STATISTICS OF HECKE EIGENVALUES FOR GL(n)

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Abstract. A two-dimensional central limit theorem for the eigenvalues of GL(n) Hecke-Maass cusp forms is newly derived. The covariance matrix is diagonal and hence verifies the statistical independence between the real and imaginary parts of the eigenvalues. We also prove a central limit theorem for the number of weighted eigenvalues in a compact region of the complex plane, and evaluate some moments of eigenvalues for the Hecke operator $T_p$ which reveal interesting interferences.

1. Introduction

In the literature there are fruitful results for the statistics of Hecke eigenvalues in the GL(2) case. Let $S_k$ be the space of holomorphic modular forms of even weight $k$ for $SL_2(\mathbb{Z})$, and $T_m$ be the $m$th Hecke operators. For any prime $p$, let $\lambda_f(p)$ be the Hecke eigenvalue of $T_p$ for the primitive form $f$ in $S_k$. The family $\mathcal{F} := \{\lambda_f(p) : p \in \mathbb{P}, f \in H\}$ shows interesting statistical behavior, where $\mathbb{P}$ denotes the set of all primes and $H = \bigcup_k H_k$ is the union of the sets $H_k$ of primitives forms in $S_k$. The famous Sato-Tate conjecture (already settled for this case) asserts that for fixed $f \in H_k$,

$$\lim_{x \to \infty} \text{Prob}_{\mathbb{P}_x}(a < \lambda_f(p) < b) = \int_a^b d\mu_{\text{ST}} := \frac{1}{2\pi} \int_a^b \sqrt{4 - x^2} \, dx$$

for any interval $(a, b)$, where $\text{Prob}_{\mathbb{P}_x}$ is the counting probability and $\mathbb{P}_x = \{p \in \mathbb{P} : p \leq x\}$. Serre [18] and Conrey et al. [5] independently showed that for fixed prime $p$,

$$\lim_{k \to \infty} \text{Prob}_{H_k}(a < \lambda_f(p) < b) = \frac{p + 1}{2\pi} \int_a^b \frac{\sqrt{4 - x^2}}{(p^{1/2} + p^{-1/2})^2 - x^2} \, dx.$$

The study of statistical behaviour of number-theoretic functions has a long history. The famous Erdős-Kac Theorem (cf. [1]) asserts the central limit behaviour for the prime divisors of integers: $\text{Prob}_{|\mathbb{P}|(1,x)}(\sum_{p \leq n} \delta_{p|n} \log_2 n / \sqrt{\log_2 n} < b)$ tends to the standard normal distribution as $x \to \infty$, where $\delta_{p|n} = 1$ if $p$ is a prime divisor of $n$ or 0 otherwise, and $\log_2 n := \log \log n$. Central limit theorem is also observed in $\mathcal{F}$. In [15], Nagoshi established that

$$\lim_{x \to \infty} \text{Prob}_{H_k}(a < \frac{1}{\pi(x)} \sum_{p \leq x} \lambda_f(p) < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, dx \tag{1.1}$$

where $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. ($\pi(x) = |\mathbb{P}_x| \sim x / \log x$.) The counterpart for the level aspect is shown in the work of Cho and Kim [4]. Very recently, following

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\textsuperscript{11}That is, $\text{Prob}_{\Omega}(\ldots) := |\{w \in \Omega : \ldots\}|/|\Omega|$.
the work of Faifman and Rudnick [6], Prabhu and Sinha [17] obtained a central limit theorem for the frequency: for \( k = k(x) \) satisfying \( \frac{\log k}{\sqrt{x \log x}} \to \infty \) as \( x \to \infty \) and for any integral \( I \subset [-2, 2] \),

\[
\lim_{x \to \infty} \text{Prob}_{H_k} \left( a < \frac{N_I(f, x) - \pi(x) \mu_{ST}(I)}{\sqrt{\pi(x)(\mu_{ST}(I) - \mu_{ST}(I)^2)}} < b \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, dx
\]

where \( N_I(f, x) := |\{ p \in \mathbb{P}_x : \lambda_f(p) \in I \}| \) and \( \mu_{ST}(I) \) is the measure of \( I \) with respect to the Sato-Tate measure. Pertinent investigations for other arithmetic objects were carried out in [12], [22] and [3], for example.

In this paper we attempt to extend the above investigations to the \( GL(n) \) case and obtain new results. When \( n \geq 3 \), the Hecke eigenvalues are not necessarily real. For prime \( p \), the (normalized) eigenvalue of \( T_p \) may be expressed as \( A_\phi(p, 1, \ldots, 1) \) where \( \phi \) is an associated eigenfunction. We still write \( T_m \) for the \( m \)th Hecke operator. Using the Hecke relation and some consequences of – a recent great progress due to Matz and Templier – automorphic Plancherel density theorem, we experimented the moments of \( \sum_{p \leq x} A_\phi(p, 1, \ldots, 1) \) and the real or imaginary part. Let \( 3\mathbb{G} \) be the set of all Hecke-Maass cusp forms \( \phi \) for \( GL(n, \mathbb{R}) \) whose Langlands parameters \( \mu_\phi \) are purely imaginary (in \( \mathbb{C}^n \)) and distant from the origin at most \( t \) in Euclidean norm. Write

\[
\langle F \rangle_t := \frac{1}{|3\mathbb{G}|} \sum_{\phi \in 3\mathbb{G}} F(\phi).
\]

We found that for any \( t = t(x) \) such that \( \frac{\log t}{\log x} \to \infty \) as \( x \to \infty \),

\[
\lim_{x \to \infty}\left\langle \left( \frac{1}{\sqrt{\pi(x)}} \sum_{p \leq x} A_\phi(p, 1, \ldots, 1) \right)^r \right\rangle_t = 0 \quad \text{for } r = 1, 2
\]

while

\[
\lim_{x \to \infty}\left\langle \left( \frac{1}{\sqrt{\pi(x)}} \sum_{p \leq x} \mathbb{R}e A_\phi(p, 1, \ldots, 1) \right)^r \right\rangle_t = \begin{cases} 
0 & \text{if } r = 1, \\
\frac{1}{2} & \text{if } r = 2.
\end{cases}
\]

(and the same result holds for \( \mathbb{I}m A_\phi(p, 1, \ldots, 1) \)). This infers that the real part and imaginary part of \( A_\phi(p, 1, \ldots, 1) \) are probably uncorrelated.

The first result justifies the uncorrelation as well as gives a central limit theorem for general eigenvalues \( A_\phi(p^k) \). For \( k = (k_1, \ldots, k_{n-1}) \), we let \( A_\phi(p^k) := A_\phi(p^{k_1}, \ldots, p^{k_{n-1}}) \).

**Theorem 1.1.** Let \( 0 \neq k = (k_1, \ldots, k_{n-1}) \in \mathbb{N}_0^{n-1} \). Suppose \( \Psi(x) \) is any increasing function that tends to infinity as \( x \to \infty \) and let \( t = t(x) \geq \exp(\Psi(x) \log x) \).

\( k \neq k’ \): For any rectangular box \( D = (a, b) + i(c, d) \) of \( \mathbb{C} \), we have

\[
\lim_{x \to \infty} \text{Prob}_{3\mathbb{G}} \left( \frac{1}{\sqrt{\pi(x)}} \sum_{p \leq x} A_\phi(p^k) \in D \right) = \frac{1}{\pi} \int_c^d \int_a^b e^{-(x^2+y^2)} \, dxdy.
\]

\( k = k’ \): In this case we have \( A_\phi(p^k) \in \mathbb{R} \), and for any interval \( (a, b) \),

\[
\lim_{x \to \infty} \text{Prob}_{3\mathbb{G}} \left( a < \frac{1}{\sqrt{\pi(x)}} \sum_{p \leq x} A_\phi(p^k) < b \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, dx.
\]

Here \( k’ := (k_{n-1}, \ldots, k_1) \) for \( k = (k_1, \ldots, k_{n-1}) \).
Remark 1. Write
\[ Z_k^\phi(x) := \pi(x)^{-1/2} \sum_{p \leq x} A_\phi(p^k) \]
and let \( 0 \neq k \in \mathbb{N}_0^{n-1} \). Suppose \( t = t(x) \) satisfies the condition in Theorem 1.1.

(i) For all integers \( r \geq 0 \), we have
\[
\lim_{x \to \infty} \langle (\Re Z_k^\phi(x))^r \rangle_t = \lim_{x \to \infty} \langle (3m Z_k^\phi(x))^r \rangle_t = \frac{1}{\pi} \int_{\mathbb{R}^2} x^r e^{-(x^2+y^2)} \, dx dy = \delta_{2|r} \cdot \frac{r!}{2^r(r/2)!} ;
\]

(ii) Theorem 1.1 (1) remains valid if \( D \) is replaced by any borel set, and hence the associated random variable is circularly symmetric Gaussian. The moduli \( |Z_k^\phi(x)| \) and the phases \( \arg(Z_k^\phi(x)) \), \( \phi \in \mathcal{H}_t \), are Rayleigh distributed and uniformly distributed, respectively, as \( x \to \infty \) (cf. [9, §3.7.1, p.145]). Thus for any real \( r \geq 0 \),
\[
\lim_{x \to \infty} \langle |Z_k^\phi(x)|^r \rangle_t = \Gamma(1 + \frac{r}{2}).
\]

Part (b) of Remark 1 (i) explains the vanishing of (1.4); together with Remark 1 (ii), one observes the cancellation among the arguments of \( \sum_{p \leq x} A_\phi(p, 1, \ldots, 1) \) over \( \phi \) (in the sense that it is suppressed by \( \sqrt{\pi(x)} \)). However, if the weight \( \pi(x)^{1/2} \) in (1.4) is reduced to \( \pi(x)^{1/n} \), we shall observe crests – positive interferences – for suitable amplifications. This phenomenon is revealed in the moment result below.

The case \( k = (1, 0, \cdots, 0) \) recover (1.5) and (1.4).

Theorem 1.2. Let \( m \in \mathbb{N}_0 \), and \( t = t(x) \) satisfying \( \frac{\log t}{\log x} \to \infty \) as \( x \to \infty \). We have
\[
\lim_{x \to \infty} \langle \left( \frac{1}{\pi(x)^{1/2}} \sum_{p \leq x} A_\phi(p, 1, \ldots, 1) \right)^m \rangle_t = \begin{cases}
\frac{m!}{n!m/n \cdot \left( \frac{m}{n} \right)!} & \text{if } n \mid m, \\
0 & \text{if } n \nmid m.
\end{cases}
\]

Naturally it is desired to consider the moments without averaging over primes \( p \).

Theorem 1.3. Let \( m \in \mathbb{N}_0 \). Then,
\[
\lim_{t \to \infty} \langle A_\phi(p, 1, \cdots, 1)^m \rangle_t = \begin{cases}
(1 + O_n(p^{-1})) \cdot m! \prod_{i=0}^{n-1} \frac{i!}{(\ell + i)!} & \text{if } m = n\ell, \\
0 & \text{if } m \nmid \ell.
\end{cases}
\]

Note that \( \prod_{i=0}^{n-1} i!/(\ell + i)! = G(1+n)G(1+\ell)/G(1+n+\ell) \) in terms of the Barnes G-function \( G(z) \) whose value at \( z = k + 1 \) is \( G(1+k) = 1 \cdot 2! \cdot 3! \cdots (k-1)! \).

The final result here is related to the studies in [6] and [17]. The frequency \( N_f(f, x) \) in (1.2) is considered in [17] but the method seems not easy to be adapted in our case. Instead we consider the smooth weighted frequency and get a central limit theorem.
Theorem 1.4. Let \( 0 \neq k = (k_1, \ldots, k_{n-1}) \in \mathbb{N}_0^{n-1} \) and \( \varphi \) be a real-valued compact supported function on the complex plane. Suppose \( t = t(x) \geq \exp(x^\Delta) \) where \( \Delta \in (0,1) \) is any fixed number. For any interval \((a,b)\),
\[
\lim_{x \to \infty} \text{Prob}_{\varphi}(a < \frac{N_\varphi(\phi, x) - \pi(x)\mu_\varphi}{\sqrt{\pi(x)\sigma^2_\varphi}} < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} \, dx
\]
where \( N_\varphi(\phi, x) = \sum_{p \leq x} \varphi(A_\phi(p^k)) \) and (see Section 2 for the definitions)
\[
\mu_\varphi = \int_{T_0/\mathfrak{S}_n} \varphi(S_k) \, d\mu_{ST} \quad \text{and} \quad \sigma^2_\varphi = \int_{T_0/\mathfrak{S}_n} (\varphi(S_k) - \mu_\varphi)^2 \, d\mu_{ST}.
\]

**Notation.** \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\} \) and \( i = \sqrt{-1} \). A vector is underlined or written in bold face, a bold vector (e.g. \( k \)) will have \( n-1 \) coordinates. A partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}_0^n \) satisfies \( \lambda_1 \geq \cdots \geq \lambda_n \) by definition, which is not underlined though \( \lambda \) is a vector. We write \( |v| := \sum_j v_j \) for a vector \( v := (v_1, \ldots, v_m) \in \mathbb{N}_0^m \), and moreover, \( \|k\| := \sum (n-j)k_j \) for \( k = (k_1, \ldots, k_{n-1}) \in \mathbb{N}_0^{n-1} \). An \( m \)-tuple \((a, \cdots, a)\) may be abbreviated as \( a_m \). The Kronecker delta \( \delta_\ast \) equals 1 if \( \ast \) holds and 0 otherwise. The \( O \)-symbol \( O_\ast \) and vinogradov symbol \( \ll_\ast \) are used whenever their dependence on \( \ast \) would be emphasized.

**Organization and method.** The automorphic Plancherel density theorem of Matz and Templier [14] with Casselman-Shalika formula manifests the statistical law underlying the Hecke eigenvalues for \( GL(n) \) in terms of the Schur polynomials and Plancherel measures. Section 2 provides a background on the Schur polynomial and a preparation – Lemma 2.1 below. Section 3 discusses Hecke-Maass cusp forms and their eigenvalues. The key ingredients, i.e. the statistical law from [14] and the integrals of degenerate Schur polynomials in [13], will be summarized therein and applied to prove Theorems 1.2 and 1.3. In Section 4, we derive the central limit behaviour in a broader context, with the prototype from Section 3, using the continuity theorem in Probability theory. This is new to [4], [6], [17], [21] where the moment method is applied; here we do not evaluate explicitly the main terms of higher moments. Theorems 1.1 and 1.4 are then proved in Section 5 with the tools in Sections 3 and 4.

2. Degenerate Schur polynomials and the Sato-Tate measure

Let \( k = (k_1, \ldots, k_{n-1}) \in \mathbb{N}_0^{n-1} \). The degenerate Schur polynomial \( S_k \) is defined as
\[
S_k(x_1, x_2, \cdots, x_n) := \det \left( \frac{x_j^{\sum_{i=1}^{n-1}(k_i+1)}}{x_j^{\sum_{i=1}^{n-1}}} \right)_{1 \leq i,j \leq n}
\]
(cf. [10, p.233]) which is different from the common Schur polynomial \( s_\lambda \) (cf. [8, Appendix A]),
\[
s_\lambda(x_1, \cdots, x_n) := \det \left( \frac{x_j^{\lambda_i+n-1}}{x_j^{n-1}} \right)_{1 \leq i,j \leq n}
\]
for partition \( \lambda = (\lambda_1, \cdots, \lambda_n) \). In [13, §7], we work out some of their connections and properties.
If \( \lambda = \iota(k) := (k_1 + \cdots + k_{n-1}, k_1 + \cdots + k_{n-2}, \ldots, k_1, 0) \), then
\[
(2.3) \quad S_k(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_n).
\]
Conversely, if \( k = j(\lambda) := (\lambda_{n-1} - \lambda_n, \ldots, \lambda_1 - \lambda_2) \), then
\[
(2.4) \quad s_\lambda(x_1, \ldots, x_n) = (x_1 \cdots x_n)^\lambda S_k(x_1, \ldots, x_n).
\]
Note \( |\lambda| := \sum_i \lambda_i = \sum (n - i)k_i =: ||k|| \) in (2.3), and \( ||k|| = |\lambda| - n\lambda_n \) in (2.4). For example,
\[
S_0 = s_0 = 1, \quad s_{(c, \ldots, c)}(x_1, \ldots, x_n) = (x_1 \cdots x_n)^c
\]
for \( c \in \mathbb{N}_0 \), and with a little calculation, we have
\[
S_{(0, n-1)}(x_1, \ldots, x_n) = s_{(1, 0, \ldots, 0)}(x_1, \ldots, x_n) = x_1 + \cdots + x_n.
\]

The Schur polynomials \( s_\lambda(x_1, \ldots, x_n) \) form an orthonormal basis for the vector space of symmetric polynomials in \( x_1, \ldots, x_n \) with respect to some inner products. One choice is \( (, ) \) defined as follows: Confining each \( x_i \) to the unit circle \( S^1 \) of \( \mathbb{C} \), a Schur polynomial is a function on the space \( U(n)^2 \) of conjugacy classes in \( U(n) \). Note that \( U(n)^2 \cong S^{1n}/S_n \) where \( S_n \) is the symmetric group of order \( n \). The inner product \( (, ) \) is induced by the pushforward measure on \( U(n)^2 \), cf. [13, §7.2]. Thus for any two partitions \( \lambda, \mu \),
\[
(2.5) \quad (s_\lambda, s_\mu) := \int_{U(n)^2} s_\lambda \overline{s_\mu} d\mu_{U(n)^2}
\]
\[
= \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} s_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_n}) s_\mu(e^{i\theta_1}, \ldots, e^{i\theta_n}) |\det(e^{i(n-i)\theta_j})|^2 d\theta_1 \cdots d\theta_n
\]
\[
= \delta_{\lambda=\mu}.
\]
Moreover the product \( s_\lambda s_\mu \) of any two Schur polynomials is a linear combination of Schur polynomials, following from the Littlewood-Richardson rule. The degenerate Schur polynomial may be regarded as the restriction of a Schur polynomial (from \( U(n)^2 \)) to \( SU(n)^2 \), the space of conjugacy classes in \( SU(n) \). Analogously to \( d\mu_{U(n)^2} \), we have a measure \( d\mu_{ST} \), called the Sato-Tate measure, on \( SU(n)^2 \). Consequently, we have an inner product \( \langle, \rangle \) defined as
\[
(2.6) \quad \langle S_k, S_{k'} \rangle := \int_{SU(n)^2} S_k \overline{S_{k'}} d\mu_{ST} = \delta_{k=k'},
\]
and ([13, Lemma 7.1 (2)]) the Littlewood-Richardson rule,
\[
(2.7) \quad S_k \cdot S_{k'} = \sum_\xi d^\xi_{kk'} S_\xi
\]
where \( d^\xi_{kk'} \)’s are nonnegative integers and the summation runs over \( \xi \in \mathbb{N}_0^{n-1} \) satisfying \( ||\xi|| \leq ||k|| + ||k'|| \) and \( ||\xi|| \equiv ||k|| + ||k'|| \mod n \). (Recall \( ||k|| := \sum_i (n - i)k_i \).)

**Lemma 2.1.** For \( m \in \mathbb{N}_0 \), let
\[
I_k(m) := \int_{SU(n)^2} S_k^m d\mu_{ST}.
\]
We have (i) \( I_k(m) = 0 \) if \( n \nmid m||k|| \), and (ii) for every \( \ell \in \mathbb{N}_0 \),
\[
I_{(0, n-1)}(n\ell) = (n\ell)! \prod_{i=0}^{n-1} \frac{i!}{(\ell + i)!}.
\]
Remark 2. One may express $I_{(0,n-1)}(m)$ into $\int_{SU(n)} \text{tr}(U)^m dU$ and boil it down to Frobenius’s formula, cf. Chapters 4 and 6 in [8].

Proof. By (2.7), it is seen that $S_k^m = \sum_\xi c_\xi S_\xi$ where $c_0 = 0$ if $n \nmid m\|k\|$. (i) follows readily as $I_k(m) = \langle S_k^m, S_0 \rangle$.

Similarly, for (ii) we have

$$I_{(0,n-1)}^{(i\ell)}(n\ell) = \langle S_{(0,n-1)}^{n\ell}, S_0 \rangle = d_0$$

where $S_{(0,n-1)}^{n\ell} = \sum_\xi d_\xi S_\xi$. By (2.3), it follows that

$$S_{(0,n-1)}^{n\ell} = s_{(1,0,n-1)}^{n\ell} = \sum_\mu f_\mu S_\mu$$

From (2.4) $s_\lambda = S_k$ on $SU(n)^2$, and by (2.6), we see that $\langle s_\mu, S_0 \rangle = 0$ if $\mu$ is non-constant, i.e. $\mu \neq (c, \cdots, c)$ where $c \in \mathbb{N}_0$. Thus,

$$d_0 = \sum_{\mu=(c,\cdots,c)} f_\mu = \sum_{c\geq 0} \langle s_{(1,0,n-1)}^{n\ell}, S_{(c,\cdots,c)} \rangle$$

by (2.5). As $s_{(1,0,n-1)}(x_1, \cdots, x_n)^{n\ell} = (x_1 + \cdots + x_n)^{n\ell}$, the inner product

$$(s_{(1,0,n-1)}^{n\ell}, S_{(c,\cdots,c)})$$

$$= \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} (e^{i\theta_1} + \cdots + e^{i\theta_n})^{n\ell} e^{-i\theta_1} \cdots e^{-i\theta_n} | \text{det}(e^{i((n-i)\theta_j)}) |^2 d\theta_1 \cdots d\theta_n$$

$$= \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1! \cdots r_n! n!(2\pi)^n} \sum_{\sigma_\pi \in \mathcal{S}_n} \text{sgn}(\sigma)\text{sgn}(\pi)$$

$$\times \int_{[0,2\pi]^n} e^{i(r_1-c+\sigma(1)-\pi(1))\theta_1} \cdots e^{i(r_n-c+\sigma(n)-\pi(n))\theta_n} d\theta_1 \cdots d\theta_n$$

$$= \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1! \cdots r_n! n!} \sum_{\sigma_\pi \in \mathcal{S}_n} \text{sgn}(\sigma)\text{sgn}(\pi)$$

where $(\ast)$ denotes the constraint given by the linear system

$$\left\{ \begin{array}{c} r_1 + \sigma(1) = \pi(1) + c, \\ \vdots \\ r_n + \sigma(n) = \pi(n) + c. \end{array} \right.$$ 

Adding up the equations yields $nc = n\ell$. The inner product is zero unless $c = \ell$. In this case, we move out the summation over $\sigma$ and apply a relabeling to obtain

$$(s_{(1,0,n-1)}^{n\ell}, S_{(\ell,\cdots,\ell)}) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1! \cdots r_n!} \sum_{\pi \in \mathcal{S}_n} \text{sgn}(\pi^{\sigma^{-1}})$$

$$= \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1! \cdots r_n!} \sum_{\pi \in \mathcal{S}_n} \text{sgn}(\pi)$$
where (**) and (***) are respectively the linear systems
\[\begin{align*}
&\begin{cases}
  r_{\sigma(1)} + \sigma(1) = \pi(1) + \ell \\
  \vdots \\
  r_{\sigma(n)} + \sigma(n) = \pi(n) + \ell 
\end{cases}
\quad \text{and} \quad \\
&\begin{cases}
  r_1 = \pi(1) + \ell - 1 \\
  \vdots \\
  r_n = \pi(n) + \ell - n 
\end{cases}
\]

Recall $1/m! = 1/\Gamma(m + 1)$ for non-negative integers $m$ and $\Gamma(s)^{-1}$ has zeros at negative integers. Hence we set $1/m! := 0$ for negative integer $m$ and may write
\[
\sigma_{n,\ell}(1,0_n,\cdots,\ell) = (n\ell)! \sum_{\pi \in S_n} \frac{\text{sgn}(\pi)}{(\ell + \pi(1) - 1)! \cdots (\ell + \pi(n) - n)!} 
= (n\ell)! \det \left( \frac{1}{(\ell + j - i)!} \right)_{n \times n} = (n\ell)! \prod_{i=0}^{n-1} \frac{i!}{(\ell + i)!}.
\]

The last equality follows from
\[
\det \left( \frac{1}{(\ell + j - i)!} \right)_{n \times n} = n^{-1} \prod_{j=0}^{n-1} \frac{1}{(\ell + j)!} 
= \prod_{j=0}^{n-1} \frac{1}{(\ell + j)!} \times 
\]
and an induction on $n$ for the last determinant which equals, after subtracting the $i$th row with $(i + 1)$th row,
\[
(n - 1)! 
\]

\[
\begin{align*}
&\begin{cases}
  \prod_{j=1}^{n-2} (\ell + j) & \prod_{j=2}^{n-2} (\ell + j) & \cdots & \prod_{j=n-2}^{n-1} (\ell + j) & 1 \\
  \prod_{j=0}^{n-3} (\ell + j) & \prod_{j=2}^{n-3} (\ell + j) & \cdots & \prod_{j=n-3}^{n-2} (\ell + j) & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  \prod_{j=3-n}^{0} (\ell + j) & \prod_{j=4-n}^{0} (\ell + j) & \cdots & \prod_{j=3-n}^{0} (\ell + j) & 1 \\
  0 & 0 & \cdots & 0 & \prod_{j=0}^{n-2} (\ell + j) \\
\end{cases} 
\end{align*}
\]
Moreover, for any \( \chi \), \( \in \mathbb{R} \) and \( h^n := G/(K \cdot \mathbb{R}^k) \). We denote by \( L^2(\Gamma \backslash h^n) \) the Hilbert space of square integrable functions on \( \Gamma \). Let \( \mathbb{R} \) be the Hecke ring with respect to \( \Gamma \) and \( \Delta \) where \( \Delta \) is the semigroup of all integral matrices in \( \Gamma \) whose determinants are positive. Hecke-Maass cusp forms are (nonzero) common eigenfunctions of all \( T \in \mathbb{R} \) in \( L^2(\Gamma \backslash h^n) \) (that satisfy some conditions), and they form an orthonormal basis \( \mathcal{H}^2 = \{ \phi_j \} \) for \( L^2_{\text{cusp}}(\Gamma \backslash h^n) \), the subspace of cusp forms in \( L^2(\Gamma \backslash h^n) \). Each \( \phi_j \) is associated with a Langlands parameter \( \mu_\phi \in \mathfrak{a}^*_C \approx \{ z \in \mathbb{C}^n : \sum z_i = 0 \} \). For \( t \geq 1 \), we let

\[
\mathfrak{H}_t := \{ \phi \in \mathcal{H}^2 : \| \mu_\phi \|_2 \leq t, \mu_\phi \in i\mathfrak{a}^* \}
\]

where \( \| \cdot \|_2 \) is the standard Euclidean norm, and \( i\mathfrak{a}^* \subset \mathfrak{a}^*_C \) is isomorphic to \( i\mathbb{R}^n \).

For \( N \in \mathbb{N} \), the Hecke operator \( T_N \) in \( \mathbb{R} \) is defined as

\[
T_N := N^{(n-1)/2} \sum_{m_0 m_1 \cdots m_{n-1} = N} \Gamma \left( \begin{array}{c} m_0 \cdots m_{n-1} \\ m_0 m_1 \cdots m_{n-1} = N \\ \vdots \\ m_0 \end{array} \right) \Gamma
\]

where the summation runs over \( m_0, \ldots, m_{n-1} \in \mathbb{N} \) satisfying \( m_0 m_1^{n-1} \cdots m_{n-1} = N \). For a Hecke-Maass cusp form \( \phi \), its (Hecke) eigenvalue under \( T_m \) is the normalized Fourier coefficient \( A_\phi(m, 1, \ldots, 1) \) of \( \phi \), i.e.

\[
T_m \phi = A_\phi(m, 1, \ldots, 1) \phi.
\]

The Hecke eigenvalues are multiplicative: in fact, for \( (m_1 \cdots m_{n-1}, m'_1 \cdots m'_{n-1}) = 1 \),

\[
A_\phi(m_1, \ldots, m_{n-1}) A_\phi(m'_1, \ldots, m'_{n-1}) = A_\phi(m_1 m'_1, \ldots, m_{n-1} m'_{n-1}).
\]

Moreover, for any \( k = (k_1, \ldots, k_n) \in \mathbb{N}^{n-1} \) and prime \( p \),

\[
A_\phi(p^k) := A_\phi(p^{k_1}, p^{k_2}, \ldots, p^{k_{n-1}}) = S_k(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \ldots, \alpha_{\phi,n}(p))
\]

where \( S_k \) is the (degenerate) Schur polynomial and \( \alpha_{\phi}(p) := (\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \ldots, \alpha_{\phi,n}(p)) \) is the Satake parameter associated to \( \phi \). The Satake parameter satisfies \( \prod_{i=1}^n \alpha_{\phi,i}(p) = 1 \) and

\[
\{ \alpha_{\phi,1}(p), \ldots, \alpha_{\phi,n}(p) \} = \{ \alpha_{\phi,1}(p)^{-1}, \ldots, \alpha_{\phi,n}(p)^{-1} \} \quad \text{(as multisets)}.
\]

Recall \( k^i = (k_{n-i}, \ldots, k_1) \) if \( k = (k_1, \ldots, k_n) \). Then we have

\[
A_\phi(p^k) = A_\phi(p^{k_{n-1}}, \ldots, p^{k_1}) = \overline{A_\phi(p^k)}
\]

and \( A_\phi(p^k) \in \mathbb{R} \) if \( k = k^i \).

Recently Matz and Templier [14] established an automorphic Plancherel density theorem with error term for \( GL(n) \) governing the distribution of \( \alpha_{\phi}(p) \). For every prime \( p \), define the Plancherel measure \( d\mu_p \) on \( SU(n)^2 \) by

\[
d\mu_p := \prod_{i=1}^n (1 - p^{-1}) \prod_{1 \leq i < j \leq n} (1 - p^{-1} e^i(\theta_j - \theta_i))^{-1} d\mu_{ST},
\]

when \( SU(n)^2 \) is identified with \( T_0/\mathfrak{g}_n \) where \( T_0 = \{(e^{i\theta_1}, \ldots, e^{i\theta_n}) : \prod e^{i\theta_i} = 1 \} \) is a subset of \( (S^1)^n \).
3.1. Key propositions. The results below are developed in [13] and the key for Proposition 3.2 is the work of Matz and Templier in [14].

Proposition 3.1. We have (i) $d\mu_p = (1 + O_n(p^{-1}))d\mu_{ST}$,

(ii) $\int_{T_0/\mathbb{S}_n} S_k \, d\mu_{ST} = \delta_{k=0}$ and (iii) $\int_{T_0/\mathbb{S}_n} S_k \, d\mu_p = 0$ if $\|k\| \neq 0 \bmod n$.

Proof. (i) follows easily from (3.4). (ii) is a special case of (2.6) while (iii) is shown in Proposition 7.4 (1) of [13].

Proposition 3.2. Let $k_p, k'_p \in \mathbb{N}_0$ for each prime $p$. Suppose both $k_p$ and $k'_p \neq 0$ only for finitely many $p$'s. Then for some positive constant $L > 0$ such that for any $t \geq 1$,

$$\frac{1}{|\mathcal{F}_t|} \sum_{\phi \in \mathcal{F}_t} \prod_p A_\phi(p^{k_p}) A_\phi(p^{k'_p}) = \prod_p \int_{T_0/\mathbb{S}_n} S_{k_p} \overline{S_{k'_p}} \, d\mu_p + O(t^{-1/2} \prod_p L|k_p + k'_p|)$$

where $|\mathcal{F}_t| = (1 + o(t^{-1/2}))\Lambda(t) \asymp t^d$ (and $d = \frac{1}{2}(n + 1) - 1$).

Proof. It follows from a theorem of Matz and Templier, cf. Theorem 1.3 in [14] and Proposition 7.5 in [13].

Corollary 3.3. Let $k_p, k'_p \in \mathbb{N}_0$ and $u_p, v_p \in \mathbb{N}_0$ for each prime $p$. Assume $u_p, v_p \neq 0$ for finitely many primes. Then for some positive constant $L$,

$$\frac{1}{|\mathcal{F}_t|} \sum_{\phi \in \mathcal{F}_t} \prod_p A_\phi(p^{k_p}) A_\phi(p^{k'_p}) = \prod_p \int_{T_0/\mathbb{S}_n} S_{k_p} \overline{S_{k'_p}} \, d\mu_p + O(t^{-1/2} \prod_p (C_{k_p} L|k_p|)^{u_p} (C_{k'_p} L|k'_p|)^{v_p})$$

where $1 \leq C_k := S_k(1, \cdots, 1) \leq (1 + |k|)^{n^2 - n}$.

Proof. By the Littlewood-Richardson rule (2.7), we have

$$\prod_p A_\phi(p^{k_p}) A_\phi(p^{k'_p}) = \prod_p S_{k_p} \overline{S_{k'_p}} = \prod_p A_\phi(p^{[k_p]}) A_\phi(p^{[k'_p]})$$

Apply Proposition 3.2 to $|\mathcal{F}_t|^{-1} \sum_{\phi \in \mathcal{F}_t} \prod_p A_\phi(p^{[k_p]}) A_\phi(p^{[k'_p]})$. A backward process yields the desired main term. The cumulation of the error terms leads to a term

$$\ll t^{-1/2} \sum_{\xi, \eta \bmod p \text{ primes}} \prod_p \int_{T_0/\mathbb{S}_n} S_{k_p} \overline{S_{k'_p}} \, d\mu_p + O(t^{1/2} \prod_p L|k_p + k'_p|)$$

by $|\xi| \leq n$, $|\eta| \leq n$, and $|\xi| \leq n$, $|\eta| \leq n$. Our result follows since $\sum_{\xi} d_{k_p} \leq S_{k_p}(1, \cdots, 1)^{u_p}$. Note $1 \leq S_k(1, \cdots, 1) \leq (1 + |k|)^{n^2 - n}$, $\forall k$ (cf. [13, Lemma 7.1 (1)]).
3.2. Proof of Theorems 1.2 and 1.3. We may consider $A_\phi(1, \ldots, 1, p)$ in lieu by (3.3) and firstly prove Theorem 1.3. As $\|e\| = 1$ if $e = (0_{n-2}, 1)$. By (1.3) and Corollary 3.3, the left-side equals
\[
\int_{T_{\nu}/S_n} S_{S_n}^m \, d\mu_p + o(1) \quad \text{as } t \to \infty.
\]
If $n \nmid m$, then by (2.7), $S_{S_n}^m$ is a linear combination of $S_{\xi}$ where $\|\xi\| = m\|e\| = m \mod n$ and thus the integral will vanish by Proposition 3.1 (iii). Otherwise, we apply Proposition 3.1 (i) and Lemma 2.1 to get the result.

Now we turn to Theorem 1.2. Let $e = (0_{n-2}, 1)$. We express
\[
\left( \sum_{p \leq x} A_\phi(p^e) \right)^m = \sum_{1 \leq \ell \leq m} \sum_{r_1, \ldots, r_\ell \geq 1} \frac{m!}{r_1! \cdots r_\ell!} \sum_{\text{distinct}} A_\phi(p_1^{r_1}) \cdots A_\phi(p_\ell^{r_\ell}).
\]
By Corollary 3.3, the average of $A_\phi(p_1^{r_1}) \cdots A_\phi(p_\ell^{r_\ell})$ over $\phi \in \mathcal{H}_t$ is
\[
\prod_{i=1}^j \int_{T_{\nu}/S_n} S_{r_i}^{n \log x} \, d\mu_p + O(t^{-1/2}c^m x^{mL}).
\]
The main term is zero unless $n|m$, $\forall 1 \leq i \leq j$. The $O$-term is $\ll x^{-1}$, in light of $\frac{\log t}{\log x} \to \infty$, and hence tends to 0 as $x \to \infty$. The case of $n \nmid m$ follows plainly, noting $n|m$ if $n|r_i$, $\forall 1 \leq i \leq j$.

When $s_i := r_i/n \in \mathbb{N}$ for all $i$, $m = \sum_i r_i$ is divisible by $n$. Write $m = n\ell$. Then $\ell = \sum_i s_i$, so the value of $j$ is at most $\ell$, and all $s_i = 1$ if $j = \ell$. Clearly, with Proposition 3.1 (i), the multiple sum over primes may be written as
\[
\Sigma(n_{s_1}, \ldots, n_{s_j})(x) := \sum_{p_1, \ldots, p_j \leq x} \prod_{i=1}^j \int_{T_{\nu}/S_n} S_{e_{n_{s_i}}}^{n \log x} \, d\mu_p,
\]
\[
= \begin{cases} O_m(\pi(x)^j) & \text{if } j < \ell, \\
\pi(x)^j \int_{T_{\nu}/S_n} S_{n \log x} \, d\mu_p \ell + O_m(\pi(x)^{\ell-1} \log_2 x) & \text{if } j = \ell.
\end{cases}
\]
The integral in the second case equals 1 because $I_e(n) = 1$ by Lemma 2.1. The result follows readily, since for $m = n\ell$,
\[
\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \left( \frac{1}{\pi(x)^{1/n}} \sum_{p \leq x} A_\phi(p^e) \right)^m = \sum_{1 \leq \ell \leq m} \sum_{r_1, \ldots, r_\ell \geq 1} \frac{m!}{r_1! \cdots r_\ell!} \frac{\Sigma(n_{s_1}, \ldots, n_{s_j})(x)}{\ell! \cdot \pi(x)^\ell}
\]
up to the addition of a term $O(x^{-1})$.

4. CENTRAL LIMIT BEHAVIOUR

Let $\{X_i\} \in (0, \infty)$ and $\{T_i\} \in (0, \infty)$ be two collections of finite sets such that $X_i \subseteq X_j$ (resp. $T_i \subseteq T_j$) for $i \leq j$, and both $\mathcal{X} = \bigcup_i X_i$ and $\mathcal{T} = \bigcup_i T_i$ are infinity. Given a family of objects $\{A_\phi(p) : \phi \in \mathcal{T}, p \in \mathcal{X}\}$ and a family of independent complex random variables $\{A_p : p \in \mathcal{X}\}$ over possibly different probability spaces.\(^{12}\)

\(^{12}\)For our main concern, the measurable space is $S^m/\mathcal{S}_n$, the (complex) random variable $A_p$ is (induced from) the function $S_k$ on the measure space $(S^m/\mathcal{S}_n, d\mu_p)$. 

Then there exists a function $x$ where

$$
(1) \frac{1}{\sqrt{|x|}} \sum_{p \in x} |E[A_p]| \to 0 \text{ as } x \to \infty,
$$

$$
(II) \frac{1}{|x|} \sum_{p \in x} E[A_p^2] \to \zeta \text{ as } x \to \infty, \text{ for some constant } \zeta \in \mathbb{C},
$$

$$
(III) \frac{1}{|x|} \sum_{p \in x} E[|A_p|^2] \to v \text{ as } x \to \infty, \text{ for some constant } v > 0,
$$

$$
(IV) E[|A_p|^r] \leq c_0^r \text{ for all } r \geq 0 \text{ and all } p \in X, \text{ for some constant } c_0 \geq 1.
$$

**Theorem 4.1.** Let $a_\phi(p)$ and $A_p$ be defined as above. Suppose the above conditions (I)-(IV) for $\{A_p\}$ holds, and for any $x > 0$,

$$
(4.1) \frac{1}{|T|} \sum_{\phi \in T} \prod_{p \in x} a_\phi(p)u_p a_\phi(p)^\dagger \xrightarrow{t \to \infty} \prod_{p \in x} E[A_p u_p A_p^\dagger]
$$

for any $u_p, v_p \in \mathbb{N}_0$ ($p \in X$). Define

$$
(4.2) Z_x(\phi) = \frac{1}{\sqrt{|x|}} \sum_{p \in x} a_\phi(p).
$$

Then there exists a function $T_A(x)$ satisfying $T_A(x) \to \infty$ as $x \to \infty$ so that for $t = t(x) \geq T_A(x)$, we have the following.

(i) $v^2 - |\zeta|^2 > 0$ : For any continuous bounded function $h : \mathbb{C} \to \mathbb{R}$,

$$
\frac{1}{|T|} \sum_{\phi \in T} h(Z_x(\phi)) \xrightarrow{x \to \infty} \frac{1}{\pi} \frac{1}{\sqrt{\det K}} \int h(z)e^{-\frac{1}{2}z^\dagger \zeta^{-1} \zeta \frac{1}{2}z}dz \wedge d\zeta
$$

where $z = (z \ \bar{z})^T$ lies in $\mathbb{C}^2$, $z^\dagger = (\bar{z} \ z)$ is the conjugate transpose of $z$ and

$$
K = \begin{pmatrix} \nu & \zeta \\ \zeta & \upsilon \end{pmatrix}.
$$

(ii) $\zeta = ve^{i\theta}$ for some $\theta \in [0, 2\pi)$ : For any bounded continuous $h : \mathbb{R} \to \mathbb{R}$,

$$
\frac{1}{|T|} \sum_{\phi \in T} h(\Re(e^{-i\theta/2}Z_x(\phi))) \xrightarrow{x \to \infty} \frac{1}{2\pi \upsilon} \int h(x)e^{-x^2/(2\upsilon)}dx.
$$

**Remark 3.** (a) The function $T_A(x)$ in Theorem 4.1 is determined in (4.11).

(b) Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we may write

$$
\frac{1}{\pi} \frac{1}{\sqrt{\det K}} \int h(z)e^{-\frac{1}{2}z^\dagger \zeta^{-1} \zeta \frac{1}{2}z}dz \wedge d\zeta = \frac{1}{2\pi} \frac{1}{\sqrt{|C|}} \int_{\mathbb{R}^2} h(x, y)e^{-\frac{1}{2}x^2|C|^{-1}x}dx dy
$$

where $z = (x \ y)^T$ denotes vectors in $\mathbb{R}^2$, and

$$
C = \begin{pmatrix} \nu + \Re \zeta & \Re \zeta \\ \Im \zeta & -\Re \zeta \end{pmatrix}.
$$

**Theorem 4.1 (i)** is equivalent to that for any open rectangle $D := (a, b) + i(c, d) \subset \mathbb{C}$,

$$
\lim_{x \to \infty} \text{Prob}_T(Z_x(\phi) \in D) = \frac{1}{2\pi} \frac{1}{\sqrt{|C|}} \int_a^b \int_c^d e^{-\frac{1}{2}x^2|C|^{-1}x}dx dy
$$

where $t = t(x) \geq T_A(x)$. 
(c) Theorem 4.1 (ii) implies that for any open interval \((a, b)\),
\[
\lim_{x \to \infty} \text{Prob}_x \left( a < \Re \left( e^{-i\theta/2} Z_x(\phi) \right) < b \right) = \frac{1}{2\pi} \frac{1}{\sqrt{b-a}} \int_a^b e^{-x^2/(2v)} \, dx
\]
where \(t = t(x) \geq T_\Lambda(x)\).

(d) If \(\{a_\phi(p)\} \subset \mathbb{R}\), then for \(t \geq T_\Lambda(x)\),
\[
\frac{1}{|\Omega|} \sum_{\phi \in \Omega} h(Z_x(\phi)) \to \frac{1}{2\pi \sqrt{v}} \int h(x)e^{-x^2/(2v)} \, dx
\]
for any bounded continuous \(h : \mathbb{R} \to \mathbb{R}\). In this case \(\Re \left( e^{-i\theta/2} Z_x(\phi) \right) = Z_x(\phi)\).

**Remark 4.** Indeed, Conditions (I)-(IV) are sufficient to establish the central limit theorem for the family \(\{A_p : p \in \mathcal{X}\}\) of independent random variables. This can be seen from the characteristic function in (4.18) with the continuity theorem. Moreover, the law of iterated logarithm is valid under a condition slightly stronger than (I):

\((I)'\) There exists \(\delta > 0\) such that
\[
\frac{1}{\sqrt{|\mathcal{X}|}} \sum_{p \in \mathcal{X}} |\mathbb{E}[A_p]| = O((\log |\mathcal{X}|)^{-1-\delta})
\]
where the implied \(O\)-constant is independent of \(x\).

Under Conditions (I)', (II)-(IV), both
\[
\limsup_{x \to \infty} \frac{\Re \sum_{p \in \mathcal{X}_x} A_p}{\sqrt{2v|\mathcal{X}_x| \log_2 |\mathcal{X}_x|}} = \limsup_{x \to \infty} \frac{\text{Im} \sum_{p \in \mathcal{X}_x} A_p}{\sqrt{2v|\mathcal{X}_x| \log_2 |\mathcal{X}_x|}} = 1 \text{ almost surely.}
\]
This follows from the Berry-Esseen inequality, cf. [19, §7.6], and [16, Theorem] or the corollary after [7, Theorem 1]. (See [2, §5] for the case that \(\mathbb{E}[A_p] = 0\) for all \(p\)).

Next we consider the central limit behaviour for the frequency. Let \(\varphi \in \mathcal{C}_0^\infty(\mathbb{C})\) be a real-valued function. (The prototype is a smooth function enveloping the characteristic function over a square.) Given the families \(\{b_\phi(p) : \phi \in \mathcal{T}, p \in \mathcal{X}\}\) (of some objects) and \(\{B_p : p \in \mathcal{X}\}\) (of independent random variables). We obtain, under some conditions, the central limit theorem for \(\{\varphi(b_\phi(p))\}\).

**Theorem 4.2.** Let \(B_p, p \in \mathcal{X}\), be independent random variables that satisfy Conditions (I)-(IV) (as in Theorem 4.1). Moreover, for some real-valued smooth compactly supported function \(\varphi\) on \(\mathbb{C}\),
\[
\frac{1}{|\mathcal{X}|} \sum_{p \in \mathcal{X}_x} \left| \mathbb{E}[\varphi(B_p)] - \mu \right| \to 0 \quad \text{and} \quad \frac{1}{|\mathcal{X}|} \sum_{p \in \mathcal{X}_x} \mathbb{E}[|\varphi(B_p)|^2] \to \nu \quad \text{as} \ x \to \infty,
\]
where \(\mu \in \mathbb{R}\) and \(\nu > \mu^2\). Suppose \(\{b_\phi(p) : \phi \in \mathcal{T}, p \in \mathcal{X}\}\) satisfies that for any \(u_p, v_p \in \mathbb{N}_0 (p \in \mathcal{X})\),
\[
\frac{1}{|\Omega|} \sum_{\phi \in \Omega} \prod_{p \in \mathcal{X}_x} b_\phi(p)^{u_p} b_\phi(p)^{v_p} \to \prod_{p \in \mathcal{X}_x} \mathbb{E}[B_p^{u_p} B_p^{v_p}].
\]
Define
\[
\varphi_{\mathcal{X}_x}(\phi) := \frac{\sum_{p \in \mathcal{X}_x} \varphi(b_\phi(p)) - |\mathcal{X}_x| \mu}{\sqrt{|\mathcal{X}_x|}}.
\]
There exists a function \( T_B(x) \) satisfying \( T_B(x) \to \infty \) as \( x \to \infty \) such that for \( t \geq T_B(x) \),
\[
\frac{1}{|q|} \sum_{\phi \in g_i} h(\mathcal{Z}_x(\phi)) \to \frac{1}{2\pi \eta} \int h(u)e^{-u^2/(2\eta^2)} \, du
\]
where \( \eta^2 = \nu - \mu^2 \) and \( h : \mathbb{C} \to \mathbb{R} \) is any bounded continuous function.

Remark 5. The smooth compactly supported function \( \varphi \) is advantageous to the analytic approach. For instance, in [6] and [17], the theory of Beurling-Selberg polynomials are invoked to deal with the characteristic function (over an interval). Beurling-Selberg polynomials are trigonometric polynomials which seems less tractable in the \( GL(n) \) case.

4.1. Preparation. We start with a lemma.

Lemma 4.3. Let \( \{v_p\}_{p \in \mathcal{X}} \) be a bounded sequence in \( \mathbb{C} \), say, \( |v_p| \leq \Upsilon \) for all \( p \). Under the assumption (I)-(IV) for \( A_p \), we have that for all sufficiently large \( x \geq x_0 \) and any integer \( 1 \leq M, N \leq |\mathcal{X}_x| \),
\[
\frac{1}{|\mathcal{X}_x|^{(M+N)/2}} \left| \mathbb{E} \left[ \left( \sum_{p \in \mathcal{X}_x} v_p A_p \right)^M \left( \sum_{p \in \mathcal{X}_x} v_p A_p \right)^N \right] \right| \leq (9c_0 \Upsilon)^{M+N} \left( \frac{(M + N)^{M+N}}{|\mathcal{X}_x|^{1/2}} + (M + N)^{(M+N)/2} \right).
\]

Proof. Since
\[
\left( \sum_{p \in \mathcal{X}_x} v_p A_p \right)^M = \sum_{1 \leq u \leq M} \sum_{\alpha_1, \ldots, \alpha_u \geq 1} \frac{M!}{\prod_{1 \leq j \leq u} \alpha_j!} \sum_{\substack{1 \leq p_1, \ldots, p_u \in \mathcal{X}_x \text{ distinct}}} v_{p_1}^{\alpha_1} \cdots v_{p_u}^{\alpha_u} A_{p_1}^{\alpha_1} \cdots A_{p_u}^{\alpha_u},
\]
where the rightmost sum runs over \( (p_1, \ldots, p_u) \in \mathcal{X}_x^u \) of distinct entries (i.e. \( p_i \neq p_j \) for every \( 1 \leq i \neq j \leq u \)), we deduce that
\begin{equation}
(4.5)
\end{equation}
\[
\mathbb{E} \left[ \left( \sum_{p \in \mathcal{X}_x} v_p A_p \right)^M \left( \sum_{p \in \mathcal{X}_x} v_p A_p \right)^N \right] = \sum_{1 \leq u \leq M} \sum_{\substack{\alpha, \beta \in \mathbb{N}_0^u, |\alpha| = M, |\beta| = N}} C(M, N, \alpha, \beta) \cdot \mathbb{E}[S_x(\alpha)S_x(\beta)]
\]
where
\begin{align}
(4.6) & \quad C(M, N, \alpha, \beta) = \frac{M!N!}{(\prod_{1 \leq j \leq u} \alpha_j!) (\prod_{1 \leq j \leq v} \beta_j!)} \cdot \frac{1}{u!v!}; \\
(4.7) & \quad \mathbb{E}[S_x(\alpha)S_x(\beta)] = \sum_{p_1, \cdots, p_u \in \mathcal{X}_x \text{ distinct}} \sum_{q_1, \cdots, q_v \in \mathcal{X}_x \text{ distinct}} v_{p_1}^{\alpha_1} \cdots v_{p_u}^{\alpha_u} v_{q_1}^{\beta_1} \cdots v_{q_v}^{\beta_v} \mathbb{E}[A_{p_1}^{\alpha_1} \cdots A_{p_u}^{\alpha_u} A_{q_1}^{\beta_1} \cdots A_{q_v}^{\beta_v}].
\end{align}

Now let \( 0 \leq i \leq M \) and \( 0 \leq j \leq N \) (and \( M, N \leq |\mathcal{X}_x| \)). The tuple \((u, v, \alpha, \beta, a, b)\) is said to be \((i, j)\)-admissible or simply admissible if the following are fulfilled:

- \( i \leq u \leq M \) and \( 0 \leq j \leq v \leq N \),
- \( \alpha = (\alpha_1, \ldots, \alpha_u) \in \mathbb{N}^u \) and \( \beta = (\beta_1, \ldots, \beta_v) \in \mathbb{N}^v \) where \( |\alpha| + |\beta| \leq M + N \), \( \alpha_1 = \cdots = \alpha_i = 1 = \beta_1 = \cdots = \beta_j \) and all other components \( \alpha_r, \beta_s \) are at least 2,
- \( a = (a_{i+1}, \cdots, a_u) \) with \( 0 \leq a_r \leq \alpha_r \) and \( b = (b_{j+1}, \cdots, b_v) \) with \( 0 \leq b_s \leq \beta_s \).
Introduce the notation

\[ 14 \] Y.-K. LAU, M.H. NG & Y. WANG

(4.8) \( \beta_{i,j}(\alpha, \beta, a, b) \)

\[
:= \sum_{p_1, \ldots, p_u \in \mathcal{X}_u \atop \text{distinct}} \sum_{q_1, \ldots, q_v \in \mathcal{X}_v \atop \text{distinct}} \left| \mathbb{E}[A_{p_1} \cdots A_{p_u} \overline{A}_{q_1} \cdots A_{q_v}] \cdot \prod_{r=1}^{u} A_{p_r}^{\alpha_r} A_{p_r}^{-\alpha_r} \prod_{s=j+1}^{v} A_{q_s}^\beta - b_s A_{q_s}^{b_s} \right|.
\]

Here, the empty product means 1 as usual. Clearly (after relabeling the running indices) we have

\[
\left| \mathbb{E}[S_x(\alpha)\overline{S_x}(\beta)] \right| \leq T^{M+N} \beta_{i,j}(\alpha, \beta, a, b)
\]

for some \( i, j, a, b \). Our goal is to show: for admissible \((u, v, \alpha, \beta, a, b)\),

\[ 15 \]

\[
\beta_{i,j}(\alpha, \beta, a, b) \leq c_0^{M+N} |\mathcal{X}_x|^{u+v-i-j}(9|\mathcal{X}_x|(M + N))^{(i+j)/2}
\]

for all \( x \geq x_0 \), where \( x_0 \) is a large enough fixed number. Note that \( u, v \) represent the number of components of \( \alpha \) and \( \beta \).

When \( i = j = 0 \) (i.e. \( \alpha_1, \ldots, \alpha_u, \beta_1, \ldots, \beta_v \geq 2 \)), we have

\[
\beta_{0,0}(\alpha, \beta, a, b) \leq \sum_{r=1}^{u} \sum_{p \in \mathcal{X}_x} \mathbb{E}[|A_p|^\alpha] \cdot \prod_{s=1}^{v} \sum_{q \in \mathcal{X}_x} \mathbb{E}[|A_q|^\beta] \leq c_0^{u+v} |\mathcal{X}_x|^{u+v}
\]

by Condition (IV), so (4.9) holds for \( i = j = 0 \). We may proceed with induction on \((i, j)\). Given \( \beta_{i,j}(\alpha, \beta, a, b) \) with \( i \geq 1 \). We shift the summation over \( p_1 \) in (4.8) to the innermost and split into two pieces according as \( p_1 \in \{q_1, \ldots, q_v\} \) or not. For \( p_1 \) is distinct from \( p_2, \ldots, p_u, q_1, \ldots, q_v \), the latter case is obviously

\[
\leq \beta_{i-1,j}(\alpha^+, \beta, a, b) \sum_{p \in \mathcal{X}_x} \mathbb{E}[|A_p|] \leq |\mathcal{X}_x|^{1/2} \beta_{i-1,j}(\alpha^-, \beta, a, b)
\]

for all \( x \geq x_0 \), by (1), where \( x_0 \) is some suitably large number and \( \alpha^-(\alpha_2, \ldots, \alpha_u) \). Hence by induction hypothesis, it is

\[
\leq |\mathcal{X}_x|^{1/2} c_0^{M+N} |\mathcal{X}_x|^{u+1+v-(i-1)+(9|\mathcal{X}_x|(M + N))^{(i-1+j)/2}
\]

\[
= c_0^{M+N} |\mathcal{X}_x|^{u+v-i-j}(9|\mathcal{X}_x|(M + N))^{(i+j)/2} \frac{1}{3(M+N)1/2},
\]

the last fraction of which is \(< 1/3 \). For the former case (i.e. \( p_1 = q_1, \ldots \text{ or } q_v \)), \( \beta_{i,j}(\alpha, \beta, a, b) \) is bounded by

\[
\sum_{1 \leq r \leq v} \sum_{p_2, \ldots, p_u \in \mathcal{X}_u \atop \text{distinct}} \sum_{q_1, \ldots, q_v \in \mathcal{X}_v \atop \text{distinct}} \left| \mathbb{E}[A_{p_2} \cdots A_{p_u} \overline{A}_{q_1} \cdots A_{q_v}] \cdot \prod_{r=1}^{u} A_{p_r}^{\alpha_r} A_{p_r}^{-\alpha_r} \prod_{s=j+1}^{v} A_{q_s}^\beta - b_s A_{q_s}^{b_s} \right| \leq j\beta_{i-1,j-1}(\alpha^+, \beta + \epsilon_j, a, b^+) + (v - j)\beta_{i-1,j}(\alpha^-, \beta + \epsilon_v, a, b)
\]

after relabeling, where \( \alpha^- = (\alpha_2, \ldots, \alpha_u) \), \( b^+ = (1, b_{j+1}, \ldots, b_v) \) and \( \epsilon_x \) denotes the \( r \)th standard coordinate vector whose \( r \)th component is 1 and 0 otherwise. Note that
\[ |a^-| + |\beta + c_\epsilon| = |a| + |\beta|. \]

It is
\[
\begin{align*}
&\leq j_0^{M+N} |\mathcal{X}_x|^{u+v-i-j+1} (9|\mathcal{X}_x|(M + N))^{(i+j)/2-1} \\
&\quad + (N-j) j_0^{M+N} |\mathcal{X}_x|^{u+v-i-j} (9|\mathcal{X}_x|(M + N))^{(i+j)/2-1/2} \\
&= c_0^{M+N} |\mathcal{X}_x|^{u+v-i-j} (9|\mathcal{X}_x|(M + N))^{(i+j)/2} \\
&\quad \left( \frac{j}{9(M + N)} + \frac{N-j}{3(|\mathcal{X}_x|(M + N))^{1/2}} \right)
\end{align*}
\]

where the two summands in the bracket are respectively < 1/3 for \( N \leq |\mathcal{X}_x| \).

The argument (of shifting the summation over \( p_1 \)) holds for \( j = 0 \). Altogether, we infer inductively (4.9) for \( 0 \leq i \leq u, j = 0 \). Applying the same argument to \( q_1 \) and so on, we obtain all the other cases.

By (4.7) and (4.9), we get
\[
|E[S_x(\alpha)\overline{S_x(\beta)}]| \leq (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{u+v-(i+j)/2} (M + N)^{(i+j)/2}
\]

for some \( 0 \leq i \leq u, 0 \leq j \leq v \) satisfying \( i + 2(u - i) \leq M, j + 2(v - j) \leq N \) (which follow from \( |a| = M \) and \( |\beta| = N \) respectively). If \( u - \frac{1}{2} < M/2 \) or \( v - \frac{1}{2} < N/2 \), then the right-side is
\[
\leq (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N-1)/2} (M + N)^{(u+v)/2},
\]
or otherwise, it equals \( (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} (M + N)^{u+v-(M+N)/2} \). Putting these and (4.6) into (4.5), the expression on the left-side of (4.5) has its modulus
\[
\begin{align*}
\leq (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} &\left( |\mathcal{X}_x|^{-1/2} + (M + N)^{-1-(M+N)/2} \right) \\
&\times \sum_{1 \leq u \leq M} \frac{(M + N)^{u+v}}{u!v!} \sum_{\frac{M}{2} \leq j \leq M \atop \frac{N}{2} \leq j \leq N} \frac{M!N!}{(\prod_{1 \leq j \leq u} \alpha_j)! (\prod_{1 \leq j \leq v} \beta_j)!} \\
&\leq (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} \left( |\mathcal{X}_x|^{-1/2} + (M + N)^{-1-(M+N)/2} \right) \sum_{1 \leq u \leq M \atop 1 \leq v \leq N} \frac{(M + N)^{u+v}}{u!v!} u^{M} v^{N} \\
&\leq (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} \left( \frac{(M + N)^{M+N}}{|\mathcal{X}_x|^{1/2}} + (M + N)^{(M+N)/2} \right).
\end{align*}
\]

The desired result follows. \( \square \)

### 4.2. Proof of Theorem 4.1.

Firstly consider the case \( v^2 > |\zeta|^2 \). By Lévy’s continuity theorem (cf. [20, 23]), it suffices to show that the characteristic function
\[
\frac{1}{|T|} \sum_{\phi \in \mathfrak{H}} e^{i \text{Re} (\tau Z_x(\phi))} \xrightarrow{x \to \infty} e^{-\frac{1}{4} v^2 |\tau|^2 - \frac{1}{2} \text{Re} (\tau^2 \zeta)}
\]

pointwisely in \( \tau \in \mathbb{C} \) where \( t \geq T_\lambda(x) \) and the function \( T_\lambda(x) \) is chosen such that for all \( t \geq T_\lambda(x) \),
\[
\frac{1}{|T|} \sum_{\phi \in \mathfrak{H}} \prod_{p \in \mathcal{X}_x} a_p(p)^{u_p} a_\phi(p)^{v_p} = \prod_{p \in \mathcal{X}_x} \text{E}[A_p^{u_p} \overline{A_p}^{v_p}] + O_{a,b}(|\mathcal{X}_x|^{-(a+b)/2-1})
\]

where \( u_p, v_p \in \mathbb{N}_0 \) satisfy \( \sum_p u_p = a, \sum_p v_p = b \) and the implied \( O \)-constant depends at most on \( a, b \).
Let \( \tau \in \mathbb{C} \) be fixed, and \( \varepsilon > 0 \) be any arbitrarily small number. We express the left-hand side of (4.10) into

\[
\frac{1}{|T|} \sum_{\phi \in T} e^{i \Re(\tau Z_{x}(\phi))} = M_{N}(\tau) + E_{N}(\tau)
\]

with the power series of \( \exp(x) \) and binomial theorem, where

\[
M_{N}(\tau) = \sum_{0 \leq a + b \leq 2N} \frac{\tau^{a+b}}{a!b!} \left( \frac{i}{2} \right)^{a+b} \frac{1}{|T|} \sum_{\phi \in T} Z_{x}(\phi)^{a} \overline{Z_{x}(\phi)}^{b}
\]

and

\[
|E_{N}(\tau)| \leq 3 \frac{|\tau|^{2N}}{(2N)!} \frac{1}{|T|} \sum_{\phi \in T} |Z_{x}(\phi)|^{2N}.
\]

Write \( x = |X_{x}| \), then \( |u| = \sum_{p \in X_{x}} u_{p} \) for a tuple \( u \in \mathbb{N}_{0}^{N} \). We have

\[
Z_{\phi}(x)^{a} = \frac{1}{|X_{x}|^{a/2}} \sum_{u \in \mathbb{N}_{0}^{N}, \|u\|_{1} = a} \frac{a!}{\prod_{p \in X_{x}} u_{p}! \prod_{p \in X_{x}} a_{\phi}(p)^{u_{p}}}
\]

(where \( \prod_{p \in X_{x}} \) is a product of at most \( a \) terms). Thus by (4.11), for \( a + b \leq 2N \),

\[
\frac{1}{|T|} \sum_{\phi \in T} Z_{x}(\phi)^{a} \overline{Z_{x}(\phi)}^{b} = \frac{1}{|X_{x}|^{(a+b)/2}} \sum_{u \in \mathbb{N}_{0}^{N}, \|u\|_{1} = a} \frac{a!}{\prod_{p \in X_{x}} u_{p}! \prod_{p \in X_{x}} a_{\phi}(p)^{u_{p}}} \left( \sum_{p \in X_{x}} A_{p} \right)^{a} \left( \sum_{p \in X_{x}} \overline{A}_{p} \right)^{b} + O_{N}(|X_{x}|^{-1})
\]

(4.16)

where the implied \( O_{N} \)-constant depends at most on \( N \). Inserting (4.16) into (4.14) and (4.13) respectively, we firstly obtain

\[
E_{N}(\tau) = \frac{O(|\tau|^{2N})}{(2N)!|X_{x}|^{N}} \mathbb{E} \left( \left| \sum_{p \in X_{x}} A_{p} \right|^{2N} \right) + O_{N}(|X_{x}|^{-1} e^{|\tau|}).
\]

It has to be emphasized that the first implied \( O \)-constant is absolute (i.e. independent of \( N \)). Secondly,

\[
M_{N}(\tau) = \sum_{0 \leq a + b \leq 2N} \frac{\tau^{a+b}}{a!b!} \left( \frac{i}{2 \sqrt{|X_{x}|}} \right)^{a+b} \mathbb{E} \left( \left| \sum_{p \in X_{x}} A_{p} \right|^{a} \left( \sum_{p \in X_{x}} \overline{A}_{p} \right)^{b} \right) + O_{N}(|X_{x}|^{-1} e^{|\tau|}).
\]

Hence we infer from (4.12) that

\[
\frac{1}{|T|} \sum_{\phi \in T} e^{i \Re(\tau Z_{x}(\phi))} = \mathbb{E} \left[ \exp \left( \frac{i}{\sqrt{|X_{x}|}} \Re \left( \tau \sum_{p \in X_{x}} A_{p} \right) \right) \right] + \frac{1}{|X_{x}|^{N}} \mathbb{E} \left( \left| \sum_{p \in X_{x}} A_{p} \right|^{2N} \right) + O_{N}(e^{|\tau|})
\]

(4.17)
If $M = N \leq |X_x|$, then by Lemma 4.3, the second summand on the right-hand side is
\[
\leq (c|\tau|)^{2N} \left( \frac{(2N)^{2N}}{(2N)!} \cdot |X_x|^{1/2} + \frac{(2N)^N}{(2N)!} \right) \leq (c'|\tau|)^{2N}(|X_x|^{-1/2} + N^{-N})
\]
by Stirling’s formula, for some absolute constants $c, c' > 1$.

Choose $N = N(\varepsilon, \tau) \geq 10c_0$ and $x_0 = x_0(\varepsilon, \tau, N)$ such that for all $x \geq x_0$,
\[
(c'|\tau|)^{2N}(|X_x|^{-1/2} + N^{-N}) + \left| O_N \left( \frac{|\tau|}{|X_x|} \right) \right| \leq \varepsilon.
\]
It remains to treat the first summand in (4.17), whose logarithm is expressed into

\[
\log \prod_{p \in X_x} \mathbb{E} \left[ \exp \left( \frac{i}{\sqrt{|X_x|}} \text{Re}(\tau A_p) \right) \right]
\]
by the independence of $A_p$’s. Expanding $\mathbb{E}[\cdots]$ (as $c_0|\tau| < |X_x|^{1/8}$) into
\[
1 + \frac{i}{\sqrt{|X_x|}} \mathbb{E} \left[ \text{Re}(\tau A_p) \right] - \frac{1}{2|X_x|} \mathbb{E} \left[ (\text{Re}(\tau A_p))^2 \right] + \mathbb{E}[|A_p|^3] O \left( \frac{|\tau|^3}{|X_x|^{3/2}} \right)
\]
we conclude with (i) that (4.18) equals
\[
- \frac{1}{2|X_x|} \sum_{p \in X_x} \mathbb{E} \left[ (\text{Re}(\tau A_p))^2 \right] + o(1) = -\frac{1}{8}(v\tau^2 + \varepsilon\tau^2 + 2v|\tau|^2) + o(1)
\]
by (II) and (III), where $o(1) \to 0$ as $x \to \infty$. Consequently, the discrepancy between the right-hand-side of (4.17) (with $t \geq T_\Lambda(x)$) and the function
\[ e^{-\frac{1}{2}(v|\tau|^2 + \text{Re}(\tau^2))} \]
is at most $2\varepsilon$, for all $x \geq x_1(\varepsilon, \tau)$, which yields (4.10).

Next we consider Case (ii) which is equivalent to $v^2 = |\varsigma|^2$. The result will follows from
\[
\frac{1}{|T_\ell|} \sum_{\phi \in T_\ell} e^{i\lambda \text{Re}(\bar{Z}_x(\phi))} e^{-\frac{1}{2}uv^2}\lambda^2
\]
where $\lambda \in \mathbb{R}$ and $\bar{Z}_x(\phi) = e^{-i\phi/2}Z_x(\phi)$. As $\lambda \text{Re}(\bar{Z}_x(\phi)) = \text{Re}(\tau Z_x(\phi))$ with $\tau = \lambda e^{i\phi/2}$, we repeat the computation (4.12)-(4.17) and the subsequent estimates with this $\tau$. The main term is $e^{-\frac{1}{2}uv\lambda^2}$ since, in this case,
\[
\mathbb{E}[(\text{Re}(\tau A_p))^2] = \lambda^2 (e^{-i\phi} \mathbb{E}[A_p^2] + e^{i\phi} \mathbb{E}[\overline{A_p^2}] + 2 \mathbb{E}[|A_p|^3]) = 4uv\lambda^2.
\]

4.3. Proof of Theorem 4.2. Let $Y = |X_x|^\delta$ where $\delta \in (0, \frac{1}{4})$ is any fixed (small) number, and $M = ((c_0 + 1)Y)^4 \leq |X_x|$. Choose $T_\Lambda(x)$ such that for all $t \geq T_\Lambda(x)$,
\[
(19) \quad \frac{1}{|T_\ell|} \sum_{\phi \in T_\ell} \prod_{p \in X_x} b_\phi(p)^{u_p} b_{\overline{\phi}}(p)^{v_p} = \prod_{p \in X_x} \mathbb{E}[B_p^{u_p}, \overline{B_p^{v_p}}] + O(|X_x|^{-M})
\]
where $u_p, v_p \in \mathbb{N}_0$ satisfy $\sum_p (u_p + v_p) \leq M$. The implied $O$-constant is uniform in $M$ and $x$. 
Now we set
\begin{equation}
(4.20) \quad a_{\phi}(p) = \varphi(b_{\phi}(p)) - \mu \quad \text{and} \quad A_{p} = \varphi(B_{p}) - \mu.
\end{equation}
Plainly $A_{p}$s satisfy Conditions (I), (II) (which is now identical to (III)) and (IV) in Theorem 4.1 in view of (4.3) and the boundedness of $\varphi$. Next we show that Equation (4.11) holds for $t \geq T_{B}(x)$. (As $a_{\phi}(p)$ is real, all $v_{\phi}$ may be taken as 0.)

Let $u_{p} \in \mathbb{N}_{0}$, $p \in X$, such that $\sum_{p \in X} u_{p} = a$. We may only consider sufficiently large $x$ so that $Y := |X_{x}|^{\mu} \geq a + 1$. Now,
\begin{equation}
(4.21) \quad \frac{1}{|T|} \sum_{\phi \in T} \prod_{p \in X} a_{\phi}(p)^{u_{p}} = \frac{1}{|T|} \sum_{\phi \in T} \prod_{p \in X} (\varphi(b_{\phi}(p)) - \mu)^{u_{p}}.
\end{equation}
As $\varphi \in C_{0}^{\infty}$, its Fourier transform\(^{13}\) $\hat{\varphi}$ decays rapidly: $\hat{\varphi}(\tau) \ll_{r} |\tau|^{r}$ for all $|\tau| \geq 1$ and $r \geq 1$. Then
\begin{equation}
\varphi(b_{\phi}(p)) = \varphi_{Y}(b_{\phi}(p)) + O_{a, \delta}(|X_{x}|^{-a - 1})
\end{equation}
where
\begin{equation}
\varphi_{Y}(b_{\phi}(p)) = (2\pi)^{-2} \int \hat{\varphi}_{Y}(\tau) \mu_{Y}^{1}Re(\tau_{b}(p))
\end{equation}
with $\hat{\varphi}_{Y} = \hat{\varphi} \cdot \chi_{C,Y}$ and $\chi_{C,Y}$ is the characteristic function over $\{\tau \in \mathbb{C} : |\tau| \leq Y\}$.

Let $\mathcal{P}_{x} = \{p \in \mathcal{P} : u_{p} \geq 1\}$. Note that $|\mathcal{P}_{x}| \leq a$. We infer that
\begin{equation}
(4.22) \quad \prod_{p \in X} (\varphi(b_{\phi}(p)) - \mu)^{u_{p}} = \prod_{p \in \mathcal{P}_{x}} (\varphi_{Y}(b_{\phi}(p)) - \mu)^{u_{p}} + O_{a, \delta}(|X_{x}|^{-a - 1}).
\end{equation}
In the following $i$, $j$ and $k$ will denote tuples of nonnegative integers ordered by $p \in \mathcal{P}_{x}$. Applying binomial expansion, we write
\begin{equation}
(4.23) \quad \prod_{p \in X} (\varphi_{Y}(b_{\phi}(p)) - \mu)^{u_{p}} = \sum_{0 \leq i_{p} \leq u_{p}, \forall p \in \mathcal{P}_{x}} C_{(i)} \int e^{i_{p}Re(w_{x}(\phi))} \cdot \prod_{p \in \mathcal{P}_{x}} \prod_{\tau \in \mathcal{T}_{\ell,p}} \hat{\varphi}_{Y}(\tau_{\ell,p})
\end{equation}
where the integral sign denotes a multiple integral of at most $a$ folds,
\begin{equation}
C_{(i)}(\mu) = \prod_{p \in \mathcal{P}_{x}} \frac{u_{p}!}{(2\pi)^{2u_{p}} \cdot i_{p}!(u_{p} - i_{p})!}
\end{equation}
and
\begin{equation}
(4.24) \quad w_{x}(\phi) = \sum_{p \in \mathcal{P}_{x}} \omega_{p} b_{\phi}(p) \quad \text{with} \quad \omega_{p} = \sum_{\ell=1}^{u_{p}} \tau_{\ell,p}.
\end{equation}
Use the expansion
\begin{equation}
(4.25) \quad e^{iRe(w_{x}(\phi))} = \sum_{0 \leq \alpha + \beta \leq 2M} \frac{1}{\alpha! \beta!} \left( \frac{1}{2} \right)^{\alpha + \beta} w_{x}(\phi)^{\alpha} w_{x}(\phi)^{\beta} + O\left( \frac{1}{(2M)!}|w_{x}(\phi)|^{2M} \right)
\end{equation}
where the implied $O$-constant is at most 3. Inserting into (4.23), (4.22) and then (4.21) and shifting the sum over $\phi$ to inside, we are led to evaluate
\begin{equation}
\frac{1}{(2M)!} \int_{T} \left| w_{x}(\phi) \right|^{2M} \quad \text{and} \quad \frac{1}{|T|} \sum_{\phi \in T} \left| w_{x}(\phi) \right|^{2M}
\end{equation}

\(^{13}\)Here we have defined $\hat{\varphi}(\tau) := \int_{C} \varphi(z) e^{-iRe(\tau)z} \frac{1}{2} dz \wedge d\overline{z}$, cf. [11, Chapter VII].
for $0 \leq \alpha + \beta \leq 2M$. Recall $\sum_{p \in \mathcal{P}_x} u_p = a$ and $i_p \leq u_p$. For the former sum, we only give an upper estimate: by Hölder’s inequality and (4.24),

$$|w_x(\phi)|^{2M} \leq \sum_{p \in \mathcal{P}_x} |b_\phi(p)|^{2M} \left( \sum_{p \in \mathcal{P}_x} |\omega_p|^{2M/(2M-1)} \right)^{2M-1},$$

$$\leq a^{4M} Y^{2M} \sum_{p \in \mathcal{P}_x} |b_\phi(p)|^{2M},$$

thus, by (4.19) and $M \geq (c_0 Y + a)^4$ (in view of the choice of $M$),

$$\left( \frac{1}{2M!} \frac{1}{|T|} \right) \sum_{\phi \in \mathcal{H}_t} |w_x(\phi)|^{2M} \leq \frac{a}{(2M)!} (c_0 a^2 Y)^{2M} \leq |X_x|^{-a-1},$$

recalling $|X_x| \geq (a + 1)^{1/\delta}$. The latter sum is

$$\frac{1}{|T|} \sum_{\phi \in \mathcal{H}_t} w_x(\phi)^\alpha w_x(\phi)^\beta$$

$$\begin{align*}
= \alpha! \beta! \sum_{\frac{1}{2} \sum p_i = \alpha} \prod_{p \in \mathcal{P}_x} \omega_p^{k_p} \prod_{p \in \mathcal{P}_x} \omega_p^{k_p} |T| \sum_{\phi \in \mathcal{H}_t} b_\phi(p)^{2\alpha} b_\phi(p)^{2\beta} \\
= \mathbb{E} \left[ \left( \sum_{p \in \mathcal{P}_x} |\omega_p B_p| \right)^\alpha \left( \sum_{p \in \mathcal{P}_x} |\omega_p B_p| \right)^\beta \right] + O((aY)^{a+\beta}|X_x|^{-M})
\end{align*}$$

by (4.19) and the facts $\sum_p |\omega_p| \leq Y \sum_p i_p \leq aY$ for $\sum_p i_p \leq \sum_p u_p = a$. Consequently, we get by (4.25) and (4.26),

$$\frac{1}{|T|} \sum_{\phi \in \mathcal{H}_t} e^{i \text{Re}(w_x(\phi))} = \mathbb{E} \left[ e^{i \text{Re}(\sum_{p \in \mathcal{P}_x} |\omega_p B_p|)} \right] + O_a(|X_x|^{-a-1}).$$

As

$$\int \prod_{p \in \mathcal{P}_x} \prod_{\ell = 1}^{i_p} |\hat{\varphi}_Y(\tau_{\ell,p})| \leq \|\hat{\varphi}\|_{L^1} \prod_{p \in \mathcal{P}_x} \prod_{\ell = 1}^{i_p} |\hat{\varphi}_Y(\tau_{\ell,p})|,$$

it follows from (4.21) and (4.20) that

$$\frac{1}{|T|} \sum_{\phi \in \mathcal{H}_t} \prod_{p \in \mathcal{P}_x} a_\phi(p)^{u_p} = \sum_{i_p = u_p} \prod_{p \in \mathcal{P}_x} \mathbb{E} \left[ e^{i \text{Re}(\sum_{p \in \mathcal{P}_x} |\omega_p B_p|)} \right] \prod_{p \in \mathcal{P}_x} \prod_{\ell = 1}^{i_p} \hat{\varphi}_Y(\tau_{\ell,p})$$

$$+ O_a \left( |X_x|^{-a-1} \prod_{p \in \mathcal{P}_x} (\|\hat{\varphi}\|_{L^1} + |\mu|)^{i_p} \right).$$
The $O$-term is $\ll_a |X_\alpha|^{-a-1}$. Reverting the steps in (4.22)-(4.23), the main term is
\[
\sum_{i_p \leq u_p, v_p \in \mathbb{P}_x} C_i(\mu) \prod_{p \in \mathbb{P}_x} \mathbb{E} \left[ \left( (2\pi)^{-2} \int \tilde{f}_Y(\tau) e^{i \Re(\tau B_p)} \right)^{i_p} \right]
\]
\[
= \mathbb{E} \left[ \prod_{p \in \mathbb{P}_x} (\varphi(B_p) - \mu)^{u_p} \right] + O_a(|\mathbb{X}_x|^{-a-1})
\]
\[
= \prod_{p \in \mathbb{P}_x} \mathbb{E}[A_p^{u_p}] + O_a(|\mathbb{X}_x|^{-a-1}),
\]
which implies readily (4.11). Hence we can apply Theorem 4.1 (ii), actually Remark 3 (c), to $a_\varphi(p)$ and $A_p$ in (4.20) to conclude the result.

5. Proofs of Theorem 1.1 and 1.4

We shall make use of Theorems 4.1 and 4.2, and Remark 3 (b) and (c).

Let $\mathbb{X}_x = \{ p \leq x : p \text{ prime} \}$ and $\mathcal{T}_t = \mathfrak{T}_t$ in (3.1). For every prime $p$, the Plancherel measure $d\mu_p$ may be regarded as a probability measure on the space $SU(n)^2 \cong T_0/\mathfrak{G}_n$. Given $k \in \mathbb{N}^{-1}_0$, the degenerate Schur polynomial $S_k$ on the probability space $(T_0/\mathfrak{G}_n, \mathcal{B}, \mu_p)$ (where $\mathcal{B}$ is the $\sigma$-algebra generated by Borel sets) induces a random variable $A_p$. Then $\{A_p : p \in \mathbb{X}_x\}$ is a collection of independent complex random variables. Moreover, by Proposition 3.1 (i),
\[
d\mu_p = (1 + O_n(p^{-1}))d\mu_{ST},
\]
thus for $k \neq 0$,
\[
\mathbb{E}[A_p] = \int_{T_0/\mathfrak{G}_n} S_k d\mu_p = (1 + O(p^{-1})) \int_{T_0/\mathfrak{G}_n} S_k d\mu_{ST} \ll p^{-1}
\]
\[
\mathbb{E}[A_p^2] = (1 + O(p^{-1})) \int_{T_0/\mathfrak{G}_n} S_k^2 d\mu_{ST} \ll p^{-1} \quad \text{if } k \neq k'
\]
\[
\mathbb{E}[|A_p|^r] = (1 + O(p^{-1})) \int_{T_0/\mathfrak{G}_n} S_k^r d\mu_{ST} = 1 + O(p^{-1})
\]
\[
\mathbb{E}[|A_p|^r] \leq \max_{x \in \mathbb{T}_0} |S_k(x)|^r \leq c_0 \quad (r \geq 0)
\]
for some constant $c_0 > 0$. Clearly Conditions (I)-(IV) are fulfilled with $\zeta = 0$ and $\upsilon = 1$. Set $a_\varphi(p) = S_k(\alpha_\varphi(p)) = A_\varphi(p^k)$. The left-side of (4.1) is
\[
\frac{1}{|\mathfrak{T}_t|} \sum_{\phi \in \mathfrak{T}_t} \prod_{p \leq x} A_\varphi(p^k)^{u_p} A_\varphi(p^k)^{v_p}
\]
and hence (4.11) holds with $T_A(x) = \exp(\Psi(x) \log x)$ by Corollary 3.3, where $\Psi(x)$ is any increasing function satisfying $\Psi(x) \to \infty$ as $x \to \infty$. The choice of $T_A(x)$ assures that the $O$-term in Corollary 3.3,
\[
l^{-1/2}C_k \sum_p (u_p + v_p) \log \prod_p (u_p + v_p) \ll a, b \quad \text{if } t \geq T_A(x), \quad \sum_p u_p = a \quad \text{and} \quad \sum_p v_p = b.
\]
(Here the $L$ and $\|k\|$ are fixed.)

Let $B_p$ be the random variable $A_p$, and $b_\varphi(p) = A_\varphi(p^k)$. Define
\[
\mu := \int_{T_0/\mathfrak{G}_n} \varphi(S_k) d\mu_{ST} \quad \text{and} \quad \nu := \int_{T_0/\mathfrak{G}_n} \varphi(S_k)^2 d\mu_{ST}.
\]
By Proposition 3.2 (i) again, we get $E[\varphi(B_p)] = \mu(1 + O(p^{-1}))$ and $E[\varphi(B_p)^2] = \nu(1 + O(p^{-1}))$. In this case, we need to fulfill (4.19) and the $O$-term in Corollary 3.3 is

$$\ll t^{-1/2} \exp \left( M \log(C_k x^{L[\|k\|]}) \right) \ll \exp \left( - M \log \pi(x) \right)$$

where $M = ((c_0 + 1)\pi(x)^{\delta})^4$, if $\delta = \Delta/5$ and $t \geq \exp(x^{\Delta})$. The proof is complete after a change of variable $u/\eta \mapsto u$.

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