# FINSLER METRICS AND KOBAYASHI HYPERBOLICITY OF THE MODULI SPACES OF CANONICALLY POLARIZED MANIFOLDS

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**Abstract.** *We show that the base complex manifold of an effectively parametrized holomorphic family of compact canonically polarized complex manifolds admits a smooth invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant. As a consequence, we show that such base manifold is Kobayashi hyperbolic.*

*Keywords*: Kobayashi hyperbolicity, moduli space, Kähler-Einstein metrics.

*Mathematics Subject Classification (2000)*: 32G13, 32Q20, 32Q45

## 1. Introduction

In the study of the moduli space $\mathcal{M}_g$ (and the Teichmüller space $\mathcal{T}_g$) of compact Riemann surfaces of genus $g \geq 2$, the Weil-Petersson metric plays an important role, and it has been widely studied. In particular, Ahlfors ([Ah1], [Ah2]) showed that the Weil-Petersson metric on $\mathcal{T}_g$ is a Kähler metric whose Ricci and holomorphic sectional curvatures are negative. Royden [R] later proved that the holomorphic sectional curvature of the Weil-Petersson metric is bounded away from zero. Subsequently Wolpert [Wo] showed that the Weil-Petersson metric is of holomorphic sectional curvature bounded above by $-\frac{1}{2\pi(g-1)}$. One immediate consequence of Wolpert's result is $\mathcal{M}_g$ is Kobayashi hyperbolic. It is interesting and natural to ask whether similar results hold for the moduli spaces of higher dimensional manifolds.

An $n$-dimensional compact complex manifold $M$ is said to be *canonically polarized* if its canonical line bundle $K_M$ is ample. It follows from results of Aubin [Au] and Yau [Y] that every compact complex manifold with ample canonical line bundle admits a Kähler-Einstein metric of negative Ricci curvature, which is unique up to a positive multiplicative constant. As such,

one can identify the moduli space of canonically polarized manifolds with that of Kähler-Einstein manifolds of negative Ricci curvature (see [NS], [V] and the references therein for existence and quasi-projectivity results on the moduli space of canonically polarized manifolds). The first breakthrough in the computation of the curvature for the Weil-Petersson metric on the moduli space of such higher dimensional manifolds is given by Siu [S2], which we recall here briefly. Let $\pi : \mathcal{X} \to S$ be a holomorphic family of compact canonically polarized complex manifolds over a complex manifold $S$, i.e., $\pi : \mathcal{X} \to S$ is a surjective holomorphic map of maximal rank between two complex manifolds $\mathcal{X}$ and $S$, and each fiber $M_t := \pi^{-1}(t)$, $t \in S$, is a compact complex manifold such that $K_{M_t}$ is ample. When the family $\pi : \mathcal{X} \to S$ is *effectively parametrized* (i.e., the Kodaira-Spencer map $\rho_t : T_t S \to H^1(M_t, TM_t)$ is injective for each $t \in S$), the Weil-Petersson metric on $S$ induced from the Kähler-Einstein metrics on the fibers is a non-degenerate Kähler metric (cf. (2.5)). In [S2], Siu computed the curvature of the Weil-Petersson metric arising from such families (see also [Sch1] for a simplified formula under the additional assumption that the Kodaira-Spencer map $\rho_t : T_t S \to H^1(M_t, TM_t)$ is surjective for each $t \in S$). It turns out that in general, one cannot decide the sign of the holomorphic sectional curvature of the Weil-Petersson metric except in some restrictive cases, say, when $H^2(M_t, \bigwedge^2 TM_t) = 0$ for all fibers $M_t$ of the family. Nonetheless, we show in this article that the base manifold of any effectively parametrized holomorphic family of canonically polarized manifolds admits a Finsler metric with appropriate curvature property, which will imply that such base manifold is necessarily Kobayashi hyperbolic. We state our main result as follows:

**Theorem 1.** *Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of compact canonically polarized complex manifolds over a complex manifold $S$. Then $S$ admits a $C^\infty$ $\mathrm{Aut}(\pi)$-invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant. As a consequence, $S$ is Kobayashi hyperbolic.*

We refer the reader to Section 3 for the definition of an "$\mathrm{Aut}(\pi)$-invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant". We also recall that a complex manifold (or more generally a complex space) $X$ is said to be *Kobayashi hyperbolic* if its Kobayashi pseudo-distance function $d_X$ is a distance function on $X$ (i.e., $d_X(x,y) > 0$ for all $x \neq y \in X$). Here $d_X$ can be characterized as the largest among all the pseudo-distance functions $\delta_X$ on $X$ satisfying $\delta_X(f(a), f(b)) \leq d_\Delta(a, b)$ for all holomorphic maps $f : \Delta \to X$ and $a, b \in \Delta$, where $\Delta$ is the unit disc

in $\mathbb{C}$ and $d_\Delta$ is the hyperbolic distance function on $\Delta$ (see e.g. [Kob] for other equivalent definitions of $d_X$).

We remark that Theorem 1 improves an earlier result of Viehweg and Zuo [VZ], which implies that $S$ is Brody hyperbolic; that is, there exists no non-constant holomorphic function from the complex plane $\mathbb{C}$ to $S$ (see also [Kov1], [Kov2] and [Mi] for related algebraic versions of such result, namely that algebraic morphisms from abelian varieties or $\mathbb{C}^*$ to $S$ are necessarily constant, and that when $S \subset \mathbb{P}^1$, the cardinality of $\mathbb{P}^1 \setminus S$ is at least three). Here we recall the well-known fact that a complex manifold (or more generally a complex space) $X$ is necessarily Brody hyperbolic if it is Kobayashi hyperbolic, and these two notions of hyperbolicity coincide when $X$ is compact. Nonetheless, there are examples of non-compact Brody hyperbolic complex manifolds which are not Kobayashi hyperbolic (see e.g. [Kob, p. 104] for such an example). The approach in [VZ] depends on positivity results for direct images of certain associated sheaves, and it is quite different from ours. We also remark that as in [VZ], Theorem 1 can be regarded as a result on the moduli stacks associated to the coarse moduli spaces of canonically polarized manifolds. As suggested by one of the referees, we will indicate some underlying parallel ingredients in the respective approaches of [VZ] and this paper (see Remark 10 at the end of this paper).

We describe briefly our approach as follows. The starting point is the curvature computation of the usual Weil-Petersson metric $h_1$ in [S2] (see the curvature formula in (2.6) in §2). We may regard this as the first level computation. The curvature expression of $h_1$ encompasses a good term which is negative and a bad term which is non-negative. We observe that the bad term can be expressed as a ratio $h_2/h_1$, where $h_2$ is some Finsler pseudometric on the the tangent space $TS$ of the parameter space $S$, which is induced through the diagonal embedding of $TS$ into the symmetric product $S^2(TS)$ endowed with a generalized Weil-Petersson Finsler pseudometric (which, for simplicity, is also denoted here by $h_2$). The second level computation is the technical derivation of the curvature of $h_2$. A prototype of this computation is the first level computation which was done in [S2]. The key point of our argument is to group the resulting curvature terms of $h_2$ into a good term involving $h_2/h_1$ and a bad term involving $h_3/h_2$, where $h_3$ can be interpreted as another Finsler pseudometric on $TS$ arisen similarly. The process is repeated. Hence for each $\ell \geq 1$, we construct at the $\ell$-th level a generalized Weil-Petersson Finsler pseudometric $h_\ell$ on $S$ measuring the $\ell$-th symmetric power of a tangent vector on the base and corresponding to the $\ell$-th composition of the Kodaira-Spencer map associated to a given tangent vector. We derive the key estimate that the curvature of $h_\ell$ is expressed as

the sum of a good term involving $h_\ell/h_{\ell-1}$ and a bad term involving $h_{\ell+1}/h_\ell$ (see Proposition 6 in §8). Our strategy is to control the bad term at the $(\ell-1)$-th level by the good term at the $\ell$-th level. This process terminates after a finite number of steps because of the following simple observation: Since $h_\ell$ is given by the $L^2$ norms of the harmonic representatives of the cohomology classes in $H^\ell(M_t, \wedge^\ell TM_t)$ corresponding to the image of the $\ell$-th iteration of the Kodaira-Spencer map, it follows that the bad term at the $n$-th level must vanish, where $n = \dim_\mathbb{C} M_t$. We remark that the iterated Kodaira-Spencer maps (and similar cohomological vanishing results as mentioned above) play an important role in the study of variation of Hodge structures, and they have also been used in [Mi], [Kov2] and [VZ]. To carry out our plan, we construct the final Finsler metric $h$ as a suitable finite linear combination of the $h_\ell^{1/\ell}$'s. From a simple direct computation which corresponds to a Gauss equation type argument, we show that the curvature of $h$ is bounded from above by a linear combination involving the $h_\ell$'s and their curvatures. Finally by adjusting the coefficients of the $h_\ell^{1/\ell}$'s in the definition of $h$ carefully pertaining to the comparison of arithmetic and geometric means, we show that the curvature estimates of the $h_\ell$'s at various levels can be combined together to conclude that the holomorphic sectional curvature of $h$ is bounded above by a negative constant (see Proposition 7 in §9).

We may break up our proof of Theorem 1 into three steps. In terms of the above description, the first step of the current paper is a direct generalization of the curvature formula for $\ell = 1$ to the cases of higher values of $\ell$, resulting in Proposition 4 in §8. The proof of this step follows closely the original formulation of Siu [S2]. The second step is to observe that the first term on the right hand side of the expression in Proposition 4 allows us to use a telescopic argument to estimate the bad term of the curvature of the generalized Weil-Petersson metric $h_\ell$ in terms of the good term in the curvature expression of $h_{\ell+1}$. The third step is the careful choice of a suitable combination of the $h_\ell$'s to make sure that a negative upper bound of the holomorphic sectional curvature can be obtained. For the sake of a clear, self-contained presentation, we include all necessary details in the computations.

The approach in this article is motivated in part from [SY1] and [SY2], in which higher order jets and appropriate Schwarz lemma are used to handle situations where the use of the first order jet is not sufficient for hyperbolicity of the manifold. Nonetheless, in this article, instead of higher order jets, we make use of symmetric powers of the first order jet of the base manifold $S$

and their cohomological images along the fibers arising from the Kodaira-Spencer map. The expression of the curvature estimates in Proposition 6 given in terms of a good and a bad term, is motivated by Ahlfors work on associated curves ([Ah3]) and also the proof of Schwarz Lemma in ([SY1], Lemma 4.4.1). The formulation of Proposition 6 is crucial for a telescopic argument in the proof of Proposition 7.

After we had completed this work, we were drawn to attention of a recent preprint (arXiv: 1002.4858) by Schumacher (which has appeared subsequently as [Sch2]), which, among other results, gives rise to Finsler metrics of negative holomorphic sectional curvature on relatively compact subsets of $S$ (see [Sch2, Proposition 14]). But this does not lead to Kobayashi hyperbolicity or Brody hyperbolicity of $S$ itself, except in the case when $S$ is compact (see Remark 11 at the end of this paper for more retrospective remarks on the respective approaches of the two papers).

The organization of this paper is as follows. In Section 2, we give some background materials and introduce some notations. In Section 3, we introduce the generalized Weil-Petersson Finsler pseudometrics, whose curvatures are computed in Sections 4-8. In Section 9, we give the construction of the desired Finsler metric, which leads to the Kobayashi hyperbolicity of $S$.

## 2. Background materials and the Weil-Petersson metric

Let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of $n$-dimensional compact canonically polarized complex manifolds over an $m$-dimensional complex manifold $S$. Let $M_t := \pi^{-1}(t)$ for each $t \in S$. Since the canonical line bundle $K_{M_t}$ of each $M_t$ is ample, it follows from a well-known result of Yau [Y] that $M_t$ admits a Kähler-Einstein metric $g(t)$ of constant Ricci curvature $k < 0$. It is easy to see that $k$ can be chosen to be independent of $t \in S$, and with such a choice of $k$, $g(t)$ is uniquely determined and $g(t)$ varies smoothly with $t$. Denote the Kähler form of $g(t)$ by $\omega(t)$ for each $t \in S$. Consider the relative canonical line bundle on $\mathcal{X}$

given by $K_{\mathcal{X}|S} := K_{\mathcal{X}} \otimes (\pi^* K_S)^{-1}$, so that $K_{\mathcal{X}|S}\big|_{M_t} = K_{M_t}$ for each $t \in S$. The volume forms associated to the $\omega(t)$'s defines a Hermitian metric $\lambda$ on $K_{\mathcal{X}|S}^{-1}$, and one obtains a $d$-closed $(1,1)$-form on $\mathcal{X}$ given by

$$(2.1) \qquad\qquad \omega_{\mathcal{X}} := \frac{2\pi}{k}\, c_1(K_{\mathcal{X}|S}^{-1}, \lambda)$$

such that $\omega_{\mathcal{X}}\big|_{M_t} = \omega(t)$ for each $t \in S$.

We will adopt the following notation throughout this article, unless stated otherwise. We will use $(z,t) = (z^1, \cdots, z^n, t^1, \cdots, t^m)$ to denote local holomorphic coordinate functions on some coordinate open subset of $\mathcal{X}$, so that $\pi$ corresponds to the coordinate projection map $(z,t) \to t$, and $t = (t^1, \cdots, t^m)$ also forms local holomorphic coordinate functions on some coordinate open subset of $S$. As such, for fixed $t$, $z = (z^1, \cdots, z^n)$ also forms local holomorphic coordinate functions on some open subset of the fiber $M_t$. We will index components of tensors on $M_t$ in the holomorphic tangential directions by Greek alphabets $\alpha$, $\beta$, etc (with the range $1, 2, \cdots, n$), while those in the complexified tangential directions are indexed by lower case Latin letters $a, b, c, d$, etc (with the range $1, 2, \cdots, n, \bar{1}, \bar{2}, \cdots, \bar{n}$). On the other hand, the components of tensors along the base directions will be indexed by the letters $i$, $j$ (with the range $1, 2, \cdots, m$), etc. We also adopt the Einstein summation notation for indices along the fibers. We denote $\partial_\alpha := \frac{\partial}{\partial z^\alpha}$ and $\partial_{\bar{\alpha}} := \frac{\partial}{\partial \bar{z}^\alpha}$ for $\alpha = 1, \cdots, n$, and $\partial_i := \frac{\partial}{\partial t^i}$ for $i = 1, \cdots, m$, etc.

The Ricci tensor of $g(t)$ is locally given by $R_{\alpha\bar{\beta}}(t) = -\partial_\alpha \partial_{\bar{\beta}} \log(\det(g_{\gamma\bar{\delta}}(t)))$, and the Kähler-Einstein condition means that $R_{\alpha\bar{\beta}}(t) = k g_{\alpha\bar{\beta}}(t)$ on each $M_t$. When no confusion arises, we sometimes drop the parameter $t$, and we simply write $R_{\alpha\bar{\beta}}$ for $R_{\alpha\bar{\beta}}(t)$, etc. We also write the $(1,1)$-form in (2.1) as $\omega = \sqrt{-1} g_{I\bar{J}}(z,t) dw^I \wedge d\bar{w}^J$, where $w$ can be $z$ or $t$ and the indices $I$, $J$ can be $i$ or $\alpha$, etc. In particular, one has $g_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}(t)$ along each fiber $M_t$.

Next we recall the 'horizontal lifting' of vector fields as defined by Schumacher in [Sch1]. First one notes that the orthogonal complement of $\mathrm{Ker}(\pi_* : T\mathcal{X} \to TS)$ in $T\mathcal{X}$ with respect to $\omega$ defines a smooth 'horizontal' vector subbundle $T^H \mathcal{X} \subset T\mathcal{X}$. For $t \in S$ and a local tangent vector field $u$ (of type $(1,0)$) on an open subset $U$ of $S$, one easily sees that there exists a unique lifting of $u$ to a smooth vector field $v_u$ (of type $(1,0)$) on $\pi^{-1}(U)$ such that $\pi_* v_u = u$ and $v_u(z,t) \in T^H \mathcal{X}$ for each $(z,t) \in \pi^{-1}(U)$. Such $v_u$ is called the *horizontal lifting* of $u$ (with respect to $\omega$). With respect to the family $\pi : \mathcal{X} \to S$, let $\rho_t : T_t S \to H^1(M_t, TM_t)$ denote the associated Kodaira-Spencer map for each $t \in S$. For each fixed $t \in U$, it follows from standard deformation theory that $\Phi(u(t)) := \bar{\partial} v_u\big|_{M_t} \in \mathcal{A}^{0,1}(M_t)$ is a Kodaira-Spencer representative of $\rho_t(u(t))$, i.e., $\rho_t(u(t)) = \big[\Phi(u(t))\big]$ in

$H^1(M_t, TM_t)$. By [Sch1, p. 342, Proposition 1.1], one knows that $\Phi(u(t))$ is harmonic with respect to the $\bar{\partial}$-Laplacian $\Box = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ on $M_t$. In particular, the horizontal lifting $v_u$ of $u$ is actually a (special type of) 'canonical lifting' in the sense of Siu in [S2], which refers to any lifting of $u$ such that $\Phi(u(t))$ is the harmonic representative of $\rho_t(u(t))$ for each $t$. When $u = \partial/\partial t^i$ is a coordinate vector field, we will simply denote its horizontal lifting by $v_i := v_{\partial/\partial t^i}$ and the associated harmonic Kodaira-Spencer representative by $\Phi_i := \Phi(\partial/\partial t^i)$. Write $\Phi_i = (\Phi_i)^\alpha_{\bar{\beta}}\partial_\alpha \otimes d\bar{z}^\beta$. It is easy to see that $v_i$ and the $(\Phi_i)^\alpha_{\bar{\beta}}$'s are given locally by

(2.2) $$v_i = \partial_i + v_i^\alpha \partial_\alpha, \quad \text{where } v_i^\alpha := -g^{\bar{\beta}\alpha}g_{i\bar{\beta}}, \text{ and}$$

(2.3) $$(\Phi_i)^\alpha_{\bar{\beta}} = \partial_{\bar{\beta}}v_i^\alpha = -\partial_{\bar{\beta}}(g^{\bar{\gamma}\alpha}g_{i\bar{\gamma}}).$$

(see [Sch1, p. 342, equation (1.2)]). Here $g^{\bar{\beta}\alpha}$ denotes the components of the inverse of $g_{\alpha\bar{\beta}}$ (and not that of $g_{I\bar{J}}$, which may not be invertible). For a given tensor $T$ of covariant degree 1 and of contravariant degree 1, we recall that the components (along the fiber direction) of its Lie derivative $\mathcal{L}_{v_i}T$ with respect to $v_i$ are given locally by

(2.4) $$(\mathcal{L}_{v_i}T)^b_a = \partial_i(T^b_a) + T^b_c\partial_a v_i^c - T^c_a\partial_c v_i^b$$

(see e.g. [S2, p. 268]), and similar formula holds for tensors of higher degree. We recall that the Weil-Petersson metric $h^{(WP)} = \sum_{i,j=1}^n h^{(WP)}_{i\bar{j}}dt^i \otimes d\bar{t}^j$ on $S$ is defined by

(2.5) $$h^{(WP)}_{i\bar{j}}(t) := \int_{M_t} \langle \Phi_i, \Phi_j \rangle \frac{\omega^n}{n!}, \quad \text{where } \langle \Phi_i, \Phi_j \rangle := (\Phi_i)^\gamma_{\bar{\alpha}}\overline{(\Phi_j)^\delta_{\bar{\beta}}}g_{\gamma\bar{\delta}}g^{\bar{\alpha}\beta}$$

denotes the pointwise Hermitian inner product on tensors, possibly of mixed types, with respect to $\omega$ (we note that the definition of $h^{(WP)}$ in [S2, p. 273] differs from (2.5) by a factor of 4). We remark that it follows from the assumption on the injectivity of $\rho_t$ that $h^{(WP)}$ is positive definite on each $T_tS$. It follows from Koiso's result [Koi] that $h^{(WP)}$ is Kähler. Let $R^{(WP)}$ denote the curvature tensor of $h^{(WP)}$. By [S2, p. 296], the components of $R^{(WP)}$ with respect to normal coordinates (of $h^{(WP)}$) at a point $t \in S$ are given by

$$(2.6) \qquad R_{i\bar{j}k\bar{\ell}}^{(WP)}(t) = k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_i, \Phi_j \rangle) \cdot \langle \Phi_k, \Phi_\ell \rangle \frac{\omega^n}{n!}$$

$$+ k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_k, \Phi_j \rangle) \cdot \langle \Phi_i, \Phi_\ell \rangle \frac{\omega^n}{n!}$$

$$+ k \int_{M_t} \langle (\Box - k)^{-1} \mathcal{L}_{v_i} \Phi_k, \mathcal{L}_{v_j} \Phi_\ell \rangle \frac{\omega^n}{n!}$$

$$+ \int_{M_t} \langle H(\Phi_i \oslash \Phi_k), H(\Phi_j \oslash \Phi_\ell) \rangle \frac{\omega^n}{n!}.$$

Here by normal coordinates of $h^{(WP)}$ at the point $t \in S$, we mean $h_{i\bar{j}}^{(WP)}(t) = \delta_{ij}$, and $\partial_k h_{i\bar{j}}^{(WP)}(t) = \partial_{\bar{k}} h_{i\bar{j}}^{(WP)}(t) = 0$ (see [S2, p. 275]). Also, $H(\Phi_i \oslash \Phi_k)$ is some harmonic $\wedge^2 T^{1,0} M_t$-valued $(0,2)$-form constructed from $\Phi_i$ and $\Phi_k$ (see (3.3), (3.4) and (3.10) for the general definition).

**Remark 1.** *As remarked in [S2, p. 297], when $i = j$ and $k = \ell$, the first on the right hand side of (2.6) is negative, while the second and third terms are semi-negative. However, the fourth term is semi-positive, which hitherto poses a big obstacle in trying to deduce hyperbolicity properties of the moduli space by using the Weil-Petersson metric (except under some restrictive conditions amounting to the vanishing of the fourth term).*

## 3. **Generalized Weil-Petersson Finsler pseudo-metrics**

Throughout Section 3, we let $\pi : \mathcal{X} \to S$ be an effectively parametrized holomorphic family of $n$-dimensional compact canonically polarized complex manifolds over an $m$-dimensional complex manifold $S$ as in Theorem 1. Let $M_t := \pi^{-1}(t)$ for each $t \in S$. In this section, we are going to construct some Finsler pseudo-metrics on $S$ via constructions similar to (2.5). To facilitate our subsequent discussion, we first recall some standard definitions.

A *Finsler pseudo-metric* $h$ on the complex manifold $S$ is simply a continuous function $h : TS \to \mathbb{R}$ such that $h(u) \geq 0$ for all $u \in TS$ and $h(cu) = |c|h(u)$ for all $u \in TS$ and $c \in \mathbb{C}$. If, in addition, $h(u) > 0$ for all $0 \neq u \in TS$, then we say that $h$ is a *Finsler metric* on $S$. A Finsler pseudo-metric $h$ is said to be $C^\infty$ (resp. $C^\ell$ for a non-negative integer $\ell$) if for any open subset $U \subset S$ and any non-vanishing $C^\infty$ section $u_t$ of $TS\big|_U$, $h(u_t)$ is a $C^\infty$ (resp. $C^\ell$) function on $U$. For a $C^2$ Finsler metric $h$ on $S$, a point $t \in S$

and a non-zero tangent vector $u \in T_t S$, the holomorphic sectional curvature $K(u)$ of $h$ in the direction $u$ is simply given by

$$(3.1) \qquad K(u) = \sup_R K(R, h\big|_R)(t),$$

where the supremum is taken over all local one-dimensional complex submanifolds $R$ of $S$ satisfying $t \in R$ and $T_t R = \mathbb{C}u$, and $K(R, h\big|_R)(t)$ is the sectional curvature of (the Riemannian metric) $(R, h\big|_R)$ at $t$ (cf. (9.13)). We say that the holomorphic sectional curvature of the Finsler metric $h$ on $S$ is bounded above by a negative constant if there exists a constant $C > 0$ such that $K(u) < -C$ for all $0 \neq u \in TS$. We remark that in the special case when the Finsler metric $h$ arises as the length function of a Hermitian metric, the holomorphic sectional curvature of $h$ (as a Finsler metric) agrees with that of the associated Hermitian metric. For the family $\pi : \mathcal{X} \to S$ as above, we say that a Finsler pseudometric (or Finsler metric) $h$ on $S$ is $Aut(\pi)$-invariant if $f^*h = h$ for any pair of automorphisms $(F, f) \in \mathrm{Aut}(\mathcal{X}) \times \mathrm{Aut}(\mathcal{S})$ satisfying $f \circ \pi = \pi \circ F$. Here $\mathrm{Aut}(\mathcal{X})$ denotes the group of self-biholomorphisms on $\mathcal{X}$, etc.

Next we introduce some definitions on the fibers $M_t$'s of $\pi : \mathcal{X} \to S$. For integers $p, q, r, s \geq 0$ and $t \in S$, let $\Phi \in \mathcal{A}^{0,p}(\wedge^r TM_t)$ and $\Psi \in \mathcal{A}^{0,q}(\wedge^s TM_t)$ be given by

$$(3.2)$$

$$\Phi = \frac{1}{p!r!} \sum_{\substack{1 \leq \alpha_1, \cdots, \alpha_r \leq n \\ 1 \leq \beta_1, \cdots, \beta_p \leq n}} \Phi^{\alpha_1 \cdots \alpha_r}_{\bar\beta_1 \cdots \bar\beta_p} \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_p},$$

$$\Psi = \frac{1}{q!s!} \sum_{\substack{1 \leq \gamma_1, \cdots, \gamma_s \leq n \\ 1 \leq \delta_1, \cdots, \delta_q \leq n}} \Psi^{\gamma_1 \cdots \gamma_s}_{\bar\delta_1 \cdots \bar\delta_q} \partial_{\gamma_1} \wedge \cdots \wedge \partial_{\gamma_s} \otimes d\bar z^{\delta_1} \wedge \cdots \wedge d\bar z^{\delta_q} \quad \text{with}$$

$$\Phi^{\sigma(\alpha_1) \cdots \sigma(\alpha_r)}_{\overline{\tau(\beta_1)} \cdots \overline{\tau(\beta_p)}} = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau) \cdot \Phi^{\alpha_1 \cdots \alpha_r}_{\bar\beta_1 \cdots \bar\beta_p}, \quad \text{for all } \sigma \in \mathfrak{S}_r, \, \tau \in \mathfrak{S}_p, \text{ etc.}$$

Here $\mathfrak{S}_p$ denotes the permutation group on $p$ elements, and $\mathrm{sgn}(\sigma)$ denotes the signature of the permutation $\sigma$, etc. Now we define $\Phi \oslash \Psi \in \mathcal{A}^{0,p+q}(\wedge^{r+s} TM_t)$ given by

$$(3.3) \quad \Phi \oslash \Psi := \frac{1}{p!q!r!s!} \sum \Phi^{\alpha_1 \cdots \alpha_r}_{\bar\beta_1 \cdots \bar\beta_p} \cdot \Psi^{\gamma_1 \cdots \gamma_s}_{\bar\delta_1 \cdots \bar\delta_q} \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \wedge \partial_{\gamma_1} \wedge \cdots \wedge \partial_{\gamma_s}$$
$$\otimes d\bar z^{\beta_1} \wedge \cdots \wedge d\bar z^{\beta_p} \wedge d\bar z^{\delta_1} \wedge \cdots \wedge d\bar z^{\delta_q},$$

where the summation is taken over all $1 \leq \alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_p, \gamma_1, \cdots, \gamma_s, \delta_1, \cdots, \delta_q \leq n$. Thus the operator $\oslash$ means taking wedge product on the

level of forms as well as that of tangent vectors. It is easy to check that

$$(3.4) \qquad \Phi \oslash \Psi = \frac{1}{(p+q)!(r+s)!} \sum_{A,B} \left( \Phi \oslash \Psi \right)_{B}^{A} \partial_A \otimes d\bar{z}^B \quad \text{with}$$

$$\left( \Phi \oslash \Psi \right)_{B}^{A} = \sum_{\substack{\sigma \in \mathfrak{S}_{r+s} \\ \tau \in \mathfrak{S}_{p+q}}} \frac{\text{sgn}(\sigma) \cdot \text{sgn}(\tau)}{p!q!r!s!} \Phi_{\tau(\beta_1)\cdots\tau(\beta_p)}^{\sigma(\alpha_1)\cdots\sigma(\alpha_r)} \cdot \Psi_{\tau(\beta_{p+1})\cdots\tau(\beta_{p+q})}^{\sigma(\alpha_{r+1})\cdots\sigma(\alpha_{r+s})},$$

where $A = (\alpha_1, \cdots, \alpha_{r+s})$, $B = (\beta_1, \cdots, \beta_{p+q})$, $\partial_A := \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_{r+s}}$, $d\bar{z}^B := d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_{p+q}}$, and the summation in the first line of (3.4) runs through all integral values of $A$, $B$ satisfying $1 \le \alpha_1, \cdots, \alpha_{r+s}, \beta_1, \cdots, \beta_{p+q} \le n$. We will skip the easy checking that for any $\Phi \in \mathcal{A}^{0,p}(\wedge^r TM_t)$, $\Psi \in \mathcal{A}^{0,q}(\wedge^s TM_t)$ and $\Upsilon \in \mathcal{A}^{0,c}(\wedge^d TM_t)$, one has

$$(3.5) \qquad \Phi \oslash \Psi = (-1)^{pq+rs} \Psi \oslash \Phi,$$

$$(3.6) \qquad \overline{\partial}(\Phi \oslash \Psi) = \overline{\partial}\Phi \oslash \Psi + (-1)^p \Phi \oslash \overline{\partial}\Psi, \quad \text{and}$$

$$(3.7) \qquad \Phi \oslash (\Psi \oslash \Upsilon) = (\Phi \oslash \Psi) \oslash \Upsilon$$

In particular, we may write $\Phi \oslash \Psi \oslash \Upsilon$ unambiguously.

**Remark 2.** *(i) When $p = r$ and $q = s$, one has $\Phi \oslash \Psi = \Psi \oslash \Phi$.*
*(ii) When $q = s = 0$ (so that $\Psi$ is a scalar-valued function on $M_t$), $\Phi \oslash \Psi$ is simply given by pointwise multiplication of $\Phi$ by the function $\Psi$.*
*(iii) From (3.6), one easily sees that if $\Phi$ and $\Psi$ are $\overline{\partial}$-closed, then $\Phi \oslash \Psi$ is also $\overline{\partial}$-closed. If, in addition, either $\Phi$ or $\Psi$ is $\overline{\partial}$-exact, then $\Phi \oslash \Psi$ is $\overline{\partial}$-exact. In particular, the operator $\oslash$ induces a homomorphism on the associated cohomology groups, which we denote by the same symbol. Explicitly, we have*

$$\oslash : H^{0,p}(\wedge^r TM_t) \otimes H^{0,q}(\wedge^s TM_t) \to H^{0,p+q}(\wedge^{r+s} TM_t)$$

*given by*

$$[\Phi] \oslash [\Psi] := [\Phi \oslash \Psi]$$

*for any classes $[\Phi] \in H^{0,p}(\wedge^r TM_t)$ and $[\Psi] \in H^{0,q}(\wedge^s TM_t)$ represented by $\Phi \in \mathcal{A}^{0,p}(\wedge^r TM_t)$ and $\Psi \in \mathcal{A}^{0,q}(\wedge^s TM_t)$ respectively.*

For a cohomology class $\mu \in H^{0,p}(\wedge^r TM_t)$, we denote by $H(\mu)$ the unique harmonic representative of $\mu$. In particular, for any $\overline{\partial}$-closed representative $\Phi(\in \mathcal{A}^{0,p}(\wedge^r TM_t))$ of $\mu$, one easily sees that $H(\mu) = H(\Phi)$, where $H(\Phi)$ denotes the harmonic projection of $\Phi$ (with respect to $\omega(t)$).

For the rest of this section, we fix an integer $\ell$ satisfying $1 \le \ell \le n$. Let $\Phi, \Psi \in \mathcal{A}^{0,\ell}(\wedge^\ell TM_t)$ with components as given in (3.2) (with $p = q = s =$

$r = \ell$). Their pointwise inner product is given by

$$(3.8) \qquad \langle \Phi, \Psi \rangle := \frac{1}{(\ell!)^2} \Phi^{\alpha_1 \cdots \alpha_\ell}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} \overline{\Psi^{\alpha'_1 \cdots \alpha'_\ell}_{\overline{\beta}'_1 \cdots \overline{\beta}'_\ell}} g_{\alpha_1 \overline{\alpha}'_1} \cdots g_{\alpha_\ell \overline{\alpha}'_\ell} g^{\beta_1 \overline{\beta}'_1} \cdots g^{\beta_\ell \overline{\beta}'_\ell},$$

and their $L^2$-inner product on $M_t$ is given by

$$(3.9) \qquad (\Phi, \Psi) = \int_{M_t} \langle \Phi, \Psi \rangle \frac{\omega^n}{n!}.$$

We denote by $\|\Phi\|_2 := \sqrt{(\Phi, \Phi)}$ the fiberwise $L^2$-norm of $\Phi$. Then for each $t \in S$ and $u_1, \ldots, u_\ell, u'_1, \ldots, u'_\ell \in T_t S$, we define, in terms of (3.9),

$$(3.10) \qquad (u_1 \otimes \cdots \otimes u_\ell, u'_1 \otimes \cdots \otimes u'_\ell)_{WP}$$
$$:= (H(\rho_t(u_1) \odot \cdots \odot \rho_t(u_\ell)), H(\rho_t(u'_1) \odot \cdots \odot \rho_t(u'_\ell)))$$
$$= (H(\Phi(u_1) \odot \cdots \odot \Phi(u_\ell)), H(\Phi(u'_1) \odot \cdots \odot \Phi(u'_\ell))).$$

Here each $\Phi(u_i)$ is the harmonic representative of $\rho_t(u_i)$ as given in Section 2. It is easy to see that (3.10) extends to a positive semi-definite Hermitian bilinear form on $\otimes^\ell T_t S$, which varies smoothly in $t$. We simply call it the *generalized Weil-Petersson pseudo-metric on $\otimes^\ell TS$*.

Now for each $t \in S$ and $u \in T_t S$, we define

$$(3.11) \qquad \|u\|_{WP,\ell} := (\underbrace{u \otimes \cdots \otimes u}_{\ell-\text{times}}, \underbrace{u \otimes \cdots \otimes u}_{\ell-\text{times}})_{WP}^{\frac{1}{2\ell}}.$$

It is easy to see that each $\| \cdot \|_{WP,\ell}$ is a Finsler pseudo-metric on $S$, i.e., $\|u\|_\ell \geqslant 0$ and $\|cu\|_{WP,\ell} = |c| \|u\|_{WP,\ell} \geqslant 0$ for all $c \in \mathbb{C}$ and $u \in TS$. We simply call $\| \cdot \|_{WP,\ell}$ the *$\ell$-th generalized Weil-Petersson Finsler pseudo-metric on $S$*.

**Remark 3.** *(i) We remark that $\| \|_{WP,1}$ is simply the norm function of the Weil-Petersson metric defined in (2.5), and is positive definite under the assumption that each $\rho_t$ is injective.*
*(ii) For a pair of automorphisms $(F, f) \in Aut(\mathcal{X}) \times Aut(\mathcal{S})$ satisfying $f \circ \pi = \pi \circ F$, one easily sees that the restriction of $F$ to the fibers are isometries with respect to the Kähler-Einstein metrics on the fibers, i.e., one has $(F|_{M_t})^* g(f(t)) = g(t)$ for all $t \in S$. This follows readily from the $Aut(M_t)$-invariance of the Kähler-Einstein metric $g(t)$ on each $M_t$. As a consequence, one easily sees that each $\| \|_{WP,\ell}$ is $Aut(\pi)$-invariant.*
*(iii) In Section 9, we will use the $\| \|_{WP,\ell}$'s to construct a Finsler metric on $S$ whose holomorphic sectional curvature is bounded above by a negative constant. For this purpose, we will need to compute $\sqrt{-1} \partial \overline{\partial} \log \|u\|^2_{WP,\ell}$, which is the main content of the next few sections.*

## 4. **Computation of Curvature**

In Sections 4-8, we are going to study the generalized Weil-Petersson Finsler pseudometrics on $S$. More specifically, we will estimate the holomorphic sectional curvatures of the restrictions of these pseudometrics to local one-dimensional complex submanifolds of $S$ (at those points where the restrictions are non-degenerate). In the process, we will make some computations of considerable independent interest and in a slightly more general setting.

We fix a coordinate open subset $U \subset S$ with coordinate functions $t = (t^1, \ldots, t^m)$ such that the origin $t = 0$ lies in $U$. For each $t \in S$ and each coordinate tangent vector $\frac{\partial}{\partial t^i}$, we recall the horizontal lifting $v_i$ and the harmonic representative $\Phi_i$ of $\rho_t(\frac{\partial}{\partial t^i})$ on $M_t$ as given in (2.2) and (2.3) respectively. Fix an integer $\ell$ satisfying $1 \leqslant \ell \leqslant n$, and let $J = (j_1, \ldots, j_\ell)$ be an $\ell$-tuple of integers satisfying $1 \leqslant j_d \leqslant m$ for each $1 \leqslant d \leqslant \ell$. We denote by

$$(4.1) \qquad \Psi_J := H(\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}) \in \mathcal{A}^{0,\ell}(\wedge^\ell TM_t)$$

the harmonic projection of $\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}$. As $t$ varies, we still denote the resulting family of tensors by $\Psi_J$ (suppressing its dependence on $t$), when no confusion arises. We are going to compute $\partial_i \partial_{\bar{i}} \log \|\Psi_J\|_2^2$ (as a function on $U$) wherever $\Psi_J \not\equiv 0$ on $M_t$. For this purpose, we will need to consider families of tensors on the fibers (or in short, relative tensors) arising from restrictions of tensors on $\mathcal{X}$ to the fibers. We will sometimes adopt the semi-colon notation to denote covariant derivatives of tensors on $M_t$, so that $(\Phi_i)_{\alpha;\gamma}^\beta := \nabla_\gamma (\Phi_i)_\alpha^\beta \ (= (\nabla_{\frac{\partial}{\partial z^\gamma}} \Phi_i)_\alpha^\beta)$, etc. Also the raising and lowering of indices for components of tensors on $M_t$ are with respect to $\omega(t)$, so that $(\Phi_i)_{\overline{\alpha},\overline{\beta}} = g_{\gamma\overline{\beta}}(\Phi_i)_{\overline{\alpha}}^\gamma$, etc, unless stated otherwise. First we have

**Lemma 1.** *(i)* $[v_i, \partial_{\overline{\alpha}}] = -(\Phi_i)_{\overline{\alpha}}^\beta \partial_\beta$.
*(ii) For a smooth $(n,n)$-form $\Upsilon$ on $\mathcal{X}$, one has*

$$\frac{\partial}{\partial t^i} \int_{M_t} \Upsilon = \int_{M_t} \mathcal{L}_{v_i} \Upsilon \quad and \quad \frac{\partial}{\partial \overline{t}^i} \int_{M_t} \Upsilon = \int_{M_t} \mathcal{L}_{\overline{v_i}} \Upsilon.$$

*(iii)* $[v_i, \overline{v_j}] = g^{\overline{\gamma}\alpha} \partial_{\overline{\gamma}}(g_{v_i \overline{v_j}}) \partial_\alpha - g^{\overline{\beta}\gamma} \partial_\gamma (g_{v_i \overline{v_j}}) \partial_{\overline{\beta}}$.
*(iv)* $(\Phi_i)_{\overline{\alpha},\overline{\beta}} = (\Phi_i)_{\overline{\beta},\overline{\alpha}}$ *for all $\alpha, \beta$.*
*(v)* $\mathcal{L}_{v_i}(g_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta) = (\Phi_i)_{\overline{\beta},\overline{\gamma}} d\overline{z}^\beta \wedge d\overline{z}^\gamma = 0$. *In particular, one has* $\mathcal{L}_{v_i}(\omega^n) = 0$ *(as relative tensor).*
*Here $[\cdot, \cdot]$ denote the Lie bracket of two vector fields.*

*Proof.* (i) follows readily from (2.2) and (2.3). (ii), (iii), (iv) and (v) can be found [Sch1, Lemma 2.1], [Sch1, Lemma 2.6], [Sch1, Proposition 1.1] and [Sch1, Lemma 2.2] respectively. □

Next we generalize the constructions in (3.8) and (3.9) to (relative) tensors of mixed type. Let $\mathcal{A}^\ell(M_t)$ (resp. $\mathcal{A}^{q,p}(M_t)$) denote the space of $C^\infty$ $\ell$-forms (resp. $(q,p)$-forms) on $M_t$. It is easy to see that there exists a unique pointwise Hermitian bilinear pairing $\langle\ ,\ \rangle_1$ on $\mathcal{A}^\ell(M_t)$ satisfying the identity on $(n,n)$-forms on $M_t$ given by

$$\langle\phi,\psi\rangle_1 \frac{\omega(t)^n}{n!} = (-1)^{\frac{\ell(\ell-1)}{2}}\phi \wedge \overline{\psi} \wedge \frac{\omega(t)^{n-\ell}}{(n-\ell)!} \quad \text{for } \phi,\psi \in \mathcal{A}^\ell(M_t).$$

Together with the identity $\mathcal{L}_{v_i}(\phi\wedge\psi) = (\mathcal{L}_{v_i}\phi)\wedge\psi + \phi\wedge(\mathcal{L}_{v_i}\psi)$ and Lemma 1(v), one easily checks that

$$(4.2) \qquad\qquad \mathcal{L}_{v_i}\langle\phi,\psi\rangle_1 = \langle\mathcal{L}_{v_i}\phi,\psi\rangle_1 + \langle\phi,\mathcal{L}_{\overline{v_i}}\psi\rangle_1.$$

Now we consider the decomposition $\mathcal{A}^\ell(M_t) = \oplus_{q+p=\ell}\mathcal{A}^{q,p}(M_t)$, and let $\mathcal{C}_{W,1}$ be the corresponding (linear) Weil operator on $\mathcal{A}^\ell(M_t)$ which acts by scalar multiplication by $(\sqrt{-1})^{q-p}$ on each summand $\mathcal{A}^{q,p}(M_t)$. It is easy to check that the (positive definite) $L^2$-inner product on $\mathcal{A}^\ell(M_t)$ with respect to $\omega(t)$ is given by

$$(4.3) \qquad\qquad (\phi,\psi) = \int_{M_t}\langle\mathcal{C}_{W,1}(\phi),\psi\rangle_1 \frac{\omega^n}{n!}.$$

Next we consider the space $C^\infty(\wedge^{\ell'}T^{\mathbb{C}}M_t)$ (resp. $C^\infty(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$) of $C^\infty$ $\ell'$-complexified vector fields (resp. $(r,s)$-vector fields) on $M_t$, where $T^{\mathbb{C}}M_t = TM_t\otimes_{\mathbb{R}}\mathbb{C}$ denotes the complexified tangent bundle of $M_t$. With respect to the decomposition $C^\infty(\wedge^{\ell'}T^{\mathbb{C}}M_t) = \oplus_{r+s=\ell'}C^\infty(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$, we denote by $\mathcal{C}_{W,2}$ the corresponding Weil operator on $C^\infty(\wedge^{\ell'}T^{\mathbb{C}}M_t)$ given by scalar multiplication by $(\sqrt{-1})^{r-s}$ on each summand $C^\infty(\wedge^r TM_t\wedge\overline{\wedge^s TM_t})$. Then by using the standard identity $\mathcal{L}_{v_i}(\phi(\eta)) = (\mathcal{L}_{v_i}\phi)(\eta) + \phi(\mathcal{L}_{v_i}\eta)$ for $\phi \in \mathcal{A}^{\ell'}(M_t)$ and $\eta \in C^\infty(\wedge^{\ell'}T^{\mathbb{C}}M_t)$, one easily sees that $\langle\ ,\ \rangle_1$ (with $\ell$ replaced by $\ell'$) induces a Hermitian bilinear pairing $\langle\ ,\ \rangle_2$ on $C^\infty(\wedge^{\ell'}T^{\mathbb{C}}M_t)$ satisfying a Leibniz rule similar to (4.2) and such that the (positive definite) $L^2$-inner product on $C^\infty(\wedge^{\ell'}T^{\mathbb{C}}M_t)$ with respect to $\omega(t)$ can be defined in terms of $\langle\ ,\ \rangle_2$ and $\mathcal{C}_{W,2}$ as in (4.3) (with the subscript 1 replaced by 2). Finally we consider the space $\mathcal{A}^\ell(\wedge^{\ell'}T^{\mathbb{C}}M_t)$ with decomposition $\mathcal{A}^\ell(\wedge^{\ell'}T^{\mathbb{C}}M_t) = \oplus_{q+p=\ell,r+s=\ell'}\mathcal{A}^{q,p}(\wedge^r TM_t\wedge\overline{\wedge^s TM_t})$ and corresponding Weil operator $\mathcal{C}_W$ given by scalar multiplication by $(\sqrt{-1})^{q-p+r-s}$ on each summand $\mathcal{A}^{q,p}(\wedge^r TM_t \wedge \overline{\wedge^s TM_t})$. As before, we denote the (positive definite) $L^2$-inner product and the corresponding $L^2$-norm on $\mathcal{A}^\ell(\wedge^{\ell'}T^{\mathbb{C}}M_t)$ with respect to $\omega(t)$ by $(\ ,\ )$ and $\|\ \|_2$ respectively. Then one easily checks that the

tensor product of $\langle \ , \ \rangle_1$ with $\langle \ , \ \rangle_2$ gives rise to a Hermitian bilinear pairing $\langle \ , \ \rangle$ on $\mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t)$ such that for all $\Upsilon, \Upsilon' \in \mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t)$, one has

$$(4.4) \qquad \mathcal{L}_{v_i}\langle \Upsilon, \Upsilon'\rangle = \langle \mathcal{L}_{v_i}\Upsilon, \Upsilon'\rangle + \langle \Upsilon, \mathcal{L}_{\overline{v_i}}\Upsilon'\rangle, \quad \text{and}$$

$$(4.5) \qquad (\Upsilon, \Upsilon') = \int_{M_t} \langle \mathcal{C}_W(\Upsilon), \Upsilon'\rangle \frac{\omega^n}{n!}.$$

**Remark 4.** *We note that the expression $\langle \mathcal{C}_W(\ ), \ \rangle$ in (4.5) is the pointwise (positive definite) inner product on $\mathcal{A}^\ell(\wedge^{\ell'} T^{\mathbb{C}} M_t)$ induced by $\omega(t)$. Also, for a given integer $\ell$, $\mathcal{C}_W$ simply restricts to the identity map on $\mathcal{A}^{0,\ell}(\wedge^\ell T M_t)$, so that the formulas in (3.9) and (4.5) agree with each other.*

For our application in Section 9, we will be interested in the expression $\partial_i \partial_{\bar{i}} \log \|\Psi_J\|_2^2$ with $j_1 = j_2 = \cdots = j_\ell = i$. Nonetheless, we will consider here $\partial_i \partial_{\bar{i}} \log \|\Psi_J\|_2^2$ in the general case when the $j_\ell$'s can be different. Note that

$$
\begin{aligned}
\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 &= \partial_i \left(\frac{\partial_{\bar{i}}\|\Psi_J\|_2^2}{\|\Psi_J\|_2^2}\right) \\
(4.6) \qquad &= \frac{\partial_i \partial_{\bar{i}}\|\Psi_J\|_2^2}{\|\Psi_J\|_2^2} - \frac{(\partial_i\|\Psi_J\|_2^2)(\partial_{\bar{i}}\|\Psi_J\|_2^2)}{\|\Psi_J\|_2^4}.
\end{aligned}
$$

From direct computation using Lemma 1(i) and (v), (4.4) and (4.5) (noting that $\mathcal{C}_W(\Psi_J) = \Psi_J$ (cf. Remark 4)), one has

$$
\begin{aligned}
\partial_i \|\Psi_J\|_2^2 &= \frac{\partial}{\partial t^i} \int_{M_t} \langle \Psi_J, \Psi_J\rangle \frac{\omega^n}{n!} \\
(4.7) \qquad &= \int_{M_t} \langle \mathcal{L}_{v_i}\Psi_J, \Psi_J\rangle \frac{\omega^n}{n!} + \int_{M_t} \langle \Psi_J, \mathcal{L}_{\overline{v_i}}\Psi_J\rangle \frac{\omega^n}{n!}.
\end{aligned}
$$

We will see from Lemma 3 in Section 5 that the component of $\mathcal{L}_{\overline{v_i}}\Psi_J$ in $\mathcal{A}^{0,\ell}(\wedge^\ell T M_t)$ is $\overline{\partial}$-exact on $M_t$. Together with the harmonicity of $\Psi_J$, it follows that

$$(4.8) \qquad \int_{M_t} \langle \Psi_J, \mathcal{L}_{\overline{v_i}}\Psi_J\rangle \frac{\omega^n}{n!} = 0$$

as a function on the base manifold. Thus we have

$$\partial_i \|\Psi_J\|_2^2 = \int_{M_t} \langle \mathcal{L}_{v_i}\Psi_J, \Psi_J\rangle \frac{\omega^n}{n!}, \quad \text{and similarly,}$$

$$
\begin{aligned}
\partial_i \partial_{\bar{i}}\|\Psi_J\|_2^2 = \partial_{\bar{i}}\partial_i\|\Psi_J\|_2^2 &= \frac{\partial}{\partial \bar{t}^i} \int_{M_t} \langle \mathcal{L}_{v_i}\Psi_J, \Psi_J\rangle \frac{\omega^n}{n!} \\
(4.9) \qquad &= \int_{M_t} \langle \mathcal{L}_{\overline{v_i}}\mathcal{L}_{v_i}\Psi_J, \Psi_J\rangle \frac{\omega^n}{n!} + \int_{M_t} \langle \mathcal{L}_{v_i}\Psi_J, \mathcal{L}_{v_i}\Psi_J\rangle \frac{\omega^n}{n!}.
\end{aligned}
$$

Upon differentiating the complex conjugate of (4.8), one gets, as in (4.9),

$$0 \; = \; \frac{\partial}{\partial t^i} \int_{M_t} \langle \mathcal{L}_{\overline{v_i}} \Psi_J, \Psi_J \rangle \frac{\omega^n}{n!}$$

$$(4.10) \qquad = \; \int_{M_t} \langle \mathcal{L}_{v_i} \mathcal{L}_{\overline{v_i}} \Psi_J, \Psi_J \rangle \frac{\omega^n}{n!} + \int_{M_t} \langle \mathcal{L}_{\overline{v_i}} \Psi_J, \mathcal{L}_{\overline{v_i}} \Psi_J \rangle \frac{\omega^n}{n!}.$$

Together with (4.9) and the identity $\mathcal{L}_{\overline{v_i}} \mathcal{L}_{v_i} = \mathcal{L}_{v_i} \mathcal{L}_{\overline{v_i}} + \mathcal{L}_{[\overline{v_i}, v_i]}$, one has

$$(4.11) \qquad \partial_i \partial_{\overline{i}} \|\Psi_J\|_2^2 = I + II + III, \quad \text{where}$$

$$(4.12) \qquad I := - \int_{M_t} \langle \mathcal{L}_{\overline{v_i}} \Psi_J, \mathcal{L}_{\overline{v_i}} \Psi_J \rangle \frac{\omega^n}{n!},$$

$$II := \int_{M_t} \langle \mathcal{L}_{[\overline{v_i}, v_i]} \Psi_J, \Psi_J \rangle \frac{\omega^n}{n!} = (\mathcal{L}_{[\overline{v_i}, v_i]} \Psi_J, \Psi_J),$$

$$III := \int_{M_t} \langle \mathcal{L}_{v_i} \Psi_J, \mathcal{L}_{v_i} \Psi_J \rangle \frac{\omega^n}{n!} = (\mathcal{L}_{v_i} \Psi_J, \mathcal{L}_{v_i} \Psi_J).$$

Here the last equality in the second line of (4.12) follows from the fact that only the component of $\mathcal{L}_{[\overline{v_i}, v_i]} \Psi_J$ in $\mathcal{A}^{0,\ell}(\wedge^\ell T M_t)$ (on which $\mathcal{C}_W$ is the identity mapping) will contribute towards the integral in that line. Likewise, the last equality in the third line of (4.12) follows from the fact that $\mathcal{L}_{v_i} \Psi_J \in \mathcal{A}^{0,\ell}(\wedge^\ell T M_t)$, which can be verified easily by a direct calculation using (2.4). In the next few sections, we will compute the terms $I, II$ and $III$ separately.

## 5. Computation of $I$

For the computation of the expression $I$ in (4.12), we begin with some preliminary discussions. For a relative tensor $\Upsilon \in \oplus_{p,q,r,s} \mathcal{A}^{q,p}(\wedge^r T M_t \wedge \overline{\wedge^s T M_t})$, we denote by $\Upsilon_{(r,s)}^{(q,p)}$ the component of $\Upsilon$ in $\mathcal{A}^{q,p}(\wedge^r T M_t \wedge \overline{\wedge^s T M_t})$.

**Lemma 2.** *Let $K \in \mathcal{A}^{0,p}(\wedge^r T M_t)$ be a relative tensor. Then we have*

$$(5.1) \qquad \overline{\partial}((\mathcal{L}_{\overline{v_i}} K)_{(r,0)}^{(0,p)}) = (\mathcal{L}_{\overline{v_i}}(\overline{\partial} K))_{(r,0)}^{(0,p+1)}.$$

*Proof.* By linearity, we just need to verify (5.1) for the special case when $K$ is locally given by a single term, i.e., $K = f \, \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_p}$ for some function $f$ and some integers $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_p$. Then

$$\overline{\partial} K = (\partial_{\overline{\sigma}} f) \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\overline{z}^\sigma \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_p}.$$

Hence

$$(5.2) \quad (\mathcal{L}_{\overline{v}_i}(\overline{\partial}K))^{(0,p+1)}_{(r,0)}$$

$$\begin{aligned}
= \ & (\partial_{\overline{i}}\partial_{\overline{\sigma}}f + \overline{v_i^{\gamma}}\partial_{\gamma}\partial_{\overline{\sigma}}f)\partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\overline{z}^{\sigma} \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_p} \\
& + (\partial_{\overline{\sigma}}f)(\overline{\partial_{\gamma}v_i^{\sigma}})\partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\overline{z}^{\gamma} \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_p} \\
& + \sum_{\ell=1}^{p}(\partial_{\overline{\sigma}}f) \cdot (\overline{\partial_{\gamma}v_i^{\beta_\ell}})\partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \\
& \qquad\qquad \otimes d\overline{z}^{\sigma} \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_{\ell-1}} \wedge d\overline{z}^{\overline{\gamma}} \wedge d\overline{z}^{\beta_{\ell+1}} \wedge \cdots \wedge d\overline{z}^{\beta_p}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
& (\mathcal{L}_{\overline{v}_i}K)^{(0,p)}_{(r,0)} \\
= \ & (\partial_{\overline{i}}f + \overline{v_i^{\gamma}}\partial_{\overline{\gamma}}f)\partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_p} \\
& + \sum_{\ell=1}^{p}f(\overline{\partial_{\overline{\gamma}}v_i^{\beta_\ell}})\partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_{\ell-1}} \wedge d\overline{z}^{\overline{\gamma}} \wedge d\overline{z}^{\beta_{\ell+1}} \wedge \cdots \wedge d\overline{z}^{\beta_p}.
\end{aligned}$$

Hence
(5.3)

$$\begin{aligned}
& \overline{\partial}((\mathcal{L}_{\overline{v}_i}K)^{(0,p)}_{(r,0)}) \\
= \ & (\partial_{\overline{\sigma}}\partial_{\overline{i}}f + \overline{\partial_{\sigma}v_i^{\gamma}}\partial_{\overline{\gamma}}f + \overline{v_i^{\gamma}}\partial_{\overline{\sigma}}\partial_{\overline{\gamma}}f)\partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \otimes d\overline{z}^{\sigma} \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_p} \\
& + \sum_{\ell=1}^{p}((\partial_{\overline{\sigma}}f) \cdot (\overline{\partial_{\gamma}v_i^{\beta_\ell}}) + f\,\overline{\partial_{\sigma}\partial_{\gamma}v_i^{\beta_\ell}}) \cdot \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_r} \\
& \qquad\qquad \otimes d\overline{z}^{\sigma} \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_{\ell-1}} \wedge d\overline{z}^{\overline{\gamma}} \wedge d\overline{z}^{\beta_{\ell+1}} \wedge \cdots \wedge d\overline{z}^{\beta_p}.
\end{aligned}$$

We may now compare the right hand sides of the identities (5.2) and (5.3). The first and the fourth terms of (5.2) corresponds to the first and the fourth terms of (5.3) respectively. The second term of (5.2) corresponds to the third term of (5.3) and vice versa. The fifth term of the right hand side of (5.3) vanishes, since $\partial_{\sigma}\partial_{\gamma}v_i^{\gamma\ell}$ is symmetric in $\sigma, \gamma$, but $d\overline{z}^{\sigma} \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_{\ell-1}} \wedge d\overline{z}^{\overline{\gamma}} \wedge d\overline{z}^{\beta_{\ell+1}} \wedge \cdots \wedge d\overline{z}^{\beta_p}$ is anti-symmetric in $\sigma, \gamma$. The lemma follows. $\square$

**Lemma 3.** *The relative tensor* $(\mathcal{L}_{\overline{v}_i}\Psi_J)^{(0,\ell)}_{(\ell,0)}$ *is* $\overline{\partial}$*-exact on each* $M_t$.

*Proof.* Since $\Psi_J$ is the harmonic projection of $\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}$ on each $M_t$, it follows that

$$(5.4) \qquad\qquad \Psi_J = \Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell} + \overline{\partial}K$$

for some relative tensor $K \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell}TM_t)$. Thus

$$(\mathcal{L}_{\overline{v}_i}\Psi_J)^{(0,\ell)}_{(\ell,0)} = (\mathcal{L}_{\overline{v}_i}(\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}))^{(0,\ell)}_{(\ell,0)} + (\mathcal{L}_{\overline{v}_i}(\overline{\partial}K))^{(0,\ell)}_{(\ell,0)}.$$

By a direct calculation similar to Lemma 2, one easily sees that

$$(5.5) \qquad (\mathcal{L}_{\overline{v}_i}(\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell}))^{(\ell,0)}_{(0,\ell)}$$

$$= \sum_{s=1}^{\ell} \Phi_{j_1} \oslash \cdots \oslash \Phi_{j_{s-1}} \oslash (\mathcal{L}_{\overline{v}_i}\Phi_{j_s})^{(1,0)}_{(0,1)} \oslash \Phi_{j_{s+1}} \oslash \cdots \oslash \Phi_{j_\ell}.$$

By [S2, p. 281-282], for each $j_s$, there exists a relative tensor $K_{j_s} \in \mathcal{A}^{0,1}(TM_t)$ such that $(\mathcal{L}_{\overline{v}_i}\Phi_{j_s})^{(1,0)}_{(0,1)} = \overline{\partial}K_{j_s}$ on each $M_t$. Note that each relative tensor $\Phi_{j_s}, 1 \leqslant s \leqslant \ell$, is harmonic and thus $\overline{\partial}$-closed on each $M_t$. Thus by Remark 2(iii), each term of the right hand side of (5.5) is $\overline{\partial}$-exact on $M_t$. Hence $(\mathcal{L}_{\overline{v}_i}(\Phi_{j_1} \oslash \cdots \oslash \Phi_{j_\ell})$ is $\overline{\partial}$-exact on each $M_t$. By Lemma 2, $(\mathcal{L}_{\overline{v}_i}(\overline{\partial}K))^{(\ell,0)}_{(0,\ell)}$ is also $\overline{\partial}$-exact on each $M_t$. Thus $(\mathcal{L}_{\overline{v}_i}\Psi_J)^{(\ell,0)}_{(0,\ell)}$ is $\overline{\partial}$-exact on each $M_t$. $\qquad\square$

Let $\overline{\Phi_i} \cdot \Psi_J \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell-1}TM_t)$ be the relative tensor with components given by

$$(5.6) \qquad (\overline{\Phi_i} \cdot \Psi_J)^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta}_1\cdots\overline{\beta}_{\ell-1}} = \overline{(\Phi_i)^{\sigma}_{\overline{\gamma}}} \cdot (\Psi_J)^{\gamma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\overline{\beta}_1\cdots\overline{\beta}_{\ell-1}}.$$

**Lemma 4.** *Let $\Phi_i$ and $\Psi_J$ be as in (4.1). Then for any $\Upsilon \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell-1}TM_t)$, we have*

$$(5.7) \qquad \langle \overline{\Phi_i} \cdot \Psi_J, \Upsilon \rangle = \langle \Psi_J, \Phi_i \oslash \Upsilon \rangle.$$

*Proof.* To prove (5.7), we need to verify that, in terms of normal coordinates,

$$(5.8)$$
$$\frac{1}{((\ell-1)!)^2}\overline{(\Phi_i)^{\sigma}_{\overline{\gamma}}}(\Psi_J)^{\gamma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\overline{\beta}_1\cdots\overline{\beta}_{\ell-1}}\cdot\overline{\Upsilon^{\beta_1\cdots\beta_{\ell-1}}_{\overline{\alpha}_1\cdots\overline{\alpha}_{\ell-1}}} = \frac{1}{(\ell!)^2}(\Psi_J)^{\gamma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\overline{\beta}_1\cdots\overline{\beta}_{\ell-1}}\cdot\overline{(\Phi_i \oslash \Upsilon)^{\sigma\beta_1\cdots\beta_{\ell-1}}_{\gamma\alpha_1\cdots\alpha_{\ell-1}}}$$

(cf. Remark 4). Note that

$$(\Phi_i \oslash \Upsilon)^{\sigma\beta_1\cdots\beta_{\ell-1}}_{\gamma\alpha_1\cdots\alpha_{\ell-1}} = \frac{1}{((\ell-1)!)^2} \sum_{\tau,\kappa\in\mathfrak{S}_\ell} \mathrm{sgn}(\tau)\mathrm{sgn}(\kappa) \cdot (\Phi_i)^{\tau(\sigma)}_{\kappa(\gamma)}\Upsilon^{\tau(\beta_1)\cdots\tau(\beta_{\ell-1})}_{\kappa(\alpha_1)\cdots\kappa(\alpha_{\ell-1})},$$

where $\mathfrak{S}_\ell$ is the set of all permutations of $\ell$ elements. Thus the right hand side of (5.8) is equal to

$$\frac{1}{(\ell!)^2} \cdot \frac{1}{((\ell-1)!)^2} \sum_{\tau,\kappa\in\mathfrak{S}_\ell} \mathrm{sgn}(\tau)\mathrm{sgn}(\kappa)(\Psi_J)^{\gamma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\overline{\beta}_1\cdots\overline{\beta}_{\ell-1}}\overline{(\Phi_i)^{\tau(\sigma)}_{\kappa(\gamma)}\Upsilon^{\tau(\beta_1)\cdots\tau(\beta_{\ell-1})}_{\kappa(\alpha_1)\cdots\kappa(\alpha_{\ell-1})}}$$

$$= \frac{1}{(\ell!)^2} \cdot \frac{1}{((\ell-1)!)^2} \sum_{\tau,\kappa\in\mathfrak{S}_\ell} (\Psi_J)^{\kappa(\gamma)\kappa(\alpha_1)\cdots\kappa(\alpha_{\ell-1})}_{\tau(\sigma)\tau(\beta_1)\cdots f(\beta_{\ell-1})}\overline{(\Phi_i)^{\tau(\sigma)}_{\kappa(\gamma)}\Upsilon^{\tau(\beta_1)\cdots\tau(\beta_{\ell-1})}_{\kappa(\alpha_1)\cdots\kappa(\alpha_{\ell-1})}}$$

$$= \frac{1}{(\ell!)^2} \cdot \frac{(\ell!)^2}{((\ell-1)!)^2}(\Psi_J)^{\gamma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\overline{\beta}_1\cdots\overline{\beta}_{\ell-1}}\overline{(\Phi_i)^{\sigma}_{\overline{\gamma}}\Upsilon^{\beta_1\cdots\beta_{\ell-1}}_{\overline{\alpha}_1\cdots\overline{\alpha}_{\ell-1}}},$$

where the numerator $(\ell!)^2$ arises from the pairs $(\tau,\kappa) \in \mathfrak{S}_\ell \times \mathfrak{S}_\ell$. This verifies (5.8). $\qquad\square$

**Lemma 5.** *We have* $\overline{\partial}^*(\overline{\Phi_i} \cdot \Psi_J) = 0$.

*Proof.* For any $\Upsilon \in \mathcal{A}^{0,\ell-2}(\wedge^{\ell-1} TM_t)$, we have

$$
\begin{aligned}
(\overline{\partial}^*(\overline{\Phi_i} \cdot \Psi_J), \Upsilon) &= (\overline{\Phi_i} \cdot \Psi_J, \overline{\partial}\Upsilon) \\
&= \int_{M_t} \langle \overline{\Phi_i} \cdot \Psi_J, \overline{\partial}\Upsilon \rangle \frac{\omega^n}{n!} \quad \text{(cf. (4.5) and Remark 4)} \\
&= \int_{M_t} \langle \Psi_J, \Phi_i \oslash \overline{\partial}\Upsilon \rangle \frac{\omega^n}{n!} \quad \text{(by Lemma 4)} \\
&= \int_{M_t} \langle \Psi_J, \overline{\partial}(\Phi_i \oslash \Upsilon) \rangle \frac{\omega^n}{n!} \quad \text{(since } \overline{\partial}\Phi_i = 0) \\
&= (\overline{\partial}^*\Psi_J, \Phi_i \oslash \Upsilon) \\
&= 0 \quad \text{(since } \Psi_J \text{ is harmonic),}
\end{aligned}
$$

which gives the lemma. $\qquad\square$

For $\Upsilon \in \mathcal{A}^{0,p}(\wedge^r TM_t)$, we denote $\overline{D_2}^*\Upsilon \in \mathcal{A}^{0,p}(\wedge^{r-1} TM_t)$, given by

$$
(\overline{D_2}^*\Upsilon)^{\alpha_1 \cdots \alpha_{r-1}}_{\overline{\beta_1} \cdots \overline{\beta_p}} = -\nabla_\sigma \Upsilon^{\sigma\alpha_1 \cdots \alpha_{r-1}}_{\overline{\beta_1} \cdots \overline{\beta_p}}.
$$

(cf. [S2, p.288] and Section 7). Following the argument of [S2, p. 280-281], we have

**Lemma 6.** *The tensor* $\overline{D_2}^*((\mathcal{L}_{\overline{v}_i}\Psi_J)^{(0,\ell)}_{(\ell,0)})$ *is* $\overline{\partial}$-*exact. Explicitly, we have*

$$
(5.9) \qquad \nabla_\sigma(\mathcal{L}_{\overline{v}_i}\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} = (\overline{\partial}(\overline{\Phi_i} \cdot \Psi_J))^{\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}}.
$$

*Proof.* To verify (5.9), we first note that

$$
(\mathcal{L}_{\overline{v}_i}\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} = (\partial_{\overline{i}} + \overline{v_i^\gamma}\partial_{\overline{\gamma}})(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} + \sum_{s=1}^{\ell} \overline{\partial_{\beta_s} v_i^\gamma}(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_{s-1}}\overline{\gamma}\overline{\beta_{s+1}} \cdots \overline{\beta_\ell}},
$$

In normal coordinates, the first partial derivatives of the metric tensor all vanish. Upon interchanging the order of the partial derivatives and using (2.3), we have

$$
\begin{aligned}
&\nabla_\sigma(\mathcal{L}_{\overline{v}_i}\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} \\
=\ & (\partial_\sigma\partial_{\overline{i}} + \overline{\partial_\sigma v_i^\gamma}\partial_{\overline{\gamma}} + \overline{v_i^\gamma}\partial_\sigma\partial_{\overline{\gamma}})(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} \\
& + \sum_{s=1}^{\ell}\{\overline{\partial_\sigma\partial_{\beta_s} v_i^\gamma}(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_{s-1}}\overline{\gamma}\overline{\beta_{s+1}} \cdots \overline{\beta_\ell}} + \overline{\partial_{\beta_s} v_i^\gamma}\partial_\sigma(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_{s-1}}\overline{\gamma}\overline{\beta_{s+1}} \cdots \overline{\beta_\ell}}\} \\
=\ & (\partial_{\overline{i}}\partial_\sigma + \overline{(\Phi_i)^\gamma_\sigma}\partial_{\overline{\gamma}} + \overline{v_i^\gamma}\partial_{\overline{\gamma}}\partial_\sigma)(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} \\
& + \sum_{s=1}^{\ell}\{\partial_{\overline{\beta_s}}\overline{(\Phi_i)^\gamma_\sigma}(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_{s-1}}\overline{\gamma}\overline{\beta_{s+1}} \cdots \overline{\beta_\ell}} + \overline{\partial_{\beta_s} v_i^\gamma}\partial_\sigma(\Psi_J)^{\sigma\alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta_1} \cdots \overline{\beta_{s-1}}\overline{\gamma}\overline{\beta_{s+1}} \cdots \overline{\beta_\ell}}\}.
\end{aligned}
$$

The first, the third and the fifth terms vanish from the identity $\overline{\partial}^* \Psi = 0$ (as a relative tensor). Hence we have

$$\nabla_\sigma (\mathcal{L}_{\overline{v}_i} \Psi_J)^{\sigma \alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} = \overline{(\Phi_i)^{\overline{\gamma}}_{\overline{\sigma}}} \partial_{\overline{\gamma}} (\Psi_J)^{\sigma \alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} + \sum_{s=1}^{\ell} \partial_{\overline{\beta}_s} \overline{(\Phi_i)^{\overline{\gamma}}_{\overline{\sigma}}} \cdot (\Psi_J)^{\sigma \alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta}_1 \cdots \overline{\beta}_{s-1} \overline{\gamma} \overline{\beta}_{s+1} \cdots \overline{\beta}_\ell},$$

which, together with the identity $\overline{\partial} \Psi_J = 0$, is easily seen to be equal to the right hand side of (5.9). $\qquad\square$

Now we proceed to compute $I$. First we note from (2.3) and (2.4) that

$$\begin{aligned}
\mathcal{L}_{\overline{v}_i} \Psi_J =& (\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell,0)} \\
& + \frac{1}{(\ell-1)!\ell!} \sum \overline{(\Phi_i)^{\sigma}_{\overline{\delta}}} (\Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\overline{\sigma} \overline{\beta}_1 \cdots \overline{\beta}_{\ell-1}} \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_\ell} \otimes dz^\delta \wedge d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_{\ell-1}} \\
& - \frac{1}{(\ell-1)!\ell!} \sum \overline{(\Phi_i)^{\overline{\gamma}}_{\overline{\sigma}}} (\Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1} \sigma}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_{\ell-1}} \wedge \partial_{\overline{\gamma}} \otimes d\overline{z}^{\beta_1} \wedge \cdots \wedge d\overline{z}^{\beta_\ell},
\end{aligned}$$

where the last term involves the use of the equality $(\Psi_J)^{\sigma \alpha_1 \cdots \alpha_{\ell-1}}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} \partial_{\overline{\gamma}} \wedge \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_{\ell-1}} = (\Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1} \sigma}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} \partial_{\alpha_1} \wedge \cdots \wedge \partial_{\alpha_{\ell-1}} \wedge \partial_{\overline{\gamma}}$. Let $\overline{\Phi_i} \searrow \Psi_J \in \mathcal{A}^{1,\ell-1}(\wedge^\ell TM_t)$ and $\overline{\Phi_i} \nearrow \Psi_J \in \mathcal{A}^{0,\ell}(\wedge^{\ell-1} TM_t \wedge \overline{TM_t})$ be given by

$$(5.10) \qquad (\overline{\Phi_i} \searrow \Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\delta \overline{\beta}_1 \cdots \overline{\beta}_{\ell-1}} := \overline{(\Phi_i)^{\sigma}_{\overline{\delta}}} (\Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\overline{\sigma} \overline{\beta}_1 \cdots \overline{\beta}_{\ell-1}} \quad \text{and}$$

$$(\overline{\Phi_i} \nearrow \Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1} \overline{\gamma}}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell} := \overline{(\Phi_i)^{\overline{\gamma}}_{\overline{\sigma}}} (\Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1} \sigma}_{\overline{\beta}_1 \cdots \overline{\beta}_\ell}$$

respectively, so that we have $(\mathcal{L}_{\overline{v}_i} \Psi_J)^{(1,\ell-1)}_{(\ell,0)} = \overline{\Phi_i} \searrow \Psi_J$ and $(\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell-1,1)} = -\overline{\Phi_i} \nearrow \Psi_J$. Note that from consideration of type, one has

$$\mathcal{C}_W((\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell,0)}) = (\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell,0)}, \quad \mathcal{C}_W(\overline{\Phi_i} \searrow \Psi_J) = -\overline{\Phi_i} \searrow \Psi_J, \quad \text{and}$$

$$\mathcal{C}_W(\overline{\Phi_i} \nearrow \Psi_J) = -\overline{\Phi_i} \nearrow \Psi_J.$$

Together with (4.5), it follows readily that

$$\int_{M_t} \langle \mathcal{L}_{\overline{v}_i} \Psi_J, \mathcal{L}_{\overline{v}_i} \Psi_J \rangle \frac{\omega^n}{n!} = ((\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell,0)}, (\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell,0)})$$

$$(5.11) \qquad\qquad - (\overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J) - (\overline{\Phi_i} \nearrow \Psi_J, \overline{\Phi_i} \nearrow \Psi_J).$$

To compute the first term on the right hand side of (5.11), we first recall from Lemma 3 that there exists some $K \in \mathcal{A}^{0,\ell-1}(\wedge^\ell TM_t)$ such that

$$(5.12) \qquad\qquad \overline{\partial} K = (\mathcal{L}_{\overline{v}_i} \Psi_J)^{(0,\ell)}_{(\ell,0)}.$$

**Lemma 7.** *Let $K$ be as in (5.12). Suppose that $\overline{\partial}^* K = 0$. Then*

$$\overline{D_2}^* K = -\square(\square - k)^{-1}(\overline{\Phi_i} \cdot \Psi_J).$$

*Proof.* The proof is similar to [S2, p.282]. First we have

$$\nabla_\sigma(\overline{\partial}K)^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}$$

$$= \nabla_\sigma\Big(\sum_{s=1}^{\ell}(-1)^{s+1}\nabla_{\overline{\beta_s}}K^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}\Big)$$

$$= \sum_{s=1}^{\ell}(-1)^{s+1}\Big[\nabla_{\overline{\beta_s}}\nabla_\sigma K^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}$$

$$\qquad\qquad + \sum_{r=1,r\neq s}^{\ell} R_{\sigma\overline{\beta_s}}{}^{\overline{\gamma}}{}_{\overline{\beta_r}} K^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{r-1}\gamma\beta_{r+1}}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}$$

$$(5.13)\qquad + R_{\sigma\overline{\beta_s}\tau}{}^{\sigma} K^{\tau\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}} + \sum_{r=1}^{\ell-1} R_{\sigma\overline{\beta_s}\gamma}{}^{\alpha_r} K^{\sigma\alpha_1\cdots\alpha_{r-1}\gamma\alpha_{r+1}\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}\Big].$$

The second term on the right hand side of (5.13) is zero, because of the symmetry of $R_{\sigma\overline{\beta_s}}{}^{\overline{\gamma}}{}_{\overline{\beta_r}}$ in $r$ and $s$ and the skew-symmetry of the expression $(-1)^{s+1}K^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{r-1}\gamma\beta_{r+1}}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}$ in $r$ and $s$. The fourth term on the right hand side of (5.13) is also zero, because of the symmetry of $R_{\sigma\overline{\beta_s}\gamma}{}^{\alpha_r}$ in $\sigma$ and $\gamma$ and the skew symmetry of $K^{\sigma\alpha_1\cdots\alpha_{r-1}\gamma\alpha_{r+1}\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}$ in $\sigma,\gamma$. Together with (5.12) and the identity $R_{\sigma\overline{\beta_s}\tau}{}^{\sigma} = k\delta_{\beta_s\tau}$ from Kähler-Einstein condition, we have

$$\nabla_\sigma(\mathcal{L}_{\overline{v_i}}\Psi_J)^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} = \sum_{s=1}^{\ell}(-1)^{s+1}\nabla_{\overline{\beta_s}}\nabla_\sigma K^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}$$

$$(5.14)\qquad\qquad\qquad + k\sum_{s=1}^{\ell}(-1)^{s+1}K^{\beta_s\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}.$$

Combining (5.14) with Lemma 6, we have

$$(5.15)\quad (\overline{\partial}(\overline{\Phi_i}\cdot\Psi_J))^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} = -(\overline{\partial}(\overline{D_2}^*K))^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} + k\,\Gamma^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}, \quad \text{where}$$

$$(5.16)\qquad\qquad \Gamma^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} := \sum_{s=1}^{\ell}(-1)^{s+1}K^{\beta_s\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}\beta_{s+1}}\cdots\overline{\beta_\ell}}.$$

In particular, the tensor $\Gamma \in \mathcal{A}^{0,\ell}(\wedge^{\ell-1}TM_t)$ with components given as in (5.16) is $\overline{\partial}$-exact. Thus we may write $\Gamma = \overline{\partial}\widetilde{F}$ for some $\widetilde{F} \in \mathcal{A}^{0,\ell-1}(\wedge^{\ell-1}TM_t)$. Without loss of generality, we may choose $\widetilde{F}$ such that $\overline{\partial}^*\widetilde{F} = 0$. Upon rewriting (5.15), we have

$$(5.17)\qquad\qquad\qquad \overline{\partial}(\overline{\Phi_i}\cdot\Psi_J + \overline{D_2}^*K - k\widetilde{F}) = 0.$$

In normal coordinates, we have, from (5.16),

$$(\overline{\partial}^*\Gamma)^{\alpha_1\cdots\alpha_{\ell-1}}_{\beta_1\cdots\beta_{\ell-1}} = -(\partial_\sigma\Gamma)^{\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\beta_1\cdots\beta_{\ell-1}}$$

$$= -\partial_\sigma K^{\sigma\alpha_1\cdots\alpha_{\ell-1}}_{\beta_1\cdots\beta_{\ell-1}} + \sum_{s=1}^{\ell-1}(-1)^{s+1}\partial_\sigma K^{\beta_s\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\beta_1\cdots\overline{\beta_{s-1}}\beta_{s+1}\cdots\beta_{\ell-1}}.$$

Since $\overline{\partial}^*K = 0$, it follows that $\partial_\sigma K^{\beta_s\alpha_1\cdots\alpha_{\ell-1}}_{\overline{\sigma}\beta_1\cdots\overline{\beta_{s-1}}\beta_{s+1}\cdots\overline{\beta_{\ell-1}}} = 0$. Thus we have $(\overline{\partial}^*\Gamma)^{\alpha_1\cdots\alpha_{\ell-1}}_{\beta_1\cdots\beta_{\ell-1}} = (\overline{D_2}^*K)^{\alpha_1\cdots\alpha_{\ell-1}}_{\beta_1\cdots\beta_{\ell-1}}$. Together with (5.17), we have

$$(5.18) \qquad \overline{\partial}(\overline{\Phi_i}\cdot\Psi_J + \overline{\partial}^*\Gamma - k\widetilde{F}) = 0.$$

Let

$$(5.19) \qquad Q := \overline{\Phi_i}\cdot\Psi_J + \overline{\partial}^*\Gamma - k\widetilde{F}.$$

By Lemma 5, $\overline{\Phi_i}\cdot\Psi_J$ is $\overline{\partial}^*$-exact. Recall also that $\overline{\partial}^*\widetilde{F} = 0$. Thus all the three terms on the right hand side of (5.19) are $\overline{\partial}^*$-closed. Hence we have $\overline{\partial}^*Q = 0$. Together with (5.18), it follows that $Q$ is harmonic. Since $\Gamma = \overline{\partial}\widetilde{F}$ and $\overline{\partial}^*\widetilde{F} = 0$, one easily sees that

$$\Box\widetilde{F} = (\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*)\widetilde{F} = \overline{\partial}^*\Gamma,$$

which, together with (5.19), gives

$$(5.20) \qquad Q = \overline{\Phi_i}\cdot\Psi_J + (\Box - k)\widetilde{F}.$$

Let $F := \widetilde{F} + \frac{Q}{k}$. Then it follows from (5.20) and the harmonicity of $Q$ that

$$\overline{\Phi_i}\cdot\Psi_J + (\Box - k)F = 0.$$

Thus, $F = -(\Box - k)^{-1}(\overline{\Phi_i}\cdot\Psi_J)$. Hence we have

$$\overline{D_2}^*K = \overline{\partial}^*\Gamma = \Box\widetilde{F} = \Box F = -\Box(\Box - k)^{-1}(\overline{\Phi_i}\cdot\Psi_J). \quad \Box$$

Our main result in this section is the following

**Proposition 1.** *We have*

$$\int_{M_t}\langle\mathcal{L}_{\overline{v_i}}\Psi_J, \mathcal{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!} = k((\Box - k)^{-1}(\overline{\Phi_i}\cdot\Psi_J), \overline{\Phi_i}\cdot\Psi_J) + (\overline{\Phi_i}\cdot\Psi_J, \overline{\Phi_i}\cdot\Psi_J)$$

$$- (\overline{\Phi_i}\searrow\Psi_J, \overline{\Phi_i}\searrow\Psi_J) - (\overline{\Phi_i}\nearrow\Psi_J, \overline{\Phi_i}\nearrow\Psi_J).$$

*Proof.* First we compute $((\mathcal{L}_{\overline{v_i}}(\Psi_J)^{(0,\ell)}_{(\ell,0)}, (\mathcal{L}_{\overline{v_i}}(\Psi_J)^{(0,\ell)}_{(\ell,0)})$. Let $K$ be as in (5.12), so that $\overline{\partial}K = (\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}$. Without loss of generality, we may choose $K$

so that $\overline{\partial}^* K = 0$, so that Lemma 7 is applicable to $K$. Now, in normal coordinates, we have

$$\int_{M_t} (\mathcal{L}_{\overline{v_i}}\Psi_J)^{\lambda_1\cdots\lambda_\ell}_{\alpha_1\cdots\alpha_\ell} \overline{(\mathcal{L}_{\overline{v_i}}\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\lambda_1\cdots\lambda_\ell}} \frac{\omega^n}{n!}$$

$$= \int_{M_t} (\overline{\partial}K)^{\lambda_1\cdots\lambda_\ell}_{\alpha_1\cdots\alpha_\ell} \overline{(\mathcal{L}_{\overline{v_i}}\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\lambda_1\cdots\lambda_\ell}} \frac{\omega^n}{n!}$$

$$= -\int_{M_t} K^{\lambda_1\cdots\lambda_\ell}_{\alpha_2\cdots\alpha_\ell} \overline{\nabla_{\alpha_1}(\mathcal{L}_{\overline{v_i}}\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\lambda_1\cdots\lambda_\ell}} \frac{\omega^n}{n!}$$

$$= -\int_{M_t} K^{\lambda_1\cdots\lambda_\ell}_{\alpha_2\cdots\alpha_\ell} \overline{(\overline{\partial}(\overline{\Phi}_i \cdot \Psi_J))^{\alpha_2\cdots\alpha_\ell}_{\lambda_1\cdots\lambda_\ell}} \frac{\omega^n}{n!} \quad \text{(by Lemma 6)}$$

$$= \int_{M_t} \nabla_{\alpha_1} K^{\alpha_1\lambda_2\cdots\lambda_\ell}_{\alpha_2\cdots\alpha_\ell} \overline{(\overline{\Phi}_i \cdot \Psi_J)^{\alpha_2\cdots\alpha_\ell}_{\lambda_2\cdots\lambda_\ell}} \frac{\omega^n}{n!}$$

$$= \int_{M_t} (\square(\square - k)^{-1}(\overline{\Phi}_i \cdot \Psi_J))^{\lambda_1\cdots\lambda_\ell}_{\alpha_2\cdots\alpha_\ell} \overline{(\overline{\Phi}_i \cdot \Psi_J)^{\alpha_2\cdots\alpha_\ell}_{\lambda_2\cdots\lambda_\ell}} \frac{\omega^n}{n!} \quad \text{(by Lemma 7)}.$$

This implies that

$$((\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}, (\mathcal{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}) = (\square(\square - k)^{-1}(\overline{\Phi}_i \cdot \Psi_J), \overline{\Phi}_i \cdot \Psi_J)$$

$$= k((\square - k)^{-1}(\overline{\Phi}_i \cdot \Psi_J), \overline{\Phi}_i \cdot \Psi_J) + (\overline{\Phi}_i \cdot \Psi_J, \overline{\Phi}_i \cdot \Psi_J).$$

Together with (5.11), one obtains the proposition readily.                $\square$

## 6. Computation of $II$

We recall the following

**Lemma 8.** *([Sch1, Lemma 2.8) One has*

$$(\square - k)(\langle v_i, v_j \rangle) = \langle \Phi_i, \Phi_j \rangle.$$

Similar to [Sch1, Lemma 2.7], one has

**Proposition 2.**

$$(\mathcal{L}_{[\overline{v_i}, v_i]}\Psi_J, \Psi_J) = -(\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) - k((\square - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle).$$

*Proof.* By Lemma 1(iii) and direct calculation, one has

$$(6.1) \quad (\mathcal{L}_{[\overline{v_i}, v_i]}\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} = -\langle v_i, v_i \rangle^{;\sigma}\partial_\sigma(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}} + \langle v_i, v_i \rangle^{;\overline{\delta}}\partial_{\overline{\delta}}(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}$$

$$+ \sum_{s=1}^{\ell} \partial_\gamma(\langle v_i, v_i \rangle^{;\alpha_s})(\Psi_J)^{\alpha_1\cdots\alpha_{s-1}\gamma\alpha_{s+1}\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}$$

$$+ \sum_{s=1}^{\ell} \partial_{\overline{\beta_s}}(\langle v_i, v_i \rangle^{;\overline{\delta}})(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}}\delta\overline{\beta_{s+1}}\cdots\overline{\beta_\ell}}.$$

By pairing the first term on the right hand side of (6.1) with $\Psi_J$, we have

$$-\langle v_i, v_i\rangle^{;\sigma}\partial_\sigma(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}} = -\langle v_i, v_i\rangle^{;\sigma}\partial_\sigma\Big((\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}\Big)$$

$$(6.2) \qquad\qquad\qquad + \langle v_i, v_i\rangle^{;\sigma}(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{\partial_{\overline{\sigma}}(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}.$$

Since $\overline{\partial}\Psi_J = 0$, we have

$$(6.3)\qquad \partial_{\overline{\sigma}}(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}} = \sum_{s=1}^{\ell}(-1)^{s+1}\partial_{\overline{\alpha_s}}(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_{s-1}}\overline{\sigma}\overline{\alpha_{s+1}}\cdots\overline{\alpha_\ell}}.$$

This is substituted into the last term of (6.2). We also substitute (6.3) (with the running index $\sigma$ replaced by $\delta$) into the second term on the right hand side of (6.1). Then one easily sees that the $L^2$-pairing of the resulting expression of (6.1) with $\Psi_J$ is given by

$$\int_{M_t}(\mathcal{L}_{[\overline{v_i}, v_i]}\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}\frac{\omega^n}{n!} = II_1 + II_2 + II_3 + II_4 + II_5,$$

where

$$II_1 : = -\int_{M_t}\langle v_i, v_i\rangle^{;\sigma}\partial_\sigma\Big((\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}\Big)\frac{\omega^n}{n!},$$

$$II_2 : = \sum_{s=1}^{\ell}(-1)^{s+1}\int_{M_t}\langle v_i, v_i\rangle^{;\sigma}(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{\partial_{\overline{\alpha_s}}(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_{s-1}}\overline{\alpha_{s+1}}\cdots\overline{\alpha_\ell}}}\frac{\omega^n}{n!},$$

$$II_3 : = \sum_{s=1}^{\ell}(-1)^{s+1}\int_{M_t}\langle v_i, v_i\rangle^{;\overline{\delta}}\partial_{\overline{\beta_s}}(\Psi_J)_{\overline{\beta_1}\cdots\overline{\beta_{s-1}}\delta\overline{\beta_{s+1}}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}\frac{\omega^n}{n!},$$

$$II_4 : = \sum_{s=1}^{\ell}\int_{M_t}\partial_\gamma(\langle v_i, v_i\rangle^{;\alpha_s})(\Psi_J)^{\alpha_1\cdots\alpha_{s-1}\gamma\alpha_{s+1}\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}\frac{\omega^n}{n!},$$

$$II_5 : = \sum_{s=1}^{\ell}\int_{M_t}\partial_{\overline{\beta_s}}(\langle v_i, v_i\rangle^{;\overline{\delta}})(\Psi_J)^{\alpha_1\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_{s-1}}\delta\overline{\beta_{s+1}}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}}\frac{\omega^n}{n!}.$$

Upon integrating by parts, one easily sees that

$$II_1 = -\int_{M_t}(\square\langle v_i, v_i\rangle)\cdot\langle\Psi_J, \Psi_J\rangle\frac{\omega^n}{n!}.$$

In $II_4$ and for each fixed $s$, we rename the running index $\alpha_s$ by $\sigma$ and then replace $\gamma$ by $\alpha_s$. This gives

$$II_4 = \sum_{s=1}^{\ell}\int_{M_t}\partial_{\alpha_s}(\langle v_i, v_i\rangle^{;\sigma})(\Psi_J)^{\alpha_1\cdots\alpha_{s-1}\alpha_s\alpha_{s+1}\cdots\alpha_\ell}_{\overline{\beta_1}\cdots\overline{\beta_\ell}}\overline{(\Psi_J)^{\beta_1\cdots\beta_\ell}_{\overline{\alpha_1}\cdots\overline{\alpha_{s-1}}\sigma\overline{\alpha_{s+1}}\cdots\overline{\alpha_\ell}}}\frac{\omega^n}{n!}.$$

Then one easily seen that

$$II_2 + II_4 = \int_{M_t} (\Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\beta_1 \cdots \beta_\ell} \overline{(\overline{\partial}\Upsilon)^{\beta_1 \cdots \beta_\ell}_{\alpha_1 \cdots \alpha_\ell}} \frac{\omega^n}{n!}, \quad \text{where}$$

$$\Upsilon^{\beta_1 \cdots \beta_\ell}_{\alpha_1 \cdots \alpha_{\ell-1}} := \overline{\langle v_i, v_i \rangle^{;\sigma}} (\Psi_J)^{\beta_1 \cdots \beta_\ell}_{\sigma \alpha_1 \cdots \alpha_{\ell-1}}.$$

Since $\overline{\partial}^* \Psi_J = 0$, it follows that $II_2 + II_4 = 0$. Similarly, one checks that

$$II_3 + II_5 = \int_{M_t} (\overline{\partial}\widehat{\Upsilon})^{\alpha_1 \cdots \alpha_\ell}_{\beta_1 \cdots \beta_\ell} \overline{(\Psi_J)^{\beta_1 \cdots \beta_\ell}_{\alpha_1 \cdots \alpha_\ell}} \frac{\omega^n}{n!} = 0, \quad \text{where}$$

$$\widehat{\Upsilon}^{\alpha_1 \cdots \alpha_\ell}_{\beta_1 \cdots \beta_{\ell-1}} := \langle v_i, v_i \rangle^{;\overline{\delta}} (\Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\overline{\delta}\beta_1 \cdots \beta_{\ell-1}}.$$

Summarizing the above discussion and using Lemma 8, one has

$$
\begin{aligned}
& (\mathcal{L}_{[\overline{v_i}, v_i]} \Psi_J, \Psi_J) \\
= {}& -(\Box \langle v_i, v_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\
= {}& -(\Box (\Box - k)^{-1} \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\
= {}& -(\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) - k((\Box - k)^{-1} \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle). \quad \Box
\end{aligned}
$$

## 7. Computation of $III$

Our main result in this section is the following

**Proposition 3.** *We have*

$$
\begin{aligned}
(\mathcal{L}_{v_i} \Psi_J, \mathcal{L}_{v_i} \Psi_J) = {}& -k((\Box - k)^{-1} (\mathcal{L}_{v_i} \Psi_J), \mathcal{L}_{v_i} \Psi_J) + (\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J) \\
& - (H(\Phi_i \oslash \Psi_J), H(\Phi_i \oslash \Psi_J)).
\end{aligned}
$$

We are going to prove Proposition 3 by generalizing the arguments in [S2, p. 287-295]. Let $\ell$ be a fixed integer satisfying $1 \leqslant \ell \leqslant n$. Similar to [S2, p. 288], we denote by $X^{(\ell)}$ the space of (relative) tensors $\Xi \in \mathcal{A}(\overline{\otimes^\ell T^* M_t} \otimes \overline{\otimes^\ell T^* M_t})$ with components $\Xi_{\overline{\alpha_1} \cdots \overline{\alpha_\ell}, \overline{\beta_1} \cdots \overline{\beta_\ell}}$ possessing the following three properties:

(P-i) $\Xi_{\overline{\alpha_1} \cdots \overline{\alpha_\ell}, \overline{\beta_1} \cdots \overline{\beta_\ell}}$ is skew-symmetric in any pair of indices $\alpha_i, \alpha_j$ for $i < j$, i.e.,

$$\Xi_{\overline{\alpha_1} \cdots \overline{(\alpha_j)_i} \cdots \overline{(\alpha_i)_j} \cdots \overline{\alpha_\ell}, \overline{\beta_1} \cdots \overline{\beta_\ell}} = -\Xi_{\overline{\alpha_1} \cdots \overline{\alpha_\ell}, \overline{\beta_1} \cdots \overline{\beta_\ell}},$$

where $\overline{(\alpha_i)_j}$ means that the $i$-th index $\overline{\alpha_i}$ is replaced by $\overline{\alpha_j}$, etc.

(P-ii) $\Xi_{\overline{\alpha_1} \cdots \overline{\alpha_\ell}, \overline{\beta_1} \cdots \overline{\beta_\ell}}$ is symmetric in the two $\ell$-tuples of indices $(\overline{\alpha_1}, \cdots, \overline{\alpha_\ell})$ and $(\overline{\beta_1}, \cdots, \overline{\beta_\ell})$, i.e.,

$$\Xi_{\overline{\alpha_1} \cdots \overline{\alpha_\ell}, \overline{\beta_1} \cdots \overline{\beta_\ell}} = \Xi_{\overline{\beta_1} \cdots \overline{\beta_\ell}, \overline{\alpha_1} \cdots \overline{\alpha_\ell}}.$$

(P-iii) For given indices $\alpha_1, \cdots, \alpha_{\ell-1}$, and $\beta_1, \cdots, \beta_{\ell+1}$, one has

$$\sum_{\nu=1}^{\ell+1} (-1)^\nu \Xi_{\overline{\alpha_1}\cdots\overline{\alpha_{\ell-1}}\overline{\beta_\nu},\overline{\beta_1}\cdots\widehat{\overline{\beta_\nu}}\cdots\overline{\beta_{\ell+1}}} = 0,$$

where $\widehat{\overline{\beta_\nu}}$ means that the index $\overline{\beta_\nu}$ is omitted.

As in [S2, p. 289], for $s = 1, 2$, we let $\overline{D_s}$ denote the operator $X^{(\ell)}$ given by taking $\overline{\partial}$ to the $s$-th $\ell$-tuple of skew-symmetric indices, and we let $\overline{D_s}^*$ denote the adjoint operator of $\overline{D_s}$. Also, we denote $\square_s = \overline{D_s}^*\overline{D_s} + \overline{D_s}\,\overline{D_s}^*$, and we denote by $H_s$ the harmonic projection operator on $X^{(\ell)}$ with respect to $\square_s$. The Green's operator on $X^{(\ell)}$ with respect to $\square_s$ is denoted by $G_s$.

**Lemma 9.** *For any $\Xi \in X^{(\ell)}$, we have*

*(a)* $\overline{D_1 D_2}\Xi = \overline{D_2 D_1}\Xi$,
*(b)* $\overline{D_1}^*\overline{D_2}\Xi = \overline{D_2 D_1}^*\Xi$,
*(c)* $\overline{D_1}^*\overline{D_2}^*\Xi = \overline{D_2}^*\overline{D_1}^*\Xi$,
*(d)* $\overline{D_1 D_2}^*\Xi = \overline{D_2}^*\overline{D_1}\Xi$,
*(e)* $\square_1\Xi \in X^{(\ell)}$,
*(f)* $\square_1\Xi = \square_2\Xi$, *and*
*(g) if* $\overline{D_1}\Xi = 0$, *then* $(\square_1 - k)^{-1}\overline{D_2}^*\Xi = \overline{D_2}^* G_2\Xi$.

*Proof.* The proofs of the above properties of $X^{(\ell)}$ follow mutatis mutandis from those in [S2, p. 289-292], which treated the case when $\ell = 2$. We will leave the details to the reader. $\square$

**Remark 5.** *(i) Let $Y^{(\ell)}$ denote the space of smooth covariant tensors $\Xi$ with two $\ell$-tuples of skew-symmetric indices of anti-holomorphic type, i.e., the components of $\Xi \in Y^{(\ell)}$ are of the form $\Xi_{\overline{\alpha_1}\cdots\overline{\alpha_\ell},\overline{\beta_1}\cdots\overline{\beta_\ell}}$ and they satisfy (P-i). Let $X^{(\ell)\perp}$ denote the orthogonal complement of $X^{(\ell)}$ in $Y^{(\ell)}$ with respect to the $L^2$-inner product on $M_t$. Then it follows readily from Lemma 9(e) that $\square_1\Xi \in X^{(\ell)\perp}$ if $\Xi \in X^{(\ell)\perp}$. Thus the spectral decomposition of $Y^{(\ell)}$ with respect to $\square_1$ induces a corresponding orthogonal decomposition of $X^{(\ell)}$. Then it follows easily that $H_1(\Xi) \in X^{(\ell)}$ if $\Xi \in X^{(\ell)}$.*
*(ii) One easily sees from Lemma 9(f) that $G_1\Xi = G_2\Xi$ (and thus also $H_1(\Xi) = H_2(\Xi)$) for $\Xi \in X^{(\ell)}$.*

Let $\Phi_i, \Psi_J$ (with $|J| = \ell$) be as in (4.1). By lowering indices of these objects, we obtain corresponding covariant tensors, which will be denoted by the same symbols (when no confusion arises). For example, $\Psi_J$ also denotes the covariant tensor with components given by

$$(\Psi_J)_{\overline{\alpha_1}\cdots\overline{\alpha_\ell},\overline{\beta_1}\cdots\overline{\beta_\ell}} = g_{\gamma_1\overline{\beta_1}}\cdots g_{\gamma_\ell\overline{\beta_\ell}}(\Psi_J)_{\overline{\alpha_1}\cdots\overline{\alpha_\ell}}^{\gamma_1\cdots\gamma_\ell}.$$

**Lemma 10.** *For each $1 \leqslant \ell \leqslant n$, we have $\Psi_J \in X^{(\ell)}$ and $\Phi_i \oslash \Psi_J \in X^{(\ell+1)}$.*

*Proof.* We are going to prove Lemma 10 by induction on $\ell$. Note that when $\ell = 1$, property (P-i) is void, while property (P-ii) and property (P-iii) coincide. By Lemma 1(iv), this common property is satisfied by $\Psi_J = \Phi_{j_1}$, where $J = (j_1)$. Thus $\Psi_J = \Phi_{j_1} \in X^{(1)}$. Moreover, as mentioned in [S2, p. 289], a simple direct verification shows that $\Phi_i \oslash \Psi_{j_1} = \Phi_i \oslash \Phi_{j_1} \in X^{(2)}$. Now we make the inductive assumption that $\Psi_{J'} \in X^{(\ell-1)}$, when $|J'| = \ell-1$ and $\Phi_i \oslash \Psi_{J'} \in X^{(\ell)}$. Then when $J = (j_1, \ldots, j_\ell)$ with $|J| = \ell$, we have, upon lowering indices, $\Psi_J = H_1(\Phi_{j_1} \oslash \Psi_{J'})$, where $J' = (j_2, \ldots, j_\ell)$. By inductive assumption, since $|J'| = \ell - 1$, we have $\Phi_i \oslash \Psi_{J'} \in X^{(\ell)}$. Together with Remark 5(i), it follows that $\Psi_J \in X^{(\ell)}$ as well. Thus, it remains to show that $\Phi_i \oslash \Psi_J \in X^{(\ell+1)}$ (upon lowering indices). Since $\Phi_i \oslash \Psi_J$ is a $\wedge^{\ell+1} TM_t$-valued $(0, \ell+1)$-form, it is easy to see that upon lowering indices, $\Phi_i \oslash \Psi_J$ possesses property (P-i). We have deduced from the inductive assumption that $\Psi_J \in X^{(\ell)}$, and thus it possesses property (P-ii). Together with symmetry property of $\Phi_i$ in Lemma 1(iv), one easily sees that $\Phi_i \oslash \Psi_J$ possesses property (P-ii). Next we are going to verify property (P-iii) for $\Phi_i \oslash \Psi_J$. For fixed indices $\alpha_1, \ldots, \alpha_\ell, \beta_0, \ldots, \beta_{\ell+1}$, and in terms of normal coordinates, we consider the expression

$$
\begin{aligned}
A &= \sum_{\nu=0}^{\ell+1} (-1)^\nu (\Phi_i \oslash \Psi_J)_{\overline{\beta_0}\cdots\widehat{\overline{\beta_\nu}}\cdots\overline{\beta_{\ell+1}},\overline{\beta_\nu}\overline{\alpha_1}\cdots\overline{\alpha_\ell}} \\
&= \sum_{\nu=0}^{\ell+1} (-1)^\nu (\Phi_i \oslash \Psi_J)^{\beta_0\cdots\widehat{\beta_\nu}\cdots\beta_{\ell+1}}_{\overline{\beta_\nu}\overline{\alpha_1}\cdots\overline{\alpha_\ell}} \\
&= \sum_{\nu=0}^{\ell+1} (-1)^\nu \sum_{\sigma,\tau\in\mathfrak{G}_{\ell+1}} \frac{\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)}{(1!)^2(\ell!)^2} (\Phi_i)^{\sigma(\beta_0)}_{\tau(\beta_\nu)} (\Psi_J)^{\sigma(\beta_1)\cdots\sigma(\beta_{\nu-1})\sigma(\beta_{\nu+1})\cdots\sigma(\beta_{\ell+1})}_{\tau(\alpha_1)\cdots\tau(\alpha_\ell)}.
\end{aligned}
$$

Let $\mathfrak{G}'_{\ell+1}$ (resp. $\mathfrak{G}''_{\ell+1}$) be the subset of $\mathfrak{G}_{\ell+1}$ consisting of those permutations which fix the first object (resp. does not fix the first object), so that

$$(7.1) \qquad\qquad \mathfrak{G}_{\ell+1} = \mathfrak{G}'_{\ell+1} \amalg \mathfrak{G}''_{\ell+1}.$$

Then we may write

$$(\ell!)^2 \cdot A = B + C, \qquad \text{where}$$

$$
B := \sum_{\tau\in\mathfrak{G}'_{\ell+1}} \mathrm{sgn}(\tau)\Big( \sum_{\sigma\in\mathfrak{G}_{\ell+1}} \sum_{\nu=0}^{\ell+1} (-1)^\nu \mathrm{sgn}(\sigma)(\Phi_i)^{\sigma(\beta_0)}_{\beta_\nu} (\Psi_J)^{\sigma(\beta_1)\cdots\sigma(\beta_{\nu-1})\sigma(\beta_{\nu+1})\cdots\sigma(\beta_{\ell+1})}_{\tau(\alpha_1)\cdots\tau(\alpha_\ell)} \Big),
$$

$$
C := \sum_{\tau\in\mathfrak{G}''_{\ell+1}} \mathrm{sgn}(\tau)\Big( \sum_{\sigma\in\mathfrak{G}_{\ell+1}} \sum_{\nu=0}^{\ell+1} (-1)^\nu \mathrm{sgn}(\sigma)(\Phi_i)^{\sigma(\beta_0)}_{\overline{*}} (\Psi_J)^{\sigma(\beta_1)\cdots\sigma(\beta_{\nu-1})\sigma(\beta_{\nu+1})\cdots\sigma(\beta_{\ell+1})}_{\overline{*}\cdots\overline{\beta_\nu}\cdots\overline{*}} \Big).
$$

Here, each $*$ denotes some $\alpha_i$ determined by $\tau$. For a given $\nu$, by considering those $\sigma \in \mathfrak{G}_{\ell+1}$ such that $\sigma(\beta_0) = \beta_\mu$ $(\neq \beta_\nu)$, we have

$$B = \sum_{\tau \in \mathfrak{G}'_{\ell+1}} \mathrm{sgn}(\tau) \cdot \ell! \cdot \Big( \sum_{\nu=0}^{\ell+1} \sum_{\mu=0}^{\nu-1} (-1)^{\nu+\mu} (\Phi_i)_{\beta_\nu}^{\beta_\mu} (\Psi_J)_{\tau(\alpha_1)\cdots\tau(\alpha_\ell)}^{\beta_0\cdots\widehat{\beta_\mu}\cdots\widehat{\beta_\nu}\cdots\beta_{\ell+1}}$$

$$+ \sum_{\nu=0}^{\ell+1} \sum_{\mu=\nu+1}^{\ell+1} (-1)^{\nu+\mu-1} (\Phi_i)_{\beta_\nu}^{\beta_\mu} (\Psi_J)_{\tau(\alpha_1)\cdots\tau(\alpha_\ell)}^{\beta_0\cdots\widehat{\beta_\nu}\cdots\widehat{\beta_\mu}\cdots\beta_{\ell+1}} \Big).$$

From the symmetry property of $\Phi_i$ (cf. Lemma 1(iv)), it is easily to see that each term of the first double summation above is matched by a corresponding term of the second double summation, and it follows that $B = 0$. Similarly, one has

$$C = \sum_{\tau \in \mathfrak{G}''_{\ell+1}} \mathrm{sgn}(\tau) \cdot \ell! \cdot \sum_{\nu=0}^{\ell+1} \Big( \sum_{\mu=0}^{\nu-1} (-1)^{\nu+\mu} (\Phi_i)_{*}^{\beta_\mu} (\Psi_J)_{\overline{*}\cdots\overline{\beta_\nu}\cdots\overline{*}}^{\beta_0\cdots\widehat{\beta_\mu}\cdots\widehat{\beta_\nu}\cdots\beta_{\ell+1}}$$

$$+ \sum_{\mu=\nu+1}^{\ell+1} (-1)^{\nu+\mu-1} (\Phi_i)_{*}^{\beta_\mu} (\Psi_J)_{\overline{*}\cdots\overline{\beta_\nu}\cdots\overline{*}}^{\beta_0\cdots\widehat{\beta_\nu}\cdots\widehat{\beta_\mu}\cdots\beta_{\ell+1}} \Big)$$

$$= 0,$$

where the last equality follows from property (P-iii) for $\Psi_J$, upon re-grouping the terms with common factor $(\Phi_i)_{*}^{\beta_\mu}$. $\qquad\square$

**Lemma 11.** *We have*
*(i)* $\overline{D_2}^*(\Phi_i \oslash \Psi_J) = \overline{D_1}(\mathcal{L}_{v_i} \Psi_J),$
*(ii)* $\overline{\partial}(\Phi_i \oslash \Psi_J) = 0,$ *and*
*(iii)* $\overline{\partial}^*(\mathcal{L}_{v_i} \Psi_J) = 0.$

*Proof.* The proof of (i) is similar to [S2, p. 288], and the proof of (iii) is similar to [S2, p. 286]. (ii) follows from Remark 2(ii) and the $\overline{\partial}$-closedness of $\Phi_i$ and $\Psi_J$. $\qquad\square$

Now we are ready to give the proof of Proposition 3 by following the arguments as in [S2, p. 292-293].

*Proof of Proposition 3.* First we have

$$(7.2) \qquad (\mathcal{L}_{v_i} \Psi_J, \mathcal{L}_{v_i} \Psi_J)$$

$$= ((\square - k)(\square - k)^{-1}(\mathcal{L}_{v_i} \Psi_J), \mathcal{L}_{v_i} \Psi_J)$$

$$= (\square(\square - k)^{-1}(\mathcal{L}_{v_i} \Psi_J), \mathcal{L}_{v_i} \Psi_J) - k((\square - k)^{-1}(\mathcal{L}_{v_i} \Psi_J), \mathcal{L}_{v_i} \Psi_J).$$

Now we consider the first term on the right hand side of (7.2). Upon lowering indices, it is given by

$$
(\Box_1(\Box_1 - k)^{-1}(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J)
$$
$$
= \; ((\Box_1 - k)^{-1}\Box_1(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J)
$$
$$
= \; ((\Box_1 - k)^{-1}\overline{D_1}^*\overline{D_1}(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J)
$$
$$
\quad (\text{since } \overline{D_1}^*(\mathcal{L}_{v_i}\Psi_J) = 0 \text{ by Lemma 11(iii)})
$$
$$
= \; (\overline{D_1}^*(\Box_1 - k)^{-1}\overline{D_2}^*(\Phi_i \oslash \Psi_J), \mathcal{L}_{v_i}\Psi_J) \quad (\text{by Lemma 11(i)})
$$
$$
= \; ((\Box_1 - k)^{-1}\overline{D_2}^*(\Phi_i \oslash \Psi_J), \overline{D_1}(\mathcal{L}_{v_i}\Psi_J))
$$
$$
= \; (\overline{D_2}^* G_2(\Phi_i \oslash \Psi_J), \overline{D_2}^*(\Phi_i \oslash \Psi_J))
$$
$$
\quad (\text{by Lemma 10, Lemma 11(i), (ii) and Lemma 9(g)})
$$
$$
= \; (\overline{D_2 D_2}^* G_2(\Phi_i \oslash \Psi_J), \Phi_i \oslash \Psi_J)
$$
$$
= \; (\Box_2 G_2(\Phi_i \oslash \Psi_J), \Phi_i \oslash \Psi_J) \quad (\text{since } G_2 \overline{D_2} = \overline{D_2} G_2
$$
$$
\quad \text{and } \overline{D_2}(\Phi_i \oslash \Psi_J) = 0 \text{ (by Lemma 11(ii) and Lemma 10)})
$$
$$
= \; (\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J) - (H_1(\Phi_i \oslash \Psi_J), H_1(\Phi_i \oslash \Psi_J))
$$
$$
\quad (\text{since } H_2 = H_1 \text{ on } X^{(\ell+1)} \text{ by Remark 5(ii)}).
$$

Upon raising indices and together with (7.2), one obtains Proposition 3 readily. □

## 8. The curvature estimates

First we have the following

**Lemma 12.** *One has*

$$
(\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J) \;=\; (\overline{\Phi_i} \cdot \Psi_J, \overline{\Phi_i} \cdot \Psi_J) + (\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle)
$$
$$
- (\overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J) - (\overline{\Phi_i} \nearrow \Psi_J, \overline{\Phi_i} \nearrow \Psi_J).
$$

*Proof.* Recall that

$$
(\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J)
$$
$$
= \; \frac{1}{((\ell+1)!)^2} \int_{M_t} (\Phi_i \oslash \Psi_J)^{\alpha_1 \cdots \alpha_{\ell+1}}_{\beta_1 \cdots \beta_{\ell+1}} \overline{(\Phi_i \oslash \Psi_J)^{\beta_1 \cdots \beta_{\ell+1}}_{\alpha_1 \cdots \alpha_{\ell+1}}} \frac{\omega^n}{n!}
$$
$$
= \; \frac{1}{((\ell+1)!)^2} \int_{M_t} \sum_{\sigma,\tau,\sigma',\tau' \in \mathfrak{S}_{\ell+1}} \frac{\mathrm{sgn}(\sigma)\mathrm{sgn}(\tau)\mathrm{sgn}(\sigma')\mathrm{sgn}(\tau')}{(\ell!)^4}
$$
$$
\cdot (\Phi_i)^{\sigma(\alpha_1)}_{\tau(\beta_1)} (\Psi_J)^{\sigma(\alpha_2) \cdots \sigma(\alpha_{\ell+1})}_{\tau(\beta_2) \cdots \tau(\beta_{\ell+1})} \overline{(\Phi_i)^{\tau'(\beta_1)}_{\sigma'(\alpha_1)} (\Psi_J)^{\tau'(\beta_2) \cdots \tau'(\beta_{\ell+1})}_{\sigma'(\alpha_2) \cdots \sigma'(\alpha_{\ell+1})}} \frac{\omega^n}{n!}.
$$

By writing $\sigma' = \sigma'' \circ \sigma$, $\tau' = \tau'' \circ \tau$ (so that $\text{sgn}(\sigma)\text{sgn}(\sigma') = \text{sgn}(\sigma'')$, etc), one easily sees that

$$(\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J)$$

$$= \frac{1}{((\ell+1)!)^2} \int_{M_t} \sum_{\sigma,\tau \in \mathfrak{S}_{\ell+1}} (\Phi_i)^{\sigma(\alpha_1)}_{\tau(\beta_1)} (\Psi_J)^{\sigma(\alpha_2)\cdots\sigma(\alpha_{\ell+1})}_{\tau(\beta_2)\cdots\tau(\beta_{\ell+1})}$$

$$\cdot \Big( \sum_{\sigma'',\tau'' \in \mathfrak{S}_{\ell+1}} \frac{\text{sgn}(\sigma'')\text{sgn}(\tau'')}{(\ell!)^4} \overline{(\Phi_i)^{\tau''(\tau(\beta_1))}_{\sigma''(\sigma(\alpha_1))} (\Psi_J)^{\tau''(\tau(\beta_2))\cdots\tau''(\tau(\beta_{\ell+1}))}_{\sigma''(\sigma(\alpha_2))\cdots\sigma''(\sigma(\alpha_{\ell+1}))}} \Big) \frac{\omega^n}{n!}$$

$$= \int_{M_t} (\Phi_i)^{\alpha_1}_{\beta_1} (\Psi_J)^{\alpha_2\cdots\alpha_{\ell+1}}_{\beta_2\cdots\beta_{\ell+1}}$$

$$\cdot \Big( \sum_{\sigma'',\tau'' \in \mathfrak{S}_{\ell+1}} \frac{\text{sgn}(\sigma'')\text{sgn}(\tau'')}{(\ell!)^4} \overline{(\Phi_i)^{\tau''(\beta_1)}_{\sigma''(a_1)} (\Psi_J)^{\tau''(\beta_2)\cdots\tau''(\beta_{\ell+1})}_{\sigma''(\alpha_2)\cdots\sigma''(\alpha_{\ell+1}))}} \Big) \frac{\omega^n}{n!}$$

(from symmetry of the expression in $\sigma, \tau$).

Next we consider the partition $\mathfrak{S}_{\ell+1} = \mathfrak{S}'_{\ell+1} \amalg \mathfrak{S}''_{\ell+1}$ as given in (7.1). Then we may write

$$(\Phi_i \oslash \Psi_J, \Phi_i \oslash \Psi_J) = I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}'_{\ell+1})} + I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}''_{\ell+1})} + I_{(\mathfrak{S}''_{\ell+1}, \mathfrak{S}'_{\ell+1})} + I_{(\mathfrak{S}''_{\ell+1}, \mathfrak{S}''_{\ell+1})},$$

where

$$I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}'_{\ell+1})} := \int_{M_t} (\Phi_i)^{\alpha_1}_{\beta_1} (\Psi_J)^{\alpha_2\cdots\alpha_{\ell+1}}_{\beta_2\cdots\beta_{\ell+1}}$$

$$\cdot \Big( \sum_{\sigma \in \mathfrak{S}'_{\ell+1}} \sum_{\tau \in \mathfrak{S}'_{\ell+1}} \frac{\text{sgn}(\sigma)\text{sgn}(\tau)}{(\ell!)^4} \overline{(\Phi_i)^{\tau(\beta_1)}_{\sigma(a_1)} (\Psi_J)^{\tau(\beta_2)\cdots\tau(\beta_{\ell+1})}_{\sigma(\alpha_2)\cdots\sigma(\alpha_{\ell+1})}} \Big) \frac{\omega^n}{n!},$$

$$I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}''_{\ell+1})} := \int_{M_t} (\Phi_i)^{\alpha_1}_{\beta_1} (\Psi_J)^{\alpha_2\cdots\alpha_{\ell+1}}_{\beta_2\cdots\beta_{\ell+1}}$$

$$\cdot \Big( \sum_{\sigma \in \mathfrak{S}'_{\ell+1}} \sum_{\tau \in \mathfrak{S}''_{\ell+1}} \frac{\text{sgn}(\sigma)\text{sgn}(\tau)}{(\ell!)^4} \overline{(\Phi_i)^{\tau(\beta_1)}_{\sigma(a_1)} (\Psi_J)^{\tau(\beta_2)\cdots\tau(\beta_{\ell+1})}_{\sigma(\alpha_2)\cdots\sigma(\alpha_{\ell+1})}} \Big) \frac{\omega^n}{n!},$$

and $I_{(\mathfrak{S}''_{\ell+1}, \mathfrak{S}'_{\ell+1})}$, $I_{(\mathfrak{S}''_{\ell+1}, \mathfrak{S}''_{\ell+1})}$ are defined similarly. Note that $|\mathfrak{S}'_{\ell+1}| = \ell!$ and $|\mathfrak{S}''_{\ell+1}| = \ell \cdot \ell!$. Now for each $\sigma \in \mathfrak{S}'_{\ell+1}$, (as a permutation on $(\alpha_1, \ldots, \alpha_{\ell+1})$), one has $\sigma(\alpha_1) = \alpha_1$ and $\text{sgn}(\sigma) = \text{sgn}(\sigma|_{(\alpha_2,\cdots,\alpha_{\ell+1})})$. Thus,

$$I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}'_{\ell+1})} := \frac{1}{(\ell!)^4} \int_{M_t} (\Phi_i)^{\alpha_1}_{\beta_1} (\Psi_J)^{\alpha_2\cdots\alpha_{\ell+1}}_{\alpha_2\cdots\alpha_{\ell+1}}$$

$$\cdot \Big( \sum_{\sigma \in \mathfrak{S}'_{\ell+1}} \sum_{\tau \in \mathfrak{S}'_{\ell+1}} \overline{(\Phi_i)^{\beta_1}_{a_1} (\Psi_J)^{\beta_2\cdots\beta_{\ell+1}}_{\beta_1\cdots\beta_{\ell+1}}} \Big) \frac{\omega^n}{n!}$$

$$= \frac{1}{(\ell!)^4} \cdot (\ell!)^2 \int_{M_t} (\Phi_i)^{\alpha_1}_{\beta_1} \overline{(\Phi_i)^{\beta_1}_{a_1}} \cdot (\Psi_J)^{\alpha_2\cdots\alpha_{\ell+1}}_{\beta_2\cdots\beta_{\ell+1}} \overline{(\Psi_J)^{\beta_2\cdots\beta_{\ell+1}}_{\alpha_2\cdots\alpha_{\ell+1}}} \Big) \frac{\omega^n}{n!}$$

$$= (\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \quad \text{(cf. (3.8) and (3.9))}.$$

Next we consider $I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}''_{\ell+1})}$. For each $\sigma \in \mathfrak{S}'_{\ell+1}$ and $\tau \in \mathfrak{S}''_{\ell+1}$ (so that $\sigma(\alpha_1) = \alpha_1$ and $\tau(\beta_1) = \beta_\mu$ with $\mu \neq 1$), one easily sees as before that

$$(\Phi_i)^{\alpha_1}_{\beta_1}(\Psi_J)^{\alpha_2 \cdots \alpha_{\ell+1}}_{\beta_2 \cdots \beta_{\ell+1}} \cdot \mathrm{sgn}(\sigma)\mathrm{sgn}(\tau) \cdot \overline{(\Phi_i)^{\tau(\beta_1)}_{\sigma(a_1)}(\Psi_J)^{\tau(\beta_2) \cdots \tau(\beta_{\ell+1})}_{\sigma(\alpha_2) \cdots \sigma(\alpha_{\ell+1})}}$$

$$= \quad (-1)^{\mu-2}(\Phi_i)^{\alpha_1}_{\beta_1}(\Psi_J)^{\alpha_2 \cdots \alpha_{\ell+1}}_{\beta_\mu \beta_2 \cdots \widehat{\beta_\mu} \cdots \beta_{\ell+1}} \cdot (-1)^{\mu-1} \cdot \overline{(\Phi_i)^{\beta_\mu}_{\alpha_1}(\Psi_J)^{\beta_1 \cdots \widehat{\beta_\mu} \cdots \beta_{\ell+1}}_{\alpha_2 \cdots \alpha_{\ell+1}}}$$

$$= \quad -\ell!(\ell-1)!\langle \overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J \rangle \quad (\text{cf. (3.8) and (5.10)}).$$

Thus,

$$\begin{aligned} I_{(\mathfrak{S}'_{\ell+1}, \mathfrak{S}''_{\ell+1})} &:= \quad \frac{1}{(\ell!)^4} \cdot \ell! \cdot (\ell \cdot \ell!) \cdot (-\ell!(\ell-1)!) \cdot (\langle \overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J \rangle) \\ &= \quad -(\overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J). \end{aligned}$$

Similarly, one easily checks that

$$\begin{aligned} I_{(\mathfrak{S}''_{\ell+1}, \mathfrak{S}'_{\ell+1})} &= \quad -(\overline{\Phi_i} \nearrow \Psi_J, \overline{\Phi_i} \nearrow \Psi_J), \quad \text{and} \\ I_{(\mathfrak{S}''_{\ell+1}, \mathfrak{S}''_{\ell+1})} &= \quad (\overline{\Phi_i} \cdot \Psi_J, \overline{\Phi_i} \cdot \Psi_J), \end{aligned}$$

and the lemma follows readily.                                                   $\square$

**Proposition 4.** *We have*

$$\begin{aligned} &\partial_i \overline{\partial}_i \log \|\Psi_J\|_2^2 \\ =\ & \frac{1}{\|\Psi_J\|_2^2} \big( -k((\square-k)^{-1}(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) - k((\square-k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\ & \qquad -k((\square-k)^{-1}(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) - \big|(\mathcal{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\big|^2 \\ & \qquad -(H(\Phi_i \oslash \Psi_J), H(\Phi_i \oslash \Psi_J)) \big). \end{aligned}$$

*Proof.* The proposition follows readily by combining (4.6), (4.9), (4.11), (4.12), Proposition 1, Proposition 2, Proposition 3 and Lemma 12.     $\square$

**Proposition 5.** *We have*

$$(8.1) \quad \partial_i \overline{\partial}_i \log \|\Psi_J\|_2^2 \ \geqslant \ \frac{1}{\|\Psi_J\|_2^2} \big( -k((\square-k)^{-1}(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) \\ -k((\square-k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\ -(H(\Phi_i \oslash \Psi_J), H(\Phi_i \oslash \Psi_J)) \big).$$

*Proof.* By considering the spectral decomposition of $\mathcal{L}_{v_i}\Psi_J$ with respect to $\square$, one easily sees that
$$(8.2)$$
$$-k((\square-k)^{-1}(\mathcal{L}_{v_i}\Psi_J), \mathcal{L}_{v_i}\Psi_J) \geqslant (H(\mathcal{L}_{v_i}\Psi_J), H(\mathcal{L}_{v_i}\Psi_J)) = \|H(\mathcal{L}_{v_i}\Psi_J)\|_2^2.$$

On the other hand, since $\Psi_J$ is harmonic, one clearly has

$$(8.3) \qquad \|H(\mathcal{L}_{v_i}\Psi_J)\|_2 \geqslant \left|(\mathcal{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\right|.$$

By combining (8.2), (8.3) and Proposition 4, one obtains Proposition 5 easily.
$\square$

For a positive integer $\ell$, we define the relative tensor

$$(8.4) \qquad H^{(\ell)} := H(\underbrace{\Phi_i \oslash \cdots \oslash \Phi_i}_{\ell-\text{times}}),$$

so that $H^{(\ell)} = \Psi_J$ with $J$ given by the $\ell$-tuple $(i, i, \cdots, i)$. Note that $H^{(\ell)}$ actually depends on $i$, but for simplicity, this is suppressed in the notation.

**Lemma 13.** *For each $\ell \geq 1$, one has*
*(i)  $H(\Phi_i \oslash H^{(\ell-1)}) = H^{(\ell)}$.*
*(ii)  $(\overline{\Phi_i} \cdot H^{(\ell)}, H^{(\ell-1)}) = \|H^{(\ell)}\|_2^2$.*
*Here we adopt the convention that $H^{(0)}$ is the constant function 1.*

*Proof.* From (5.4), (3.6) and the definition that $H^{(0)}$ is the constant function 1, one easily sees that $\Phi_i \oslash H^{(\ell-1)} - \underbrace{\Phi_i \oslash \cdots \oslash \Phi_i}_{\ell-\text{times}}$ is $\overline{\partial}$-exact (resp. zero) when $\ell \geq 2$ (resp. $\ell = 1$ (cf. Remark 2(ii))). Together with the fact that the harmonic projection of a $\overline{\partial}$-exact form is zero, one obtains (i) immediately. Next we see from Lemma 4 that

$$(8.5) \qquad \langle \overline{\Phi_i} \cdot H^{(\ell)}, H^{(\ell-1)} \rangle = \langle H^{(\ell)}, \Phi_i \oslash H^{(\ell-1)} \rangle.$$

Upon integrating both sides of (8.5) over $M_t$ and using the harmonicity of $H^{(\ell)}$ and (i), one obtains (ii) immediately. $\square$

**Remark 6.** *One easily sees from Lemma 13(i) that on $M_t$, $H^{(\ell)} \equiv 0 \implies H^{(\ell')} \equiv 0$ for all $\ell' > \ell$. Equivalently, $\|H^{(\ell)}\|_2 = 0 \implies \|H^{(\ell')}\|_2 = 0$ for all $\ell' > \ell$.*

Now we state the main result in this section which is to be used later on to prove hyperbolicity of $S$.

**Proposition 6.** *Let $i, \ell, H^{(\ell)}$ be as in (8.4). Suppose $\|H^{(\ell)}\|_2 > 0$ (so that $\|H^{(\ell-1)}\|_2 > 0$ (cf. Remark 6)). Then we have*

$$(8.6) \qquad \partial_i \overline{\partial_i} \log \|H^{(\ell)}\|_2^2 \geq \frac{\|H^{(\ell)}\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2}.$$

**Remark 7.** *As in Lemma 13, we adopt in Proposition 6 the convention that $H^{(0)} \equiv 1$, so that by (2.1), $\|H^{(0)}\|_2^2 = Vol(M_t) = \frac{(2\pi)^n K_{M_t}^n}{k^n n!}$, which is independent of $t \in S$ (since $K_{M_t}^n$ is determined by a fixed class in $H^{2n}(M, \mathbb{Z})$, where $M$ is the underlying topological manifold of the $M_t$'s).*

*Proof of Proposition 6.* We are going to apply Proposition 5 (with $\Psi_J$ given by $H^{(\ell)}$). First we consider the second term on the right hand side of (8.1). By Lemma 8 (and with $v_i$ as given there), one has

$$(8.7) \quad -k((\square - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle H^{(\ell)}, H^{(\ell)} \rangle) = -k(\langle v_i, v_i \rangle, \langle H^{(\ell)}, H^{(\ell)} \rangle) \geq 0,$$

since the integrand is pointwise non-negative on $M_t$. (In fact, since $\langle \Phi_i, \Phi_i \rangle$ is a nonnegative-valued and non-identically-zero real-analytic function on $M_t$, it follows from the arguments in [Siu2, p. 297-298] that $-k((\square - k)^{-1}\langle \Phi_i, \Phi_i \rangle$ is also a nonnegative-valued and non-identically-zero real-analytic function; and together with such property of $\langle H^{(\ell)}, H^{(\ell)} \rangle$, one easily sees that $-k((\square - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle H^{(\ell)}, H^{(\ell)} \rangle) > 0$.) For the last term of (8.1), we also note from Lemma 13(i) that $H(\Phi_i \oslash H^{(\ell)}) = H^{(\ell+1)}$. Thus Proposition 5 implies that

$$(8.8) \quad \partial_i \overline{\partial_i} \log \|H^{(\ell)}\|_2^2$$
$$\geq \frac{1}{\|H^{(\ell)}\|_2^2}\Big( -k((\square - k)^{-1}(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) - \|H^{(\ell+1)}\|_2^2 \Big).$$

For the first term above, we note that $\overline{\Phi_i} \cdot H^{(\ell)}$ and $H^{(\ell-1)}$ are both in $\mathcal{A}^{0,\ell-1}(\wedge^{\ell-1}TM_t)$, and $H^{(\ell-1)}$ is harmonic. Now we take an orthonormal basis of $\mathcal{A}^{0,\ell-1}(\wedge^{\ell-1}TM_t)$ consisting of eigensections of $\square$ and with $\frac{1}{\|H^{(\ell-1)}\|_2}H^{(\ell-1)}$ as one of the (harmonic) basis elements. Then by considering the spectral decomposition of $\overline{\Phi_i} \cdot H^{(\ell)}$ with respect to $\square$, one easily sees that

$$-k((\square - k)^{-1}(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) \geq \Big| \Big(\overline{\Phi_i} \cdot H^{(\ell)}, \frac{H^{(\ell-1)}}{\|H^{(\ell-1)}\|_2}\Big) \Big|^2$$
$$= \frac{\|H^{(\ell)}\|_2^4}{\|H^{(\ell-1)}\|_2^2} \quad \text{(by Lemma 13(ii))}.$$

Together with (8.8), one obtains (8.6) readily.                      $\square$

**Remark 8.** *We remark that in the special case when $\ell = 1$, the first two terms of (8.1) coincide. Then one easily sees from the proof of Proposition 6 that (8.6) in this special case can be strengthened so that the following inequality holds:*

$$(8.9) \qquad \partial_i \overline{\partial_i} \log \|H^{(1)}\|_2^2 \geq 2 \cdot \frac{\|H^{(1)}\|_2^2}{\|H^{(0)}\|_2^2} - \frac{\|H^{(2)}\|_2^2}{\|H^{(1)}\|_2^2}.$$

*In the case of families of Riemann surfaces (i.e., when $n = 1$), one easily checks that (8.9) leads readily to the same upper bound given by Wolpert [Wo, Lemma 4.6] and mentioned in the beginning of this article. For the sake of coherence and clarity in our subsequent discussion, we will only use (8.6) (without incorporating (8.9)), which will already be sufficient for our purpose.*

## 9. Finsler metric and Kobayashi hyperbolicity

Let $\pi : \mathcal{X} \to S$ be an effectively parametrized family of canonically polarized manifolds as in Theorem 1. As before, we let $M_t = \pi^{-1}(M_t)$ for $t \in S$, and denote $n = \dim_{\mathbb{C}} M_t$ and $m = \dim_{\mathbb{C}} S$. Without loss of generality, we assume that $n \geq 2$. We are going to construct a (non-degenerate) Finsler metric on $S$, whose holomorphic sectional curvature is bounded above by a negative constant. This will establish the Kobayashi hyperbolicity of $S$ readily. First we make some preparations.

Let $N$ be a fixed positive integer satisfying $N \geq n$, and recall from Remark 7 the following constant (independent of $t$) given by

$$(9.1) \qquad A := \mathrm{Vol}(M_t) = \frac{(2\pi)^n K_{M_t}^n}{k^n n!}.$$

We first consider the following two sequences of positive numbers $\{C_\ell\}_{1 \leq \ell \leq n}$ and $\{a_\ell\}_{1 \leq \ell \leq n}$ given by

$$(9.2) \qquad C_1 := \min\left\{1, \frac{1}{A}\right\}, \quad C_\ell = \frac{C_{\ell-1}}{3} = \frac{C_1}{3^{\ell-1}}, \quad 2 \leq \ell \leq n,$$

$$(9.3) \qquad a_1 := 1, \quad a_\ell = \left(\frac{3a_{\ell-1}}{C_1}\right)^N = \left(\frac{3}{C_1}\right)^{\frac{N(N^{\ell-1}-1)}{N-1}}, \quad 2 \leq \ell \leq n.$$

**Lemma 14.** *Let $N \geq n \geq 2$, $A$ and $\{C_\ell\}_{1 \leq \ell \leq n}$ and $\{a_\ell\}_{1 \leq \ell \leq n}$ be as above, and let $\kappa$ be an integer satisfying $1 \leq \kappa \leq n$. Then for all real numbers $x_1, \cdots, x_\kappa > 0$, we have*

$$(9.4) \qquad \frac{a_1 x_1^{N+1}}{A} + \sum_{\ell=2}^{\kappa} \left(\frac{a_\ell}{\ell} \cdot \frac{x_\ell^{N+\ell}}{x_{\ell-1}^{\ell-1}} - \frac{a_{\ell-1}}{\ell-1} \cdot x_{\ell-1}^{N-\ell+1} x_\ell^\ell\right) \geq C_\kappa \cdot \sum_{\ell=1}^{\kappa} x_\ell^{N+1}.$$

*(When $\kappa = 1$, the first summation in (9.4) is understood to be zero.)*

*Proof.* We are going to prove the inequality in (9.4) by induction on $\kappa$. When $\kappa = 1$, (9.4) follows readily from the definition of $a_1$ and $C_1$. For $\kappa \geq 2$, let $T_\kappa$ denote the left hand side of (9.4). Then one has

$$(9.5) \qquad T_\kappa = T_{\kappa-1} + \nu_\kappa, \quad \text{where } \nu_\kappa := \frac{a_\kappa}{\kappa} \cdot \frac{x_\kappa^{N+\kappa}}{x_{\kappa-1}^{\kappa-1}} - \frac{a_{\kappa-1}}{\kappa-1} \cdot x_{\kappa-1}^{N-\kappa+1} x_\kappa^\kappa.$$

Together with the induction hypothesis (that (9.4) holds with $\kappa$ replaced by $\kappa - 1$), we have

$$(9.6) \qquad T_\kappa \geq C_{\kappa-1} \cdot \sum_{\ell=1}^{\kappa-1} x_\ell^{N+1} + \nu_\kappa.$$

First we consider the case when $x_\kappa \leq \mu_\kappa x_{\kappa-1}$, where $\mu_\kappa := \left(\frac{(\kappa-1)C_\kappa}{a_{\kappa-1}}\right)^{\frac{1}{\kappa}}$. From (9.2) and (9.3), one easily sees that $\mu_\kappa \leq 1$. Thus in this case, we have, from (9.5) and the definition of $\mu_\kappa$,

$$\nu_\kappa \geq -\frac{a_{\kappa-1}}{\kappa-1} \cdot x_{\kappa-1}^{N-\kappa+1} x_\kappa^\kappa \geq -\frac{a_{\kappa-1}}{\kappa-1} \cdot \mu_\kappa^\kappa x_{\kappa-1}^{N+1} = -C_\kappa x_{\kappa-1}^{N+1} \quad \text{and} \quad x_{\kappa-1} \geq x_\kappa.$$

Together with (9.6) and the equality $C_{\ell-1} = 3C_\ell$ (cf. (9.2)), we have,

$$T_\kappa \geq 3C_\kappa \cdot \sum_{\ell=1}^{\kappa-1} x_\ell^{N+1} - C_\kappa x_{\kappa-1}^{N+1} \geq C_\kappa \cdot \sum_{\ell=1}^{\kappa} x_\ell^{N+1},$$

where the last inequality follows from the inequality $x_{\kappa-1} \geq x_\kappa$ (so that $C_\kappa x_{\ell-1}^{N+1} \geq C_\kappa x_\ell^{N+1}$). Now it remains to consider the other case when $x_\kappa \geq \mu_\kappa x_{\kappa-1}$. Substituting this into (9.5), one gets

$$(9.7) \qquad \nu_\kappa - C_\kappa x_\kappa^{N+1} \geq \left(\frac{a_\kappa}{\kappa} \cdot \mu_\kappa^{\kappa-1} - \frac{a_{\kappa-1}}{\kappa-1} \cdot \frac{1}{\mu_\kappa^{N-\kappa+1}} - C_\kappa\right) x_\kappa^{N+1}$$

$$= \left(a_\kappa - \frac{\kappa a_{\kappa-1}}{\kappa-1} \cdot \frac{1}{\mu_\kappa^N} - \frac{\kappa C_\kappa}{\mu_\kappa^{\kappa-1}}\right) \frac{\mu_\kappa^{\kappa-1} x_\kappa^{N+1}}{\kappa}.$$

Since $\mu_\kappa \leq 1$, we have, from the definition of $\mu_\kappa$,

$$a_\kappa - \frac{\kappa a_{\kappa-1}}{\kappa-1} \cdot \frac{1}{\mu_\kappa^N} - \frac{\kappa C_\kappa}{\mu_\kappa^{\kappa-1}} \geq a_\kappa - \frac{\kappa a_{\kappa-1}}{\kappa-1} \cdot \left(\frac{a_{\kappa-1}}{(\kappa-1)C_\kappa}\right)^{\frac{N}{\kappa}} - \frac{\kappa C_\kappa}{\mu_\kappa^\kappa}$$

$$\geq a_\kappa - \frac{\kappa a_{\kappa-1}}{\kappa-1} \cdot \left(\frac{3^{\kappa-1} a_{\kappa-1}}{(\kappa-1)C_1}\right)^{\frac{N}{\kappa}} - \frac{\kappa a_{\kappa-1}}{\kappa-1}$$

$$\geq a_\kappa - \frac{3^N a_{\kappa-1}^N}{C_1^N}$$

$$\geq 0 \quad \text{(by (9.2))},$$

where the second last inequality can be verified readily by using the inequalities $N \geq n \geq \kappa \geq 2$ and $a_{\kappa-1} \geq 1 \geq C_1$. Substituting this into (9.7), we get $\nu_\kappa \geq C_\kappa x_\kappa^{N+1}$. Together with (9.6) and the fact that $C_{\kappa-1} \geq C_\kappa$, one obtains (9.4) readily. $\qquad\square$

The following lemma is well-known and follows from a straightforward computation (see also [Sch2, Lemma 8]).

**Lemma 15.** *Let $U$ be a complex manifold, and $\phi_\ell$, $1 \le \ell \le r$, be positive $C^2$ functions on $U$. Then*

$$(9.8) \qquad \sqrt{-1}\partial\overline{\partial}\log\Big(\sum_{\ell=1}^{r}\phi_\ell\Big) \ge \frac{\sum_{\ell=1}^{r}\phi_\ell\sqrt{-1}\partial\overline{\partial}\log\phi_\ell}{\sum_{j=1}^{r}\phi_j}.$$

From now on, we fix $N = n!$, and let $\{C_\ell\}_{1\le\ell\le n}$ and $\{a_\ell\}_{1\le\ell\le n}$ be the corresponding sequences as given in (9.2) and (9.3). Now we define a function $h : TS \to \mathbb{R}$ given by

$$(9.9) \qquad h(u) = \Big(\sum_{\ell=1}^{n} a_\ell \|u\|_{WP,\ell}^{2N}\Big)^{\frac{1}{2N}} \quad \text{for } u \in T_tS \text{ and } t \in S.$$

Here $\| \ \|_{WP,\ell}$ is as defined in (3.11).

**Lemma 16.** *$h$ is a $Aut(\pi)$-invariant $C^\infty$ Finsler metric on $S$.*

**Proof.** It is obvious that $h(cu) = |c|h(u)$ for all $c \in \mathbb{C}$ and $u \in TS$. Moreover, one sees from Remark 3 that $h(u) > 0$ if $u \ne 0$. Thus $h$ is a Finsler metric on $S$. Next we note that the $Aut(\pi)$-invariance of $h$ follows readily from that of the $\| \ \|_{WP,\ell}$'s (cf. Remark 3). To verify the smoothness of $h$, we take a $C^\infty$ local section $u$ of $TS\big|_U$ over an open subset $U$ of $S$ such that $u_t \ne 0$ for each $t \in U$ (here $u_t$ denotes the value of $u$ at $t$), then for each $1 \le \ell \le n$, $\|u_t\|_{WP,\ell}^{2\ell}$ is a $C^\infty$ function in $t$, since it is given in (3.10) as an integral with the integrand varying smoothly in $t$. For each integer $\ell$ satisfying $1 \le \ell \le n$, since $N/\ell = n!/\ell$ is still a positive integer, it follows that $\|u_t\|_{WP,\ell}^{2N} = (\|u_t\|_{WP,\ell}^{2\ell})^{\frac{N}{\ell}}$ is still a $C^\infty$ function in $t$ (even at points $t$ where $\|u_t\|_{WP,\ell}^{2N} = 0$). Together with the fact that $h(u_t) > 0$ for each $t \in U$, it follows readily that $h(u_t)$ is a $C^\infty$ function in $t$. $\qquad\square$

**Remark 9.** *From the proof of Lemma 16, it is easy to see that as long as the positive integer $N$ in (9.9) is divisible by $1, 2, \cdots, n$, the resulting Finsler metric is still $C^\infty$.*

Let $u \in TS$ and $\ell$ be an integer satisfying $1 \le \ell \le n$. Similar to (8.4), we denote

$$(9.10) \qquad H^{(\ell)}(u) := H(\underbrace{\Phi(u) \oslash \cdots \oslash \Phi(u)}_{\ell-\text{times}}),$$

where $\Phi(u)$ is the harmonic representative of $\rho_t(u)$ as in Section 2. This gives rise to a function $r : \mathbb{P}TS \to \mathbb{Z}$ given by

$$(9.11) \qquad r([u]) := \max\{\ell \,\big|\, H^{(\ell)}(u) \ne 0\} \quad \text{for } 0 \ne u \in TS,$$

where $[u]$ denotes the class of $u$ in $\mathbb{P}TS$. Since $\rho_t$ is injective for each $t \in S$, it follows that $1 \leq r([u]) \leq n$ for each $[u] \in \mathbb{P}TS$. Now we let $R$ be a local one-dimensional complex submanifold of $S$. Then it is easy to see that $r$ induces a function $r_R : R \to \mathbb{Z}$ given by

$$(9.12) \qquad r_R(t) := r([u_t]) \quad \text{for } t \in R,$$

where $u_t$ is any non-zero vector in $T_t R$. Let $\kappa$ be an integer satisfying $1 \leq \kappa \leq n$. We say that a point $t_o \in R$ is a $\kappa$-*stable point of $R$* if there exists an open neighborhood $\mathcal{U}_{t_o}$ of $t_o$ in $R$ such that $r_R(t) = \kappa$ for all $t \in \mathcal{U}_{t_o}$. We also recall that the sectional curvature $K(R, h\big|_R)(t_o)$ of $h\big|_R$ at a point $t_o \in R$ is given by

$$(9.13) \qquad K(R, h\big|_R)(t_o) = -\frac{\partial_t \partial_{\bar{t}} \log((h(\frac{\partial}{\partial t}))^2)}{(h(\frac{\partial}{\partial t}))^2}\bigg|_{t=t_o},$$

where $t$ denotes a local holomorphic coordinate function on some open subset of $R$ containing $t_o$.

**Proposition 7.** *Let $R$ be a local one-dimensional complex submanifold of $S$, and let $t_o \in R$ be a $\kappa$-stable point of $R$ for some integer $1 \leq \kappa \leq n$. Let $h$ be the Finsler metric on $S$ as given in (9.9). Then*

$$K(R, h\big|_R)(t_o) \leq -\frac{C_\kappa}{\kappa^{\frac{1}{N}} a_\kappa^{1+\frac{1}{N}}},$$

*where $a_\kappa$ and $C_\kappa$ are as in (9.3) and (9.2) (with $N = n!$).*

*Proof.* Since $t_o$ is a $\kappa$-stable point of $R$, there exists an open neighborhood $\mathcal{U}$ of $t_o$ in $R$ such that for all $0 \neq u \in T\mathcal{U}$, one has $H^{(\ell)}(u) = 0$ (and thus $\|u\|_{WP,\ell} = 0$) for all $\ell > \kappa$, and $H^{(\ell)}(u) \neq 0$ for all $1 \leq \ell \leq \kappa$ (cf. Remark 6). In particular, the Finsler metric $h$ on $\mathcal{U}$ can be written as

$$(9.14) \qquad h(u) = \Big( \sum_{\ell=1}^{\kappa} a_\ell \|u\|_{WP,\ell}^{2N} \Big)^{\frac{1}{2N}} \quad \text{for all } u \in T\mathcal{U}$$

(recalling that $N = n!$). Shrinking $\mathcal{U}$ if necessary, we may assume that $\mathcal{U}$ is an open coordinate subset of $R$ with coordinate function $t$. To compute (9.13), we denote, as in (8.4), the relative tensor on the fibers over $\mathcal{U}$ given by $H^{(\ell)} := H^{(\ell)}(\frac{\partial}{\partial t})$ for each $\ell$. Together with the definition of $\| \; \|_{WP,\ell}$ in (3.11), we may further rewrite (9.14) as

$$(9.15) \qquad h(\frac{\partial}{\partial t}) = \Big( \sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \Big)^{\frac{1}{2N}} \quad \text{on } \mathcal{U}.$$

Then we have

$$(9.16) \quad \partial_t \partial_{\bar t} \log((h(\tfrac{\partial}{\partial t}))^2) = \frac{1}{N} \cdot \partial_t \partial_{\bar t} \log\Big( \sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \Big)$$

$$\geq \frac{1}{N} \cdot \frac{\sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \cdot \partial_t \partial_{\bar t} \log\big( a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \big)}{\sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}}}$$

$$\text{(by Lemma 15)}$$

$$= \frac{B}{\sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}}}, \quad \text{where}$$

$$B := \sum_{\ell=1}^{\kappa} \frac{a_\ell}{\ell} \cdot \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \cdot \partial_t \partial_{\bar t} \log\big( \|H^{(\ell)}\|_2^2 \big).$$

By Proposition 6 (and with $\|H^{(0)}\|_2^2$ as there), we have

$$(9.17) \quad B \geq \sum_{\ell=1}^{\kappa} \frac{a_\ell}{\ell} \cdot \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \cdot \Big( \frac{\|H^{(\ell)}\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2} \Big)$$

$$= \frac{a_1 \|H^{(1)}\|_2^2}{\|H^{(0)}\|_2^2} + \sum_{\ell=2}^{\kappa} \Big( \frac{a_\ell}{\ell} \cdot \frac{\|H^{(\ell)}\|_2^{\frac{2(N+\ell)}{\ell}}}{\|H^{(\ell-1)}\|_2^2}$$

$$- \frac{a_{\ell-1}}{\ell-1} \cdot \|H^{(\ell-1)}\|_2^{\frac{2(N-\ell+1)}{\ell-1}} \cdot \|H^{(\ell)}\|_2^2 \Big)$$

$$\geq C_\kappa \cdot \sum_{\ell=1}^{\kappa} \|H^{(\ell)}\|_2^{\frac{2(N+1)}{\ell}},$$

where the second line is obtained by regrouping the terms of the first line (involving $H^{(\ell)}$ and $H^{(\ell-1)}$ for given $\ell$) and using that fact that $\|H^{(\kappa+1)}\|_2 = 0$, and the last inequality follows from Lemma 14 (with $x_\ell$ given here by $\|H^{(\ell)}\|_2^{\frac{2}{\ell}}$). By Hölder inequality and using the fact that $a_\ell \geq a_{\ell-1}$, we have

$$\sum_{\ell=1}^{\kappa} a_\ell \cdot \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \leq \Big( \sum_{\ell=1}^{\kappa} a_\ell^{N+1} \Big)^{\frac{1}{N+1}} \Big( \sum_{\ell=1}^{\kappa} \|H^{(\ell)}\|_2^{\frac{2(N+1)}{\ell}} \Big)^{\frac{N}{N+1}}$$

$$(9.18) \quad \Longrightarrow \sum_{\ell=1}^{\kappa} \|H^{(\ell)}\|_2^{\frac{2(N+1)}{\ell}} \geq \frac{1}{\big( \kappa a_\kappa^{N+1} \big)^{\frac{1}{N}}} \cdot \Big( \sum_{\ell=1}^{\kappa} a_\ell \|H^{(\ell)}\|_2^{\frac{2N}{\ell}} \Big)^{\frac{N+1}{N}}.$$

Combining (9.16), (9.17) and (9.18), we get

$$\partial_t \partial_{\bar t} \log((h(\tfrac{\partial}{\partial t}))^2) \geq \frac{C_\kappa}{\kappa^{\frac{1}{N}} a_\kappa^{1+\frac{1}{N}}} \cdot \Big( \sum_{\ell=1}^{\kappa} \|a_\ell H^{(\ell)}\|_2^{\frac{2N}{\ell}} \Big)^{\frac{1}{N}} = \frac{C_\kappa}{\kappa^{\frac{1}{N}} a_\kappa^{1+\frac{1}{N}}} \cdot (h(\tfrac{\partial}{\partial t}))^2,$$

where the last equality follows from (9.15). Together with (9.13), one obtains the proposition readily. $\qquad \square$

**Lemma 17.** *Let $R$ be a local one-dimensional complex submanifold of $S$, and let $\mathcal{Q}_R := \{t \in R \,|\, t \text{ is a } \kappa\text{-stable point of } R \text{ for some } 1 \leq \kappa \leq n\}$. Then $\mathcal{Q}_R$ is a dense subset of $R$ (with respect to the usual topology).*

*Proof.* We take a point $t_o \in R$ and an open neighborhood $U$ of $t_o$ in $R$. Since the function $r_R$ in (9.12) takes values in the discrete set $\{1, 2, \cdots, n\}$, $r_R\big|_U$ necessarily attains maximum value, say $\kappa$, at some point $t_1 \in U$ for some $1 \leq \kappa \leq n$. Now we take a smooth non-vanishing vector field $u_t$ on some open neighborhood of $t_1$ in $U$. Then it is easy to see that $H^{(\kappa)}(u_t)$ (as defined in (9.10)) varies smoothly in $t$. Since we also have $H^{(\kappa)}(u_{t_1}) \neq 0$ (as $r_R(t_1) = \kappa$), it follows that there exists some open neighborhood $V$ of $t_1$ in $U$ such that $H^{(\kappa)}(u_t) \neq 0$ (and thus $r_R(t) \geq \kappa$) for all $t \in V$. Together with the definition of $\kappa$ as the maximum value of $r_R\big|_U$, it follows that $r_R(t) = \kappa$ for all $t \in V$. Hence $t_1 \in \mathcal{Q}_R$. Since $t_o$ and $U$ are arbitrary, one concludes that $\mathcal{Q}_R$ is dense in $R$. $\qquad\square$

We are ready to give the proof of Theorem 1 as follows:

*Proof of Theorem 1.* Let $\pi : \mathcal{X} \to S$ be as in Theorem 1, and let $n := \dim M_t$. Let $h$ be as in (9.9). From Lemma 16, we know that $h$ is an $\mathrm{Aut}(\pi)$-invariant $C^\infty$ Finsler metric on $S$. Take a point $t \in S$, and let $R$ be a local one-dimensional complex submanifold of $S$ passing through $t$ (i.e. $t \in R$). By Lemma 17, there exists a sequence of points $\{t_j\}_{j=1}^\infty$ in $\mathcal{Q}_R$ such that $\lim_{j \to \infty} t_j = t$ in $R$. In particular, each $t_j$ is a $\kappa_j$-stable point of $R$ for some integer $\kappa_j$ satisfying $1 \leq \kappa_j \leq n$. Let $\{C_\kappa\}_{1 \leq \kappa \leq n}$ and $\{a_\kappa\}_{1 \leq \kappa \leq n}$ be as in (9.2) and (9.3) (with $N = n!$). By Proposition 7, we have, for each $j$,

$$K(R, h\big|_R)(t_j) \leq -\frac{C_{\kappa_j}}{\kappa_j^{\frac{1}{N}} a_{\kappa_j}^{1+\frac{1}{N}}} \leq -\frac{C_n}{n^{\frac{1}{N}} a_n^{1+\frac{1}{N}}},$$

where the last inequality follows from the facts that $C_\kappa$ decreases with $\kappa$ while $a_\kappa$ increases with $\kappa$. Together with the fact that $h\big|_R$ is $C^\infty$ (cf. Lemma 16), one concludes readily that

$$K(R, h\big|_R)(t) \leq -\frac{C_n}{n^{\frac{1}{N}} a_n^{1+\frac{1}{N}}},$$

where the above upper bound is a negative constant independent of $t$ and $R$. Hence the holomorphic sectional curvature of the Finsler metric $h$ on $S$ is bounded above by a negative constant. Finally it is well-known (and follows from standard arguments involving the usual Ahlfors lemma) that the existence of a Finsler metric $h$ on $S$ with the above curvature property implies readily that $S$ is Kobayashi hyperbolic (cf. e.g. [Kob, p. 112, Theorem 3.7.1]). $\qquad\square$

**Remark 10.** *We indicate here some underlying parallel ingredients in the respective approaches of [VZ] and this paper. We recall that by taking direct images of the exterior powers of the relative tangent bundle $T_{\mathcal{X}|S}$, one obtains the Higgs bundle $(\bigoplus_{i=0}^{n} R^i \pi_* \wedge^i T_{\mathcal{X}|S}, \bigoplus_{i=0}^{n} \rho_i)$, where the Higgs field $\rho_i : T_S \otimes R^i \pi_* \wedge^i T_{\mathcal{X}|S} \to R^{i+1} \pi_* \wedge^{i+1} T_{\mathcal{X}|S}$ is given by the Kodaira-Spencer map. For each $p \geq 0$, the composition of the $\rho_i$'s, $i = 0, 1, \cdots, p-1$, also gives rise the p-th iterated Kodaira-Spencer map $\rho^{(p)} : S^p T_S \to R^p \pi_* \wedge^p T_{\mathcal{X}|S}$ (see e.g. [VZ]). Denote by $p_0$ the maximal number such that $\rho^{(p_0)}$ is not the zero map on $S^p T_S$. Then as pointed out by one of the referees, a key ingredient in deriving the (Brody or Kobayashi) hyperbolicity of S is to show that the locally free part of the image $\mathcal{F}^{(p_0)} := \rho^{(p_0)}(S^{p_0} T_S) \subset R^{p_0} \pi_* \wedge^{p_0} T_{X/S}$ is negatively curved (in certain sense) with respect to certain Hermitian metric. In [VZ], the above Higgs bundle is embedded in a logarithmic system of Hodge bundles associated to the Hodge filtration of an auxiliary variation of polarized Hodge structures constructed by taking the middle dimensional relative de Rham cohomlogy on the cyclic cover of $\mathcal{X}$ ramified along a generic section of suitable multiple of the relative canonical sheaf. Under such embedding, $F^{(p_0)}$ lies in the kernel of the Kodaira-Spencer map from the corresponding Hodge bundle, and the kernel is negatively curved from a well-known curvature computation of Hodge metric by Griffiths (see e.g. [Gri]). In this paper, the corresponding ingredient is the negativity of the curvature of $F^{(p_0)}$ with respect to the $p_0$-th Weil-Petersson pseudometric, which can be seen readily from Proposition 6 (upon letting $\ell = p_0$ in Proposition 6 and noting that the last term of (8.6) is zero when $\ell = p_0$). To derive the Kobayashi hyperbolicity of S, one actually needs to consider all the components of the Higgs bundle from $i = 0$ to $i = p_0$, which is manifested in the Finsler metric in (9.9) (noting that the terms $\|u\|_{WP,\ell}$ in (9.9) are zero for all $\ell > p_0$).*

**Remark 11.** *Finally we give some retrospective remarks on the respective approaches of [Sch2] and this paper. As mentioned earlier, the curvature computation for the Weil-Petersson metric of a family of higher dimensional manifolds began with the paper of Siu in [S2], where Proposition 4 for $\ell = |J| = 1$ was formulated and proved. The result of Siu in [S2] was reformulated and reproved by Schumacher in [Sch1]. Both Proposition 4 of this paper and Theorem V of [Sch2] are generalizations of the result of [S2] to $\ell > 1$ following Siu's approach to various extent. This corresponds to the first step mentioned in the introduction. In this step, our formulation of Proposition 4 works directly for our purpose, and our approach and grouping of terms actually follow closely the original approach of [S2]. We provide*

*sufficient details to make the presentation clear and readily verifiable to the readers.*

*In our second step as described in Section 1, we have utilized the first term on the right hand side of the expression in Proposition 4 (i.e., $-k((\square - k)^{-1}(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J))$ to achieve the estimates in (8.6). This is crucial for us to start a telescopic argument to handle the bad term in (8.6) inductively on $\ell$ and set up the stage for the choice of constants for (9.10) in our third step. In contrast to our work here, [Sch2] utilizes the term corresponding to the second term on the right hand side of the expression in our Proposition 4 (i.e., $-k((\square - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle))$. As such it only leads to an upper bound of the holomorphic sectional curvature depending on the base point $x$ in the family, which yields a result on hyperbolicity only if the base manifold is compact.*

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