FOURIER COEFFICIENTS OF CUSP FORMS OF HALF-INTEGRAL WEIGHT

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Abstract. Let $\mathfrak{f}$ be a cusp form of half integral weight whose Fourier coefficients $\mathfrak{a}_{\mathfrak{f}}(n)$ are all real. We study the sign change problem of $\mathfrak{a}_{\mathfrak{f}}(n)$, when n runs over some specific sets of integers. Lower bounds of the best possible order of magnitude are established for the number of those coefficients that have the same signs. These give an improvement on some recent results of Bruinier & Kohnen [2] and Kohnen [13].

1. Introduction

Owing to different reasons, the problem of sign changes of Hecke eigenvalues of integral weight cusp forms has attracted many attentions [10, 15, 7, 14, 19, 17]. One motivation is to delve the analogy with (real) Dirichlet characters. Real Dirichlet characters admit only ± 1 other than 0; however these eigenvalues (when properly normalized) vary in the range from -2 to 2. A reasonable parallel one may consider is the pattern of the signs. Such an investigation has a long history in the case of real characters, like the problem of the least quadratic non-residue. The work [17] provides a comprehensive discussion in the context of modular forms. In light of the theory of half integral weight forms in Shimura [23], Waldspurger [24], Kohnen-Zagier [16] and Kohnen [11, 12], etc, the half integral weight forms are closely related to integral weight cusp forms and hence it is naturally important to study the analogous sign-change problems. The case of half integral weight cusp forms, although looking like a formal extension, is somehow more subtle. A reason is that the Fourier coefficients of a half integral weight cuspidal Hecke eigenform in general are not plainly multiplicative (cf. [4, page 783]). In [2], Bruinier & Kohnen studied the sign changes of the Fourier coefficient $\mathfrak{a}_{\mathfrak{f}}(n)$ of a half integral weight cusp form $\mathfrak{f}$ for specific sequences of integers n, which also stimulates this work.

Throughout we let $k \geqslant 1$ be an integer and assume $N \geqslant 4$ to be divisible by 4. Fix any Dirichlet character χ modulo N. We write $S_{k+1/2}(N,\chi)$ for the space of cusp forms of weight $k+1/2$ for the group $\Gamma_0(N)$ with character χ (cf. [23]). The space $S_{3/2}(N,\chi)$ contains unary theta functions. Let $S^*_{3/2}(N,\chi)$ be the orthogonal complement with respect to the Petersson scalar product of the subspace generated by these theta functions (cf. [23, Section 4] and [3, Section 4]). For convenience, we put $S^*_{k+1/2}(N,\chi) = S_{k+1/2}(N,\chi)$ when $k \geqslant 2$. Each $\mathfrak{f} \in S^*_{k+1/2}(N,\chi)$ has a Fourier

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expansion

$$\mathfrak{f}(z) = \sum_{n\geqslant 1} \mathfrak{a}_{\mathfrak{f}}(n)\mathrm{e}^{2\pi i n z} \qquad (z \in \mathbb{H}), \tag{1.1}$$

on the complex upper half plane $\mathbb{H}$. Let $t \geqslant 1$ be a squarefree integer. The Shimura correspondance [23] lifts $\mathfrak{f}$ to a cusp form f_t of weight $2k$ for the group $\Gamma_0(N/2)$ with character χ^2. Also it gives a vital relation between their Fourier coefficients,

$$a_{f_t}(n) := \sum_{d\mid n} \chi_{t,N}(d) d^{k-1} \mathfrak{a}_{\mathfrak{f}}\left(t\frac{n^2}{d^2}\right), \tag{1.2}$$

where $\chi_{t,N}$ denotes the character

$$\chi_{t,N}(d) := \chi(d)\left(\frac{(-1)^k t}{d}\right)$$

and

$$f_t(z) := \sum_{n\geqslant 1} a_{f_t}(n)\mathrm{e}^{2\pi i n z} \qquad (z \in \mathbb{H}). \tag{1.3}$$

(f_t is called the Shimura lift of $\mathfrak{f}$ associated to t.) Furthermore if $\mathfrak{f}$ is a Hecke eigenform, then so is the Shimura lift. In fact, in this case

$$f_t = a_{\mathfrak{f}}(t) f \tag{1.4}$$

where f is a normalized ($a_f(1) = 1$) Hecke eigenform independent of t.

Let $\mathfrak{f} \in S^*_{k+1/2}(N, \chi_0)$ be a cusp form with trivial character χ_0, squarefree level and real coefficients $\mathfrak{a}_{\mathfrak{f}}(n)$. Suppose that $\mathfrak{f}$ lies in the plus space, that is, $\mathfrak{a}_{\mathfrak{f}}(n) = 0$ when $(-1)^k n \equiv 2, 3 \,(\mathrm{mod}\, 4)$, see [16, 12]. Bruinier & Kohnen [2] gave the conjectures

$$\lim_{x\to\infty} \frac{|\{n \leqslant x : \mathfrak{a}_{\mathfrak{f}}(n) \gtrless 0\}|}{|\{n \leqslant x : \mathfrak{a}_{\mathfrak{f}}(n) \neq 0\}|} = \frac{1}{2} \tag{1.5}$$

and

$$\lim_{x\to\infty} \frac{|\{|d| \leqslant x : d \text{ fundamental discriminant}, \mathfrak{a}_{\mathfrak{f}}(|d|) \gtrless 0\}|}{|\{|d| \leqslant x : d \text{ fundamental discriminant}, \mathfrak{a}_{\mathfrak{f}}(|d|) \neq 0\}|} = \frac{1}{2} \tag{1.6}$$

with empirical evidence, which may be, however, out of the present reach. Alternatively, they considered the change in signs of $\mathfrak{a}_{\mathfrak{f}}(n)$ when n runs over specific sets of integers, such as $\{tn^2\}_{n\in\mathbb{N}}$, $\{tp^{2\nu}\}_{\nu\in\mathbb{N}}$ and $\{tn_t^2\}_{t\,\text{squarefrees}}$. Here t is a positive squarefree integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$, p denotes any fixed prime and n_t is an integer determined by t (cf. [2, Theorems 2.1 and 2.2]). Amongst other things, their approach comprises a well-known robust analytic tool - Landau's theorem on Dirichlet series.

Our first result gives an improvement to [2, Theorem 2.1] and [13, Theorem], exploiting tools in analytic number theory in connection with Rankin-Selberg L-functions.

Theorem 1. *Let $k \geqslant 1$ be an integer, $N \geqslant 4$ an integer divisible by 4 and χ be a Dirichlet character modulo N. Suppose that $\mathfrak{f} \in S^*_{k+1/2}(N, \chi)$ and $t \geqslant 1$ is a squarefree integer such that $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$. Assume that the sequence $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n\in\mathbb{N}}$ is real. Then $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n\in\mathbb{N}}$ has infinitely many sign changes. More specifically there is a small constant $\alpha = \alpha(\mathfrak{f}, t) > 0$ such that for all sufficiently large x, i.e. $x \geqslant x_0(\mathfrak{f}, t)$,*

$\mathfrak{a}_{\mathfrak{f}}(tp^2)$ *has (at least) one sign-change when* p *runs through primes in the interval* $[x^\alpha, x]$.

Our proof shows an alternative (other than [13]) to remove the hypothesis on the non-vanishing of $L(s, \chi_{t,N})$ on $(0,1)$ (Chowla's conjecture if $\chi_{t,N}$ is quadratic), see Theorem 2.1 of [2]. This conjecture asserts that $L(s, \chi_{t,N})$ has no zeros in the interval $(0,1)$. Kohnen [13] removed the hypothesis by refining the argument of [2]. However as in [2], the method did not produce a quantitative result. In this regard Theorem 1 goes further and in fact, the proof here applies to the finer sequence of primes, that is, we narrow down to the infinitely many sign changes in $\{\mathfrak{a}_{\mathfrak{f}}(tp^2)\}_{p \text{ primes}}$ (instead of $\{\mathfrak{a}_{\mathfrak{f}}(tn^2)\}_{n\in\mathbb{N}}$).

The form $\mathfrak{f}$ in Theorem 1 is not assumed to be a Hecke eigenform. Imposing this assumption, if the Shimura lift f_t, or equivalently f in (1.4) when $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$, is not of CM type (see Remark 1), we can tell more in the next theorems. A salience now is the retrieve of multiplicativity. More precisely, for any fixed (squarefree) t and Hecke eigenform $\mathfrak{f}$, the arithmetic function $n \mapsto \mathfrak{a}_{\mathfrak{f}}(tn^2)$ is multiplicative in the following sense (cf. [23, (1.18)]):

$$\mathfrak{a}_{\mathfrak{f}}(tm^2)\mathfrak{a}_{\mathfrak{f}}(tn^2) = \mathfrak{a}_{\mathfrak{f}}(t)\mathfrak{a}_{\mathfrak{f}}(tm^2n^2) \quad \text{if} \quad (m,n) = 1. \tag{1.7}$$

The condition of a Hecke eigenform $\mathfrak{f}$ is indispensable in our argument, as we start with (1.7). These results are clearly the best possible in order of magnitude.

Theorem 2. *Let* $k \geqslant 1$ *be an integer,* $N \geqslant 4$ *an integer divisible by* 4 *and* χ *be a real Dirichlet character modulo* N. *Suppose that* $\mathfrak{f} \in S^*_{k+1/2}(N,\chi)$ *is a Hecke eigenform and* $t \geqslant 1$ *is a squarefree integer such that* $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$. *Assume that its Shimura lift is not of CM type. Then we have*

$$\sum_{\substack{n\leqslant x,\, n \text{ is squarefree} \\ (n,Nt)=1,\, \mathfrak{a}_{\mathfrak{f}}(tn^2)\gtrless 0}} 1 \gg_{\mathfrak{f},t} x \tag{1.8}$$

for $x \geqslant x_0(\mathfrak{f}, t)$. *If* $N/2$ *is squarefree, the assumption of a non-CM Shimura lift will automatically hold and hence can be omitted.*

Remark 1. A Hecke eigenform f is of CM type if there exists a non-trivial Dirichlet character φ such that $\lambda_f(p) = \varphi(p)\lambda_f(p)$ for all primes p in a set of primes of density 1 (see [22, Section 7.2]). Here and in the sequel the Vinogradov symbol $f(x) \ll g(x)$ (or $g(x) \gg f(x)$) means $|f(x)| \leqslant Cg(x)$ for all sufficiently large $x \geqslant x_0$, where C is a positive constant. We also write $\ll_*$ or $\gg_*$ to stress the dependence of the implied constants on $*$.

The following result refines [2, Theorem 2.2].

Theorem 3. *Let* $k \geqslant 1$ *be an integer,* $N \geqslant 4$ *an integer divisible by* 4 *and* χ *be a real Dirichlet character modulo* N. *Suppose that* $\mathfrak{f} \in S^*_{k+1/2}(N,\chi)$ *is a Hecke eigenform, and* t *is a positive squarefree integer for which* $\mathfrak{a}_{\mathfrak{f}}(t) \neq 0$. *For any prime* $p \nmid N$, *define* $\theta_f(p) \in [0,\pi]$ *by the relation* $\lambda_f(p) = 2\cos\theta_f(p)$ *where* $\lambda_f(p)p^{k-1/2}$ *is the* p*-th Fourier coefficient of* f *in (1.4). We have the following results where* $\varepsilon = 1$ *or* -1 *in Case (ii)-(iv).*

Case (i). $\theta_f(p) = 0$. *Then* $\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})$ *has the same sign as* $\mathfrak{a}_{\mathfrak{f}}(t)$, *for all* $\nu \geqslant 0$.

Case (ii). $\theta_f(p)=\pi$. *Then*

$$\sum_{\substack{\nu\leqslant x\\ \varepsilon\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})>0}} 1\sim\frac{1}{2}x\qquad(x\to\infty). \tag{1.9}$$

Case (iii). $\theta_f(p)/(2\pi)=m/n\in(0,1/2)$ *is rational with* $(m,n)=1$. *Then*

$$\sum_{\substack{\nu\leqslant x\\ \varepsilon\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})\geqslant(\sqrt{3}/2-1/\sqrt{p})p^{\nu(k-1/2)}/\sin\theta_f(p)}} 1\geqslant\frac{1}{n}x+O_{\mathfrak{f}}(1)\qquad(x\to\infty). \tag{1.10}$$

Case (iv). $\theta_f(p)/(2\pi)\in(0,1/2)$ *is irrational. Then*

$$\sum_{\substack{\nu\leqslant x\\ \varepsilon\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})\geqslant(c-1/\sqrt{p})p^{\nu(k-1/2)}/\sin\theta_f(p)}} 1\geqslant\Big(\frac{1}{2}-\frac{\arcsin c}{\pi}\Big)x+o(x)\qquad(x\to\infty) \tag{1.11}$$

for any $c\in(1/\sqrt{p},1)$.

Remark 2. Cases (i) and (ii) can happen for at most finitely many primes p only. Indeed if we let K_f be the number field generated by all the Fourier coefficients $a_f(n)$ of f, then the total number of primes p for which $0\neq\cos\theta_{f_t}(p)\in\mathbb{Q}$ cannot exceed r where $2^r\|[K_f:\mathbb{Q}]$, i.e., 2^r is the greatest power of two that divides the degree of K_f over $\mathbb{Q}$. This follows from the proof of [2, Remark 2.3]: if $0\neq\lambda_f(p)\in\mathbb{Q}$, then $\sqrt{p}\in K_f$ for $a_f(p)=\lambda_f(p)p^{k-1/2}$, whence our assertion follows by the fact $[\mathbb{Q}(\sqrt{p_1},\dots,\sqrt{p_t}):\mathbb{Q}]=2^t$ for distinct primes $p_1,\dots,p_t$.

For other values $\alpha\in(-2,2)$, the Sato-Tate conjecture suggests that $\lambda_f(p)=\alpha$ holds only for a zero density of primes p. When $\alpha=0$, it is shown to be true in Serre [22]; though in this case ($\alpha=0$) and f is a non CM form, one may anticipate, parallel to Lehmer's conjecture in [20], that no prime p for $\lambda_f(p)=0$ exists. Another resemble question is (the analogue of) Lang-Trotter conjecture - the primes for which $\lambda_f(p)p^{(k-1/2)}=\alpha$ is of zero density, for any α.

The positive proportion of integers from $\{tn^2\}_{n\in\mathbb{N}}$ (resp. $\{tp^{2\nu}\}_{\nu\in\mathbb{N}}$) on which $\mathfrak{a}_{\mathfrak{f}}(tn^2)$ (resp. $\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})$) are of the same sign, shown in Theorems 2 and 3, encourages our belief in Conjecture (1.5). Finally we would remark that for the sequence $\{\mathfrak{a}_{\mathfrak{f}}(tp^2)\}_{p\,\text{primes}}$, there is also a positive density (over the set of primes) of sign changes.

Theorem 4. *Let* $k\geqslant 1$ *be an integer,* $N\geqslant 4$ *an integer divisible by* 4 *and* χ *be a real Dirichlet character modulo* N. *Suppose that* $\mathfrak{f}\in S^*_{k+1/2}(N,\chi)$ *is a Hecke eigenform and* $t\geqslant 1$ *is squarefree such that* $\mathfrak{a}_{\mathfrak{f}}(t)\neq 0$. *Then we have*

$$\sum_{\substack{p\leqslant x\\ \varepsilon\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^2)>1.68p^{k-1/2}}} 1\gg_{\mathfrak{f}}\frac{x}{\log x} \tag{1.12}$$

for $x\geqslant x_0(\mathfrak{f})$ *and* $\varepsilon=\pm1$.

This is a direct application of (4.1) (with $\nu=1$) below and that there exists a positive density of primes for which $\lambda_f(p)>1.681$ and $\lambda_f(p)<-1.681$ respectively, shown in [9, Theorem 4.10].

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2. The proof of Theorem 1

We begin with the basic theory of Atkin-Lehner on primitive forms (or newforms), and results following from the L-functions attached to these forms. Let $S_\kappa(L,\psi)$ be the space of holomorphic cusp forms of integral weight $\kappa \geqslant 2$ and nebentypus ψ for $\Gamma_0(L)$. The nebentypus ψ is a Dirichlet character mod L whose conductor is denoted by L^*. In an attempt to diagonalize all Hecke operators over $S_\kappa(L,\psi)$, Atkin and Lehner figured out the orthogonal complement $S_\kappa^{\rm new}(L,\psi)$ (with respect to Petersson inner product) of the subspace generated by all forms $g(\ell z)$ (called oldforms) where $g \in S_\kappa(M,\psi_M)$ is of strict lower level $M|L$ (with $L^*|M$) and the nebentypus ψ_M induced by ψ (more precisely, by the unique primitive character ψ^* mod L^* that induces ψ). The integer ℓ runs over all divisors of L/M. We call f a primitive form (or newform) if $f \in S_\kappa^{\rm new}(L,\psi)$ is a common eigenfunction of all Hecke operators (including the involution) and its first coefficient equals one. Denote by $\mathrm{H}_\kappa^*(L,\psi)$ the set of all primitive forms of weight κ, level L and nebentypus ψ. By the theory of primitive forms (see [1] or [8, §14.7]), we have the decomposition

$$S_\kappa(L,\psi) = \bigoplus_{\substack{M|L\\ M\equiv 0 (\bmod L^*)}} \bigoplus_{f\in \mathrm{H}_\kappa^*(M,\psi_M)} \mathrm{Span}_{\mathbb{C}}\big\{f_{|\ell} : \ell \mid (L/M)\big\}, \tag{2.1}$$

where ψ_M is the Dirichlet character mod M induced from ψ and $f_{|\ell}(z) := f(\ell z)$. The outermost direct sum runs over M and $\mathrm{Span}_{\mathbb{C}} S$ denotes the subspace generated by elements in S over $\mathbb{C}$.

The primitive forms give rise to a special basis for $S_\kappa(L,\psi)$, and above all, their associated L-functions satisfy a functional equation and admit an Euler product factorization. More generally a primitive form f corresponds uniquely to an irreducible unitary cuspidal representation π of $GL_2(\mathbb{A}_{\mathbb{Q}})$ (whose ∞-component π_∞ is a discrete series), and they have the same L-functions, i.e. $L(s,f) = L(s,\pi)$, up to normalization by a scalar. There are many fruitful results in the context of L-functions for automorphic representations, for example, the nonvanishing on the line $\Re e\, s = 1$ and the zero-free regions. Our first lemma is also one of their consequences.

Lemma 2.1. *Let $f \in \mathrm{H}^*_\kappa(M,\psi)$ whose n-th coefficient is $\lambda_f(n)n^{(\kappa-1)/2}$. Then as $x\to\infty$,*

$$\sum_{\substack{p\leqslant x\\ p\nmid M}} \frac{\lambda_f(p)}{p} \ll 1, \tag{2.2}$$

$$\sum_{\substack{p\leqslant x\\ p\nmid M}} \frac{|\lambda_f(p)|^2}{p} = \log\log x + O(1). \tag{2.3}$$

Moreover, if $g \neq f$ is another primitive form (of any level), then

$$\sum_{\substack{p\leqslant x\\ p\nmid M}} \frac{\lambda_f(p)\overline{\lambda_g(p)}}{p} \ll 1. \tag{2.4}$$

The implied constants in $\ll$ or O-symbol depend on the form f in (2.2) and (2.3), and both f and g in (2.4).

These results are proved using analytic techniques and the theory of $GL_2\times GL_2$ Rankin-Selberg L-functions. Indeed the general case of $GL_m \times GL_{m'}$ were treated by, for instance, Rudnick-Sarnak, Liu-Ye, Liu-Wang-Ye, etc. Here we shall prove the lemma with the tools in [8].

Proof. The key ingredient is the Prime Number Theorem for general L-functions. Suppose $L(s,F)$ is an L-function (defined as in [8, p.94]) that has no zero, except possibly a simple real zero β, for $s=\sigma+it$ in the region

$$\sigma \geqslant 1 - \frac{c_F}{\log(|t|+3)} \tag{2.5}$$

where $c_F>0$ is a constant depending on F only. We further assume $\beta<1$ if it exists. Write

$$-\frac{L'}{L}(s,F) = \sum_{n\geqslant 1}\frac{\Lambda_F(n)}{n^s} \qquad (\sigma>1),$$

where the sum ranges over prime powers ($n=p^\nu$) only. Then by [8, Theorem 5.13], we have the formula

$$\sum_{n\leqslant x}\Lambda_F(n) = rx + O_F\big(x\mathrm{e}^{-c'_F\sqrt{\log x}}\big) \tag{2.6}$$

where r denotes the order of the possible pole of $L(s,F)$ at $s=1$, and $c'_F>0$ is a constant whose value depends on F. We state (2.6) in this weak form just for simplicity.

Now consider the L-functions $L(s,f)$ and $L(s,f\times\overline{g})$ (which are respectively $L(f,s)$ and $L(f\otimes\overline{g},s)$ in [8]). By Theorems 5.39 and 5.44 of [8], we obtain (2.6) for these two L-functions. (Note that here $\overline{g}$ is interpreted as the primitive form $\overline{g(-\overline{z})}$.) Moreover, $r=0$ when $F=f$ or $F=f\times\overline{g}$ with $f\neq g$. In case $F=f\times\overline{f}$, we have $r=1$. Furthermore we note that

$$\Lambda_f(p)=\lambda_f(p)\log p, \qquad \Lambda_{f\times\overline{g}}(p)=\lambda_f(p)\overline{\lambda_g(p)}\log p$$

and

$$|\Lambda_f(p^\nu)|\leqslant 2\log p, \qquad |\Lambda_{f\times\overline{g}}(p^\nu)|\leqslant 4\log p$$

with Deligne's inequality. By (2.6), we deduce that

$$\sum_{p\leqslant x}\frac{\lambda_f(p)}{p}=\sum_{n\leqslant x}\frac{\Lambda_f(n)}{n\log n}-\sum_{p\leqslant x}\sum_{\nu\geqslant 2}\frac{\Lambda_f(p^\nu)}{p^\nu\log p^\nu}.$$

The double sum is obviously $O(1)$, and the sum over n is, by partial integration, equal to

$$\begin{aligned}\int_2^x\frac{1}{t\log t}\,\mathrm{d}\Big(\sum_{n\leqslant t}\Lambda_f(n)\Big)&=r\int_2^x\frac{\mathrm{d}t}{t\log t}+\int_2^x\frac{\mathrm{d}O\big(t\mathrm{e}^{-c_f'\sqrt{\log t}}\big)}{t\log t}\\&=r\log\log x+O(1).\end{aligned}$$

Apparently we have the same conclusion for $F=f\times\overline{g}$. Only a finite number of primes divide M, so we may drop the corresponding terms without a significant loss. Our proof is complete by invoking $r=1$ for (2.3) and 0 for (2.2) or (2.4). □

Lemma 2.2. *Let $k\geqslant 1$ be an integer, $N\geqslant 4$ an integer divisible by 4 and χ be a Dirichlet character modulo N. Suppose that $\mathfrak{f}\in S^*_{k+1/2}(N,\chi)$ and $t\geqslant 1$ is a squarefree integer such that $\mathfrak{a}_\mathfrak{f}(t)\neq 0$. Assume that the sequence $\{\mathfrak{a}_\mathfrak{f}(tn^2)\}_{n\in\mathbb{N}}$ is real. Then*

$$\sum_{\substack{p\leqslant x\\ p\nmid N}}\frac{\mathfrak{a}_\mathfrak{f}(tp^2)}{p^{k+1/2}}\ll_{\mathfrak{f},t}1, \tag{2.7}$$

$$\sum_{\substack{p\leqslant x\\ p\nmid N}}\frac{\mathfrak{a}_\mathfrak{f}(tp^2)^2}{p^{2k}}=C_{\mathfrak{f},t}\log\log x+O_{\mathfrak{f},t}(1)\qquad(x\to\infty), \tag{2.8}$$

where the positive constant $C_{\mathfrak{f},t}$ and the implied constants depend on $\mathfrak{f}$ and t.

Proof. Applying the Möbius inversion formula to (1.2), we derive that

$$\mathfrak{a}_\mathfrak{f}(tn^2)=\sum_{d\mid n}\mu(d)\chi_{t,N}(d)d^{k-1}a_{f_t}\Big(\frac{n}{d}\Big), \tag{2.9}$$

where $\mu(d)$ is the Möbius function and $a_{f_t}(n)$ is the n-th coefficient of f_t. Write $a_{f_t}(n)=\lambda_{f_t}(n)n^{k-1/2}$, the formula (2.9) is reformulated as

$$\frac{\mathfrak{a}_\mathfrak{f}(tn^2)}{n^{k-1/2}}=\sum_{d\mid n}\frac{\mu(d)\chi_{t,N}(d)}{\sqrt{d}}\lambda_{f_t}\Big(\frac{n}{d}\Big). \tag{2.10}$$

Taking $n=p$ a prime, it follows that

$$\frac{\mathfrak{a}_\mathfrak{f}(tp^2)}{p^{k-1/2}}=\lambda_{f_t}(p)-\frac{\chi_{t,N}(p)}{\sqrt{p}}\mathfrak{a}_\mathfrak{f}(t), \tag{2.11}$$

as $\lambda_{f_t}(1)=\mathfrak{a}_\mathfrak{f}(t)$.

Now we apply (2.1) to $S_{2k}(N/2,\chi^2)$, and obtain an basis

$$\bigcup_{\substack{M\mid(N/2)\\ M\equiv 0(\mathrm{mod}\,(N/2)^*)}}\Big\{f_{|\ell}\ :\ \ell\Big|\frac{(N/2)}{M},\ f\in\mathrm{H}^*_{2k}\big(M,(\chi^2)_M\big)\Big\}$$

where $(N/2)^*$ is the conductor of χ^2, and $(\chi^2)_M$ is the character mod M induced by χ^2. Hence each $f \in S_{2k}(N/2, \chi^2)$ is uniquely expressed as

$$f(z) = \sum_i \sum_{\ell | (N/(2M_i))} c_{i,\ell} f_i(\ell z)$$

where $f_i \in \mathrm{H}_{2k}^*(M_i, (\chi^2)_{M_i})$ is primitive of level M_i (and $(N/2)^* \mid M_i \mid (N/2)$) and $c_{i,\ell}$'s are scalars depending on f. Note that M_i's take the same value for those f_i's of the same level. Through their Fourier expansions we see that for any prime $p \nmid N$,

$$\lambda_{f_t}(p) = \sum_i c_i \lambda_{f_i}(p)$$

where $c_i := c_{i,1}$ and not all c_i's equal zero for

$$0 \neq a_{\mathfrak{f}}(t) = \lambda_{f_t}(1) = \sum_i c_i$$

(whence $f_t \not\equiv 0$). In view of (2.11), it follows that

$$\frac{\mathfrak{a}_{\mathfrak{f}}(tp^2)}{p^{k-1/2}} = \sum_i c_i \lambda_{f_i}(p) - \frac{\chi_{t,N}(p)}{\sqrt{p}} \mathfrak{a}_{\mathfrak{f}}(t) \tag{2.12}$$

and under the assumption $\mathfrak{a}_{\mathfrak{f}}(tn^2) \in \mathbb{R}$, we infer

$$\begin{aligned} \frac{\mathfrak{a}_{\mathfrak{f}}(tp^2)^2}{p^{2k-1}} &= \sum_i |c_i|^2 \big|\lambda_{f_i}(p)\big|^2 + \sum_{i \neq j} c_i \overline{c_j} \lambda_{f_i}(p) \overline{\lambda_{f_j}(p)} \\ &\quad + |\mathfrak{a}_{\mathfrak{f}}(t)|^2 \frac{|\chi_{t,N}(p)|^2}{p} - 2\Re e \sum_i \frac{\overline{c_i \lambda_{f_i}(p)} \chi_{t,N}(p)}{\sqrt{p}} \mathfrak{a}_{\mathfrak{f}}(t). \end{aligned}$$

Imposing the weight p^{-1} to these two formulas and summing over $p \leqslant x$ except for the prime factors of N, we conclude that

$$\sum_{\substack{p \leqslant x \\ p \nmid N}} \frac{\mathfrak{a}_{\mathfrak{f}}(tp^2)}{p^{k+1/2}} = \sum_i c_i \sum_{p \leqslant x} \frac{\lambda_{f_i}(p)}{p} + O_{\mathfrak{f},t}(1)$$

and

$$\sum_{\substack{p \leqslant x \\ p \nmid N}} \frac{\mathfrak{a}_{\mathfrak{f}}(tp^2)^2}{p^{2k}} = \sum_i |c_i|^2 \sum_{\substack{p \leqslant x \\ p \nmid N}} \frac{|\lambda_{f_i}(p)|^2}{p} + \sum_{i \neq j} c_i \overline{c_j} \sum_{\substack{p \leqslant x \\ p \nmid N}} \frac{\lambda_{f_i}(p) \overline{\lambda_{f_j}(p)}}{p} + O_{\mathfrak{f},t}(1).$$

Set $C_{\mathfrak{f},t} := \sum_i |c_i|^2 > 0$ (as some $c_i \neq 0$), the desired results follow plainly with Lemma 2.1. $\square$

Now we are ready to finish the proof of Theorem 1. With Deligne's bound, we deduce from (2.12) that

$$\big|\mathfrak{a}_{\mathfrak{f}}(tp^2) p^{-(k-1/2)}\big| \leqslant 2 \sum_i |c_i| + |\mathfrak{a}_{\mathfrak{f}}(t)| =: C'_{\mathfrak{f},t}.$$

Assume all $\mathfrak{a}_{\mathfrak{f}}(tp^2)$ are of the same sign for $y \leqslant p \leqslant x$ with $p \nmid N$. Then,

$$\sum_{\substack{y \leqslant p \leqslant x \\ p \nmid N}} \frac{\mathfrak{a}_{\mathfrak{f}}(tp^2)^2}{p^{2k}} \leqslant C'_{\mathfrak{f},t} \sum_{\substack{y \leqslant p \leqslant x \\ p \nmid N}} \frac{|\mathfrak{a}_{\mathfrak{f}}(tp^2)|}{p^{k+1/2}} = C'_{\mathfrak{f},t} \left| \sum_{\substack{y \leqslant p \leqslant x \\ p \nmid N}} \frac{\mathfrak{a}_{\mathfrak{f}}(tp^2)}{p^{k+1/2}} \right|. \tag{2.13}$$

Immediately (2.8) implies that the L.H.S. of (2.13) equals

$$C_{\mathfrak{f},t} \log\left(\frac{\log x}{\log y}\right) + O_{\mathfrak{f},t}(1),$$

but from (2.7), the R.H.S. of (2.13) is $O_{\mathfrak{f},t}(1)$ for all $x \geqslant y \geqslant 2$. This is impossible if $y = x^{\alpha}$ with a small constant $\alpha = \alpha(\mathfrak{f}, t) > 0$.

3. The proof of Theorem 2

The next lemma comes from the first part of Theorem 15 in Serre [22], which is the key tool for our proof.

Lemma 3.1. *Let g be any Hecke eigenform of integral weight $\geqslant 2$ and of level M. Suppose $h(X) \in \mathbb{C}[X]$ is any polynomial. Write $a_g(n)$ for the n-th Fourier coefficient of g. If g is not of type CM, then*

$$\sum_{\substack{p \leqslant x \\ p \nmid M,\, a_g(p) = h(p)}} 1 \ll_{g,h,\delta} \frac{x}{(\log x)^{1+\delta}} \tag{3.1}$$

for any $\delta < \frac{1}{4}$ and all $x \geqslant 2$.

Now we are in a position to prove Theorem 2. Given a Hecke eigenform $\mathfrak{f}$, we let f be the associated Shimura lift as in (1.4). As f is a Hecke eigenform, we have by (2.1), $f(z) = \sum_{\ell | L} c_\ell f_i(\ell z)$ for some $L|(N/2)$, where f_i is a primitive form. Thus $\lambda_f(p) = \lambda_{f_i}(p)$ for all primes $p \nmid N$, and f_i is not of CM type by the assumption that f is not of CM type. (See Remark 1.)

Now it remains to prove (1.8). We let $\mathscr{P}_{Nt} := \{p : p \nmid Nt\}$ and by (2.9) and (1.4), we obtain for $p \in \mathscr{P}_{Nt}$,

$$\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^2) = a_f(p) - \chi_{t,N}(p)p^{k-1}.$$

As $\chi_{t,N}(p) = \pm 1$ for $p \in \mathscr{P}_{Nt}$ (noting that the nebentypus χ is quadratic), we split

$$\mathscr{P}_{Nt} := \{p : p \nmid Nt\} = \mathscr{P}_{Nt}^{(1)} \cup \mathscr{P}_{Nt}^{(2)}$$

where for $p \in \mathscr{P}_{Nt}^{(j)}$ $(j = 1, 2)$,

$$\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^2) = a_f(p) - \varepsilon_j p^{k-1} \tag{3.2}$$

with $\varepsilon_1 := 1$ and $\varepsilon_2 := -1$. Thus we have $\mathfrak{a}_{\mathfrak{f}}(tp^2) = 0 \ \Rightarrow\ a_f(p) = \varepsilon_j p^{k-1}$ for $j = 1$ or 2. By applying Lemma 3.1 to $g = f$, we deduce that

$$\begin{aligned}\sum_{\substack{p\leqslant x\\ p\nmid Nt,\, \mathfrak{a}_{\mathfrak{f}}(tp^2)=0}} 1 &= \sum_{1\leqslant j\leqslant 2} \sum_{\substack{p\leqslant x\\ p\in\mathscr{P}_{Nt}^{(j)},\, a_f(p)=\varepsilon_j p^{k-1}}} 1\\ &\leqslant \sum_{1\leqslant j\leqslant 2} \sum_{\substack{p\leqslant x,\, p\nmid(N/2)\\ a_f(p)=\varepsilon_j p^{k-1}}} 1 \qquad (3.3)\\ &\ll_{\mathfrak{f},t,\delta} \frac{x}{(\log x)^{1+\delta}}\end{aligned}$$

for any $\delta < \frac{1}{4}$ and all $x \geqslant 2$.

Define

$$\begin{aligned}\mathscr{B}_{\mathfrak{f}} &:= \big\{p : p \nmid Nt,\ \mathfrak{a}_{\mathfrak{f}}(tp^2) = 0\big\} \cup \{p_0\} \cup \{p : p \mid Nt\} \cup \big\{p^2 : p \nmid p_0Nt,\ \mathfrak{a}_{\mathfrak{f}}(tp^2) \neq 0\big\}\\ &=: \{b_i\}_{i\geqslant 1} \quad \text{(with increasing order)},\end{aligned}$$

where p_0 is the first prime such that $p_0 \nmid tN$ and $\mathfrak{a}_{\mathfrak{f}}(t)\mathfrak{a}_{\mathfrak{f}}(tp_0^2) < 0$. (Theorem 1 assures the existence of p_0). By virtue of (3.3), a simple integration by parts allows us to deduce

$$\begin{aligned}\sum_{\substack{p\leqslant x\\ p\nmid Nt,\, \mathfrak{a}_{\mathfrak{f}}(tp^2)=0}} \frac{1}{p} &= \int_{2-}^{x} \frac{1}{t}\,\mathrm{d}\Bigg(\sum_{\substack{p\leqslant t\\ p\nmid Nt,\, \mathfrak{a}_{\mathfrak{f}}(tp^2)=0}} 1\Bigg)\\ &\ll 1 + \int_{2}^{x} \frac{\mathrm{d}t}{t(\log t)^{1+\delta}}\\ &\ll 1.\end{aligned}$$

Thus we infer that

$$\sum_{i\geqslant 1} \frac{1}{b_i} < \infty \qquad \text{and} \qquad (b_i, b_j) = 1 \quad (i \neq j).$$

Let $\mathscr{A}_{\mathfrak{f}} := \{a_i\}_{i\geqslant 1}$ (with increasing order) be the sequence of all $\mathscr{B}_{\mathfrak{f}}$-free numbers, i.e. the integers indivisible by any element in $\mathscr{B}_{\mathfrak{f}}$. According to [6], $\mathscr{A}_{\mathfrak{f}}$ is of positive density

$$\lim_{x\to\infty} \frac{|\mathscr{A}_{\mathfrak{f}} \cap [1, x]|}{x} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{b_i}\right) > 0. \tag{3.4}$$

The definition of $\mathscr{B}_{\mathfrak{f}}$ and (1.7) yields that for all $a \in \mathscr{A}_{\mathfrak{f}}$,

$$\mathfrak{a}_{\mathfrak{f}}(ta^2) = \mathfrak{a}_{\mathfrak{f}}(t)^{1-\omega(a)} \prod_{p\mid a} \mathfrak{a}_{\mathfrak{f}}(tp^2) \neq 0$$

where $\omega(a)$ denotes the number of all distinct prime factors of a. As in [19], we shall exploit the two sets of integers

$$\mathscr{N}^{\pm} := \mathscr{A}_{\mathfrak{f}}^{\pm} \cup \{p_0 a_i : a_i \in \mathscr{A}_{\mathfrak{f}}^{\mp}\}$$

where

$$\mathscr{A}_{\mathfrak{f}}^{\pm} := \big\{a_i \in \mathscr{A}_{\mathfrak{f}} : \mathfrak{a}_{\mathfrak{f}}(ta_i^2) \gtrless 0\big\}$$

constitutes the partition

$$\mathscr{A}_{\mathfrak{f}} = \mathscr{A}_{\mathfrak{f}}^{+} \cup \mathscr{A}_{\mathfrak{f}}^{-}.$$

The upshot is the switch of signs via the multiplicativity (1.7) and the negativity of $\mathfrak{a}_{\mathfrak{f}}(t)\mathfrak{a}_{\mathfrak{f}}(tp_0^2)$ under our construction. Indeed,

$$\mathfrak{a}_{\mathfrak{f}}(t(p_0 a)^2) = \mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp_0^2)\mathfrak{a}_{\mathfrak{f}}(ta^2) \gtrless 0$$

according as $a \in \mathscr{A}_{\mathfrak{f}}^{\mp}$. Hence $\mathfrak{a}_{\mathfrak{f}}(ta^2) \gtrless 0$ and $(a, Nt) = 1$ for all $a \in \mathscr{N}^{\pm}$ and

$$\mathscr{N}_{\mathfrak{f},t}^{\pm}(x) \geqslant \left|\mathscr{N}^{\pm} \cap [1, x]\right| \geqslant \left|\mathscr{A}_{\mathfrak{f}} \cap [1, x/p_0]\right|$$

has a positive density for all $x \geqslant 1$, by (3.4). Hence (1.8) follows.

Finally let us consider the case that $N/2$ is squarefree, for which the Shimura lift is automatically not of CM type. It is because according to the proof of the Corollary of Theorem A in [21], p.30, a primitive form $g \in S_k^{\mathrm{new}}(N', \chi_0)$ whose level N' is squarefree and nebentypus χ_0 is trivial is not of CM type. Now our primitive form f_i is of level $N/2$ and a trivial nebentypus (as χ^2 is trivial when χ is a real character).

This completes the proof of Theorem 2.

4. The proof of Theorem 3

The key tool is still (2.10). We set $n = p^{\nu}$ with $p \nmid N$, then

$$\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\frac{\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})}{p^{\nu(k-1/2)}} = \lambda_f(p^{\nu}) - \frac{\chi_{t,N}(p)}{\sqrt{p}}\lambda_f(p^{\nu-1}). \tag{4.1}$$

Recall that χ is real and so is $\chi_{t,N}$. The Hecke eigenform f is independent of t and its p^{ν}-th eigenvalue $\lambda_f(p^{\nu})$ $(p \nmid N)$ is expressible as

$$\lambda_f(p^{\nu}) = \frac{\sin((\nu+1)\theta_f(p))}{\sin\theta_f(p)}$$

(with the obvious interpretation in the limiting cases $\theta_f(p) = 0, \pi$). Indeed, this follows by elementary calculations, using the definition of $\theta_f(p)$ and factoring the quadratic polynomial in the denominator of the Euler p-factor of the L-series of f.

Now we consider case by case.

Case (i). $\theta_f(p) = 0$.

We have $\lambda_f(p^{\nu}) = \nu + 1$ so each $\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu}) \geqslant 1 + \nu(1 - p^{-1/2}) > 0$.

Case (ii). $\theta_f(p) = \pi$.

Thus $\lambda_f(p^{\nu}) = (-1)^{\nu}(\nu+1)$, and in this case, (4.1) turns to

$$\frac{\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})}{p^{\nu(k-1/2)}} = \mathfrak{a}_{\mathfrak{f}}(t)(-1)^{\nu}\left\{1 + \nu\left(1 + \frac{\chi_{t,N}(p)}{\sqrt{p}}\right)\right\}.$$

It follows that half of $\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})$ are positive and half are negative, depending on the parity of ν.

For the remaining two cases, we rewrite (4.1) into

$$\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\frac{\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})}{p^{\nu(k-1/2)}}\sin\theta_f(p) = \sin((\nu+1)\theta_f(p)) - \frac{\chi_{t,N}(p)}{\sqrt{p}}\sin(\nu\theta_f(p)),$$

which easily leads to

$$
\begin{aligned}
(4.2)\qquad \sum_{\substack{\nu\leqslant x\\ \pm\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})\geqslant(c-1/\sqrt{p})p^{\nu(k-1/2)}/\sin\theta_f(p)}} 1 &\geqslant \sum_{\substack{\nu\leqslant x\\ \sin((\nu+1)\theta_f(p))\gtrless\pm c}} 1\\
&= \sum_{\substack{2\leqslant\nu\leqslant x+1\\ \sin(\nu\theta_f(p))\gtrless\pm c}} 1
\end{aligned}
$$

for any $c\in(1/\sqrt{p},\,1)$. Here the symbol $\gtrless$ is abbreviated for $\geqslant$ and $\leqslant$.

Case (iii). $\theta_f(p)/(2\pi)=m/n\in(0,1/2)$ where m and n are coprime.

For $n\geqslant 3$, we set $a_n^+=d$ when n is of form $4d$ or $4d+1$, and $a_n^+=d+1$ when $n=4d+2$ or $n=4d+3$, so $1/5\leqslant a_n^+/n\leqslant 1/3$. Besides we take $a_n^-=n-a_n^+$, then

$$
\sin\left(\frac{2\pi a_n^\pm}{n}\right)\gtrless\pm\sin\left(\frac{2\pi}{3}\right)=\pm\frac{\sqrt{3}}{2}\gtrless\pm\frac{1}{\sqrt{p}}\cdot
$$

Then we consider $\nu m\equiv a_n^\pm(\operatorname{mod} n)$, whose solutions form the arithmetic progression $\nu=\ell n+\overline{m}^{(n)}a_n^\pm$ $(\ell\in\mathbb{Z})$ where $\overline{m}^{(n)}m\equiv 1\,(\operatorname{mod} n)$. For these ν's, we have

$$
\sin(\nu\theta_f(p))=\sin\left(\frac{2\pi\nu m}{n}\right)=\sin\left(\frac{2\pi a_n^\pm}{n}\right)\gtrless\pm\sin\left(\frac{2\pi}{3}\right)=\pm\frac{\sqrt{3}}{2}\cdot
$$

Setting $c=\sqrt{3}/2$ in (4.2), we deduce that

$$
\begin{aligned}
\sum_{\substack{\nu\leqslant x\\ \pm\mathfrak{a}_{\mathfrak{f}}(t)^{-1}\mathfrak{a}_{\mathfrak{f}}(tp^{2\nu})\geqslant(\sqrt{3}/2-1/\sqrt{p})p^{\nu(k-1/2)}/\sin\theta_f(p)}} 1 &\geqslant \sum_{\substack{2\leqslant\nu\leqslant x+1\\ \sin(\nu\theta_f(p))\gtrless\pm\sqrt{3}/2}} 1\\
&\geqslant \sum_{2\leqslant\ell n+\overline{m}^{(n)}a_n^\pm\leqslant x+1} 1\\
&\geqslant \frac{1}{n}x+O_{\mathfrak{f}}(1)\qquad(x\to\infty).
\end{aligned}
$$

Case (iv). $\theta_f(p)/(2\pi)$ is irrational.

Write

$$
\nu\theta_f(p)=2\pi[\nu\theta_f(p)/(2\pi)]+2\pi\{\nu\theta_f(p)/(2\pi)\},
$$

where $[t]$ (resp. $\{t\}$) is the integral part of t (resp. fractional part). It follows that

$$
\sin(\nu\theta_f(p))=\sin(2\pi\{\nu\theta_f(p)/(2\pi)\}).
$$

Thus for any $[a,b]\subset[-1,1]$, the last sum in (4.2) becomes

$$
\begin{aligned}
\sum_{\substack{2\leqslant\nu\leqslant x+1\\ \sin(\nu\theta_f(p))\in[a,b]}} 1 &= \sum_{\substack{2\leqslant\nu\leqslant x+1\\ \sin(2\pi\{\nu\theta_f(p)/(2\pi)\})\in[a,\,b]}} 1\\
&= 2\sum_{\substack{2\leqslant\nu\leqslant x+1\\ 2\pi\{\nu(\theta_f(p)/2\pi)\}\in[\arcsin a,\,\arcsin b]}} 1.
\end{aligned}
$$

As is well known, $\{\nu(\theta_f(p)/2\pi)\}$ is distributed uniformly mod 1 if and only if $\theta_f(p)/(2\pi)$ is irrational, by Weyl's criterion (see [8, Chapter 21]). In this case, we have

$$\sum_{\substack{2\leqslant \nu\leqslant x+1\\ \sin(\nu\theta_f(p))\in[a,b]}} 1 \sim \frac{\arcsin b - \arcsin a}{\pi} x \qquad (x\to\infty). \tag{4.3}$$

Now the required result follows from (4.2) and (4.3) with the choice of $a=c$, $b=1$ or $a=-1$, $b=-c$.

This completes the proof of Theorem 3.

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