The star-shapedness of a generalized numerical range

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Abstract

Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices and $\mathcal{H}_m^n$ be the set of all $m$-tuples of $n \times n$ Hermitian matrices. For $A = (A_1, ..., A_m) \in \mathcal{H}_m^n$ and for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$, we define the $L$-numerical range of $A$ by

$$W_L(A) := \{ L(U^*A_1U, ..., U^*A_mU) : U \in \mathbb{C}^{n \times n}, U^*U = I_n \}.$$ 

In this paper, we prove that if $\ell \leq 3$, $n \geq \ell$ and $A_1, ..., A_m$ are simultaneously unitarily diagonalizable, then $W_L(A)$ is star-shaped with star center at $L \left( \frac{\text{tr} A_1}{n} I_n, ..., \frac{\text{tr} A_m}{n} I_n \right)$.

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1 Introduction

Let $\mathbb{C}^{n \times n}$ denote the set of all $n \times n$ complex matrices, and $A \in \mathbb{C}^{n \times n}$. The (classical) numerical range of $A$ is defined by

$$W(A) := \{ x^*Ax : x \in \mathbb{C}^n, x^*x = 1 \}.$$ 

The properties of $W(A)$ were studied extensively in the last few decades and many nice results were obtained; see [10, 13]. The most beautiful result is probably the Toeplitz-Hausdorff Theorem which affirmed the convexity of $W(A)$; see [12, 17]. The generalizations of $W(A)$ remain an active research area in the field.

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For any \( A \in \mathbb{C}^{n \times n} \), write \( A = A_1 + iA_2 \) where \( A_1, A_2 \) are Hermitian matrices. Then by regarding \( \mathbb{C} \) as \( \mathbb{R}^2 \), one can rewrite \( W(A) \) as

\[
W(A) := \{(x^*A_1x, x^*A_2x) : x \in \mathbb{C}^n, x^*x = 1\}.
\]

This expression motivates naturally the generalization of the numerical range to the joint numerical range, which is defined as follows. Let \( \mathcal{H}_n \) be the set of all \( n \times n \) Hermitian matrices and \( \mathcal{H}_n^m \) be the set of all \( m \)-tuples of \( n \times n \) Hermitian matrices. The joint numerical range of \( A = (A_1, \ldots, A_m) \in \mathcal{H}_n^m \) is defined as

\[
W(A) = W(A_1, \ldots, A_m) := \{(x^*A_1x, \ldots, x^*A_mx) : x \in \mathbb{C}^n, x^*x = 1\}.
\]

It has been shown that for \( m \leq 3 \) and \( n \geq m \), the joint numerical range is always convex [1]. This result generalizes the Toeplitz-Hausdorff Theorem. However, the convexity of the joint numerical range fails to hold in general for \( m > 3 \), see [1, 11, 14].

When a new generalization of numerical range is introduced, people are always interested in its convexity. Unfortunately, this nice property fails to hold in some generalizations. However, another property, namely star-shapedness, holds in some generalizations; see [5, 18]. Therefore, the star-shapedness is the next consideration when the generalized numerical ranges fail to be convex. A set \( M \) is called star-shaped with respect to a star-center \( x_0 \in M \) if for any \( 0 \leq \alpha \leq 1 \) and \( x \in M \), we have \( \alpha x + (1-\alpha)x_0 \in M \). In [15], Li and Poon showed that for a given \( m \), the joint numerical range \( W(A_1, \ldots, A_m) \) is star-shaped if \( n \) is sufficiently large.

Let \( \mathcal{U}_n \) be the set of all \( n \times n \) unitary matrices. For \( C \in \mathcal{H}_n \) and \( A = (A_1, \ldots, A_m) \in \mathcal{H}_n^m \), the joint \( C \)-numerical range of \( A \) is defined by

\[
W_C(A) := \{\text{tr}(CU^*A_1U), \ldots, \text{tr}(CU^*A_mA) : U \in \mathcal{U}_n\},
\]

where \( \text{tr}(\cdot) \) is the trace function. When \( C \) is the diagonal matrix with diagonal elements \( 1, 0, \ldots, 0 \), then \( W_C(A) \) reduces to \( W(A) \). Hence the joint \( C \)-numerical range is a generalization of the joint numerical range. In [3], Au-Yeung and Tsing generalized the convexity result of the joint numerical range to the joint \( C \)-numerical range by showing that \( W_C(A) \) is always convex if \( m \leq 3 \) and \( n \geq m \). However \( W_C(A) \) fails to be convex in general if \( m > 3 \). One may consult [6] and [7] for the study of the convexity of \( W_C(A) \). The star-shapedness of \( W_C(A) \) remains unclear for \( m > 3 \).

For \( A = (A_1, \ldots, A_m) \in \mathcal{H}_n^m \), we define the joint unitary orbit of \( A \) by

\[
\mathcal{U}_n(A) := \{U^*A_1U, \ldots, U^*A_mA : U \in \mathcal{U}_n\}.
\]

For \( C \in \mathcal{H}_m \), we consider the linear map \( L_C : \mathcal{H}_n^m \to \mathbb{R}^m \) defined by

\[
L_C(X_1, \ldots, X_m) = (\text{tr}(CX_1), \ldots, \text{tr}(CX_m)).
\]

Then the joint \( C \)-numerical range of \( A \) is the linear image of \( \mathcal{U}_n(A) \) under \( L_C \). Inspired by this alternative expression, we consider the following generalized
numerical range of $A \in \mathcal{H}_n^m$. For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ and linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$, we define
\[
W_L(A) = L(U_n(A)) := \{L(U^*A_1U, ..., U^*A_mU) : U \in U_n\},
\]
and call it the $L$-numerical range of $A$, due to [4]. Because $LC$ is a special case of general linear maps $L$, the $L$-numerical range generalizes the joint $C$-numerical range and hence the classical numerical range.

In this paper, we shall study in Section two an inclusion relation of the $L$-numerical range of $m$-tuples of simultaneously unitarily diagonalizable Hermitian matrices and linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell = 2, 3$. This inclusion relation will be applied in Section three to show that the $L$-numerical ranges of $A$ under our consideration are star-shaped.

2 An Inclusion Relation for $L$-numerical Ranges

The following results follow easily from the definition of the $L$-numerical range.

**Lemma 2.1.** Let $(A_1, ..., A_m) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ be linear. Then the followings hold:

(i) $W_L(\alpha(A_1, ..., A_m) + \beta(I_n, ..., I_n)) = \alpha W_L(A_1, ..., A_m) + \beta L(I_n, ..., I_n)$ if $\alpha, \beta \in \mathbb{R}$;

(ii) $W_L(U^*A_1U, ..., U^*A_mU) = W_L(A_1, ..., A_m)$ for all unitary $U$.

In the following we shall consider those $A_1, ..., A_m$ which are simultaneously unitarily diagonalizable, i.e., there exists $U \in U_n$ such that $U^*A_1U, ..., U^*A_mU$ are all diagonal. Hence by Lemma 2.1, we assume without loss of generality that $A_1, ..., A_m$ are (real) diagonal matrices. For $d = (d_1, ..., d_n)^T \in \mathbb{R}^n$, we denote by $\text{diag}(d)$ the $n \times n$ diagonal matrix with diagonal elements $d_1, ..., d_n$. We first introduce a special class of matrices which is useful in studying the generalized numerical range; see [9, 16, 18].

An $n \times n$ real matrix $P = (p_{ij})$ is called a pinching matrix if for some $1 \leq s < t \leq n$ and $0 \leq \alpha \leq 1$,
\[
\begin{align*}
p_{ij} &= \left\{ \begin{array}{ll}
\alpha, & \text{if } (i,j) = (s, s) \text{ or } (t, t), \\
1 - \alpha, & \text{if } (i,j) = (s, t) \text{ or } (t, s), \\
1, & \text{if } i = j \neq s, t, \\
0, & \text{otherwise.}
\end{array} \right.
\end{align*}
\]

**Definition 2.2.** Assume $D = (\text{diag}(d^{(1)}), ..., \text{diag}(d^{(m)})), \hat{D} = (\text{diag}(\hat{d}^{(1)}), ..., \text{diag}(\hat{d}^{(m)}))$ where $d^{(1)}, ..., d^{(m)}, \hat{d}^{(1)}, ..., \hat{d}^{(m)} \in \mathbb{R}^n$. We say $\hat{D} \prec D$ if there exist a finite number of pinching matrices $P_1, ..., P_k$ such that $\hat{d}^{(i)} = P_1P_2\cdots P_kd^{(i)}$ for all $i = 1, ..., m$.

The following inclusion relation is the main result in this section.
Theorem 2.3. Let \( D, \tilde{D} \in \mathcal{H}_n^m \) and \( n > 2 \). If \( \tilde{D} < D \), then for any linear map \( L : \mathcal{H}_n^m \to \mathbb{R}^3 \), we have \( W_L(\tilde{D}) \subseteq W_L(D) \).

To prove Theorem 2.3, we need some lemmas. For \( \theta, \phi \in \mathbb{R} \), let \( T_{\theta, \phi} \in U_n \) be defined by

\[
T_{\theta, \phi} = \begin{pmatrix}
\cos \theta & \sin \theta e^{-\text{Tr} \phi} & 0 \\
-\sin \theta & \cos \theta e^{-\text{Tr} \phi} & 0 \\
0 & 0 & I_{n-2}
\end{pmatrix}.
\]

Lemma 2.4. Let \( D = (D_1, \ldots, D_m) \in \mathcal{H}_n^m \) be an \( m \)-tuple of diagonal matrices. Then for any linear map \( L : \mathcal{H}_n^m \to \mathbb{R}^3 \) and \( U \in U_n \), the set of points

\[
E_L(D, U) := \{ L(U^*T_{\theta, \phi}D_1T_{\theta, \phi}U, \ldots, U^*T_{\theta, \phi}D_mT_{\theta, \phi}U) : \theta \in [0, \pi], \phi \in [0, 2\pi] \}
\]

forms an ellipsoid in \( \mathbb{R}^3 \).

Proof. Note that for any \( L : \mathcal{H}_n^m \to \mathbb{R}^3 \), we can always express \( L \) as

\[
L(X_1, \ldots, X_m) = \left( \text{tr} \left( \sum_{i=1}^{m} P_iX_i \right), \text{tr} \left( \sum_{i=1}^{m} Q_iX_i \right), \text{tr} \left( \sum_{i=1}^{m} R_iX_i \right) \right)
\]

for some suitable \( P_i, Q_i, R_i \in \mathcal{H}_n, i = 1, \ldots, m \). For \( U \in U_n \), we write \( UP_i^{-1}U^* = (p_{ij}^{(i)}) \), \( UQ_iU^* = (q_{ij}^{(i)}) \), \( UR_iU^* = (r_{ij}^{(i)}) \) and \( D_i = \text{diag}(d_{11}^{(i)}, \ldots, d_{n1}^{(i)}) \), \( i = 1, \ldots, m \).

By direct computations, the first coordinate of points in \( E_L(D, U) \) is

\[
\text{tr} \left( \sum_{i=1}^{m} P_iU^*T_{\theta, \phi}D_iT_{\theta, \phi}U \right)
\]

\[
= \text{tr} \left( \sum_{i=1}^{m} D_iT_{\theta, \phi}U P_i U^* T_{\theta, \phi} \right)
\]

\[
= \frac{1}{2} \sum_{i=1}^{m} \left( d_{11}^{(i)} + d_{22}^{(i)} \right) (p_{11}^{(i)} + p_{22}^{(i)}) + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^{(i)} p_{ij}^{(i)}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{m} \left( d_{11}^{(i)} - d_{22}^{(i)} \right) (p_{11}^{(i)} - p_{22}^{(i)}) \cos 2\theta
\]

\[
+ \sum_{i=1}^{m} (d_{11}^{(i)} - d_{22}^{(i)}) \text{Re}(p_{21}^{(i)} e^{-i\phi}) \sin 2\theta.
\]

Similarly for the second and the third coordinates of points in \( E_L(D, U) \). Note that for \( a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R} \) and \( a_3, b_3, c_3 \in \mathbb{C} \), the points \( (a_1, b_1, c_1) + (a_2, b_2, c_2) \cos 2\theta + \text{Re}(a_3 e^{-i\phi}, b_3 e^{-i\phi}, c_3 e^{-i\phi}) \sin 2\theta \) form an ellipsoid in \( \mathbb{R}^3 \) when \( \theta, \phi \) run through \([0, \pi]\) and \([0, 2\pi]\) respectively. Hence \( E_L(D, U) \) is an ellipsoid in \( \mathbb{R}^3 \).

Note that \( E_L(D, U) \subseteq W_L(D) \) for any \( U \in U_n \).
Lemma 2.5. Let $D \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices with $n > 2$. Then for any linear map $L: \mathcal{H}_n^m \to \mathbb{R}^3$, there exists $V \in \mathcal{U}_n$ such that $E_L(D, V)$ defined in Lemma 2.4 degenerates (i.e., $E_L(D, V)$ is contained in a plane in $\mathbb{R}^3$).

Proof. Following the notations in Lemma 2.4 and its proof, we let $d_1 = \alpha_1 = d_1^{(i)} - d_2^{(i)}$ for $i = 1, \ldots, m$ and $P' = \sum_{i=1}^{m} \alpha_i P_i \in \mathcal{H}_n$. Since $n > 2$, by generalized interlacing inequalities for eigenvalues of Hermitian matrices (see [8]), there exist $V \in \mathcal{U}_n$ and $\alpha \in \mathbb{R}$ such that $V P' V^*$ has $\alpha I_2$ as leading $2 \times 2$ principal submatrix. For any matrix $M$, let $M_{ij}$ denote its $(i, j)$ entry. Then by taking $U = V$ in the proof of Lemma 2.4, the first coordinate of points in $E_L(D, V)$ is $a + b \cos 2 \theta + c \sin 2 \theta$ where

$$a = \frac{1}{2} \sum_{i=1}^{m} (d_1^{(i)} + d_2^{(i)})(p_{11}^{(i)} + p_{22}^{(i)}) + \sum_{i=1}^{m} \sum_{j=3}^{n} d_j^{(i)} p_{ij}$$

$$b = \frac{1}{2} \sum_{i=1}^{m} \alpha_i [(VP_i V^*)_{11} - (VP_i V^*)_{22}]$$

$$= \frac{1}{2} \left( V \left( \sum_{i=1}^{m} \alpha_i P_i \right) V^* \right)_{11} - \frac{1}{2} \left( V \left( \sum_{i=1}^{m} \alpha_i P_i \right) V^* \right)_{22}$$

$$= \frac{1}{2} (VP' V^*)_{11} - \frac{1}{2} (VP' V^*)_{22}$$

$$= \frac{1}{2} \alpha - \frac{1}{2} \alpha = 0,$$

$$c = \sum_{i=1}^{m} \alpha_i \text{Re} \left( (VP_i V^*)_{21} e^{\sqrt{-1} \phi} \right)$$

$$= \text{Re} \left[ \left( V \left( \sum_{i=1}^{m} \alpha_i P_i \right) V^* \right)_{21} e^{\sqrt{-1} \phi} \right]$$

$$= \text{Re}((VP' V^*)_{21} e^{\sqrt{-1} \phi}) = 0.$$
$f : [0, 1] \to \mathcal{U}_n$ such that $f(0) = U$ and $f(1) = V$ where $V$ is defined in Lemma 2.5 and hence $E(D, f(1))$ degenerates. By continuity, there exists $t \in [0, 1]$ such that \( L(U^* DU) \in E(D, f(t)) \subset W_L(D) \). 

Using similar techniques, one can prove that Theorem 2.3 stills holds for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $n \geq 2$. However, the following example shows that the inclusion relation in Theorem 2.3 fails to hold if $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ is linear with $\ell > 3$.

**Example 2.6.** Let $n \geq 2$, $d = (1, ..., 0)^T$, $\hat{d} = (\frac{1}{2}, \frac{1}{2}, 0, ..., 0)^T \in \mathbb{R}^n$ and let $O_k$ be the $k \times k$ zero matrix. Consider $D = (\text{diag}(\hat{d}), O_n, ..., O_n)$, $\hat{D} = (\text{diag}(\hat{d}), O_n, ..., O_n) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell \geq 4$ defined by

\[
L(X_1, ..., X_m) = (\text{tr}(PX_1), \text{tr}(QX_1), \text{tr}(RX_1), \text{tr}(SX_1), 0, ..., 0)
\]

where

\[
P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus O_{n-2}, \quad Q = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus O_{n-2},
\]
\[
R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus O_{n-2}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus O_{n-2}.
\]

Then we have $\hat{D} \prec D$ and $(1, 0, ..., 0) \in W_L(\hat{D})$, but $(1, 0, ..., 0) \notin W_L(D)$.

### 3 Star-shapedness of the L-numerical range

The $L$-numerical range may fail to be convex for linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell \geq 2$ even when $A_1, ..., A_m \in \mathcal{H}_n$ are simultaneously unitarily diagonalizable; see [2]. However, we shall show in this section that for $n > 2$, $W_L(A_1, ..., A_m)$ is always star-shaped for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^3$ and simultaneously unitarily diagonalizable $A_1, ..., A_m \in \mathcal{H}_n$. The following result is the essential element in our proof.

**Proposition 3.1.** [18] Let $\mathbb{P}_n$ be the set of all finite products of $n \times n$ pinching matrices. Then for $0 \leq \alpha \leq 1$, $\alpha I_n + (1 - \alpha)J_n$ is in the closure of $\mathbb{P}_n$ where $J_n$ is the $n \times n$ matrix with all entries equal $1/n$.

Note that for any $A \in \mathcal{H}_n^m$, $\mathcal{U}_n(A)$ is compact. Hence $W_L(A)$ is compact for all linear maps $L$.

**Theorem 3.2.** Let $D = (D_1, ..., D_m) \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices with $n > 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$, $W_L(D)$ is star-shaped with respect to star-center $L(\frac{nD_1}{n}, ..., \frac{nD_m}{n})$.

**Proof.** By Lemma 2.1, we may assume without loss of generality that $\text{tr}D_i = 0$ for $i = 1, ..., m$; otherwise we replace $D_i$ by $D_i - \frac{\text{tr}D_i}{n}I_n$. Let $D_i = \text{diag}(d^{(i)})$ where $d^{(i)} \in \mathbb{R}^n$, $i = 1, ..., m$. For any $0 \leq \alpha \leq 1$, we have $\alpha J_n + (1 - \alpha)J_n d^{(i)}$. Then for any $U \in \mathcal{U}_n$, by Proposition 3.1, Theorem 2.3 and the compactness of $W_L(D)$, we have $\alpha L(U^* DU) \in W_L(\alpha D) \subset W_L(D) = W_L(D)$ where $\overline{M}$ denotes the closure of $M$. 

\[ \square \]
For a linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$, by regarding it as a projection of some linear map $\tilde{L} : \mathcal{H}_n^m \to \mathbb{R}^3$, we deduce the following corollary easily.

**Corollary 3.3.** Let $D = (D_1, \ldots, D_m) \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices with $n \geq 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$, $W_L(D)$ is star-shaped with respect to star-center $L(\frac{\tr D_1}{n} I_n, \ldots, \frac{\tr D_m}{n} I_n)$.

**Proof.** We only need to consider the case $n = 2$. We may assume without loss of generality that $m = 1$ and $D = \text{diag}(1, -1)$. For any linear map $L : \mathcal{H}_2 \to \mathbb{R}^2$, we express it as $L(X) := (\tr(PX), \tr(QX))$ for some $P, Q \in \mathcal{H}_2$. Then we have

$$W_L(D) = \{2(x^*Px, x^*Qx) - (\tr P, \tr Q) : x \in \mathbb{C}^n, x^*x = 1\} = 2W(P, Q) - (\tr P, \tr Q),$$

which is convex and contains the origin. This implies that $W_L(D)$ is star-shaped with respect to star-center $L(\frac{\tr D_1}{n} I_2)$, which is the origin.

Note that the star-shapedness of the $L$-numerical range for linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell > 3$ remains open in the diagonal case. Moreover, for general cases of $A = (A_1, \ldots, A_m)$ where $A_1, \ldots, A_m$ are not necessarily simultaneously unitarily diagonalizable and $L : \mathcal{H}_n^m \to \mathbb{R}^2$ with $m \geq 3$, the star-shapedness of $W_L(A)$ is also unclear. However, by applying a result in [4], we can show that $L(\frac{\tr A_1}{n} I_n, \ldots, \frac{\tr A_m}{n} I_n) \in W_L(A_1, \ldots, A_m)$ for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^2$.

**Proposition 3.4** ([4], P. 23). Let $A_k = (a_{ij}^{(k)}) \in \mathcal{H}_n$, $k = 1, \ldots, m$. For $0 \leq \epsilon \leq 1$, define $A_k(\epsilon)$ as

$$A_k(\epsilon) = \begin{pmatrix}
\epsilon a_{11}^{(k)} & \epsilon a_{12}^{(k)} & \cdots & \epsilon a_{1n}^{(k)} \\
\epsilon a_{12}^{(k)} & \epsilon a_{22}^{(k)} & \cdots & \epsilon a_{2n}^{(k)} \\
\vdots & \vdots & \ddots & \vdots \\
\epsilon a_{n1}^{(k)} & \epsilon a_{n2}^{(k)} & \cdots & \epsilon a_{nn}^{(k)}
\end{pmatrix}, \quad k = 1, \ldots, m.
$$

Then $W_L(A_1(\epsilon), \ldots, A_m(\epsilon)) \subseteq W_L(A_1, \ldots, A_m)$ for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$.

**Theorem 3.5.** Let $A = (A_1, \ldots, A_m) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^2$ be linear. Then $L(\frac{\tr A_1}{n} I_n, \ldots, \frac{\tr A_m}{n} I_n) \in W_L(A)$.

**Proof.** Define $A_i(\epsilon)$ as in Proposition 3.4 and note that $\tr A_i(\epsilon) = \tr A_i$ for $i = 1, \ldots, m$. Hence by Corollary 3.3 and Proposition 3.4, we have

$$L \left( \frac{\tr A_1}{n} I_n, \ldots, \frac{\tr A_m}{n} I_n \right) \in W_L(A_1(0), \ldots, A_m(0)) \subseteq W_L(A_1, \ldots, A_m).$$

\[\square\]
References


