COHOMOLOGICAL ORIENTIFOLD DONALDSON-THOMAS INVARIANTS AS CHOW GROUPS

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Abstract. We establish a geometric interpretation of orientifold Donaldson-Thomas invariants of σ-symmetric quivers with involution. More precisely, we prove that the cohomological orientifold Donaldson-Thomas invariant is isomorphic to the rational Chow group of the moduli space of σ-stable self-dual quiver representations. As an application we prove that the Chow Betti numbers of moduli spaces of stable m-tuples in classical Lie algebras can be computed numerically. We also prove a cohomological wall-crossing formula relating semistable Hall modules for different stabilities.

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Introduction

There are by now a number of geometric interpretations of Donaldson-Thomas invariants of quivers. In various settings and levels of generality, see [16], [5], [22], [14], [21], [6]. In particular, in [14] it is proved that the cohomological Donaldson-Thomas invariant of a symmetric quiver is isomorphic to the rational Chow group of the moduli space of stable representations of the quiver. In this paper we establish an analogous result for orientifold Donaldson-Thomas invariants, giving the first geometric interpretation of these invariants. Roughly speaking, orientifold Donaldson-Thomas invariants are counting invariants of moduli spaces of quiver representations which have orthogonal or symplectic structure groups.

Date: May 21, 2016.

The slope μ Donaldson-Thomas invariant of a quiver Q with stability θ is defined in terms of the Hilbert-Poincaré series of the semistable cohomological Hall algebra $\mathcal{H}_{Q,\mu}^{\theta\text{-}ss}$ of Kontsevich and Soibelman [19]. The algebra $\mathcal{H}_{Q,\mu}^{\theta\text{-}ss}$ is a convolution algebra on the cohomology of the stack of semistable representations of slope μ. Understanding the finer structure of $\mathcal{H}_{Q,\mu}^{\theta\text{-}ss}$ has led to a better understanding of Donaldson-Thomas invariants. Similarly, when the quiver has an involution and duality structure (see Section 1.2) the orientifold Donaldson-Thomas invariant is defined in terms of the Hilbert-Poincaré series of the semistable cohomological Hall module $\mathcal{M}_{Q}^{\theta\text{-}ss}$, a left $\mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss}$-module structure on the cohomology of the stack of semistable self-dual representations [28].

Motivated by results of [14], in this paper we study a Chow theoretic version of $\mathcal{M}_{Q}^{\theta\text{-}ss}$, defined in the same way as $\mathcal{M}_{Q}^{\theta\text{-}ss}$ but with Chow groups used in place of cohomology groups. The resulting object $\mathcal{B}_{Q}^{\theta\text{-}ss}$ is a left-module over the Chow theoretic Hall algebra $\mathcal{A}_{Q,\mu=0}^{\theta\text{-}ss}$ introduced in [12]. While cohomological and Chow theoretic Hall algebras share many common features, one technical advantage of the latter is the existence of the localization exact sequence in Chow theory. Using the localization exact sequence, the problem of understanding $\mathcal{A}_{Q,\mu}^{\theta\text{-}ss}$ can be reduced to that of understanding the Chow groups of the Harder-Narasimhan strata of the affine variety of representations. This idea has been used extensively in the representation theory of quivers (see [24] for its introduction) and is central to the approach of [14]. In its original setting, the Harder-Narasimhan stratification was applied by Atiyah and Bott [1] in their study of two dimensional Yang-Mills theory with compact gauge group H. See also [20] for related work. The case in which H is a unitary group is analogous to the case of ordinary representations of quivers, but the framework is valid for arbitrary H. The approach of this paper is to import the method of Atiyah and Bott when H is an orthogonal or symplectic group. The result is the σ-Harder-Narasimhan stratification of the affine variety of self-dual representations. The main geometric properties of this stratification are summarized in Propositions 2.2 and 2.3, both of which are direct analogues of results in the ordinary case. The key result we prove using the σ-Harder-Narasimhan stratification is Proposition 2.5, which states that the mixed Hodge structure on the cohomology of the stack of semistable self-dual representations is pure of Hodge-Tate type. The proof proceeds by showing that the σ-Harder-Narasimhan stratification is equivariantly perfect in the sense of [1]. We use Proposition 2.5 to prove that the cycle map defines an isomorphism $\mathcal{B}_{Q}^{\theta\text{-}ss} \xrightarrow{\sim} \mathcal{M}_{Q}^{\theta\text{-}ss}$ which is compatible with the corresponding algebra isomorphisms $\mathcal{A}_{Q,\mu}^{\theta\text{-}ss} \xrightarrow{\sim} \mathcal{H}_{Q,\mu}^{\theta\text{-}ss}$ from [14]. See Theorem 3.1. We deduce from this result that the cycle map from the Chow group to the cohomology of the moduli space of σ-stable self-dual representations surjects onto the pure part; see Corollary 3.2. Theorem 3.4 establishes a wall-crossing formula for Chow theoretic Hall modules. More precisely, we show that as a graded vector space the trivial stability Hall module $\mathcal{B}_Q$ is determined by the module $\mathcal{B}_{Q}^{\theta\text{-}ss}$ and the algebras $\{\mathcal{A}_{Q,\mu}^{\theta\text{-}ss}\}_{\mu>0}$. At the level of Grothendieck groups, Theorem 3.4 recovers the motivic orientifold wall-crossing formula of [27].

In Section 4 we give some applications of the above results to Donaldson-Thomas theory. We prove in Theorem 4.1 that the cohomological orientifold Donaldson-Thomas invariant of a σ-symmetric quiver is isomorphic to the rational Chow group of the moduli space of σ-stable self-dual representations. An immediate corollary is that the orientifold integrality conjecture for σ-symmetric quivers holds; see Corollary 4.2. In Section 4.2 we restrict attention to the quivers L_m with one node and $m \geq 0$ loops. For particular choices of duality structures, moduli spaces of self-dual representations of L_m are moduli spaces of m-tuples in orthogonal or symplectic Lie algebras. It follows from Theorem 4.1, together with Reineke's computation of the Donaldson-Thomas invariants of L_m [25] and the freeness of $\mathcal{M}_{L_m}$ [28], that the Chow Betti numbers of moduli spaces of σ-stable self-dual representations of L_m can be computed numerically. As a final application, using Theorem 4.1 and the explicit shuffle description of the $\mathcal{M}_{L_m}$ from [28] we show in Theorem 4.3 that the Chow Betti numbers of moduli spaces of stable m-tuples in symplectic and odd orthogonal Lie algebras groups agree.

Notation. Throughout this paper we write $\otimes$ for $\otimes_{\mathbb{Q}}$, unless otherwise noted. All varieties/schemes are over the ground field $\mathbb{C}$. By the dimension of a smooth variety we always mean its complex dimension.

Acknowledgements. The authors would like to thank Ben Davison for helpful correspondences. This work grew out of a visit of M.Y. to the Hausdorff Center for Mathematics. He would like to thank Tobias Dyckerhoff for the invitation and support. H.F. was supported by the DFG SFB / Transregio 45 "Perioden, Modulräume und Arithmetik algebraischer Varietäten". M.Y. was been partially supported by the Research Grants Council of the Hong Kong SAR, China (GRF HKU 703712) and by HKU URC Conference Support for Teaching Staff (award no. 143414).

1. Background material

1.1. **Quiver representations.** Let $Q = (Q_0, Q_1)$ be a finite quiver and let $\Lambda_Q^+ = \mathbb{Z}_{\geq 0} Q_0$ be its monoid of dimension vectors. A representation (U, u) of Q is a finite dimensional Q_0-graded complex vector space $U = \bigoplus_{i \in Q_0} U_i$ together with a linear map $u_\alpha : U_i \to U_j$ for each arrow $i \xrightarrow{\alpha} j$. The dimension vector $\mathbf{dim}\, U \in \Lambda_Q^+$ of U is the tuple $(\dim U_i)_{i \in Q_0}$ and the total dimension $\dim U \in \mathbb{Z}_{\geq 0}$ is the dimension of U as a complex vector space. We also define the total dimension $\dim d$ of a dimension vector as $\sum_{i \in Q_0} d_i$.

The affine variety of representations of Q of dimension vector $d \in \Lambda_Q^+$ is

$$R_d = \bigoplus_{i \xrightarrow{\alpha} j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}).$$

The group $\mathsf{GL}_d = \prod_{i \in Q_0} \mathsf{GL}_{d_i}(\mathbb{C})$ acts linearly on R_d via change of basis. The stack $\mathbf{M}_d = [R_d / \mathsf{GL}_d]$ is the stack of representations of dimension vector d.

Write $\mathsf{Rep}_{\mathbb{C}}(Q)$ for the category of finite dimensional complex representations of Q. It is abelian and hereditary. The Euler form of $\mathsf{Rep}_{\mathbb{C}}(Q)$ descends to the bilinear form on $\Lambda_Q = \mathbb{Z} Q_0$ given by

$$\chi(d, d') = \sum_{i \in Q_0} d_i d'_i - \sum_{i \xrightarrow{\alpha} j} d_i d'_j.$$

1.2. **Self-dual quiver representations.** For an introduction to self-dual quiver representations see [7], or from the point of view of this paper, [27].

An involution of a quiver Q is a pair of involutions $\sigma : Q_k \to Q_k$, $k = 0, 1$, such that

(i) if $i \xrightarrow{\alpha} j$, then $\sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i)$, and
(ii) if $i \xrightarrow{\alpha} \sigma(i)$, then $\alpha = \sigma(\alpha)$.

A duality structure on (Q, σ) is a pair of functions $s : Q_0 \to \{\pm 1\}$ and $\tau : Q_1 \to \{\pm 1\}$ such that s is σ-invariant and $\tau_\alpha \tau_{\sigma(\alpha)} = s_i s_j$ for each arrow $i \xrightarrow{\alpha} j$.

Fix an involution and duality structure on Q. A self-dual representation of Q is a representation (M, m) with a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ having the following properties:

(i) M_i and M_j are orthogonal unless $i = \sigma(j)$,
(ii) the restriction of $\langle \cdot, \cdot \rangle$ to $M_i + M_{\sigma(i)}$ satisfies $\langle x, x' \rangle = s_i \langle x', x \rangle$, and
(iii) for each arrow $i \xrightarrow{\alpha} j$ the structure maps of M satisfy

$$\langle m_\alpha x, x' \rangle - \tau_\alpha \langle x, m_{\sigma(\alpha)} x' \rangle = 0, \qquad x \in M_i,\ x' \in M_{\sigma(j)}. \tag{1}$$

The dimension vector of a self-dual representation lies in the submonoid $\Lambda_Q^{\sigma,+} \subset \Lambda_Q^+$ of dimension vectors which are fixed by σ. Categorically, a choice of duality structure induces an exact contravariant functor $S : \mathsf{Rep}_{\mathbb{C}}(Q) \to \mathsf{Rep}_{\mathbb{C}}(Q)$ and an isomorphism of functors $\Theta : \mathbf{1}_{\mathsf{Rep}(Q)} \xrightarrow{\sim} S^2$ which satisfies $S(\Theta_U)\Theta_{S(U)} = \mathbf{1}_{S(U)}$, thus giving $\mathsf{Rep}_{\mathbb{C}}(Q)$ the structure of an abelian category with duality. In this language, a self-dual representation is a pair consisting of a representation M and an isomorphism $M \simeq S(M)$ which is symmetric with respect to Θ_M.

Let $e \in \Lambda_Q^{\sigma,+}$. Throughout the paper we will assume that e_i is even whenever $i \in Q_0^\sigma$ and $s_i = -1$. Up to isometry, the trivial representation $\mathbb{C}^e = \bigoplus_{i \in Q_0} \mathbb{C}^{e_i}$ then admits a unique self-dual structure. The affine variety of self-dual representations of dimension vector e can be identified with the subspace of R_e consisting of structure maps which satisfy equation (1). Explicitly, fixing partitions $Q_0 = Q_0^- \sqcup Q_0^\sigma \sqcup Q_0^+$ and $Q_1 = Q_1^- \sqcup Q_1^\sigma \sqcup Q_1^+$ such that Q_0^σ consists of the nodes fixed by σ and $\sigma(Q_0^-) = Q_0^+$, and similarly for Q_1, we have

$$R_e^\sigma \simeq \bigoplus_{i \xrightarrow{\alpha} j \in Q_1^+} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{e_i}, \mathbb{C}^{e_j}) \oplus \bigoplus_{i \xrightarrow{\alpha} \sigma(i) \in Q_1^\sigma} \mathrm{Bil}^{s_i \tau_\alpha}(\mathbb{C}^{e_i})$$

where $\mathrm{Bil}^{\pm 1}(\mathbb{C}^{e_i})$ denotes the vector space of symmetric $(+)$ or skew-symmetric $(-)$ bilinear forms on $\mathbb{C}^{e_i}$. The isometry group of $\mathbb{C}^e$ is

$$\mathsf{G}_e^\sigma \simeq \prod_{i \in Q_0^+} \mathsf{GL}_{e_i}(\mathbb{C}) \times \prod_{i \in Q_0^\sigma} \mathsf{G}_{e_i}^{s_i}$$

where

$$\mathsf{G}_{e_i}^{s_i} = \begin{cases} \mathsf{Sp}_{e_i}(\mathbb{C}), & \text{if } s_i = -1 \\ \mathsf{O}_{e_i}(\mathbb{C}), & \text{if } s_i = 1. \end{cases}$$

The stack of self-dual representations of dimension vector e is $\mathbf{M}_e^\sigma = [R_e^\sigma / \mathsf{G}_e^\sigma]$.

The categorical point of view makes it clear that for each representation U the vector space $\mathrm{Ext}^i(S(U), U)$ is naturally a $\mathbb{Z}_2$-representation. By $\mathrm{Ext}^i(S(U), U)^{\pm S}$ we denote the subspace of symmetric or skew-symmetric elements. The analogue of the Euler form in the self-dual setting is

$$\mathcal{E}(U) = \dim_{\mathbb{C}} \mathrm{Hom}(S(U), U)^{-S} - \dim_{\mathbb{C}} \mathrm{Ext}^1(S(U), U)^S.$$

This descends to a function $\mathcal{E} : \Lambda_Q \to \mathbb{Z}$ given by

$$\mathcal{E}(d) = \sum_{i \in Q_0^\sigma} \frac{d_i(d_i - s_i)}{2} + \sum_{i \in Q_0^+} d_{\sigma(i)} d_i - \sum_{\sigma(i) \xrightarrow{\alpha} i \in Q_1^\sigma} \frac{d_i(d_i + \tau_\alpha s_i)}{2} - \sum_{i \xrightarrow{\alpha} j \in Q_1^+} d_{\sigma(i)} d_j.$$

1.3. **Moduli spaces of quiver representations.** We recall the construction of moduli spaces of quiver representations using geometric invariant theory (GIT) [17]. Fix an element $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z})$, called a stability. A representation V of Q is called semistable if $\mu(U) \leq \mu(V)$ for all non-trivial subrepresentations $U \subset V$ and is called stable if this inequality is strict. Here $\mu(V) = \frac{\theta(\mathbf{dim}\, V)}{\dim V} \in \mathbb{Q}$ is the slope of V. There are GL_d-invariant open (possibly empty) subvarieties $R_d^{\theta\text{-}st} \subset R_d^{\theta\text{-}ss} \subset R_d$ of (semi)stable representations. The moduli scheme $\mathfrak{M}_d^{\theta\text{-}ss}$ of semistable representations is the GIT quotient $R_d /\!\!/_\theta \mathsf{GL}_d$, the stability θ determining the linearization of the group action. The stack $\mathbf{M}_e^{\theta\text{-}st} = [R_d^{\theta\text{-}st}/\mathsf{GL}_d]$ is a $\mathbb{C}^\times$-gerbe over the smooth moduli scheme $\mathfrak{M}_d^{\theta\text{-}st}$ of stable representations, which is an open subvariety of $\mathfrak{M}_d^{\theta\text{-}ss}$.

Suppose that Q has an involution and duality structure. In this case we will always assume that θ is σ-compatible, that is, $\sigma^*\theta = -\theta$. A self-dual representation M is called σ-semistable if $\mu(U) \leq \mu(M)$ for all isotropic subrepresentations $U \subset M$ and is called σ-stable if this inequality is strict. The slope of a self-dual representation is necessarily zero. The representation theoretic notion of σ-(semi)stability agrees with the corresponding notion in GIT [27, Theorem 3.7]. Hence there are G_e^σ-invariant open subschemes $R_e^{\sigma,\theta\text{-}st} \subset R_e^{\sigma,\theta\text{-}ss} \subset R_e^\sigma$. The stack $\mathbf{M}_e^{\sigma,\theta\text{-}st} = [R_e^{\sigma,\theta\text{-}st}/\mathsf{G}_e^\sigma]$ is a smooth Deligne-Mumford stack. The σ-stable moduli scheme $\mathfrak{M}_e^{\sigma,\theta\text{-}st}$ is thus an open subscheme of $\mathfrak{M}_e^{\sigma,\theta\text{-}ss} = R_e^\sigma /\!\!/_\theta \mathsf{G}_e^\sigma$ with at worst finite quotient singularities. By convention we set $\mathfrak{M}_0^{\sigma,\theta\text{-}ss} = \mathfrak{M}_0^{\sigma,\theta\text{-}st} = \mathsf{Spec}(\mathbb{C})$.

2. Harder-Narasimhan stratifications

2.1. **The σ-HN stratification.** Let Q be an arbitrary quiver with stability θ. Recall that each representation U has a unique Harder-Narasimhan (HN) filtration [24, Proposition 2.5]. This is an increasing filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_r = U$ with the property that the subquotients $U_1/U_0, \ldots, U_r/U_{r-1}$ are semistable and satisfy

$$\mu(U_1/U_0) > \mu(U_2/U_1) > \cdots > \mu(U_r/U_{r-1}).$$

Suppose now that Q has an involution and a duality structure and let M be a self-dual representation. If $U \subset M$ is an isotropic subrepresentation, then the orthogonal $U^\perp \subset M$ contains U as a subrepresentation and the quotient $U^\perp/U$ inherits from M a canonical self-dual structure. Denote by $M/\!\!/U$ the self-dual representation $U^\perp/U$.

Proposition 2.1 ([27, §3.1])**.** *Let θ be a σ-compatible stability.*

(1) A representation U is semistable of slope μ if and only if $S(U)$ is semistable of slope $-\mu$.

(2) A self-dual representation is σ-semistable if and only if it is semistable as an ordinary representation.

(3) Each self-dual representation M has a unique σ-HN filtration, that is, an isotropic filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset M \tag{2}$$

such that the subquotients $U_1/U_0, \ldots, U_r/U_{r-1}$ are semistable and satisfy

$$\mu(U_1/U_0) > \mu(U_2/U_1) > \cdots > \mu(U_r/U_{r-1}) > 0$$

and, if non-zero, $M/\!\!/U_r$ is σ-semistable.

Using Proposition 2.1, it is straightforward to show that if the σ-HN filtration of M is given by (2), then

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset U_r^\perp \subset \cdots \subset U_0^\perp = M \tag{3}$$

is the HN filtration of M (considered as an ordinary representation), after identifying U_r and $U_r^\perp$ if $M/\!\!/U_r$ is zero. Observe that the subquotients of the extended filtration (3) satisfy the symmetry conditions

$$S(U_i/U_{i-1}) \simeq U_{i-1}^\perp/U_i^\perp, \qquad i = 1, \ldots, r. \tag{4}$$

Definition. *Let $r \geq 0$. A tuple $(d^\bullet, e^\infty) \in (\Lambda_Q^+)^r \times \Lambda_Q^{\sigma,+}$ is called a σ-HN type if the following conditions hold:*

(1) each $d^1, \dots, d^r$ is non-zero,
(2) $\mu(d^1) > \cdots > \mu(d^r) > 0$, and
(3) each of $R_{d^1}^{\theta\text{-}ss}, \dots, R_{d^r}^{\theta\text{-}ss}$ and $R_{e^\infty}^{\sigma,\theta\text{-}ss}$ is non-empty.

The weight of $(d^\bullet, e^\infty)$ is defined to be $\sum_{j=1}^r (d^j + \sigma(d^j)) + e^\infty$.

Note that e^∞ is allowed to be zero in the above definition. We will write $\mathsf{HN}^\sigma(e)$ for the set of σ-HN types of weight $e \in \Lambda_Q^{\sigma,+}$. For $(d^\bullet, e^\infty) \in \mathsf{HN}^\sigma(e)$ denote by $R_{d^\bullet,e^\infty}^{\sigma,HN} \subset R_e^\sigma$ the subset of self-dual representations whose σ-HN filtration is of type $(d^\bullet, e^\infty)$. Similarly, let $R_e^{\sigma,(d^\bullet,e^\infty)} \subset R_e^\sigma$ be the subset of self-dual representations which have an isotropic filtration of type $(d^\bullet, e^\infty)$. In the latter case we do not require that the subquotients of this filtration be σ-semistable. Analogous subsets $R_{d^\bullet}^{HN}, R_d^{d^\bullet} \subset R_d$ are defined for an ordinary HN type $d^\bullet \in \mathsf{HN}(d)$; see [24].

We need variations of two results of Reineke.

Proposition 2.2 (*cf.* [24, Proposition 3.4])**.** *For each $e \in \Lambda_Q^{\sigma,+}$, the collection $\{R_{d^\bullet,e^\infty}^{\sigma,HN}\}_{(d^\bullet,e^\infty)\in\mathsf{HN}^\sigma(e)}$ defines a stratification of R_e^σ by locally closed G_e^σ-invariant smooth subschemes.*

Proof. The argument is nearly the same as [24]; we give it here for completeness. Let $(d^\bullet, e^\infty) \in \mathsf{HN}^\sigma(e)$ and let $\mathbb{C}^e$ be the trivial self-dual representation of dimension vector e. Let

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r \subset \mathbb{C}^e$$

be a Q_0-graded isotropic filtration whose subquotients have dimension vector type $(d^\bullet, e^\infty)$. Let $R_{d^\bullet,e^\infty}^\sigma \subset R_e^\sigma$ be the closed subscheme of self-dual representations which preserve $E_\bullet \subset \mathbb{C}^e$. The natural map

$$\pi^\sigma : R_{d^\bullet,e^\infty}^\sigma \to R_{d^1} \times \cdots \times R_{d^r} \times R_{e^\infty}^\sigma$$

is a trivial vector bundle. The preimage

$$R_{d^\bullet,e^\infty}^{\sigma,\theta\text{-}ss} = (\pi^\sigma)^{-1}(R_{d^1}^{\theta\text{-}ss} \times \cdots \times R_{d^r}^{\theta\text{-}ss} \times R_{e^\infty}^{\sigma,\theta\text{-}ss})$$

is open in $R_{d^\bullet,e^\infty}^\sigma$. Let $\mathsf{G}_{d^\bullet,e^\infty}^\sigma \subset \mathsf{G}_e^\sigma$ be the parabolic subgroup which stabilizes $E_\bullet \subset \mathbb{C}^e$. Both $R_{d^\bullet,e^\infty}^\sigma$ and $R_{d^\bullet,e^\infty}^{\sigma,\theta\text{-}ss}$ are $\mathsf{G}_{d^\bullet,e^\infty}^\sigma$-invariant subschemes of R_e^σ. Since the quotient $\mathsf{G}_e^\sigma / \mathsf{G}_{d^\bullet,e^\infty}^\sigma$ is projective, the action map

$$m^\sigma : \mathsf{G}_e^\sigma \times_{\mathsf{G}_{d^\bullet,e^\infty}^\sigma} R_{d^\bullet,e^\infty}^\sigma \to R_e^\sigma$$

is proper. It follows that the image of m^σ, which equals $R_e^{\sigma,(d^\bullet,e^\infty)}$, is a closed subscheme of R_e^σ. The uniqueness of σ-HN filtrations implies that the restriction of m^σ to $\mathsf{G}_e^\sigma \times_{\mathsf{G}_{d^\bullet,e^\infty}^\sigma} R_{d^\bullet,e^\infty}^{\sigma,\theta\text{-}ss}$ defines an isomorphism

$$\mathsf{G}_e^\sigma \times_{\mathsf{G}_{d^\bullet,e^\infty}^\sigma} R_{d^\bullet,e^\infty}^{\sigma,\theta\text{-}ss} \xrightarrow{\sim} R_{d^\bullet,e^\infty}^{\sigma,HN}. \tag{5}$$

In particular, $R_{d^\bullet,e^\infty}^{\sigma,HN}$ is a smooth open subscheme of $R_e^{\sigma,(d^\bullet,e^\infty)}$. □

Unlike the case of ordinary quiver representations, the σ-HN strata $R_{d^\bullet,e^\infty}^{\sigma,HN}$ need not be connected. This is in turn because the isotropic flag variety $\mathsf{G}_e^\sigma / \mathsf{G}_{d^\bullet,e^\infty}^\sigma$ need not be irreducible. More precisely, $\mathsf{G}_e^\sigma / \mathsf{G}_{d^\bullet,e^\infty}^\sigma$ fails to be connected if and only if one of its factors is a flag variety for an orthogonal group $\mathsf{O}_{2n}(\mathbb{C})$ which parameterizes isotropic flags whose largest subspace is Lagrangian; such a flag variety has two irreducible components. The following example illustrates this behaviour.

Example. Let Q be the quiver $\bullet \to \bullet \to \bullet$ with its unique involution. Fix the duality structure $s = 1$ and $\tau = -1$ and the stability $\theta = (1, 0, -1)$. Let $e = (1, 2, 1)$ and consider the σ-HN type $((1,1,0),0) \in \mathsf{HN}^\sigma(e)$. The the σ-HN stratum $R^{\sigma,HN}_{(1,1,0),0}$ consists of self-dual representations of the form

$$\mathbb{C} \xrightarrow{\left(\begin{smallmatrix}\lambda\\0\end{smallmatrix}\right)} \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}0 & -\lambda\end{smallmatrix}\right)} \mathbb{C} \qquad \text{or} \qquad \mathbb{C} \xrightarrow{\left(\begin{smallmatrix}0\\\lambda\end{smallmatrix}\right)} \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}-\lambda & 0\end{smallmatrix}\right)} \mathbb{C}$$

for some $\lambda \in \mathbb{C}^\times$. The vector space $\mathbb{C}^2$ attached to the middle node has ordered basis $\{e,f\}$ with symmetric bilinear form determined by

$$\langle e,e\rangle = 0, \qquad \langle e,f\rangle = 1, \qquad \langle f,f\rangle = 0.$$

It follows that $R^{\sigma,HN}_{(1,1,0),0}$ is a torsor for $\mathsf{O}_2(\mathbb{C})$ and thus has two irreducible components. ◁

For each $d^\bullet \in (\Lambda_Q^+)^r$ let $\mathcal{P}(d^\bullet)$ be the polygon in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$ with vertices

$$(\dim \sum_{i=1}^k d^i, \theta(\sum_{i=1}^k d^i)), \qquad k = 0, \dots, r.$$

Following [24], define a partial order on tuples of Λ_Q^+ by $d^\bullet \leq d'^\bullet$ if $\mathcal{P}(d^\bullet)$ lies on or below $\mathcal{P}(d'^\bullet)$. Note that the polygon associated to a HN type is convex. Similarly, for a σ-HN type $(d^1, \dots, d^r, e^\infty) \in (\Lambda_Q^+)^r \times \Lambda_Q^{\sigma,+}$ we define $\mathcal{P}(d^\bullet, e^\infty)$ to be the polygon attached to the HN type $(d^1, \dots, d^r, e^\infty, \sigma(d^r), \dots, \sigma(d^1))$. The polygon attached to a σ-HN type has a vertical reflection symmetry and lies in the subset $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$.

Proposition 2.3 (*cf.* [24, Proposition 3.7]). *Let $e \in \Lambda_Q^{\sigma,+}$. For each $(d^\bullet, e^\infty) \in \mathsf{HN}^\sigma(e)$ the inclusion*

$$\overline{R^{\sigma,HN}_{d^\bullet,e^\infty}} \subset \bigcup_{\substack{(d'^\bullet,e'^\infty) \in \mathsf{HN}^\sigma(e) \\ (d^\bullet,e^\infty) \leq (d'^\bullet,e'^\infty)}} R^{\sigma,HN}_{d'^\bullet,e'^\infty}$$

holds.

Proof. It is shown in the proof of [24, Proposition 3.7] that if X is a representation with HN type $c^\bullet$ and $U \subset X$ is a subrepresentation, then the point $(\dim U, \theta(U))$ lies on or below $\mathcal{P}(c^\bullet)$. In particular, this applies if X is self-dual and U is isotropic. We conclude that

$$R_e^{\sigma,(d^\bullet,e^\infty)} \subset \bigcup_{\substack{(d'^\bullet,e'^\infty) \in \mathsf{HN}^\sigma(e) \\ (d^\bullet,e^\infty) \leq (d'^\bullet,e'^\infty)}} R^{\sigma,HN}_{d'^\bullet,e'^\infty}.$$

But $\overline{R^{\sigma,HN}_{d^\bullet,e^\infty}} = R_e^{\sigma,(d^\bullet,e^\infty)}$, as follows from the proof of Proposition 2.2. □

2.2. **Equivariant perfection of the σ-HN stratification.** We show that the σ-HN filtration of R_e^σ associated to a σ-compatible stability θ is G_e^σ-equivariantly perfect in the sense of [1], [18, §2]. See [15, §4], [3, §4] for analogous results in the setting of ordinary quiver representations.

Let X be a smooth connected complex algebraic variety with an action of a reductive algebraic group G. Suppose that $\{S_i\}_{i=0}^N$ is a disjoint collection of locally closed G-invariant smooth subschemes of X such that $\bigsqcup_{i=0}^N S_i = X$ and

$$\overline{S}_i \subset \bigcup_{i \leq j} S_j$$

for each $i = 0, \dots, N$. Assume that each connected component of S_i has complex codimension δ_i in X and write $\nu_i \to S_i$ for the normal bundle of $S_i \subset X$. For each i there is an associated long exact sequence of mixed Hodge structures, the equivariant Thom-Gysin sequence:

$$\cdots \to H_{\mathsf{G}}^{k-2\delta_i}(S_i)(-\delta_i) \to H_{\mathsf{G}}^{k}(\bigcup_{i\leq j} S_j) \to H_{\mathsf{G}}^{k}(\bigcup_{i<j} S_j) \to H_{\mathsf{G}}^{k-2\delta_i+1}(S_i)(-\delta_i) \to \cdots. \tag{6}$$

Here $(-\delta_i) = - \otimes \mathbb{Q}(-1)^{\otimes \delta_i}$ with $\mathbb{Q}(-1)$ the Tate Hodge structure of weight 2. It is shown in [1] that if each of the equivariant Euler classes $\mathsf{eu}_{\mathsf{G}}(\nu_i) \in H_{\mathsf{G}}^{\bullet}(S_i)$, $i = 0, \dots, N$, is a not a zero divisor, then the long exact sequence (6) breaks into short exact sequences

$$0 \to H_{\mathsf{G}}^{k-2\delta_i}(S_i)(-\delta_i) \to H_{\mathsf{G}}^{k}(\bigcup_{i\leq j} S_j) \to H_{\mathsf{G}}^{k}(\bigcup_{i<j} S_j) \to 0. \tag{7}$$

In this situation the stratification $\{S_i\}_{i=0}^N$ is called G-equivariantly perfect.

Lemma 2.4. *Suppose that $\{S_i\}_{i=0}^N$ is a G-equivariantly perfect stratification of X. If $H_{\mathsf{G}}^{2k+1}(X) = 0$ and $H_{\mathsf{G}}^{2k}(X)$ is pure of Hodge-Tate type for all $k \in \mathbb{Z}$, then the same is true for $H_{\mathsf{G}}^{\bullet}(S_i)$, $i = 0, \dots, N$.*

Proof. This follows by induction on i using the short exact sequence (7). □

Proposition 2.5. *Let $e \in \Lambda_Q^{\sigma,+}$ and $(d^\bullet, e^\infty) \in \mathsf{HN}^\sigma(e)$. Then $H_{\mathsf{G}_e^\sigma}^{2k+1}(R_{d^\bullet,e^\infty}^{\sigma,HN}) = 0$ and $H_{\mathsf{G}_e^\sigma}^{2k}(R_{d^\bullet,e^\infty}^{\sigma,HN})$ is pure of Hodge-Tate type for all $k \in \mathbb{Z}$.*

Proof. We will apply Lemma 2.4 with $X = R_e^\sigma$, $\mathsf{G} = \mathsf{G}_e^\sigma$ and the stratification $\{R_{d^\bullet,e^\infty}^{\sigma,HN}\}_{(d^\bullet,e^\infty)\in\mathsf{HN}^\sigma(e)}$ of Proposition 2.2 ordered as in Proposition 2.3. Note that $H_{\mathsf{G}_e^\sigma}^{\bullet}(R_e^\sigma) \simeq H_{\mathsf{G}_e^\sigma}^{\bullet}$ vanishes in odd degree and is of Hodge-Tate type in degree $2k$. Let $\nu_{d^\bullet,e^\infty} \to R_{d^\bullet,e^\infty}^{\sigma,HN}$ be the normal bundle of $R_{d^\bullet,e^\infty}^{\sigma,HN} \subset R_e^\sigma$. We need to prove that $\mathsf{eu}_{\mathsf{G}_e^\sigma}(\nu_{d^\bullet,e^\infty}) \in H_{\mathsf{G}_e^\sigma}^{\bullet}(R_{d^\bullet,e^\infty}^{\sigma,HN})$ is not a zero divisor. The isomorphism (5) induces isomorphisms

$$\begin{aligned} H_{\mathsf{G}_e^\sigma}^{\bullet}(R_{d^\bullet,e^\infty}^{\sigma,HN}) &\simeq H_{\mathsf{G}_{d^\bullet,e^\infty}^\sigma}^{\bullet}(R_{d^\bullet,e^\infty}^{\sigma,\theta\text{-}ss}) \\ &\simeq H_{\mathsf{GL}_{d^1}\times\cdots\times\mathsf{GL}_{d^r}\times\mathsf{G}_{e^\infty}^\sigma}^{\bullet}(R_{d^1}^{\theta\text{-}ss}\times\cdots\times R_{d^r}^{\theta\text{-}ss}\times R_{e^\infty}^{\sigma,\theta\text{-}ss}). \end{aligned}$$

Denote by $\tilde{\nu}_{d^\bullet,e^\infty}$ the restriction of $\nu_{d^\bullet,e^\infty}$ to $R_{d^1}^{\theta\text{-}ss}\times\cdots\times R_{d^r}^{\theta\text{-}ss}\times R_{e^\infty}^{\sigma,\theta\text{-}ss}$. Then $\mathsf{eu}_{\mathsf{G}_e^\sigma}(\nu_{d^\bullet,e^\infty})$ is mapped to $\mathsf{eu}_{\mathsf{GL}_{d^1}\times\cdots\times\mathsf{GL}_{d^r}\times\mathsf{G}_{e^\infty}^\sigma}(\tilde{\nu}_{d^\bullet,e^\infty})$ under the above isomorphisms. Fix $V_j \in R_{d^j}^{\theta\text{-}ss}$, $j = 1, \dots, r$, and $N \in R_{e^\infty}^{\sigma,\theta\text{-}ss}$ and consider the orthogonal direct sum self-dual representation

$$M = \bigoplus_{j=1}^r H(V_j) \oplus N \in R_e^\sigma.$$

Here $H(V) = V \oplus S(V)$ is the hyperbolic self-dual representation on V. The fibre of $\tilde{\nu}_{d^\bullet,e^\infty}$ over M is naturally identified with the subspace of

$$\bigoplus_{1\leq i<j\leq r} \mathrm{Ext}^1(V_i, V_j) \oplus \bigoplus_{i=1}^r \mathrm{Ext}^1(V_i, N) \oplus \bigoplus_{1\leq i\leq j\leq r} \mathrm{Ext}^1(V_i, S(V_j))$$

consisting of elements whose component from $\oplus_{1\leq i\leq j\leq r}\mathrm{Ext}^1(V_i, S(V_j))$ is fixed by S. Note that we have used the isomorphisms (4).

Let $\mathsf{T} \simeq (\mathbb{C}^\times)^r$ be the diagonal torus

$$\mathsf{T} \subset \prod_{j=1}^r \mathsf{GL}_{d^j} \subset \prod_{j=1}^r \mathsf{GL}_{d^j} \times \mathsf{G}_{e^\infty}^\sigma.$$

Then T acts trivially on $R^{\theta\text{-}ss}_{d^1} \times \cdots \times R^{\theta\text{-}ss}_{d^r} \times R^{\sigma,\theta\text{-}ss}_{e^\infty}$ and acts on $\tilde{\nu}_{d^\bullet,e^\infty|_M}$ with weights

(1) $t_j - t_i$ on the summand $\mathrm{Ext}^1(V_i, V_j)$,
(2) $-t_i$ on the summand $\mathrm{Ext}^1(V_i, N)$, and
(3) $-t_j - t_i$ on the summand $\mathrm{Ext}^1(V_i, S(V_j))$.

We can therefore apply the Atiyah-Bott lemma [1, Proposition 13.4] to conclude that $\mathsf{eu}_{\mathsf{G}^\sigma_e}(\nu_{d^\bullet,e^\infty})$ is not a zero divisor. □

Here is one application of Proposition 2.5. Let $\mathbb{F}_q$ be a finite field of odd order q. In [27, Theorem 4.4] the stacky number of $\mathbb{F}_q$-rational points $\#\mathbf{M}^{\sigma,\theta\text{-}ss}_e(\mathbb{F}_q)$ was explicitly computed, the result being a rational function of q. By a similar computation, one can prove that the motive $[\mathbf{M}^{\sigma,\theta\text{-}ss}_e] \in K_0(\mathsf{St}_\mathbb{C})$ is given by the same rational function, with q replaced by the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1_\mathbb{C}]$. The new ingredient in the motivic setting is the computation of the motive of the classifying stack $[B\mathsf{G}^\sigma_e]$, a non-trivial task since G^σ_e may contain factors of the orthogonal groups O_n and so need not be a special algebraic group in the sense of Serre. In [8] it was proved that $[B\mathsf{O}_n]_{|\mathbb{L}\mapsto q} = \#B\mathsf{O}_n(\mathbb{F}_q)$. It follows from Proposition 2.5 that the specialization $\mathbb{L} \mapsto t^2$ of the expression for $[\mathbf{M}^{\sigma,\theta\text{-}ss}_e]$, and hence the formula from [27, Theorem 4.4], gives the Poincaré series of $H^\bullet(\mathbf{M}^{\sigma,\theta\text{-}ss}_e)$.

3. Chow theoretic Hall algebras and modules

3.1. **Basic definitions.** We recall the cohomological Hall algebra of Kontsevich and Soibelman along with some of its modifications.

Let Q be a quiver. Let $\mathsf{Vect}_\mathbb{Z}$ be the category of finite dimensional $\mathbb{Z}$-graded rational vector spaces. Write $D^{lb}(\mathsf{Vect}_\mathbb{Z}) \subset D(\mathsf{Vect}_\mathbb{Z})$ for the full subcategory of objects whose cohomological and $\mathbb{Z}$ degrees are bounded below. Let $D^{lb}(\mathsf{Vect}_\mathbb{Z})_{\Lambda^+_Q}$ be the category whose objects are Λ^+_Q-graded objects of $D^{lb}(\mathsf{Vect}_\mathbb{Z})$ with finite dimensional $\Lambda^+_Q \times \mathbb{Z}$-homogeneous summands and whose morphisms preserve the $\Lambda^+_Q \times \mathbb{Z}$-grading. Denote by $\{\frac{1}{2}\}$ the functor which is tensor product with the one dimensional vector space of cohomological and $\mathbb{Z}$ degree -1. Define a monoidal product $\boxtimes^{\mathsf{tw}}$ on $D^{lb}(\mathsf{Vect}_\mathbb{Z})_{\Lambda^+_Q}$ by

$$\bigoplus_{d\in\Lambda^+_Q} \mathcal{U}'_d \boxtimes^{\mathsf{tw}} \bigoplus_{d\in\Lambda^+_Q} \mathcal{U}''_d = \bigoplus_{d\in\Lambda^+_Q} \Big(\bigoplus_{\substack{(d',d'')\in\Lambda^+_Q\times\Lambda^+_Q \\ d=d'+d''}} \mathcal{U}'_{d'} \otimes \mathcal{U}''_{d''}\{(\chi(d',d'') - \chi(d'',d'))/2\}\Big).$$

Fix a stability θ. For each slope $\mu \in \mathbb{Q}$ let

$$\mathcal{H}^{\theta\text{-}ss}_{Q,\mu} = \bigoplus_{d\in\Lambda^+_{Q,\mu}} H^\bullet_{\mathsf{GL}_d}(R^{\theta\text{-}ss}_d)\{\chi(d,d)/2\} \in D^{lb}(\mathsf{Vect}_\mathbb{Z})_{\Lambda^+_{Q,\mu}}.$$

Here $\Lambda^+_{Q,\mu} = \{d \in \Lambda^+_Q \mid \mu(d) = \mu\} \cup \{0\}$ and we use singular equivariant cohomology with rational coefficients. The $\mathbb{Z}$-grading of $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu}$ is the cohomological, or equivalently Hodge theoretic weight, grading. We interpret $H^\bullet_{\mathsf{GL}_d}(R^{\theta\text{-}ss}_d)$ as the cohomology of the stack $\mathbf{M}^{\theta\text{-}ss}_d = [R^{\theta\text{-}ss}_d/\mathsf{GL}_d]$. For each $d, d' \in \Lambda^+_{Q,\mu}$ there is a correspondence

$$\begin{array}{ccccc} \mathbf{M}^{\theta\text{-}ss}_d \times \mathbf{M}^{\theta\text{-}ss}_{d'} & \xleftarrow{\pi_1\times\pi_3} & \mathbf{M}^{\theta\text{-}ss}_{d,d'} & \xrightarrow{\pi_2} & \mathbf{M}^{\theta\text{-}ss}_{d+d'} \\ (U, V/U) & \leftarrow\!\shortmid & U \subset V & \mapsto & V \end{array}$$

with $\mathbf{M}^{\theta\text{-}ss}_{d,d'} = [R^{\theta\text{-}ss}_{d,d'}/\mathsf{GL}_{d,d'}]$ the stack of flags of representations of dimension vector type (d, d'). The map $\pi_1 \times \pi_3$ is a homotopy equivalence while π_2 is proper. The composition $\pi_{2!} \circ (\pi_1 \times \pi_3)^*$ defines a product on $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu}$ making it into an associative algebra object in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$, called the slope μ semistable cohomological Hall algebra (CoHA) [19].

Suppose now that Q has a duality structure and that θ is a σ-compatible stability. With this additional data, the category $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$ becomes a left module over $(D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}, \boxtimes^{\mathsf{tw}})$ via the formula

$$\bigoplus_{d \in \Lambda_Q^+} \mathcal{U}_d \boxtimes^{S\text{-}\mathsf{tw}} \bigoplus_{e \in \Lambda_Q^{\sigma,+}} \mathcal{V}_e = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} \Big(\bigoplus_{\substack{(d',e') \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+} \\ e = H(d') + e'}} \mathcal{U}_{d'} \otimes \mathcal{V}_{e'}\{\gamma(d',e')/2\} \Big)$$

where

$$\gamma(d,e) = \chi(d,e) - \chi(e,d) + \mathcal{E}(\sigma(d)) - \mathcal{E}(d).$$

Define

$$\mathcal{M}_Q^{\theta\text{-}ss} = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} H^\bullet_{\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}ss})\{\mathcal{E}(e)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}.$$

If $d \in \Lambda_{Q,\mu=0}^+$ and $e \in \Lambda_Q^{\sigma,+}$, then there is a modified correspondence of stacks

$$\begin{array}{ccccc} \mathbf{M}_d^{\theta\text{-}ss} \times \mathbf{M}_e^{\sigma,\theta\text{-}ss} & \xleftarrow{\pi_1 \times \pi_3^\sigma} & \mathbf{M}_{d,e}^{\sigma,\theta\text{-}ss} & \xrightarrow{\pi_2^\sigma} & \mathbf{M}_{d+\sigma(d)+e}^{\sigma,\theta\text{-}ss} \\ (U, M/\!\!/U) & \leftarrow\!\shortmid & U \subset M & \mapsto & M \end{array}$$

where $\mathbf{M}_e^{\sigma,\theta\text{-}ss} = [R_e^{\sigma,\theta\text{-}ss}/\mathsf{G}_e^\sigma]$ and $\mathbf{M}_{d,e}^{\sigma,\theta\text{-}ss} = [R_{d,e}^{\sigma,\theta\text{-}ss}/\mathsf{G}_{d,e}^\sigma]$. As above, $\pi_1 \times \pi_3^\sigma$ is a homotopy equivalence and π_2^σ is proper. The composition $\pi_{2!}^\sigma \circ (\pi_1 \times \pi_3^\sigma)^*$ gives $\mathcal{M}_Q^{\theta\text{-}ss}$ the structure of left $\mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss}$-module object in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$, called the semistable cohomological Hall module (CoHM). See [28] for details. We denote by $\star$ the action of $\mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss}$ on $\mathcal{M}_Q^{\theta\text{-}ss}$.

Let $\mathsf{W}(Q)$ be the abelian group $\mathbb{Z}_2 Q_0^\sigma$. We have an exact sequence of groups

$$\Lambda_Q \xrightarrow{H} \Lambda_Q^\sigma \xrightarrow{\nu} \mathsf{W}(Q) \to 0$$

with $H(d) = d + \sigma(d)$ and ν sending a dimension vector to its parities at Q_0^σ. There is a $\mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss}$-module decomposition $\mathcal{M}_Q^{\theta\text{-}ss} = \bigoplus_{w \in \mathsf{W}(Q)} \mathcal{M}_Q^{\theta\text{-}ss}(w)$ with

$$\mathcal{M}_Q^{\theta\text{-}ss}(w) = \bigoplus_{\substack{e \in \Lambda_Q^{\sigma,+} \\ \nu(e) = w}} H^\bullet_{\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}ss})\{\mathcal{E}(e)/2\}.$$

Note that $\mathcal{M}_Q^{\theta\text{-}ss}(w)$ is trivial unless $s_i = 1$ whenever $w_i = 1$.

In the case of trivial stability, $\theta = 0$, both $\mathcal{H}_Q$ and $\mathcal{M}_Q$ have explicit combinatorial descriptions in terms of shuffle algebras and signed shuffle modules. See [19, Theorem 2] and [28, Theorem 3.3], respectively.

There are also Chow theoretic versions of $\mathcal{H}_{Q,\mu}^{\theta\text{-}ss}$ and $\mathcal{M}_Q^{\theta\text{-}ss}$, called the Chow Hall algebra (ChowHA) and module (ChowHM) and denoted by $\mathcal{A}_{Q,\mu}^{\theta\text{-}ss}$ and $\mathcal{B}_Q^{\theta\text{-}ss}$, respectively. The former was introduced and studied in [12], [13], [14]. The definitions of $\mathcal{A}_{Q,\mu}^{\theta\text{-}ss}$ and $\mathcal{B}_Q^{\theta\text{-}ss}$ are entirely similar to those of their cohomological counterparts but with rational equivariant Chow groups used in place of rational equivariant cohomology groups. For background on equivariant Chow theory the reader is referred to [10], [2]. For example, the slope μ semistable ChowHA is

$$\mathcal{A}_{Q,\mu}^{\theta\text{-}ss} = \bigoplus_{d \in \Lambda_{Q,\mu}^+} A^\bullet_{\mathsf{GL}_d}(R_d^{\theta\text{-}ss})_{\mathbb{Q}}\{\chi(d,d)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_{Q,\mu}^+}.$$

The $\mathbb{Z}$-grading of $\mathcal{A}_{Q,\mu}^{\theta\text{-}ss}$ is defined by putting the equivariant Chow group $A^k_{\mathsf{GL}_d}(R_d^{\theta\text{-}ss})$ in degree $2k$. The semistable ChowHM is defined analogously. Like $\mathcal{M}_Q^{\theta\text{-}ss}$, the $\mathcal{A}_{Q,\mu}^{\theta\text{-}ss}$-module $\mathcal{B}_Q^{\theta\text{-}ss}$ admits a decomposition labelled by $\mathsf{W}(Q)$.

3.2. **Comparison of the ChowHM and CoHM.** Given a linear algebraic group G, we write $A^\bullet_{\mathsf{G}}$ for $A^\bullet_{\mathsf{G}}(\operatorname{Spec}\mathbb{C})_{\mathbb{Q}}$ and similarly for $H^\bullet_{\mathsf{G}}$. For each $r \geq 1$ there is a graded ring isomorphism $A^\bullet_{\mathsf{GL}_r} \simeq \mathbb{Q}[x_1, \dots, x_r]$ with x_i of degree i and the equivariant cycle map $A^\bullet_{\mathsf{GL}_r} \to H^\bullet_{\mathsf{GL}_r}$ is a degree doubling ring isomorphism. In particular, $H^\bullet_{\mathsf{GL}_r}$ is concentrated in even degree. Similarly, for G an orthogonal or symplectic group of rank $r \geq 1$ we have a ring isomorphism $A^\bullet_{\mathsf{G}} \simeq \mathbb{Q}[p_1, \dots, p_r]$ with p_i of degree $2i$ and the equivariant cycle map $A^\bullet_{\mathsf{G}} \to H^\bullet_{\mathsf{G}}$ is again an isomorphism. Note that with integer coefficients the above statements are only valid for general linear and symplectic groups. Even with rational coefficients the statements are not true for the special orthogonal groups SO_{2r}. See [4], [9], [23], [26] for proofs of these statements.

In [14, Corollary 5.6] it is shown that for any stability θ and slope μ the equivariant cycle map $\mathsf{cl}_{alg} : \mathcal{A}^{\theta\text{-}ss}_{Q,\mu} \to \mathcal{H}^{\theta\text{-}ss}_{Q,\mu}$ is an isomorphism of algebra objects in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda^+_{Q,\mu}}$. We show that this isomorphism lifts to Hall modules.

Theorem 3.1. *Let θ be a σ-compatible stability. The equivariant cycle map cl_{mod} : $\mathcal{B}^{\theta\text{-}ss}_Q \to \mathcal{M}^{\theta\text{-}ss}_Q$ is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda^{\sigma,+}_Q}$. Moreover, the diagram*

$$\begin{array}{ccc} \mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0} \boxtimes^{S\text{-tw}} \mathcal{B}^{\theta\text{-}ss}_Q & \xrightarrow{\star} & \mathcal{B}^{\theta\text{-}ss}_Q \\ {\scriptstyle \mathsf{cl}_{alg} \boxtimes^{S\text{-tw}} \mathsf{cl}_{mod}} \downarrow & & \downarrow {\scriptstyle \mathsf{cl}_{mod}} \\ \mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0} \boxtimes^{S\text{-tw}} \mathcal{H}^{\theta\text{-}ss}_Q & \xrightarrow[\star]{} & \mathcal{M}^{\theta\text{-}ss}_Q \end{array}$$

commutes.

Proof. It suffices to prove the theorem when $\mathcal{B}^{\theta\text{-}ss}_Q$ and $\mathcal{M}^{\theta\text{-}ss}_Q$ are replaced with $\mathcal{B}^{\theta\text{-}ss}_Q(w)$ and $\mathcal{M}^{\theta\text{-}ss}_Q(w)$, respectively, for an arbitrary class $w \in \mathsf{W}(Q)$.

Proposition 2.5 shows that $H^{2k+1}_{\mathsf{G}^\sigma_e}(R^{\sigma,\theta\text{-}ss}_e)$ for all $k \geq 0$, so we need only prove that

$$\mathsf{cl}^e_{mod} : A^k_{\mathsf{G}^\sigma_e}(R^{\sigma,\theta\text{-}ss}_e) \to H^{2k}_{\mathsf{G}^\sigma_e}(R^{\sigma,\theta\text{-}ss}_e)$$

is a vector space isomorphism for each $k \geq 0$. Define a partial order on $\Lambda^{\sigma,+}_Q$ by $e' \leq e$ if $e'_i \leq e_i$ for all $i \in Q_0$. We proceed by induction on $e \in \Lambda^{\sigma,+}_Q$ with fixed class $w \in \mathsf{W}(Q)$. Let e be the minimal such dimension vector. Explicitly, e is zero except at those nodes $i \in Q^\sigma_0$ with $s_i = 1$, in which case e_i is zero if $w_i = 0$ and is one if $w_i = 1$. A self-dual representation of dimension vector e has no isotropic subrepresentations. Hence $R^{\sigma,\theta\text{-}ss}_e = R^\sigma_e$ and the cycle map reduces to $\mathsf{cl}^e_{mod} : A^k_{\mathsf{G}^\sigma_e} \to H^{2k}_{\mathsf{G}^\sigma_e}$, which is an isomorphism by the discussion above the theorem.

Assume that $\mathsf{cl}^{e'}_{mod}$ is an isomorphism for all $e' < e$ of class w. Let $R^{\sigma,\theta\text{-}unst}_e = R^\sigma_e - R^{\sigma,\theta\text{-}ss}_e$ be the closed subscheme of unstable self-dual representations of dimension vector e. Using Proposition 2.3 we obtain a stratification of $R^{\sigma,\theta\text{-}unst}_e$ by G^σ_e-invariant closed subschemes whose successive complements are of the form $R^{\sigma,HN}_{d^\bullet,e^\infty}$ for some $(d^\bullet, e^\infty) \in \mathsf{HN}^\sigma(e)$ different from (e). Since $R^{\sigma,\theta\text{-}ss}_e$ is smooth, there are isomorphisms

$$A^{\mathsf{G}^\sigma_e}_\bullet(R^{\sigma,\theta\text{-}ss}_e) \simeq A^{\dim R^\sigma_e - \bullet}_{\mathsf{G}^\sigma_e}(R^{\sigma,\theta\text{-}ss}_e), \qquad H^{BM,\mathsf{G}^\sigma_e}_\bullet(R^{\sigma,\theta\text{-}ss}_e) \simeq H^{2\dim R^\sigma_e - \bullet}_{\mathsf{G}^\sigma_e}(R^{\sigma,\theta\text{-}ss}_e),$$

where $H^{BM}_\bullet(-)$ denotes Borel-Moore homology with rational coefficients. Using the equivariant lift of [14, Lemma 5.3], to prove that cl^e_{mod} is an isomorphism it therefore suffices to prove that each of the cycle maps $A^\bullet_{\mathsf{G}^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty})_{\mathbb{Q}} \to H^\bullet_{\mathsf{G}^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty})$ is an isomorphism. Arguing as in the proof of Proposition 2.5, we have an isomorphism

$$A^{\bullet}_{\mathsf{G}^{\sigma}_{e}}(R^{\sigma,HN}_{d^{\bullet},e^{\infty}}) \simeq A^{\bullet}_{\mathsf{GL}_{d^1}\times\cdots\times\mathsf{GL}_{d^r}\times\mathsf{G}^{\sigma}_{e^{\infty}}}(R^{\theta\text{-}ss}_{d^1}\times\cdots\times R^{\theta\text{-}ss}_{d^r}\times R^{\sigma,\theta\text{-}ss}_{e^{\infty}}).$$

Since $e^{\infty} < e$, the inductive hypothesis implies that $\mathsf{cl}^{e^{\infty}}_{mod}$ is an isomorphism while cl^{d}_{alg} is an isomorphism for all $d \in \Lambda^+_Q$ by [14, Theorem 5.1]. As $R^{\sigma,\theta\text{-}ss}_{e^{\infty}}$ has no odd $\mathsf{G}^{\sigma}_{e^{\infty}}$-equivariant cohomology, we can apply the equivariant version of [26, Lemmas 6.1, 6.2] (with rational coefficients) to conclude that the exterior product map

$$A^{\bullet}_{\mathsf{GL}_{d^1}}(R^{\theta\text{-}ss}_{d^1})_{\mathbb{Q}} \otimes\cdots\otimes A^{\bullet}_{\mathsf{GL}_{d^r}}(R^{\theta\text{-}ss}_{d^r})_{\mathbb{Q}} \otimes A^{\bullet}_{\mathsf{G}^{\sigma}_{e^{\infty}}}(R^{\sigma,\theta\text{-}ss}_{e^{\infty}})_{\mathbb{Q}} \to A^{\bullet}_{\mathsf{GL}_{d^1}\times\cdots\times\mathsf{GL}_{d^r}\times\mathsf{G}^{\sigma}_{e^{\infty}}}(R^{\theta\text{-}ss}_{d^1}\times\cdots\times R^{\theta\text{-}ss}_{d^r}\times R^{\sigma,\theta\text{-}ss}_{e^{\infty}})_{\mathbb{Q}}$$

is an isomorphism. Similarly, the corresponding Künneth map in equivariant cohomology is an isomorphism. Compatibility of the cycle map with exterior products then implies that cl^{e}_{mod} is an isomorphism.

That cl_{mod} respects gradings is clear. That cl_{mod} respects the Hall algebra actions follows from the fact that cycle maps are covariant for proper morphisms and contravariant for morphisms of smooth varieties. □

Remark. Unlike $\mathsf{cl}_{alg} : \mathcal{A}^{\theta\text{-}ss}_{Q,\mu} \to \mathcal{H}^{\theta\text{-}ss}_{Q,\mu}$, the map $\mathsf{cl}_{mod} : \mathcal{B}^{\theta\text{-}ss}_{Q} \to \mathcal{M}^{\theta\text{-}ss}_{Q}$ is neither injective nor surjective if integer coefficients are used. However, the diagram from Theorem 3.1 remains commutative over the integers.

Corollary 3.2. *Let Q be a quiver with duality structure and σ-compatible stability θ. For each $e \in \Lambda^{\sigma,+}_Q$ the cycle map $\mathsf{cl} : A^{\bullet}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e)_{\mathbb{Q}} \to H^{\bullet}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e)$ surjects onto the pure part*

$$PH^{\bullet}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e) = \bigoplus_{k\geq 0} W_k H^k(\mathfrak{M}^{\sigma,\theta\text{-}st}_e).$$

In particular, $PH^{\bullet}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e)$ consists entirely of Hodge classes, that is,

$$W_{2k}H^{2k}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e) = W_{2k}H^{2k}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e) \cap F^k H^{2k}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e;\mathbb{C})$$

and $W_{2k+1}H^{2k+1}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e) = 0$ for all $k \geq 0$.

Proof. The proof of [5, Theorem 1.1] can be modified to show that the restriction $H^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}ss}_e) \to H^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}st}_e)$ factors through a surjection $H^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}ss}_e) \twoheadrightarrow PH^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}st}_e)$. This is proved in [28, Proposition 3.9] under the assumption that Q is σ-symmetric and $\theta = 0$, but the same argument works in general. The only new ingredient is that $H^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}ss}_e)$ vanishes in odd degree and is of Hodge-Tate type otherwise, which was proved in Proposition 2.5. For each $k \geq 0$ we therefore obtain an exact commutative diagram

$$\begin{array}{ccc} A^{k}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}ss}_e)_{\mathbb{Q}} & \twoheadrightarrow & A^{k}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}st}_e)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} \\ H^{2k}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}ss}_e) & \twoheadrightarrow & PH^{2k}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}st}_e) \end{array}$$

By Theorem 3.1 the left-hand vertical map is an isomorphism. The surjectivity of the right-hand vertical map follows. To complete the proof, note that by [10, Theorem 4] we have

$$A^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma,\theta\text{-}st}_e)_{\mathbb{Q}} \simeq A^{\bullet}(\mathbf{M}^{\sigma,\theta\text{-}st}_e)_{\mathbb{Q}} \simeq A^{\bullet}(\mathfrak{M}^{\sigma,\theta\text{-}st}_e)_{\mathbb{Q}}$$

and similarly for cohomology groups with their mixed Hodge structures. □

Remark. The analogue of Corollary 3.2 also holds for ordinary quiver moduli, with the same proof. If, in addition, we assume that Q is the double of a quiver, then for all stabilities θ the cycle map $A^\bullet(\mathfrak{M}_d^{\theta\text{-}st})_{\mathbb{Q}} \to H^\bullet(\mathfrak{M}_d^{\theta\text{-}st})$ is an isomorphism onto $PH^\bullet(\mathfrak{M}_d^{\theta\text{-}st})$. Indeed, in the case of trivial stability it is proved in [5, Theorem 2.2] that the restriction map $H^\bullet_{\mathsf{GL}_d}(R_d^{ss}) \to H^\bullet_{\mathsf{GL}_d}(R_d^{st})$ induces an isomorphism $V_{Q,d}^{\mathsf{prim}} \xrightarrow{\sim} PH^\bullet(\mathfrak{M}_d^{st})\{\chi(d,d)/2\}$, where $V_{Q,d}^{\mathsf{prim}}$ denotes the cohomological Donaldson-Thomas invariant of Q [11]. On the other hand, it is proved in [14, §9] that the cycle map defines an isomorphism $A^\bullet(\mathfrak{M}_d^{st})_{\mathbb{Q}}\{\chi(d,d)/2\} \xrightarrow{\sim} V_{Q,d}^{\mathsf{prim}}$, where $A^k(\mathfrak{M}_d^{st})_{\mathbb{Q}}$ is given $\mathbb{Z}$-degree $2k$. For a general stability, the open embedding $\mathfrak{M}_d^{st} \hookrightarrow \mathfrak{M}_d^{\theta\text{-}st}$ induces a commutative diagram

$$\begin{array}{ccc} A^k(\mathfrak{M}_d^{\theta\text{-}st})_{\mathbb{Q}} & \twoheadrightarrow & A^k(\mathfrak{M}_d^{st})_{\mathbb{Q}} \\ \downarrow{\scriptstyle \mathsf{cl}} & & \downarrow{\scriptstyle \mathsf{cl}} \\ PH^{2k}(\mathfrak{M}_d^{\theta\text{-}st}) & \longrightarrow & PH^{2k}(\mathfrak{M}_d^{st}) \end{array}$$

whose top horizontal map is an isomorphism by [14, Theorem 9.2] and whose right vertical map is an isomorphism by the discussion above. This implies that the left vertical map is injective. But by the ordinary version of Corollary 3.2 this map is also surjective. Hence all maps in the above diagram are isomorphisms.

3.3. **Restriction to σ-stable representations.** In this section we relate $\mathcal{B}_Q^{\theta\text{-}ss}$ to the rational Chow groups of moduli spaces of σ-stable self-dual representations.

Define

$$\mathcal{B}_Q^{\theta\text{-}st} = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} A^\bullet_{\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}st})_{\mathbb{Q}}\{\mathcal{E}(e)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}.$$

The open inclusions $R_e^{\sigma,\theta\text{-}st} \hookrightarrow R_e^{\sigma,\theta\text{-}ss}$ induce a surjection $\mathcal{B}_Q^{\theta\text{-}ss} \twoheadrightarrow \mathcal{B}_Q^{\theta\text{-}st}$.

Proposition 3.3. *For each $e \in \Lambda_Q^{\sigma,+}$ the kernel of the restriction $\mathcal{B}_{Q,e}^{\theta\text{-}ss} \twoheadrightarrow \mathcal{B}_{Q,e}^{\theta\text{-}st}$ is equal to*

$$\sum_{\substack{(d',e') \in \Lambda_{Q,\mu=0}^+ \times \Lambda_Q^{\sigma,+} \\ H(d')+e'=e,\, d' \neq 0}} \mathcal{A}_{Q,d'}^{\theta\text{-}ss} \star \mathcal{B}_{Q,e'}^{\theta\text{-}ss}.$$

Proof. By definition, the locus $R_e^{\sigma,\theta\text{-}ss} - R_e^{\sigma,\theta\text{-}st}$ of properly semistable self-dual representations is the union

$$\bigcup_{\substack{(d',e') \in \Lambda_{Q,\mu=0}^+ \times \Lambda_Q^{\sigma,+} \\ H(d')+e'=e,\, d' \neq 0}} R_e^{\sigma,(d',e'),\theta\text{-}ss}$$

of the subsets $R_e^{\sigma,(d',e'),\theta\text{-}ss}$ of semistable self-dual representations of dimension vector e which possess an isotropic subrepresentation of dimension vector d'. From the proof of Proposition 2.2 we know that $R_e^{\sigma,(d',e'),\theta\text{-}ss}$ is the G_e^σ-saturation of $R_{d',e'}^{\sigma,\theta\text{-}ss}$ and that $R_{d',e'}^{\sigma,\theta\text{-}ss}$ is a closed subset of $R_e^{\sigma,\theta\text{-}ss}$. We can therefore apply [14, Lemma 8.2] to conclude that

$$\bigoplus_{\substack{(d',e') \in \Lambda_{Q,\mu=0}^+ \times \Lambda_Q^{\sigma,+} \\ H(d')+e'=e,\, d' \neq 0}} A_k^{\mathsf{G}_e^\sigma}(\mathsf{G}_e^\sigma \times_{\mathsf{G}_{d',e'}^\sigma} R_{d',e'}^{\sigma,\theta\text{-}ss})_{\mathbb{Q}} \to A_k^{\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}ss})_{\mathbb{Q}} \to A_k^{\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}st})_{\mathbb{Q}} \to 0 \tag{8}$$

is an exact sequence. The affine bundles

$$\mathsf{G}^\sigma_e \times_{\mathsf{GL}_{d'} \times \mathsf{G}^\sigma_{e'}} (R^{\theta\text{-}ss}_{d'} \times R^{\sigma,\theta\text{-}ss}_{e'}) \leftarrow \mathsf{G}^\sigma_e \times_{\mathsf{GL}_{d'} \times \mathsf{G}^\sigma_{e'}} R^{\sigma,\theta\text{-}ss}_{d',e'} \to \mathsf{G}^\sigma_e \times_{\mathsf{G}^\sigma_{d',e'}} R^{\sigma,\theta\text{-}ss}_{d',e'}$$

give rise to isomorphisms

$$\begin{aligned} A^{\mathsf{G}^\sigma_e}_k(\mathsf{G}^\sigma_e \times_{\mathsf{G}^\sigma_{d',e'}} R^{\sigma,\theta\text{-}ss}_{d',e'}) &\simeq A^{2\dim R^\sigma_e - k + \chi(d',e') + \mathcal{E}(\sigma(d'))}_{\mathsf{G}^\sigma_{d',e'}}(R^{\sigma,\theta\text{-}ss}_{d',e'}) \\ &\simeq A^{2\dim R^\sigma_e - k + \chi(d',e') + \mathcal{E}(\sigma(d'))}_{\mathsf{GL}_{d'} \times \mathsf{G}^\sigma_{e'}}(R^{\theta\text{-}ss}_{d'} \times R^{\sigma,\theta\text{-}ss}_{e'}). \end{aligned}$$

It was shown in the proof of Theorem 3.1 that the exterior product map

$$A^\bullet_{\mathsf{GL}_{d'}}(R^{\theta\text{-}ss}_{d'})_{\mathbb{Q}} \otimes A^\bullet_{\mathsf{G}^\sigma_{e'}}(R^{\sigma,\theta\text{-}ss}_{e'})_{\mathbb{Q}} \to A^\bullet_{\mathsf{GL}_{d'} \times \mathsf{G}^\sigma_{e'}}(R^{\theta\text{-}ss}_{d'} \times R^{\sigma,\theta\text{-}ss}_{e'})_{\mathbb{Q}}$$

is an isomorphism. Combining these isomorphisms and taking into account the grading shifts, the leftmost map of the exact sequence (8) identifies with the action map

$$\bigoplus_{\substack{(d',e') \in \Lambda^+_{Q,\mu=0} \times \Lambda^{\sigma,+}_Q \\ H(d')+e'=e,\, d' \neq 0}} \mathcal{A}^{\theta\text{-}ss}_{Q,d'} \boxtimes^{S\text{-}\mathsf{tw}} \mathcal{B}^{\theta\text{-}ss}_{Q,e'} \to \mathcal{B}^{\theta\text{-}ss}_{Q,e}.$$

This completes the proof. □

3.4. **Chow theoretic wall-crossing formulas.** In [27] a motivic orientifold wall-crossing formula was proved using finite field Hall algebras and their representations. In this section we lift this formula to Chow theoretic and cohomological Hall modules.

By applying [14, Corollary 5.4] to the σ-HN stratification of R^σ_e we obtain an isomorphism

$$A^\bullet_{\mathsf{G}^\sigma_e}(R^\sigma_e) \simeq \bigoplus_{(d^\bullet, e^\infty) \in \mathsf{HN}^\sigma(e)} A^{\bullet - \operatorname{codim}_{R^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty})}_{\mathsf{G}^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty}). \tag{9}$$

Using the isomorphism (5) we see that

$$\operatorname{codim}_{R^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty}) = \operatorname{codim}_{R^\sigma_e}(R^\sigma_{d^\bullet,e^\infty}) - \dim \mathsf{G}^\sigma_e + \dim \mathsf{G}_{d^\bullet,e^\infty}.$$

By displaying $R^\sigma_{d^\bullet,e^\infty}$ in terms of matrices, analogous to the description of $R^\sigma_{d,e}$ given in [28, §3.1], we compute

$$\operatorname{codim}_{R^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty}) = -\sum_{1 \leq k < l \leq r} \chi(d^k, d^l) - \chi(d, e^\infty) - \mathcal{E}(\sigma(d)) \tag{10}$$

where $d = d^1 + \cdots + d^r$.

In particular, by considering the open stratum $R^{\sigma,\theta\text{-}ss}_e \subset R^\sigma_e$ we obtain from (9) a vector space splitting of the surjection $\mathcal{B}_Q \twoheadrightarrow \mathcal{B}^{\theta\text{-}ss}_Q$. We use this splitting to regard $\mathcal{B}^{\theta\text{-}ss}_Q$ as a subobject of $\mathcal{B}_Q$. In the same way, we can consider $\mathcal{A}^{\theta\text{-}ss}_Q$ as a subobject of $\mathcal{A}_Q$.

Theorem 3.4. *Let θ be a σ-compatible stability. Then the slope ordered ChowHA action map*

$$\overleftarrow{\boxtimes}^{\mathsf{tw}}_{\mu \in \mathbb{Q}_{>0}} \mathcal{A}^{\theta\text{-}ss}_{Q,\mu} \boxtimes^{S\text{-}\mathsf{tw}} \mathcal{B}^{\theta\text{-}ss}_Q \xrightarrow{\star} \mathcal{B}_Q$$

is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda^{\sigma,+}_Q}$. The analogous statement for $\mathcal{M}_Q$ also holds.

Proof. As in the proof of Theorem 3.1, we identify $A^\bullet_{\mathsf{G}^\sigma_e}(R^{\sigma,HN}_{d^\bullet,e^\infty})_{\mathbb{Q}}$ with the tensor product $A^\bullet_{\mathsf{GL}_{d^1}}(R^{\theta\text{-}ss}_{d^1})_{\mathbb{Q}} \otimes \cdots \otimes A^\bullet_{\mathsf{GL}_{d^r}}(R^{\theta\text{-}ss}_{d^r})_{\mathbb{Q}} \otimes A^\bullet_{\mathsf{G}^\sigma_{e^\infty}}(R^{\sigma,\theta\text{-}ss}_{e^\infty})_{\mathbb{Q}}$ and consider the sections of the surjections

$$\Big(\bigotimes_{k=1}^r A^\bullet_{\mathsf{GL}_{d^k}}(R_{d^k})_{\mathbb{Q}}\Big) \otimes A^\bullet_{\mathsf{G}^\sigma_{e^\infty}}(R^\sigma_{e^\infty})_{\mathbb{Q}} \twoheadrightarrow \Big(\bigotimes_{k=1}^r A^\bullet_{\mathsf{GL}_{d^k}}(R^{\theta\text{-}ss}_{d^k})_{\mathbb{Q}}\Big) \otimes A^\bullet_{\mathsf{G}^\sigma_{e^\infty}}(R^{\sigma,\theta\text{-}ss}_{e^\infty})_{\mathbb{Q}}$$

coming from the σ-HN stratification. This leads to a commutative diagram

$$\begin{array}{ccc}
\Big(\bigotimes_{k=1}^{r} A^{\bullet}_{\mathsf{GL}_{d^k}}(R^{\theta\text{-}ss}_{d^k})_{\mathbb{Q}}\Big) \otimes A^{\bullet}_{\mathsf{G}^{\sigma}_{e^\infty}}(R^{\sigma,\theta\text{-}ss}_{e^\infty})_{\mathbb{Q}} & \xrightarrow{\cong} & A^{\bullet}_{\mathsf{G}^{\sigma}_{e}}(R^{\sigma,HN}_{d^\bullet,e^\infty})_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
\Big(\bigotimes_{k=1}^{r} A^{\bullet}_{\mathsf{GL}_{d^k}}(R_{d^k})_{\mathbb{Q}}\Big) \otimes A^{\bullet}_{\mathsf{G}^{\sigma}_{e^\infty}}(R^{\sigma}_{e^\infty})_{\mathbb{Q}} & \xrightarrow[\star]{} & A^{\bullet-\mathrm{codim}_{R^{\sigma}_{e}}(R^{\sigma,HN}_{d^\bullet,e^\infty})}_{\mathsf{G}^{\sigma}_{e}}(R^{\sigma}_{d^\bullet,e^\infty})_{\mathbb{Q}}
\end{array}$$

The vertical maps are the sections arising from the σ-HN stratifications. Comparing equation (10) with the twists appearing in the monoidal products $\boxtimes^{\mathsf{tw}}$ and $\boxtimes^{S\text{-}\mathsf{tw}}$, we conclude that the ChowHA-action induces the desired isomorphism $\overleftarrow{\boxtimes}^{\mathsf{tw}}_{\mu\in\mathbb{Q}_{>0}} \mathcal{A}^{\theta\text{-}ss}_{Q,\mu} \boxtimes^{S\text{-}\mathsf{tw}} \mathcal{B}^{\theta\text{-}ss}_{Q} \xrightarrow{\star} \mathcal{B}_Q$. By Theorem 3.1 the cohomological statement follows from the Chow theoretic statement. $\square$

Special cases of Theorem 3.4, for Q a finite type quiver or the cyclic affine A_1 quiver, were proved in [28, Theorems 4.9 and 5.8] by a direct study of $\mathcal{M}_Q$.

4. Applications to orientifold Donaldson-Thomas theory

4.1. **Orientifold DT invariants are Chow groups.** Recall that a quiver Q is called symmetric if its Euler form χ is a symmetric bilinear form. If Q is symmetric, then $\boxtimes^{\mathsf{tw}}$ reduces to the untwisted monoidal product and $\mathcal{A}^{\theta\text{-}ss}_{Q,\mu}$ becomes a $\Lambda^{+}_{Q} \times \mathbb{Z}$-graded algebra. Moreover, up to a twist of the multiplication by a sign, the algebra $\mathcal{A}^{\theta\text{-}ss}_{Q,\mu}$ is supercommutative [19, §2.6].

Similarly, for a fixed duality structure, a quiver is called σ-symmetric if it is symmetric and $\sigma^*\mathcal{E} = \mathcal{E}$. In this case $\boxtimes^{S\text{-}\mathsf{tw}}$ reduces to the untwisted monoidal module structure and $\mathcal{B}^{\theta\text{-}ss}_{Q}$ is a $\Lambda^{\sigma,+}_{Q} \times \mathbb{Z}$-graded $\mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0}$-module. If $\mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0}$ is supercommutative without any twist, then $\mathcal{B}^{\theta\text{-}ss}_{Q}$ is a super $\mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0}$-module. In general it is unknown if the supercommutative twist of $\mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0}$ can be lifted to $\mathcal{B}^{\theta\text{-}ss}_{Q}$. In any case, we will not use the supercommutative twist in this paper.

Let $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0,+}$ be the augmentation ideal of $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0}$. In [28] the cohomological orientifold Donaldson-Thomas invariant of a σ-symmetric quiver was defined to be the $\Lambda^{\sigma,+}_{Q} \times \mathbb{Z}$-graded vector space

$$W^{\mathsf{prim},\theta}_{Q} = \mathcal{M}^{\theta\text{-}ss}_{Q} / \mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0,+} \star \mathcal{M}^{\theta\text{-}ss}_{Q}.$$

Denote by $W^{\mathsf{prim},\theta}_{Q,(e,l)}$ the degree $(e,l) \in \Lambda^{\sigma,+}_{Q} \times \mathbb{Z}$ summand of W^{prim}_{Q}.

Theorem 4.1. *Let Q be a σ-symmetric quiver with σ-compatible stability θ. Then*

$$W^{\mathsf{prim},\theta}_{Q,(e,l)} \simeq \begin{cases} A_{-\frac{1}{2}(l+\mathcal{E}(e))}(\mathfrak{M}^{\sigma,\theta\text{-}st}_{e})_{\mathbb{Q}} & \textit{if } l+\mathcal{E}(e) \equiv 0 \mod 2, \\ 0 & \textit{if } l+\mathcal{E}(e) \equiv 1 \mod 2. \end{cases}$$

In particular, if $W^{\mathsf{prim},\theta}_{Q,(e,l)}$ is non-trivial, then $\mathcal{E}(e) \leq l \leq -\mathcal{E}(e)$.

Proof. By Theorem 3.1 the equivariant cycle map induces an isomorphism

$$\mathcal{B}^{\theta\text{-}ss}_{Q} / \mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0,+} \star \mathcal{B}^{\theta\text{-}ss}_{Q} \xrightarrow{\sim} \mathcal{M}^{\theta\text{-}ss}_{Q} / \mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0,+} \star \mathcal{M}^{\theta\text{-}ss}_{Q}$$

in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda^{\sigma,+}_{Q}}$. Proposition 3.3 implies that

$$\mathcal{B}^{\theta\text{-}st}_{Q} \simeq \mathcal{B}^{\theta\text{-}ss}_{Q} / \mathcal{A}^{\theta\text{-}ss}_{Q,\mu=0,+} \star \mathcal{B}^{\theta\text{-}ss}_{Q}. \tag{11}$$

It follows that we have an induced isomorphism $\mathcal{B}_Q^{\theta\text{-}st} \xrightarrow{\sim} W_Q^{\mathsf{prim},\theta}$. By definition, the degree $(e,l) \in \Lambda_Q^{\sigma,+} \times \mathbb{Z}$ component $\mathcal{B}_{Q,(e,l)}^{\theta\text{-}st}$ of $\mathcal{B}_Q^{\theta\text{-}st}$ is trivial unless $l - \mathcal{E}(e)$ is even, in which case we have

$$\begin{aligned} \mathcal{B}_{Q,(e,l)}^{\theta\text{-}st} &= A_{\mathsf{G}_e^\sigma}^{\frac{1}{2}(l-\mathcal{E}(e))}(R_e^{\sigma,\theta\text{-}st})_{\mathbb{Q}} \\ &\simeq A^{\frac{1}{2}(l-\mathcal{E}(e))}(\mathbf{M}_e^{\sigma,\theta\text{-}st})_{\mathbb{Q}} \\ &\simeq A^{\frac{1}{2}(l-\mathcal{E}(e))}(\mathfrak{M}_e^{\sigma,\theta\text{-}st})_{\mathbb{Q}} \\ &\simeq A_{-\frac{1}{2}(l+\mathcal{E}(e))}(\mathfrak{M}_e^{\sigma,\theta\text{-}st})_{\mathbb{Q}}. \end{aligned}$$

These isomorphisms follow from [10, Theorem 4] together with the fact that, if non-empty, the complex dimension of $\mathfrak{M}_e^{\sigma,\theta\text{-}st}$ is $-\mathcal{E}(e)$. The final statement of the theorem also follows from this dimension formula. □

Corollary 4.2. *For each $e \in \Lambda_Q^{\sigma,+}$ the vector space $\bigoplus_{l\in\mathbb{Z}} W_{Q,(e,l)}^{\theta,\mathsf{prim}}$ is finite dimensional.*

The statement of Corollary 4.2 is known as the orientifold integrality conjecture. A direct but complicated proof of Corollary 4.2 in the case of trivial stability was given in [28, Theorem 3.4].

The motivic orientifold Donaldson-Thomas invariant of Q is defined by

$$\Omega_Q^{\sigma,\theta}(q^{\frac{1}{2}},\xi) = \sum_{(e,l)\in\Lambda_Q^{\sigma,+}\times\mathbb{Z}} \dim_{\mathbb{Q}} W_{Q,(e,l)}^{\mathsf{prim},\theta}(-q^{\frac{1}{2}})^l \xi^e \in \mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}][\![\xi]\!].$$

By Theorem 4.1 the invariant $\Omega_Q^{\sigma,\theta}$ is the generating series of shifted Chow theoretic Poincaré polynomials of $\mathfrak{M}_e^{\sigma,\theta\text{-}st}$. Again by Theorem 4.1, if we define a normalized invariant by

$$\overline{\Omega}_{Q,e}^{\sigma,\theta} = (-q^{\frac{1}{2}})^{-\mathcal{E}(e)}\Omega_{Q,e}^{\sigma,\theta},$$

then $\overline{\Omega}_{Q,e}^{\sigma,\theta} \in \mathbb{Z}_{\geq 0}[q]$ with constant term 1 and degree at most $-\mathcal{E}(e)$.

Example. A symmetric quiver Q admits an essentially unique involution σ which restricts to the identity on Q_0. Any duality structure on (Q,σ) is automatically σ-symmetric. The only σ-compatible stability is the trivial stability. In this setting we have $\overline{\Omega}_{Q,e}^{\sigma} \in \mathbb{Z}_{\geq 0}[q^2]$. Geometrically, using Theorem 4.1 and Corollary 3.2 this implies that

$$A_{-\mathcal{E}(e)-2k-1}(\mathfrak{M}_e^{\sigma,st})_{\mathbb{Q}} = 0$$

for all $k \in \mathbb{Z}$ and

$$PH^k(\mathfrak{M}_e^{\sigma,st}) = 0$$

unless $k \equiv 0 \mod 4$. ◁

Remark. Without the assumption of σ-symmetry, Proposition 3.3 still implies equation (11). However in this generality we do not expect the right-hand side to be the correct definition of the orientifold DT invariant.

4.2. **Orientifold DT invariants of loop quivers.** Let L_m be the quiver with one node and $m \geq 0$ loops. Then L_m admits a unique involution, being the identity on both nodes and arrows, and we are in the setting of the example from Section 4.1. Any stability is equivalent to the trivial stability $\theta = 0$, which is σ-compatible. A duality structure on L_m is given by a sign s and m signs τ. Let τ_+ (respectively, τ_-) be the number of the latter which are positive (negative).

Suppose that τ is identically -1, that is $\tau_- = m$. If $s = 1$, then the variety of self-dual representations is $R_e^\sigma = \mathfrak{so}_e(\mathbb{C})^{\oplus m}$ with the simultaneous adjoint action of $\mathsf{G}_e^\sigma = \mathsf{O}_e(\mathbb{C})$ while if $s = -1$, then $R_e^\sigma = \mathfrak{sp}_e(\mathbb{C})^{\oplus m}$ with the simultaneous adjoint action of $\mathsf{G}_e^\sigma = \mathsf{Sp}_e(\mathbb{C})$. Hence $\mathfrak{M}_e^{\sigma,st}$ is a moduli space of stable m-tuples in a classical Lie algebra. When $s=1$ we have a decomposition $\mathcal{B}_{L_m} = \mathcal{B}^D_{L_m} \oplus \mathcal{B}^B_{L_m}$, the summands corresponding to even and odd dimensional representations, respectively.

Theorem 4.3. *For any $m \geq 0$ and $e \geq 0$ we have an isomorphism*

$$A_\bullet((\mathfrak{sp}_{2e}^{\oplus m})^{st}/\mathsf{Sp}_{2e})_{\mathbb{Q}} \simeq A_\bullet((\mathfrak{so}_{2e+1}^{\oplus m})^{st}/\mathsf{O}_{2e+1})_{\mathbb{Q}}$$

of $\mathbb{Z}$-graded vector spaces.

Proof. Define a $\Lambda^+_{L_m} \times \mathbb{Z}$-graded ring automorphism $\phi : \mathcal{H}_{L_m} \to \mathcal{H}_{L_m}$ by $\phi(f_d) = 2^d f_d$ for $f_d \in \mathcal{H}_{L_m,d}$. Let $(\mathcal{M}^B_{L_m})_\phi$ be the ϕ-twisted module associated to $\mathcal{M}^B_{L_m}$. Explicitly, $(\mathcal{M}^B_{L_m})_\phi$ equals $\mathcal{M}^B_{L_m}$ as a graded abelian group and has $\mathcal{H}_{L_m}$-module structure

$$f_d \star_\phi g = \phi(f_d) \star g.$$

Comparing the explicit signed shuffle descriptions of $\mathcal{M}^B_{L_m}$ and $\mathcal{M}^C_{L_m}$ given in [28, Theorem 3.3], we see immediately that $(\mathcal{M}^B_{L_m})_\phi \simeq \mathcal{M}^C_{L_m}[1]$ via the identity map. Here [1] denotes $\Lambda^{\sigma,+}_{L_m}$-degree shift by one. It follows that $W^{\mathsf{prim},B}_{L_m} \simeq W^{\mathsf{prim},C}_{L_m}[1]$ as $\Lambda^+_{L_m} \times \mathbb{Z}$-graded vector spaces. Applying Theorem 4.1 completes the proof. □

Variations of Theorem 4.3 arise by choosing different duality structures on L_m. Fix $1 \leq m_0 \leq m$ and consider the duality structures

$$(s; \tau_+, \tau_-) = (-1; m - m_0, m_0), \qquad (s'; \tau'_+, \tau'_-) = (1; m_0 - 1, m - m_0 + 1).$$

The corresponding varieties of self-dual representations are

$$R^\sigma_e = \mathfrak{sp}_{2e}(\mathbb{C})^{\oplus m_0} \oplus (\bigwedge\nolimits^2 \mathbb{C}^{2e})^{\oplus m - m_0}, \qquad \mathsf{G}^\sigma_e = \mathsf{Sp}_{2e}(\mathbb{C})$$

and, in even dimensions,

$$R'^\sigma_e = \mathfrak{so}_{2e}(\mathbb{C})^{\oplus m - m_0 + 1} \oplus (\mathrm{Sym}^2 \mathbb{C}^{2e})^{\oplus m_0 - 1}, \qquad \mathsf{G}'^\sigma_e = \mathsf{O}_{2e}(\mathbb{C}).$$

The vector space $\mathbb{C}^{2e}$ denotes the fundamental representation of $\mathsf{Sp}_{2e}(\mathbb{C})$ or $\mathsf{O}_{2e}(\mathbb{C})$, as appropriate. Up to a twist by an automorphism of $\mathcal{H}_{L_m}$, the associated Hall modules $\mathcal{M}^C_{L_m}$ and $\mathcal{M}'^D_{L_m}$ are isomorphic. Arguing as in the proof of Theorem 4.3 we see that the rational Chow groups of the associated moduli spaces of σ-stable self-dual representations are isomorphic. Similarly, for $m \geq 1$ and duality structures

$$(s; \tau_+, \tau_-) = (1; 1, m-1), \qquad (s'; \tau'_+, \tau'_-) = (1; m, 0)$$

we obtain a twisted isomorphism between $\mathcal{M}^B_{L_m}$ and $\mathcal{M}'^D_{L_m}[1]$ and hence an isomorphism of Chow groups.

Finally, we explain how the invariants $\Omega^\sigma_{L_m}$, for any duality structure, can be computed. After fixing a type B, C or D, in [28, Theorem 4.6] it is proved that $\mathcal{M}_{L_m}$ is a free module with basis $W^{\mathsf{prim}}_{L_m}$ over an explicitly defined subalgebra $\widetilde{\mathcal{H}}_{L_m} \subset \mathcal{H}_{L_m}$. At the level of generating series this implies the factorization $A^\sigma_{L_m} = \widetilde{A}_{L_m} \Omega^\sigma_{L_m}$. Here

$$A^\sigma_{L_m} = \sum_{e \in \Lambda^{\sigma,+}_{L_m}} \frac{(-q^{\frac{1}{2}})^{\mathcal{E}(e)}}{\prod_{j=1}^{\lfloor \frac{e}{2} \rfloor}(1 - q^{2j})} \xi^e$$

is the parity twisted Hilbert-Poincaré series of $\mathcal{M}_{L_m}$ and e is restricted to be even or odd depending on the type. Similarly, the parity twisted Hilbert-Poincaré series $\widetilde{A}_{L_m}$ of $\widetilde{\mathcal{H}}_{L_m}$ can be written in terms of the q-Pochhammer symbol $(x;q)_\infty = \prod_{k=0}^\infty (1 - xq^k)$ and the $\mathbb{Z}_2$-equivariant DT invariants:

$$\widetilde{A}_{L_m} = \prod_{\substack{(e,k) \in \Lambda^{\sigma,+}_{L_m} \times \mathbb{Z} \\ \lambda \in \{\pm\}}} (q^{\frac{k}{2} + \delta_{-1,\lambda}} \xi^e; q^2)_\infty^{-\widetilde{\Omega}^\lambda_{L_m,(e,k)}}.$$

Explicitly, writing the motivic DT invariant of L_m as

$$\Omega_{L_m}(q^{\frac{1}{2}}, t) = \sum_{(d,k) \in \Lambda^+_{L_m} \times \mathbb{Z}} \Omega_{L_m,(d,k)} q^{\frac{k}{2}} t^d \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}][\![t]\!],$$

we have $\widetilde{\Omega}^{\pm}_{L_m,(2d+1,k)} = 0$ and

$$\widetilde{\Omega}^{+}_{L_m,(2d,k)} = \begin{cases} \Omega_{L_m,(d,k)} & \text{if } \chi(e,d) + \mathcal{E}(d) + \frac{k-\chi(d,d)}{2} \equiv 0 \mod 2, \\ 0 & \text{if } \chi(e,d) + \mathcal{E}(d) + \frac{k-\chi(d,d)}{2} \equiv 1 \mod 2 \end{cases}$$

and

$$\widetilde{\Omega}^{-}_{L_m,(2d,k)} = \begin{cases} 0 & \text{if } \chi(e,d) + \mathcal{E}(d) + \frac{k-\chi(d,d)}{2} \equiv 0 \mod 2, \\ \Omega_{L_m,(d,k)} & \text{if } \chi(e,d) + \mathcal{E}(d) + \frac{k-\chi(d,d)}{2} \equiv 1 \mod 2. \end{cases}$$

Example. Suppose that $m = 3$. Using [25, Theorem 6.8] we find that the motivic DT invariant is given by

$$\Omega_{L_3} = q^{-1}t + q^{-4}t^2 + q^{-9}(1 + q^2 + q^3)t^3 + \\ q^{-16}(1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + q^8)t^4 + O(t^5).$$

Take the duality structure $(s; \tau_+, \tau_-) = (1; 0, m)$. Then the motivic orientifold DT invariant is

$$\Omega^B_{L_3} = \xi + q^{-3}\xi^3 + q^{-10}(1 + q^2 + 2q^4)\xi^5 + \\ q^{-21}(1 + q^2 + 2q^4 + 3q^6 + 4q^8 + 4q^{10} + 4q^{12} + q^{14})\xi^7 + \\ q^{-36}(1 + q^2 + 2q^4 + 3q^6 + 5q^8 + 6q^{10} + 9q^{12} + 10q^{14} + \\ 13q^{16} + 14q^{18} + 15q^{20} + 13q^{22} + 10q^{24} + 3q^{26})\xi^9 + O(\xi^{11}).$$

Theorem 4.1 implies that the coefficient of ξ^{2e+1} in $\Omega^B_{L_3}$ is the Chow theoretic Poincaré polynomial of $(\mathfrak{so}^{\oplus 3}_{2e+1})^{st}/\mathsf{O}_{2e+1}$, which by Theorem 4.3 agrees with the Chow theoretic Poincaré polynomial of $(\mathfrak{sp}^{\oplus 3}_{2e})^{st}/\mathsf{Sp}_{2e}$. ◁

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