k}$ be a holomorphic embedding and $f(U \cap \Sigma(\mathbb{D}_{k, n}^\ell)) \subset \Sigma(\mathbb{D}_{k, m}^{\ell+m-n})$. Then the differential df maps characteristic vectors to characteristic vectors.*

Proof. Recall that $\Sigma(\mathbb{D}_{k, n}^\ell)$ and $\Sigma(\mathbb{D}_{k, n}^{n-\ell-1})$ are equivalent under an automorphism of $G_{k+1, n-k}$. Similarly, $\Sigma(\mathbb{D}_{k, m}^{\ell+m-n})$ and $\Sigma(\mathbb{D}_{k, m}^{n-\ell-1})$ are equivalent. Let $\ell' = n - \ell - 1$, then we have $k + 1 \leq \ell' < n/2$ and $f(U \cap \Sigma(\mathbb{D}_{k, n}^{\ell'})) \subset \Sigma(\mathbb{D}_{k, m}^{\ell'})$. Now the result follows from Proposition 3.4. \square

We are now in the position to prove Theorem 1.1 for $k \geq 1$.

Proof. We first prove that a local holomorphic embedding $f : U \subset G_{k+1, n-k} \rightarrow G_{k+1, m-k}$ satisfying the hypotheses extends to a standard embedding of $G_{k+1, n-k}$ into $G_{k+1, m-k}$. By Proposition 3.4, Corollary 3.5 and Theorem 3.3, we just need to check that $f(U)$ is not contained in any projective linear subspace in $G_{k+1, m-k}$. There are two kinds of projective linear subspaces in $G_{k+1, m-k}$. The first kind is the $(k + 2)$ -confined subgrassmannians. They are biholomorphic to \mathbb{P}^{k+1} . Since f is a local embedding, by counting the dimensions we see that this is impossible. The second kind can be described as follows. Let $V \subset \mathbb{C}^{m+1}$ be a k -dimensional linear subspace and W be a complementary linear subspace of V in \mathbb{C}^{m+1} . Then the set $\{E \in G_{k+1, m-k} : E = V \oplus w, w \in \mathbb{P}(W)\}$ is an $(m - k)$ -dimensional projective linear subspace in $G_{k+1, m-k}$ and we denote it by $\mathbb{P}_{V, W}^{m-k}$.

Suppose $f(U) \subset \mathbb{P}_{V, W}^{m-k}$ for some $V, W \subset \mathbb{C}^{m+1}$. We will first consider case (i) in the statement of Theorem 1.1. Recall that the maximal compact complex analytic subvarieties in $\Sigma(\mathbb{D}_{k, n}^\ell)$ and $\Sigma(\mathbb{D}_{k, m}^\ell)$ are $(\ell + 1)$ -confined subgrassmannians and are of dimension $(k + 1)(\ell - k)$ (c.f. Proposition 3.2). Since f is an embedding and $f(U \cap \Sigma(\mathbb{D}_{k, n}^\ell)) \subset \Sigma(\mathbb{D}_{k, m}^\ell) \cap \mathbb{P}_{V, W}^{m-k}$ by hypotheses, we deduce that some $(\ell + 1)$ -confined subgrassmannian in $\Sigma(\mathbb{D}_{k, m}^\ell)$ is contained in $\mathbb{P}_{V, W}^{m-k}$. But this is impossible because the intersection of any two $(k + 1)$ -planes in $\mathbb{P}_{V, W}^{m-k}$ contains V while obviously the same thing cannot happen for any non-trivial confined-subgrassmannian.

If our map f is in case (ii), we let $\ell' = n - \ell - 1$. Then we have $k + 1 \leq \ell' < \frac{n}{2}$. As explained in Proposition 3.2, $\Sigma(\mathbb{D}_{k, n}^\ell)$ and $\Sigma(\mathbb{D}_{k, n}^{\ell'})$ are equivalent under an automorphism of $G_{k+1, n-k}$ and similarly for $\Sigma(\mathbb{D}_{k, m}^{\ell+m-n})$ and $\Sigma(\mathbb{D}_{k, m}^{\ell'})$. We are then back to case (i) and hence case (ii) is also settled.

By the Identity Theorem for real analytic functions, the extension of f maps the the whole $\Sigma(\mathbb{D}_{k, n}^\ell)$ into $\Sigma(\mathbb{D}_{k, m}^\ell)$. Note that $\Sigma(\mathbb{D}_{k, n}^\ell)$ can be interpreted as the $(k + 1)$ -dimensional null-planes in \mathbb{C}^{n+1} when the latter is equipped with the standard non-degenerate Hermitian form with signature $(\ell + 1, n - \ell)$. On the other hand, the extension of f is induced by some linear map $A : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{m+1}$. With the above interpretation, we see that A must map null vectors

in \mathbb{C}^{n+1} to null vectors in \mathbb{C}^{m+1} . If we have furthermore $f(U \cap \mathbb{D}_{k,n}^\ell) \subset \mathbb{D}_{k,m}^\ell$ in case (i) (resp. $f(U \cap \mathbb{D}_{k,n}^\ell) \subset \mathbb{D}_{k,m}^{\ell+m-n}$ in case (ii)), it follows readily from Lemma 2.13 that the extension of f restricts to a proper holomorphic embedding of $\mathbb{D}_{k,n}^\ell$ into $\mathbb{D}_{k,m}^\ell$ (resp. into $\mathbb{D}_{k,m}^{\ell+m-n}$). \square

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