The star-shapedness of a generalized numerical range

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Abstract

Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices and $\mathcal{H}_n^m$ be the set of all m-tuples of $n \times n$ Hermitian matrices. For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ and for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$, we define the L-numerical range of A by

$$W_L(A) := \{L(U^*A_1U, ..., U^*A_mU) : U \in \mathbb{C}^{n\times n}, U^*U = I_n\}.$$

In this paper, we prove that if $\ell \leq 3$, $n \geq \ell$ and $A_1, ..., A_m$ are simultaneously unitarily diagonalizable, then $W_L(A)$ is star-shaped with star center at $L\left(\frac{\mathrm{tr}A_1}{n} I_n, ..., \frac{\mathrm{tr}A_m}{n} I_n\right)$.

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1 Introduction

Let $\mathbb{C}^{n\times n}$ denote the set of all $n \times n$ complex matrices, and $A \in \mathbb{C}^{n\times n}$. The (classical) numerical range of A is defined by

$$W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

The properties of $W(A)$ were studied extensively in the last few decades and many nice results were obtained; see [10, 13]. The most beautiful result is probably the Toeplitz-Hausdorff Theorem which affirmed the convexity of $W(A)$; see [12, 17]. The generalizations of $W(A)$ remain an active research area in the field.

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For any $A \in \mathbb{C}^{n\times n}$, write $A = A_1 + iA_2$ where A_1, A_2 are Hermitian matrices. Then by regarding $\mathbb{C}$ as $\mathbb{R}^2$, one can rewrite $W(A)$ as

$$W(A) := \{(x^*A_1x, x^*A_2x) : x \in \mathbb{C}^n, x^*x = 1\}.$$

This expression motivates naturally the generalization of the numerical range to the joint numerical range, which is defined as follows. Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices and $\mathcal{H}_n^m$ be the set of all m-tuples of $n \times n$ Hermitian matrices. The joint numerical range of $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ is defined as

$$W(A) = W(A_1, ..., A_m) := \{(x^*A_1x, ..., x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.$$

It has been shown that for $m \leq 3$ and $n \geq m$, the joint numerical range is always convex [1]. This result generalizes the Toeplitz-Hausdorff Theorem. However, the convexity of the joint numerical range fails to hold in general for $m > 3$, see [1, 11, 14].

When a new generalization of numerical range is introduced, people are always interested in its convexity. Unfortunately, this nice property fails to hold in some generalizations. However, another property, namely star-shapedness, holds in some generalizations; see [5, 18]. Therefore, the star-shapedness is the next consideration when the generalized numerical ranges fail to be convex. A set M is called star-shaped with respect to a star-center $x_0 \in M$ if for any $0 \leq \alpha \leq 1$ and $x \in M$, we have $\alpha x + (1-\alpha)x_0 \in M$. In [15], Li and Poon showed that for a given m, the joint numerical range $W(A_1, ..., A_m)$ is star-shaped if n is sufficiently large.

Let $\mathcal{U}_n$ be the set of all $n \times n$ unitary matrices. For $C \in \mathcal{H}_n$ and $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$, the joint C-numerical range of A is defined by

$$W_C(A) := \{(\mathrm{tr}(CU^*A_1U), ..., \mathrm{tr}(CU^*A_mU) : U \in \mathcal{U}_n\},$$

where $\mathrm{tr}(\cdot)$ is the trace function. When C is the diagonal matrix with diagonal elements $1, 0, ..., 0$, then $W_C(A)$ reduces to $W(A)$. Hence the joint C-numerical range is a generalization of the joint numerical range. In [3], Au-Yeung and Tsing generalized the convexity result of the joint numerical range to the joint C-numerical range by showing that $W_C(A)$ is always convex if $m \leq 3$ and $n \geq m$. However $W_C(A)$ fails to be convex in general if $m > 3$. One may consult [6] and [7] for the study of the convexity of $W_C(A)$. The star-shapedness of $W_C(A)$ remains unclear for $m > 3$.

For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$, we define the joint unitary orbit of A by

$$\mathcal{U}_n(A) := \{(U^*A_1U, ..., U^*A_mU) : U \in \mathcal{U}_n\}.$$

For $C \in \mathcal{H}_n$, we consider the linear map $L_C : \mathcal{H}_n^m \to \mathbb{R}^m$ defined by

$$L_C(X_1, ..., X_m) = (\mathrm{tr}(CX_1), ..., \mathrm{tr}(CX_m)).$$

Then the joint C-numerical range of A is the linear image of $\mathcal{U}_n(A)$ under L_C. Inspired by this alternative expression, we consider the following generalized numerical range of $A \in \mathcal{H}_n^m$. For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ and linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$, we define

$$W_L(A) = L(\mathcal{U}_n(A)) := \{L(U^*A_1U, ..., U^*A_mU) : U \in \mathcal{U}_n\},$$

and call it the L-numerical range of A, due to [4]. Because L_C is a special case of general linear maps L, the L-numerical range generalizes the joint C-numerical range and hence the classical numerical range.

In this paper, We shall study in Section two an inclusion relation of the L-numerical range of m-tuples of simultaneously unitarily diagonalizable Hermitian matrices and linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell = 2, 3$. This inclusion relation will be applied in Section three to show that the L-numerical ranges of A under our consideration are star-shaped.

2 An Inclusion Relation for L-numerical Ranges

The following results follow easily from the the definition of the L-numerical range.

Lemma 2.1. *Let* $(A_1, ..., A_m) \in \mathcal{H}_n^m$ *and* $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ *be linear. Then the followings hold:*

(i) $W_L(\alpha(A_1, ..., A_m) + \beta(I_n, ..., I_n)) = \alpha W_L(A_1, ..., A_m) + \beta L(I_n, ..., I_n)$ *if* $\alpha, \beta \in \mathbb{R}$*;*

(ii) $W_L(U^*A_1U, ..., U^*A_mU) = W_L(A_1, ..., A_m)$ *for all unitary* U*.*

In the following we shall consider those $A_1, ..., A_m$ which are simultaneously unitarily diagonalizable, i.e., there exists $U \in \mathcal{U}_n$ such that $U^*A_1U, ..., U^*A_mU$ are all diagonal. Hence by Lemma 2.1, we assume without loss of generality that $A_1, ..., A_m$ are (real) diagonal matrices. For $d = (d_1, ..., d_n)^T \in \mathbb{R}^n$, we denote by $\text{diag}(d)$ the $n \times n$ diagonal matrix with diagonal elements $d_1, ..., d_n$. We first introduce a special class of matrices which is useful in studying the generalized numerical range; see [9, 16, 18].

An $n \times n$ real matrix $P = (p_{ij})$ is called a pinching matrix if for some $1 \le s < t \le n$ and $0 \le \alpha \le 1$,

$$p_{ij} = \begin{cases} \alpha, & \text{if } (i,j) = (s,s) \text{ or } (t,t), \\ 1-\alpha, & \text{if } (i,j) = (s,t) \text{ or } (t,s), \\ 1, & \text{if } i = j \neq s, t, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2. *Assume* $D = (\text{diag}(d^{(1)}), ..., \text{diag}(d^{(m)}))$*,* $\hat{D} = (\text{diag}(\hat{d}^{(1)}), ..., \text{diag}(\hat{d}^{(m)}))$ *where* $d^{(1)}, ..., d^{(m)}, \hat{d}^{(1)}, ...\hat{d}^{(m)} \in \mathbb{R}^n$*. We say* $\hat{D} \prec D$ *if there exist a finite number of pinching matrices* $P_1, ..., P_k$ *such that* $\hat{d}^{(i)} = P_1P_2 \cdots P_k d^{(i)}$ *for all* $i = 1, ..., m$*.*

The following inclusion relation is the main result in this section.

Theorem 2.3. *Let $D, \hat{D} \in \mathcal{H}_n^m$ and $n > 2$. If $\hat{D} \prec D$, then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$, we have $W_L(\hat{D}) \subset W_L(D)$.*

To prove Theorem 2.3, we need some lemmas. For $\theta, \phi \in \mathbb{R}$, let $T_{\theta,\phi} \in \mathcal{U}_n$ be defined by

$$T_{\theta,\phi} = \begin{pmatrix} \cos\theta & \sin\theta e^{\sqrt{-1}\phi} & 0 \\ -\sin\theta & \cos\theta e^{\sqrt{-1}\phi} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}.$$

Lemma 2.4. *Let $D = (D_1, ..., D_m) \in \mathcal{H}_n^m$ be an m-tuple of diagonal matrices. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$ and $U \in \mathcal{U}_n$, the set of points*

$$E_L(D,U) := \{L(U^*T_{\theta,\phi}^* D_1 T_{\theta,\phi} U, ..., U^*T_{\theta,\phi}^* D_m T_{\theta,\phi} U) : \theta \in [0,\pi],\ \ \phi \in [0,2\pi]\}$$

forms an ellipsoid in $\mathbb{R}^3$.

Proof. Note that for any $L : \mathcal{H}_n^m \to \mathbb{R}^3$, we can always express L as

$$L(X_1, ..., X_m) = \left(\operatorname{tr}\left(\sum_{i=1}^m P_i X_i\right), \operatorname{tr}\left(\sum_{i=1}^m Q_i X_i\right), \operatorname{tr}\left(\sum_{i=1}^m R_i X_i\right)\right)$$

for some suitable $P_i, Q_i, R_i \in \mathcal{H}_n$, $i = 1, ..., m$. For $U \in \mathcal{U}_n$, we write $UP_iU^* = (p_{jk}^{(i)})$, $UQ_iU^* = (q_{jk}^{(i)})$, $UR_iU^* = (r_{jk}^{(i)})$ and $D_i = \operatorname{diag}(d_1^{(i)}, ..., d_n^{(i)})$, $i = 1, ..., m$. By direct computations, the first coordinate of points in $E_L(D,U)$ is

$$\begin{aligned} &\operatorname{tr}\left(\sum_{i=1}^m P_i U^* T_{\theta,\phi}^* D_i T_{\theta,\phi} U\right) \\ &= \operatorname{tr}\left(\sum_{i=1}^m D_i T_{\theta,\phi} U P_i U^* T_{\theta,\phi}^*\right) \\ &= \frac{1}{2}\sum_{i=1}^m (d_1^{(i)} + d_2^{(i)})(p_{11}^{(i)} + p_{22}^{(i)}) + \sum_{i=1}^m \sum_{j=3}^n d_j^{(i)} p_{jj}^{(i)} \\ &\quad + \frac{1}{2}\sum_{i=1}^m (d_1^{(i)} - d_2^{(i)})(p_{11}^{(i)} - p_{22}^{(i)})\cos 2\theta \\ &\quad + \sum_{i=1}^m (d_1^{(i)} - d_2^{(i)})\operatorname{Re}(p_{21}^{(i)} e^{\sqrt{-1}\phi})\sin 2\theta. \end{aligned}$$

Similarly for the second and the third coordinates of points in $E_L(D,U)$. Note that for $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $a_3, b_3, c_3 \in \mathbb{C}$, the points $(a_1, b_1, c_1) + (a_2, b_2, c_2)\cos 2\theta + \operatorname{Re}(a_3 e^{\sqrt{-1}\phi}, b_3 e^{\sqrt{-1}\phi}, c_3 e^{\sqrt{-1}\phi})\sin 2\theta$ form an ellipsoid in $\mathbb{R}^3$ when θ, ϕ run through $[0,\pi]$ and $[0,2\pi]$ respectively. Hence $E_L(D,U)$ is an ellipsoid in $\mathbb{R}^3$. □

Note that $E_L(D,U) \subset W_L(D)$ for any $U \in \mathcal{U}_n$.

Lemma 2.5. *Let $D \in \mathcal{H}_n^m$ be an m-tuple of diagonal matrices with $n > 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$, there exists $V \in \mathcal{U}_n$ such that $E_L(D, V)$ defined in Lemma 2.4 degenerates (i.e., $E_L(D, V)$ is contained in a plane in $\mathbb{R}^3$).*

Proof. Following the notations in Lemma 2.4 and its proof, we let $\alpha_i = d_1^{(i)} - d_2^{(i)}$ for $i = 1, ..., m$ and $P' = \sum_{i=1}^m \alpha_i P_i \in \mathcal{H}_n$. Since $n > 2$, by generalized interlacing inequalities for eigenvalues of Hermitian matrices (see [8]), there exist $V \in \mathcal{U}_n$ and $\alpha \in \mathbb{R}$ such that $VP'V^*$ has αI_2 as leading 2×2 principal submatrix. For any matrix M, let M_{ij} denote its (i, j) entry. Then by taking $U = V$ in the proof of Lemma 2.4, the first coordinate of points in $E_L(D, V)$ is $a + b \cos 2\theta + c \sin 2\theta$ where

$$
\begin{aligned}
a &= \frac{1}{2}\sum_{i=1}^m (d_1^{(i)} + d_2^{(i)})(p_{11}^{(i)} + p_{22}^{(i)}) + \sum_{i=1}^m \sum_{j=3}^n d_j^{(i)} p_{ii} \\
b &= \frac{1}{2}\sum_{i=1}^m \alpha_i \left[(VP_iV^*)_{11} - (VP_iV^*)_{22}\right] \\
&= \frac{1}{2}\left(V\left(\sum_{i=1}^m \alpha_i P_i\right)V^*\right)_{11} - \frac{1}{2}\left(V\left(\sum_{i=1}^m \alpha_i P_i\right)V^*\right)_{22} \\
&= \frac{1}{2}(VP'V^*)_{11} - \frac{1}{2}(VP'V^*)_{22} \\
&= \frac{1}{2}\alpha - \frac{1}{2}\alpha = 0, \\
c &= \sum_{i=1}^m \alpha_i \mathrm{Re}\left((VP_iV^*)_{21} e^{\sqrt{-1}\phi}\right) \\
&= \mathrm{Re}\left[\left(V\left(\sum_{i=1}^m \alpha_i P_i\right)V^*\right)_{21} e^{\sqrt{-1}\phi}\right] \\
&= \mathrm{Re}((VP'V^*)_{21} e^{\sqrt{-1}\phi}) = 0.
\end{aligned}
$$

Since the first coordinate of points in $E_L(D, V)$ is constant for $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, $E_L(D, V)$ degenerates. □

Proof of Theorem 2.3. Let $D = (D_1, ..., D_m) = (\mathrm{diag}(d^{(1)}), ..., \mathrm{diag}(d^{(m)}))$ and $\hat{D} = (\hat{D}_1, ..., \hat{D}_m) = (\mathrm{diag}(\hat{d}^{(1)}), ..., \mathrm{diag}(\hat{d}^{(m)}))$ where $d^{(1)}, .., d^{(m)}, \hat{d}^{(1)}, ..., \hat{d}^{(m)} \in \mathbb{R}^n$. We may further assume without loss of generality that $\hat{d}^{(i)} = Pd^{(i)}$ for all $i = 1, ..., m$ and $P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix} \oplus I_{n-2}$ with $0 \leq \alpha \leq 1$. Then we have

$$\hat{D}_i = \alpha T_{0,0}^* D_i T_{0,0} + (1-\alpha) T_{\frac{\pi}{2},0}^* D_i T_{\frac{\pi}{2},0}, \quad i = 1, ..., m.$$

For any $U \in \mathcal{U}_n$, we have $L(U^*\hat{D}U) \in \mathrm{conv}(E_L(D, U))$ where $\mathrm{conv}(\cdot)$ denotes the convex hull. By path-connectedness of $\mathcal{U}_n$, there exists a continuous function

$f : [0,1] \to \mathcal{U}_n$ such that $f(0) = U$ and $f(1) = V$ where V is defined in Lemma 2.5 and hence $E(D, f(1))$ degenerates. By continuity, there exists $t \in [0,1]$ such that $L(U^*\hat{D}U) \in E(D, f(t)) \subset W_L(D)$. $\square$

Using similar techniques, one can prove that Theorem 2.3 stills holds for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^2$ with $n \geq 2$. However, the following example shows that the inclusion relation in Theorem 2.3 fails to hold if $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ is linear with $\ell > 3$.

Example 2.6. *Let $n \geq 2$, $d = (1, ..., 0)^T$, $\hat{d} = (\frac{1}{2}, \frac{1}{2}, 0, ..., 0)^T \in \mathbb{R}^n$ and let O_k be the $k \times k$ zero matrix. Consider $D = (\mathrm{diag}(d), O_n, ..., O_n)$, $\hat{D} = (\mathrm{diag}(\hat{d}), O_n, ..., O_n) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell \geq 4$ defined by*

$$L(X_1, ..., X_m) = (\mathrm{tr}(PX_1), \mathrm{tr}(QX_1), \mathrm{tr}(RX_1), \mathrm{tr}(SX_1), 0, ..., 0)$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus O_{n-2}, \quad Q = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus O_{n-2},$$
$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus O_{n-2}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus O_{n-2}.$$

Then we have $\hat{D} \prec D$ and $(1, 0, ..., 0) \in W_L(\hat{D})$, but $(1, 0, ..., 0) \notin W_L(D)$.

3 Star-shapedness of the L-numerical range

The L-numerical range may fail to be convex for linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell \geq 2$ even when $A_1, ..., A_m \in \mathcal{H}_n$ are simultaneously unitarily diagonalizable; see [2]. However, we shall show in this section that for $n > 2$, $W_L(A_1, ..., A_m)$ is always star-shaped for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^3$ and simultaneously unitarily diagonalizable $A_1, ..., A_m \in \mathcal{H}_n$. The following result is the essential element in our proof.

Proposition 3.1. *[18] Let $\mathbb{P}_n$ be the set of all finite products of $n \times n$ pinching matrices. Then for $0 \leq \alpha \leq 1$, $\alpha I_n + (1-\alpha)J_n$ is in the closure of $\mathbb{P}_n$ where J_n is the $n \times n$ matrix with all entries equal $1/n$.*

Note that for any $A \in \mathcal{H}_n^m$, $\mathcal{U}_n(A)$ is compact. Hence $W_L(A)$ is compact for all linear maps L.

Theorem 3.2. *Let $D = (D_1, ..., D_m) \in \mathcal{H}_n^m$ be an m-tuple of diagonal matrices with $n > 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$, $W_L(D)$ is star-shaped with respect to star-center $L(\frac{\mathrm{tr}D_1}{n} I_n, ..., \frac{\mathrm{tr}D_m}{n} I_n)$.*

Proof. By Lemma 2.1, we may assume without loss of generality that $\mathrm{tr}D_i = 0$ for $i = 1, ..., m$; otherwise we replace D_i by $D_i - \frac{\mathrm{tr}D_i}{n} I_n$. Let $D_i = \mathrm{diag}(d^{(i)})$ where $d^{(i)} \in \mathbb{R}^n$, $i = 1, ..., m$. For any $0 \leq \alpha \leq 1$, we have $\alpha d^{(i)} = [\alpha I_n + (1-\alpha)J_n]d^{(i)}$. Then for any $U \in \mathcal{U}_n$, by Proposition 3.1, Theorem 2.3 and the compactness of $W_L(D)$, we have $\alpha L(U^*DU) \in W_L(\alpha D) \subset \overline{W_L(D)} = W_L(D)$ where $\overline{M}$ denotes the closure of M. $\square$

For a linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$, by regarding it as a projection of some linear map $\hat{L} : \mathcal{H}_n^m \to \mathbb{R}^3$, we deduce the following corollary easily.

Corollary 3.3. *Let $D = (D_1, ..., D_m) \in \mathcal{H}_n^m$ be an m-tuple of diagonal matrices with $n \geq 2$. Then for any linear map $L : H_n^m \to \mathbb{R}^2$, $W_L(D)$ is star-shaped with respect to star-center $L(\frac{\mathrm{tr} D_1}{n} I_n, ..., \frac{\mathrm{tr} D_m}{n} I_n)$.*

Proof. We only need to consider the case $n = 2$. We may assume without loss of generality that $m = 1$ and $D = \mathrm{diag}(1, -1)$. For any linear map $L : \mathcal{H}_2 \to \mathbb{R}^2$, we express it as $L(X) := (\mathrm{tr}(PX), \mathrm{tr}(QX))$ for some $P, Q \in \mathcal{H}_2$. Then we have

$$\begin{aligned} W_L(D) &= \{2(x^*Px, x^*Qx) - (\mathrm{tr}P, \mathrm{tr}Q) : x \in \mathbb{C}^n, x^*x = 1\} \\ &= 2W(P, Q) - (\mathrm{tr}P, \mathrm{tr}Q), \end{aligned}$$

which is convex and contains the origin. This implies that $W_L(D)$ is star-shaped with respect to star-center $L\left(\frac{\mathrm{tr}\, D}{n} I_2\right)$, which is the origin. □

Note that the star-shapedness of the L-numerical range for linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell > 3$ remains open in the diagonal case. Moreover, for general cases of $A = (A_1, ..., A_m)$ where $A_1, ..., A_m$ are not necessarily simultaneously unitarily diagonlizable and $L : \mathcal{H}_n^m \to \mathbb{R}^2$ with $m \geq 3$, the star-shapedness of $W_L(A)$ is also unclear. However, by applying a result in [4], we can show that $L(\frac{\mathrm{tr} A_1}{n} I_n, ..., \frac{\mathrm{tr} A_m}{n} I_n) \in W_L(A_1, ..., A_m)$ for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^2$.

Proposition 3.4 ([4], P. 23.). *Let $A_k = (a_{ij}^{(k)}) \in \mathcal{H}_n$, $k = 1, ..., m$. For $0 \leq \epsilon \leq 1$, define $A_k(\epsilon)$ as*

$$A_k(\epsilon) = \begin{pmatrix} a_{11}^{(k)} & \epsilon a_{12}^{(k)} & \cdots & \epsilon a_{1n}^{(k)} \\ \epsilon a_{21}^{(k)} & a_{22}^{(k)} & \cdots & \epsilon a_{2n}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon a_{n1}^{(k)} & \epsilon a_{12}^{(k)} & \cdots & a_{nn}^{(k)} \end{pmatrix}, \quad k = 1, ..., m.$$

Then $W_L(A_1(\epsilon), ..., A_m(\epsilon)) \subseteq W_L(A_1, ..., A_m)$ for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$.

Theorem 3.5. *Let $A = (A_1, ...A_m) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^2$ be linear. Then $L(\frac{\mathrm{tr} A_1}{n} I_n, ..., \frac{\mathrm{tr} A_m}{n} I_n) \in W_L(A)$.*

Proof. Define $A_i(\epsilon)$ as in Proposition 3.4 and note that $\mathrm{tr} A_i(\epsilon) = \mathrm{tr} A_i$ for $i = 1, ..., m$. Hence by Corollary 3.3 and Proposition 3.4, we have

$$L\left(\frac{\mathrm{tr} A_1}{n} I_n, ..., \frac{\mathrm{tr} A_m}{n} I_n\right) \in W_L(A_1(0), ..., A_m(0)) \subseteq W_L(A_1, ...A_m).$$

□

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