Generalized Bonnet surfaces and Lax pairs of $\mathrm{P_{VI}}$

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We build analytic surfaces in $\mathbb{R}^3(c)$ represented by the most general sixth Painlevé equation $\mathrm{P_{VI}}$ in two steps. Firstly, the moving frame of the surfaces built by Bonnet in 1867 is extrapolated to a new, second order, isomonodromic matrix Lax pair of $\mathrm{P_{VI}}$, whose elements depend rationally on the dependent variable and quadratically on the monodromy exponents θ_j. Secondly, by converting back this Lax pair to a moving frame, we obtain an extrapolation of Bonnet surfaces to surfaces with two more degrees of freedom. Finally, we give a rigorous derivation of the quantum correspondence for $\mathrm{P_{VI}}$.

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CONTENTS

I. INTRODUCTION

From the very beginning[36], two representations have coexisted for the $\mathrm{P_{VI}}$ equation. The first one,

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}=\frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]\frac{\mathrm{d}u}{\mathrm{d}x}$$
$$+\frac{u(u-1)(u-x)}{2x^2(x-1)^2}\left[\theta_\infty^2-\theta_0^2\frac{x}{u^2}+\theta_1^2\frac{x-1}{(u-1)^2}+(1-\theta_x^2)\frac{x(x-1)}{(u-x)^2}\right], \tag{1}$$

in which the four θ_j^2 are arbitrary complex constants, displays the main property of this “équation différentielle curieuse” (as Picard called it after he found it in the particular case $\theta_j=0,\ j=\infty,0,1,x$): its general solution $u(x)$ is singlevalued except at three points, conveniently put at $x=\infty,0,1$ so that x is the crossratio $(\infty,0,1,x)$.

The second representation also originates from Picard. It results from the invertible point transformation $(U,X,T)\mapsto(u,x,t)$ [we also give here its extension to the spectral parameter t, to be used later] defined by[36] (p. 298)

$$U=\frac{1}{2\omega}\int_\infty^u\frac{\mathrm{d}u}{\sqrt{u(u-1)(u-x)}},\ \frac{X}{a_{\mathrm{X}}}=\Omega=i\pi\frac{\omega'}{\omega},\ T=\frac{1}{2\omega}\int_\infty^t\frac{\mathrm{d}t}{\sqrt{t(t-1)(t-x)}}, \tag{2}$$

(with a_{X} some normalization constant), whose inverse is

$$u=\frac{\wp(2\omega U,g_2,g_3)-e_1}{e_2-e_1},\ \sqrt{u(u-1)(u-x)}=\frac{1}{2}(e_2-e_1)^{-3/2}\wp'(2\omega U,g_2,g_3),$$
$$t=\frac{\wp(2\omega T,g_2,g_3)-e_1}{e_2-e_1},\quad \sqrt{t(t-1)(t-x)}=\frac{1}{2}(e_2-e_1)^{-3/2}\wp'(2\omega T,g_2,g_3),$$
$$x=\frac{e_3-e_1}{e_2-e_1}\cdot \tag{3}$$

The new independent variable X (denoted Ω by classical authors like Halphen) is proportional to the ratio of the two half-periods ω,ω' of the elliptic function $2\omega U\mapsto u$.

The transformed ordinary differential equation (ODE) for $U(X)$, a systematic computation of which is recalled in Appendix A, was initially written by R. Fuchs[17] (Eq. (8)), then simplified by Painlevé[34] (p. 1117) by insertion of the prefactor $1/(2\omega)$ in (2), and much later rediscovered by Manin[29] and Babich and Bordag[2].

In these new coordinates (U,X), $\mathrm{P_{VI}}$ becomes quite simple,

$$\frac{\mathrm{d}^2U}{\mathrm{d}X^2}=\frac{(2\omega)^3}{\pi^2a_{\mathrm{X}}^2}\sum_{j=\infty,0,1,x}\theta_j^2\wp'(2\omega U+\omega_j,g_2,g_3), \tag{4}$$

in which the summation runs over the four half-periods ω_j of $\wp$, and $\wp'$ denotes the partial derivative of $\wp$ with respect to its first argument, taken at point $2\omega U+\omega_j$. One advantage of the elliptic representation (4) is its Hamiltonian description,

$$H(Q,P,X)=\frac{P^2}{2}+V(Q,X),\ Q=U,\ P=\frac{\mathrm{d}Q}{\mathrm{d}X},\ \frac{\mathrm{d}^2Q}{\mathrm{d}X^2}=-\frac{\partial V}{\partial Q}, \tag{5}$$
$$V(Q,X,\{\theta_j^2\})=-\frac{(2\omega)^2}{\pi^2a_{\mathrm{X}}^2}\sum_{j=\infty,0,1,x}\theta_j^2\wp(2\omega Q+\omega_j,g_2,g_3).$$

We will respectively call (u, x, t) and (U, X, T) the *rational coordinates* and *elliptic coordinates* and, depending on the context, denote the four half-periods as either $(\omega_\infty, \omega_0, \omega_1, \omega_x)$ (like in (4)), or $(0, \omega, \omega+\omega', \omega')$ (like in (2)), or $(0, \omega_1, \omega_2, \omega_3)$ (to respect the correspondence $\wp(\omega_j) = e_j, j = 1, 2, 3$ with the classical definition (A5)).

There exist various linear representations[17,21,24,31,49] of this nonlinear ODE by a Lax pair, and we restrict here to those which have minimal order, i.e. two. Their main characteristics were outlined by Poincaré[37] (p. 219) in the scalar case and by Schlesinger[39] in the matrix case, let us remind them. First of all, their singularities in the complex plane of the spectral parameter t can be restricted to be only of the Fuchsian type ("regular" singularities).

In the scalar case, which can be defined as

$$\partial_t^2\psi + \frac{S}{2}\psi = 0,\ \partial_x\psi + C\partial_t\psi - \frac{C_t}{2}\psi = 0,\ S_x + C_{ttt} + CS_t + 2C_tS = 0, \tag{6}$$

the first equation must have four Fuchsian singularities (characterized by their crossratio x since three of them can be put at predefined locations by a homography), plus, as prescribed by Poincaré[37] (p. 219), one apparent singularity, also of the Fuchsian type, located at $t = u$. Without such an apparent singularity, the system would be rigid and no nonlinear ODE would come out.

In the matrix case, defined as

$$\partial_x\psi = L\psi,\ \partial_t\psi = M\psi,\ L_t - M_x + LM - ML = 0, \tag{7}$$

the second order matrices L and M can be chosen traceless without loss of generality, and Schlesinger[39] proved that the monodromy matrix M must be the sum of four simple poles of crossratio x, and that the matrix L must be the sum of a simple pole and a regular term, i.e. with the convention $t = \infty, 0, 1, x$ for the four Fuchsian singularities,

$$L = -\frac{M_x}{t-x} + L_\infty,\ M = \frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x},\ M_\infty + M_0 + M_1 + M_x = 0. \tag{8}$$

In particular, no apparent singularity is required in the matrix case.

Neither Poincaré nor Schlesinger performed the practical computations which they had prescribed. This was first achieved in the scalar case by Richard Fuchs[17] (the son of the Lazarus Fuchs of Fuchsian equations and brother-in-law of Schlesinger), and in the matrix case by Jimbo and Miwa[24]. However, while one would expect the just mentioned matrix Lax pair to be "simpler" than the scalar one because of the unnecessity for an apparent singularity, this is not the case, as detailed in[11,26]. The reason is that, in order to unveil P_{VI} during the resolution process, no additional assumption is required in the scalar case, while in the matrix case one must make the following practical assumption: in order to implement the property established by Schlesinger that the determinants of the four residues M_j are constant (and equivalent to the four parameters of P_{VI}), one must assume a representation of the four second order traceless matrices M_j enforcing this prescription.

The present article, which is an extended version of a short note 12, contains three main results.

1. We first show that a classical problem of geometry, set up and solved by Pierre-Ossian Bonnet in 1867, yields as a by-product a new, isomonodromic, very symmetric second order matrix Lax pair of a codimension-two P_{VI}, which is easily extrapolated to the generic P_{VI}. The decisive advantage conferred by its geometric origin is that no assumption is required concerning the four residues.

2. The second result is a rigorous derivation of a nice property of $\mathrm{P_{VI}}$, unveiled by Suleimanov[42] and known as the "quantum correspondence".

3. Finally, we match the completeness property of $\mathrm{P_{VI}}$ (impossibilily to add complementary terms without losing the Painlevé property) and the completeness property of the Gauss-Codazzi equations (they completely describe the geometry) by building a solution of the Gauss-Codazzi equations in terms of the full $\mathrm{P_{VI}}$.

The paper is organized as follows.

In section II, we recall the classical analytic description of surfaces, a prerequisite to the presentation of Bonnet surfaces and their moving frame, done in section III.

Section IV is the core of the paper: we upgrade this moving frame to a second order, isomonodromic matrix Lax pair of $\mathrm{P_{VI}}$, and we compare this new Lax pair to the existing ones. Next, in section V, we establish the link with the classical second order scalar Lax pair and give a rigorous derivation of the "quantum correspondence".

Finally, in section VI, by converting back the matrix Lax pair to the moving frame of some surface, we lift Bonnet surfaces to surfaces which depend on two more degrees of freedom and are described by the full $\mathrm{P_{VI}}$.

Most results are presented in both rational coordinates (u, x, t) and elliptic coordinates (U, X, T).

II. CLASSICAL GEOMETRY OF SURFACES

As shown by Gauss in 1827, surfaces in $\mathbb{R}^3$ are characterized by the two "fundamental" quadratic forms $< \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{F} >, - < \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{N} >$, in which $\mathbf{F}(x_1, x_2)$ is the current point on the surface, $\mathrm{d}\mathbf{F}$ a vector in the tangent plane, $\mathbf{N}$ any unit vector normal to the tangent plane. In "conformal coordinates", these quadratic forms

$$\mathrm{I} =< \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{F} >= e^{\upsilon} \mathrm{d}z\ \mathrm{d}\bar{z}, \tag{9}$$

$$\mathrm{II} = - < \mathrm{d}\mathbf{F}, \mathrm{d}\mathbf{N} >= Q\ \mathrm{d}z^2 + e^{\upsilon} H \mathrm{d}z\ \mathrm{d}\bar{z} + \overline{Q}\ \mathrm{d}\bar{z}^2, \tag{10}$$

define four fields: υ real, Q, its complex conjugate $\overline{Q}$, H real, and the link with the two principal curvatures $1/R_1$ and $1/R_2$ is,

$$\begin{aligned} &\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{mean curvature} = H, \\ &\frac{1}{R_1 R_2} = \text{total (or Gaussian) curvature} = -2e^{-\upsilon}\upsilon_{z\bar{z}}. \end{aligned} \tag{11}$$

If σ denotes some moving frame defined from $\mathbf{F}$ and $\mathbf{N}$, the gradient of σ defines two square matrices $\mathbb{U}$, $\mathbb{V}$,

$$\sigma_z = \mathbb{U}\sigma,\ \ \sigma_{\bar{z}} = \mathbb{V}\sigma, \tag{12}$$

and the zero-curvature condition

$$[\partial_z - \mathbb{U}, \partial_{\bar{z}} - \mathbb{V}] = \mathbb{U}_{\bar{z}} - \mathbb{V}_z + [\mathbb{U}, \mathbb{V}[= 0, \tag{13}$$

generates a set of nonlinear partial differential equations (PDEs) involving υ, Q, $\overline{Q}$, H.

An additional parameter c can be inserted in these moving frame equations (12) if one replaces $\mathbb{R}^3$ by the three-dimensional Riemannian manifold $\mathbb{R}^3(c)$ having a constant curvature $\kappa = -c^2$. When κ is respectively negative, zero, positive, this three-dimensional manifold is respectively the hyperbolic space $\mathbb{H}^3(c)$, the Euclidean space $\mathbb{R}^3$, the sphere $\mathbb{S}^3(c)$ of radius $\kappa^{-1/2}$. The moving frame defined by

$$\sigma = \begin{cases} {}^{\mathrm{t}}(\mathbf{F}_z, \mathbf{F}_{\bar{z}}, \mathbf{N}) & (c = 0), \\ {}^{\mathrm{t}}(\mathbf{F}, \mathbf{F}_z, \mathbf{F}_{\bar{z}}, \mathbf{N}) & (c \neq 0), \end{cases} \tag{14}$$

then yields matrices $\mathbb{U}$, $\mathbb{V}$ of respective orders three ($c = 0$) and four ($c \neq 0$). Instead of them, it proves quite convenient to use the representation by second order matrices[3,41],

$$\begin{cases} \mathbb{U} = \begin{pmatrix} (1/4)\upsilon_z & -Qe^{-\upsilon/2} \\ (1/2)(H+c)e^{\upsilon/2} & -(1/4)\upsilon_z \end{pmatrix}, \\ \mathbb{V} = \begin{pmatrix} -(1/4)\upsilon_{\bar{z}} & -(1/2)(H-c)e^{\upsilon/2} \\ \overline{Q}e^{-\upsilon/2} & (1/4)\upsilon_{\bar{z}} \end{pmatrix} \end{cases} \tag{15}$$

The nonlinear PDEs generated by the zero-curvature condition (13) are known as the Gauss-Codazzi equations[50],

$$\begin{cases} \upsilon_{z\bar{z}} + \dfrac{1}{2}(H^2 - c^2)e^{\upsilon} - 2|Q|^2 e^{-\upsilon} = 0 \text{ (Gauss)}, \\ Q_{\bar{z}} - \dfrac{1}{2}H_z e^{\upsilon} = 0,\ \overline{Q}_z - \dfrac{1}{2}H_{\bar{z}} e^{\upsilon} = 0 \text{ (Codazzi)}. \end{cases} \tag{16}$$

This is a classical result due to Bonnet that any solution $(\upsilon, H, Q, \overline{Q})$ determines a unique surface up to rigid motion.

In addition to the classical conformal invariance,

$$\forall G(z):\ (z, e^{\upsilon}, H, Q) \to \left(G(z), |G'(z)|^2 e^{\upsilon}, H, G'(z)^2 Q\right), \tag{17}$$

and the scaling invariance,

$$\forall k:\ (z, e^{\upsilon}, H, Q, c) \to \left(z, k^2 e^{\upsilon}, k^{-1}H, kQ, k^{-1}c\right), \tag{18}$$

the system (16) possesses another invariance, which only exists under the condition $Q - \overline{Q} = c$, this is the involution[3] (Eq. (4.4))[5] (p. 77)[41] (§3 p. 6) defined by the permutation of the two basis vectors of the moving frame (15)

$$(\upsilon, H, Q, \overline{Q}) \to \left(-\upsilon, 2Q - c = 2\overline{Q} + c, \frac{H+c}{2}, \frac{H-c}{2}\right). \tag{19}$$

III. BONNET SURFACES AND THEIR MOVING FRAME

As an application of the newly discovered Gauss-Codazzi equations, the geometer Pierre-Ossian Bonnet set up in 1867 a natural problem (the Bonnet problem) and solved it in full generality. Since this achievement is often incompletely presented in modern articles, we find it useful to recall in Appendix B the complete proof as given by Bonnet himself.

The most interesting of the five solutions to the Bonnet problem is what is now called the *Bonnet surfaces*. Characterized in local coordinates by conditions on v, $|Q|$ and H (i.e. excluding $\arg Q$),

$$\begin{cases} e^{v}|Q|^{-2}H_{\bar z}=g_1(z)\neq 0,\ e^{v}|Q|^{-2}H_{z}=g_2(\bar z)\neq 0,\\ \dfrac{2\mathrm{d}H}{g_1\mathrm{d}z+g_2\mathrm{d}\bar z}+H^2-c^2\neq 0, \end{cases} \tag{20}$$

they depend, after a conformal transformation detailed in the Appendix, on one fixed parameter (c, at least if one considers $\mathbb{R}^3(c)$ instead of $\mathbb{R}$) and six arbitrary movable constants. Their metric v and Q are given by

$$\begin{cases} Q=2c_{\mathrm{z}}\coth 2c_{\mathrm{z}}(z-z_0)-2c_{\mathrm{z}}\coth 4c_{\mathrm{z}}\Re(z-z_0)=\dfrac{\sinh 2c_{\mathrm{z}}(\bar z-\bar z_0)}{\sinh 2c_{\mathrm{z}}(z-z_0)}\dfrac{2c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}\Re(z-z_0)},\\ \overline{Q}=2c_{\mathrm{z}}\coth 2c_{\mathrm{z}}(\bar z-\bar z_0)-2c_{\mathrm{z}}\coth 4c_{\mathrm{z}}\Re(z-z_0)=\dfrac{\sinh 2c_{\mathrm{z}}(z-z_0)}{\sinh 2c_{\mathrm{z}}(\bar z-\bar z_0)}\dfrac{2c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}\Re(z-z_0)},\\ |Q|^2=\left(\dfrac{2c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}\Re(z-z_0)}\right)^2,\\ e^{v}=4|Q|^2\dfrac{\mathrm{d}\Re(z)}{\mathrm{d}H}, \end{cases} \tag{21}$$

in which c_{z} is an arbitrary (possibly zero) complex constant, and the mean curvature H, which only depends on $\xi=\Re(z)$, obeys the third order ODE (B19), whose first integral (B20) defines a second order second degree ODE for $H=h(\xi)$. Despite being just a particular case of the ODE labeled (B,V) by Chazy[9] (p. 340), this second order ODE remained unnoticed (even by Élie Cartan[8] (p. 85)) and therefore unintegrated for nearly one century, until Bobenko and Eitner[4] expressed its general solution in terms of the sixth Painlevé equation $\mathrm{P_{VI}}$ (1). More precisely, the mean curvature H is equal to the logarithmic derivative $\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI}}$ of a τ-function of $\mathrm{P_{VI}}$. Rather than the expression built by Chazy[9] (expression t page 341), it is preferable to adopt its homographic transform by Malmquist[28],

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}&=\frac{x(x-1)}{4u(u-1)(u-x)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\\ &+\frac{1}{4x(x-1)}\left[\theta_\infty^2\left(-u+\frac{1}{2}\right)+\theta_0^2\left(-\frac{x}{u}+\frac{1}{2}\right)\right.\\ &\left.+\theta_1^2\left(\frac{x-1}{u-1}-\frac{1}{2}\right)+(\theta_x-1)^2\left(-\frac{x(x-1)}{u-x}-x+\frac{1}{2}\right)\right], \end{aligned} \tag{22}$$

for two (equivalent) reasons: (i) absence of a first degree term $\mathrm{d}u/\mathrm{d}x$, (ii) choice of θ_x to break the parity invariance of (1) in the θ_j's (Chazy chose θ_∞). Bonnet surfaces are then analytically represented as

$$x=\frac{1}{1-e^{4c_{\mathrm{z}}(z+\bar z)}},\ H=8c_{\mathrm{z}}Y,\ Y=x(x-1)\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}, \tag{23}$$

with however three constraints among the four monodromy exponents θ_j,

$$\theta_\infty=0,\ c=c_{\mathrm{z}}(\theta_1^2-\theta_0^2),\ \theta_x^2=1. \tag{24}$$

The only movable singularities of $H=h(\xi)$ are a unique movable simple pole.

The moving frame (15) of these Bonnet surfaces

$$\begin{cases} \mathbb{U}\mathrm{d}z + \mathbb{V}\mathrm{d}\bar{z} = x(x-1)\dfrac{Y''}{Y'}(c_\mathrm{z}\mathrm{d}\bar{z} - c_\mathrm{z}\mathrm{d}z)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ +\sqrt{Y'}\begin{pmatrix} 0 & -S_1\mathrm{d}z - \dfrac{Y-(\theta_0^2-\theta_1^2)/8}{Y'}4c_\mathrm{z}\mathrm{d}\bar{z} \\ S_2\mathrm{d}\bar{z} + \dfrac{Y+(\theta_0^2-\theta_1^2)/8}{Y'}4c_\mathrm{z}\mathrm{d}z & 0 \end{pmatrix}, \\ S_1 = \dfrac{2c_\mathrm{z}}{\sinh(2c_\mathrm{z}(z+\bar{z}))}\dfrac{\sinh(2c_\mathrm{z}\bar{z})}{\sinh(2c_\mathrm{z}z)},\ S_2 = \dfrac{2c_\mathrm{z}}{\sinh(2c_\mathrm{z}(z+\bar{z}))}\dfrac{\sinh(2c_\mathrm{z}z)}{\sinh(2c_\mathrm{z}\bar{z})}, \end{cases} \tag{25}$$

then defines a linear representation of the variable $\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI}}$, however with several undesired features:

: – lack of a spectral parameter,

: – nonrational dependence on z, $\bar{z}$, Y',

: – restriction to $\theta_\infty = 0$, $\theta_x^2 = 1$.

Remark. Under the involution (19), Bonnet surfaces are mapped to surfaces with a harmonic inverse mean curvature[5] (Prop. 4.7.1 page 77), which also integrate with $\mathrm{P_{VI}}$[6] and whose moving frame is

$$\begin{aligned} \mathbb{U}\mathrm{d}z + \mathbb{V}\mathrm{d}\bar{z} &= \left[-X(X-1)\frac{Y''}{Y'}(c_\mathrm{z}\mathrm{d}\bar{z} - c_\mathrm{z}\mathrm{d}z) - \frac{1}{2}\mathrm{d}\log\frac{\sinh 2c_\mathrm{z}z}{\sinh 2c_\mathrm{z}\bar{z}}\right]\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &+ \sqrt{X(X-1)Y'}\begin{pmatrix} 0 & -4c_\mathrm{z}\mathrm{d}\bar{z} - S_1\dfrac{Y+(\theta_0^2-\theta_1^2)/8}{X(X-1)Y'}\mathrm{d}z \\ 4c_\mathrm{z}\mathrm{d}z + S_2\dfrac{Y-(\theta_0^2-\theta_1^2)/8}{X(X-1)Y'}\mathrm{d}\bar{z} & 0 \end{pmatrix}. \end{aligned} \tag{26}$$

Since the transition matrix

$$P = \begin{pmatrix} 0 & g \\ -1/g & 0 \end{pmatrix},\ g = \left(\frac{\sinh(2c_\mathrm{z}\bar{z})}{\sinh(2c_\mathrm{z}z)}\right)^{1/2}, \tag{27}$$

maps (26) to (25), it is sufficient to consider the moving frame of Bonnet surfaces.

IV. FROM THE MOVING FRAME OF BONNET TO A LAX PAIR OF $\mathrm{P_{VI}}$

Let us convert the moving frame to a second order, isomonodromic matrix Lax pair for the generic $\mathrm{P_{VI}}$ for $u(x)$, i.e. with the following properties,

: – dependence on an arbitrary parameter t (the spectral parameter),

: – rational dependence on t, the independent variable x and the dependent variables $u(x)$, $u'(x)$ of $\mathrm{P_{VI}}$,

: – absence of any restriction on the four θ_j's.

The successive steps are:

1. Introduction of a spectral parameter t and creation of a rational dependence on x and t. This is not achieved by a conformal transformation (17) like for constant mean curvature surfaces, see Eq. (B8), but *via* a change of variables $(z,\bar{z}) \to (x,t)$. Indeed, as shown in[4], choosing for x and t any homographic transform of, respectively, $e^{4c_z(z+\bar{z})}$ and $e^{4c_z z}$ creates four poles in the monodromy matrix. For instance, the choice

$$x = \frac{1}{1-e^{4c_z(z+\bar{z})}},\ t = \frac{1}{1-e^{4c_z z}}, \tag{28}$$

creates the set of poles $t = \infty, 0, 1, x$,

$$\begin{cases} 4c_z \mathrm{d}z = \dfrac{\mathrm{d}t}{t(t-1)},\ 4c_z \mathrm{d}\bar{z} = \dfrac{\mathrm{d}x}{x(x-1)} - \dfrac{\mathrm{d}t}{t(t-1)}, \\ S_1 \mathrm{d}z = -\dfrac{t-x}{t(t-1)}\mathrm{d}t,\ S_2 \mathrm{d}\bar{z} = -\dfrac{\mathrm{d}x}{t-x} + \dfrac{x(x-1)}{t(t-1)(t-x)}\mathrm{d}t, \end{cases} \tag{29}$$

and therefore changes the moving frame (25) to an isomonodromic Lax pair (8) for an incomplete $\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}$ with an algebraic dependence on Y',

$$(z,\bar{z}) \to (x,t),\ \mathbb{U}\mathrm{d}z + \mathbb{V}\mathrm{d}\bar{z} = L\mathrm{d}x + M\mathrm{d}t, \tag{30}$$

$$\det M_\infty = \det M_x = 0,\ -4\det M_0 = \theta_0^2,\ -4\det M_1 = \theta_1^2. \tag{31}$$

For reference, the (x,t) representation of the Bonnet surfaces is

$$\begin{cases} H = \dfrac{c}{\kappa} 8c_z x(x-1)\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}},\ e^{-v} = \dfrac{c^2}{\kappa^2}\dfrac{\mathrm{d}}{\mathrm{d}x}[x(x-1)\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}], \\ Q = -\dfrac{\kappa}{c}4c_z(t-x),\ \overline{Q} = -\dfrac{\kappa}{c}4c_z\dfrac{x(x-1)}{t-x},\ |Q|^2 = 16\dfrac{\kappa^2}{c^2}c_z^2 x(x-1), \\ \theta_\infty = 0,\ \theta_x^2 = 1,\ \kappa = c_z(\theta_1^2 - \theta_0^2), \end{cases} \tag{32}$$

values in which c has been made arbitrary by the scaling invariance (18).

2. Switch from the $\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}$ field Y to the $\mathrm{P_{VI}}$ field u. This is done *via* the birational transformation between $\mathrm{P_{VI}}$ and its Hamiltonian $\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}$ defined by the relations[9] (p. 341)

$$Y = x(x-1)\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}, \tag{33}$$

and[32] (Table R),

$$u = x + \frac{\Theta_x x(x-1)Y'' - \left[Y + \dfrac{\theta_\infty^2 + 3\Theta_x^2}{8}(2x-1) - \dfrac{\theta_1^2-\theta_0^2}{8}\right]\left[Y' + \dfrac{\theta_\infty^2+\Theta_x^2}{4}\right]}{\left(Y' + \dfrac{(\theta_\infty+\Theta_x)^2}{4}\right)\left(Y' + \dfrac{(\theta_\infty-\Theta_x)^2}{4}\right)}$$
$$+ \frac{\dfrac{\Theta_x^2}{2}\left[Y + \dfrac{3\theta_\infty^2+\Theta_x^2}{8}(2x-1) + \dfrac{\theta_1^2-\theta_0^2}{8}\right]}{\left(Y' + \dfrac{(\theta_\infty+\Theta_x)^2}{4}\right)\left(Y' + \dfrac{(\theta_\infty-\Theta_x)^2}{4}\right)}, \tag{34}$$

in which Θ_x denotes the shifted exponent

$$\Theta_x = \theta_x - 1. \tag{35}$$

While the Hamiltonian (22) breaks the parity of only one θ_j, the birational transformation between $\mathrm{P_{VI}}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI}}$ breaks the parity of two θ_j's.
For the Bonnet constraints $\theta_\infty = 0$, $\theta_x^2 = 1$ (24), these two relations simplify to

$$Y = x(x-1)\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}},\ \frac{8Y'}{8Y+\theta_0^2-\theta_1^2} = -\frac{1}{u-x},\ \frac{Y''}{Y'} = -\frac{u'}{u-x}, \tag{36}$$

and now change the moving frame to an algebraic isomonodromic Lax pair for a codimension-two $\mathrm{P_{VI}}$,

$$\left\lbrace \begin{array}{l} L\mathrm{d}x + M\mathrm{d}t = -x(x-1)\dfrac{u'}{u-x}(c_{\mathrm{z}}\mathrm{d}\bar{z} - c_{\mathrm{z}}\mathrm{d}z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ +\sqrt{-\dfrac{Y_{\mathrm{s}}}{u-x}}\begin{pmatrix} 0 & -S_1\mathrm{d}z + \dfrac{(u-x)Y}{Y_{\mathrm{s}}}4c_{\mathrm{z}}\mathrm{d}\bar{z}\\ S_2\mathrm{d}\bar{z} - \dfrac{(u-x)Y}{Y_{\mathrm{s}}}4c_{\mathrm{z}}\mathrm{d}z & 0\end{pmatrix},\\ Y_{\mathrm{s}} = x(x-1)\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}} + \dfrac{\theta_0^2-\theta_1^2}{8}, \end{array}\right. \tag{37}$$

in which $\mathrm{d}z$ and $\mathrm{d}\bar{z}$ are assumed replaced by their values (29).

3. Removal of the algebraic dependence on Y' by a change of basis vectors defined by the transition matrix

$$P_1 = \mathrm{diag}(Y'^{1/4}, Y'^{-1/4}),\ Y' = -\frac{Y_{\mathrm{s}}}{u-x}. \tag{38}$$

The algebraic Lax pair (37) becomes rational, and its five terms as defined by (8) evaluate to

$$\left\lbrace \begin{array}{l} \theta_\infty = 0,\ \Theta_x = 0,\ Y_{\mathrm{s}} = x(x-1)\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI}} + \dfrac{\theta_0^2-\theta_1^2}{8},\\ M_\infty = \begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix},\ M_x = \begin{pmatrix}0 & 0\\ -\dfrac{Y_{\mathrm{s}}}{u-x} & 0\end{pmatrix},\\ M_0 = \dfrac{1}{2}\begin{pmatrix} -x(x-1)\dfrac{u'}{u-x} & 2u\\ \dfrac{\theta_0^2-\theta_1^2}{2} - 2Y_{\mathrm{s}} - 2\dfrac{x-1}{u-x}Y_{\mathrm{s}} & x(x-1)\dfrac{u'}{u-x}\end{pmatrix},\\ M_1 = \dfrac{1}{2}\begin{pmatrix} x(x-1)\dfrac{u'}{u-x} & -2(u-1)\\ -\dfrac{\theta_0^2-\theta_1^2}{2} + 2Y_{\mathrm{s}} + 2\dfrac{x}{u-x}Y_{\mathrm{s}} & -x(x-1)\dfrac{u'}{u-x}\end{pmatrix},\\ L_\infty = -\dfrac{u-x}{x(x-1)}M_\infty. \end{array}\right. \tag{39}$$

4. (Last step) Extrapolation to arbitrary values of the θ_j's. It is sufficient to notice that all residues in (39) are polynomials of u' of degree at most two. By only requiring the conservation of such a dependence and enforcing $M_\infty =$ constant, one immediately removes all the constraints on the θ_j's. Each term u' of (39) essentially extrapolates to the left-hand-side of a Riccati equation which defines hypergeometric solutions of P_{VI}.

The final Lax pair can be defined in two canonical forms.

A. First canonical form of the matrix Lax pair

This first canonical form is valid for any value of the θ_j's,

$$\left\{\begin{array}{l}
L = -\dfrac{M_x}{t-x} - \dfrac{u-x}{x(x-1)} M_\infty,\ M = \dfrac{M_0}{t} + \dfrac{M_1}{t-1} + \dfrac{M_x}{t-x},\\
M_\infty + M_0 + M_1 + M_x = 0,\\
M_\infty = \dfrac{1}{4}\begin{pmatrix} 2a & -4 \\ a^2 - \theta_\infty^2 & -2a \end{pmatrix},\\
M_0 = -\dfrac{1}{2(u-x)}\begin{pmatrix} e_0 & -2u(u-x) \\ \dfrac{e_0^2 - \theta_0^2 (u-x)^2}{2u(u-x)} & -e_0 \end{pmatrix},\\
M_1 = \dfrac{1}{2(u-x)}\begin{pmatrix} e_1 & -2(u-1)(u-x) \\ \dfrac{e_1^2 - \theta_1^2 (u-x)^2}{2(u-1)(u-x)} & -e_1 \end{pmatrix},\\
M_x = \dfrac{1}{2}\begin{pmatrix} -\Theta_x & 0 \\ 2M_{x,21} & \Theta_x \end{pmatrix},\\
M_{x,21} = -\dfrac{e^2 - (u-x)^2((\theta_\infty^2 + \Theta_x^2 - 2a\Theta_x)u(u-1) - \theta_0^2(u-1) + \theta_1^2 u)}{4u(u-1)(u-x)^2},\\
e = x(x-1)u' + \Theta_x u(u-1),\ \Theta_x^2 = (\theta_x - 1)^2,\\
e_0 = e - (\Theta_x - a)u(u-x),\\
e_1 = e - (\Theta_x - a)(u-1)(u-x),\\
-4\det M_j = \theta_j^2,\ j = \infty, 0, 1; -4\det M_x = \Theta_x^2,
\end{array}\right. \qquad (40)$$

in which a is an arbitrary constant, which can be set to any convenient value, such as 0, Θ_x or $\pm\theta_\infty$, by action of a constant transition matrix.

The main property of this Lax pair is ho have *exactly* the same dependence on all variables as $\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}$, Eq. (22): second degree polynomial in u', meromorphic dependence on u (the only poles being those of P_{VI}), meromorphic dependence on x (same), affine dependence on three θ_j^2's and break of the affine dependence on θ_x^2 (enjoyed by P_{VI} but not by $\dfrac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,M}}$). Another property if that the four residues are always nonzero; although not required in the matrix case (see section IV C 1 hereafter), such a property is an essential requirement in the scalar case: three Fuchsian singularities are insufficient to generate P_{VI}.

The structure of P_{VI} (u'' a second degree polynomial of u') makes it possible to find an even simpler Lax pair, whose matrices L, M would be first degree polynomials of u', so that the zero-curvature condition does not generate powers of u' higher than u'^2. Such a Lax pair does not exist in the class (8), but it does exist outside this class, see (55) hereafter.

B. Second canonical form of the matrix Lax pair

About the matrix Lax pair (8), Schlesinger[39] (p. 105) proved two results, which are indeed obeyed by (40): (i) the residue M_∞ must be a constant, (ii) the regular term L_∞ must be a scalar multiple of M_∞. He also proved that, if M_∞ is invertible, there exists a change of basis allowing one to cancel the term L_∞ in (8) and therefore to uniquely define the Lax pair.

If θ_∞ vanishes, M_∞ is a Jordan matrix and the pair (40) is final. If θ_∞ is nonzero, the transition matrix P_2P_3,

$$P_2 = \begin{pmatrix} 2 & 2 \\ a-\theta_\infty & a+\theta_\infty \end{pmatrix}, P_3 = \begin{pmatrix} g^{-1/2} & 0 \\ 0 & g^{1/2} \end{pmatrix},\ \frac{g'}{g} = \theta_\infty \frac{u-x}{x(x-1)}, \tag{41}$$

yields the second canonical form,

$$\begin{cases} \theta_\infty \neq 0:\ L = -\dfrac{M_x}{t-x},\ M = \dfrac{M_0}{t} + \dfrac{M_1}{t-1} + \dfrac{M_x}{t-x},\ \Theta_x^2 = (\theta_x-1)^2, \\ M_\infty + M_0 + M_1 + M_x = 0, \\ M_\infty = \dfrac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \\ M_{0,11} = \quad \dfrac{u-1}{N}\left[(e-\Theta_x u(u-x))^2 - (u-x)^2(\theta_0^2+\theta_\infty^2 u^2)\right], \\ M_{0,12} = \quad \dfrac{u-1}{N}\left[(e-(\Theta_x+\theta_\infty)u(u-x))^2 - (u-x)^2\theta_0^2\right] g, \\ M_{0,21} = -\dfrac{u-1}{N}\left[(e-(\Theta_x-\theta_\infty)u(u-x))^2 - (u-x)^2\theta_0^2\right] g^{-1}, \\ M_{1,11} = -\dfrac{u}{N}\left[(e-\Theta_x(u-1)(u-x))^2 - (u-x)^2(\theta_1^2+\theta_\infty^2(u-1)^2)\right], \\ M_{1,12} = -\dfrac{u}{N}\left[(e-(\Theta_x+\theta_\infty)(u-1)(u-x))^2 - (u-x)^2\theta_1^2\right] g, \\ M_{1,21} = \quad \dfrac{u}{N}\left[(e-(\Theta_x-\theta_\infty)(u-1)(u-x))^2 - (u-x)^2\theta_1^2\right] g^{-1}, \\ M_{x,11} = \quad \dfrac{1}{N}\left[e^2 - (u-x)^2\left[(\theta_\infty^2+\Theta_x^2)u(u-1) - \theta_0^2(u-1)+\theta_1^2 u)\right]\right], \\ M_{x,12} = \quad \dfrac{1}{N}\left[e^2 - (u-x)^2\left[((\Theta_x+\theta_\infty)^2)u(u-1) - \theta_0^2(u-1)+\theta_1^2 u)\right]\right] g, \\ M_{x,21} = -\dfrac{1}{N}\left[e^2 - (u-x)^2\left[((\Theta_x-\theta_\infty)^2)u(u-1) - \theta_0^2(u-1)+\theta_1^2 u)\right]\right] g^{-1}, \\ -4\det M_j = \theta_j^2,\ j=\infty,0,1; -4\det M_x = \Theta_x^2, \end{cases} \tag{42}$$

with the notation

$$\frac{g'}{g} = \theta_\infty \frac{u-x}{x(x-1)},\ e = x(x-1)u' + \Theta_x u(u-1),\ N = 4\theta_\infty u(u-1)(u-x)^2. \tag{43}$$

This Lax pair displays a nice symmetry with respect to the diagonal,

$$M_{12}(\theta_\infty) = M_{21}(-\theta_\infty). \tag{44}$$

This result puts an end to our previous attempts[11,26].

C. Comparison with existing matrix Lax pairs

Among the two other second order matrix Lax pairs of P_{VI} we are aware of, one[24] (Eq. (C.47))[27] has four Fuchsian singularities in the complex plane of its spectral parameter t, like (40) or (42), while the other[49] has four Fuchsian singularities on the torus and depends on its spectral parameter T through the function σ of Weierstrass.

1. *Matrix Lax pair of Jimbo and Miwa*

Let us start with the rational one, which has the structure (8) (four Fuchsian singularities in the complex t plane). Any isomonodromic deformation of the Fuchsian system

$$\frac{\partial}{\partial_t}\psi = \left(\frac{M_0}{t} + \frac{M_1}{t-1} + \frac{M_x}{t-x}\right)\psi,\ M_\infty = -M_0 - M_1 - M_x, \tag{45}$$

has to overcome a technical difficulty, already mentioned in the introduction, which consists in finding a "nice" representation of the property $\det M_j =$ constant. The representation chosen by Jimbo and Miwa[24] (Eq. (C.47)) can be made traceless[27] (Eq. (3.6)), this is

$$M_\infty = \frac{1}{2}\begin{pmatrix}\theta_\infty & 0\\ 0 & -\theta_\infty\end{pmatrix},\ M_j = \frac{1}{2}\begin{pmatrix} z_j & (\theta_j - z_j)u_j \\ (\theta_j + z_j)/u_j & -z_j\end{pmatrix}, j = 0, 1, x. \tag{46}$$

It defines four functions z_0, z_1, u_0, u_1 of three variables x, u, u' to be determined by the zero-curvature condition, and this results in quite intricate expressions for L_{ij}, M_{ij} as detailed in[26] (Table 1) and in[15] (p. 211). For $\theta_\infty = 0$, the Lax pair still exists although one of the four residues vanishes.

The decisive advantage of the geometric origin of the linear representation is to avoid this difficulty, and the structure of the residues of (42) is *a posteriori*

$$M_\infty = \frac{1}{2}\begin{pmatrix}\theta_\infty & 0\\ 0 & -\theta_\infty\end{pmatrix},\ M_j = \frac{f_j}{\theta_\infty}\begin{pmatrix} P_{j,11} & P_{j,12}g \\ -P_{j,21}g^{-1} & -P_{j,11}\end{pmatrix},\ j = 0, 1, x, \tag{47}$$

involving two rational functions $f_j(x,u)$ and six monic second degree polynomials of $x(x-1)u'$ whose coefficients are polynomial in (x,u).

2. *Matrix Lax pair affine in* θ_j

Let us now turn to the Lax pair defined in elliptic coordinates (U, X, T)[49]. It is affine in θ_j and it only involves one dimensionless function φ[49] (Eq. (A.10)) which mainly depends on the two dimensionless variables U, T and also on one of the four half-periods ω_j,

$$\varphi(U + \omega_j/(2\omega), T) = 2\omega \frac{\sigma(2\omega U + \omega_j + 2\omega T, g_2, g_3)}{\sigma(2\omega U + \omega_j, g_2, g_3)\sigma(2\omega T, g_2, g_3)} e^{-2\eta(2\omega U + \omega_j)T + n_j T}, \tag{48}$$

in which n_j is an integer multiple of $i\pi$ characterized by the property

$$n_j = 2(\eta\omega_j - \eta_j\omega),\ \mathrm{d}\frac{\omega_j}{2\omega} = -\frac{n_j}{2\pi^2}\frac{\mathrm{d}X}{a_{\mathrm{X}}}, \tag{49}$$

(with the classical notation $\eta_j = \zeta(\omega_j), \eta = \zeta(\omega)$).

Such a function φ is classically called[20] an elliptic function of the second kind of U (resp. T) (i.e. not doubly periodic, but multiplied by the exponential of an affine function of U (resp. T) under addition of one period).

Denoting φ' the derivative of φ with respect to its first argument, this Lax pair is

$$
\begin{aligned}
& \mathrm{d}\Psi = \mathcal{L}\Psi \mathrm{d}X + \mathcal{M}\Psi \mathrm{d}T, && (50)\\
& \mathcal{L} = \frac{1}{(2\pi)^2} \sum_{j=\infty,0,1,x} \theta_j \begin{pmatrix} 0 & \varphi'(U+\omega_j/(2\omega),T) \\ \varphi'(-U+\omega_j/(2\omega),T) & 0 \end{pmatrix}, && (51)\\
& \mathcal{M} = -\frac{1}{2} \sum_{j=\infty,0,1,x} \theta_j \begin{pmatrix} 0 & \varphi(U+\omega_j/(2\omega),T) \\ \varphi(-U+\omega_j/(2\omega),T) & 0 \end{pmatrix} \\
& \qquad + \pi^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\mathrm{d}U}{\mathrm{d}X}.
\end{aligned}
$$

Once converted to rational coordinates (see Appendix A), this Lax pair still depends on $W(u,x)$ and $W(t,x)$, with W defined in (A22). Fortunately, there exists a transition matrix

$$
\begin{aligned}
& P = \mathrm{diag}(e^{-F(u,x,t)/2}, e^{F(u,x,t)/2}), && (52)\\
& e^{-2F(u,x,t)} = \frac{\varphi(U,T)}{\varphi(-U,T)} = \frac{\sigma(2\omega(T+U),g_2,g_3)}{\sigma(2\omega(T-U),g_2,g_3)} e^{-8\eta\omega UT}, && (53)
\end{aligned}
$$

able to eliminate all W terms. The resulting Lax pair

$$
\mathrm{d}\Psi_1 = \mathcal{L}_1\Psi_1 \mathrm{d}x + \mathcal{M}_1\Psi_1 \mathrm{d}t,\ \Psi = P\Psi_1, \tag{54}
$$

has all its elements algebraic in u,x,t, affine in u',θ_j, and it displays the same remarkable symmetry between u and t as the scalar Lax pair of Fuchs,

$$
\begin{aligned}
\mathcal{L}_1 = & \frac{1}{4x(x-1)} \Big[- \frac{x(x-1)t(t-1)(u-x)u'}{\sqrt{t(t-1)(t-x)}\sqrt{u(u-1)(u-x)}(t-u)} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
& -\theta_\infty \frac{t(t-1)(u-x)}{\sqrt{t(t-1)(t-x)}\sqrt{t-u}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \theta_\infty \frac{\sqrt{u(u-1)(u-x)}}{\sqrt{t-u}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
& + \sum_{j=1}^{3} \theta_j \left(\frac{t(t-1)}{\sqrt{t(t-1)(t-x)}} - 4A_{j,+} \right) \sqrt{t-u}\sqrt{B_{j,+}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
& + \sum_{j=1}^{3} \theta_j \left(\frac{t(t-1)}{\sqrt{t(t-1)(t-x)}} - 4A_{j,-} \right) \sqrt{t-u}\sqrt{B_{j,-}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Big], \\
\mathcal{M}_1 = & \frac{1}{4\sqrt{t(t-1)(t-x)}\sqrt{u(u-1)(u-x)}} \left[-x(x-1)u' + \frac{u(u-1)(t-x)}{t-u} \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
& - \frac{(t-u)}{4\sqrt{t(t-1)(t-x)}} \left[\theta_\infty \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{j=1}^{3} \theta_j \begin{pmatrix} 0 & \sqrt{B_{j,+}} \\ \sqrt{B_{j,-}} & 0 \end{pmatrix} \right].
\end{aligned} \tag{55}
$$

The algebraic functions of (u,x,t) appearing above are

$$\begin{aligned}
A_{j,\pm} &= \frac{\wp'(2\omega T)-\wp'(\pm 2\omega U+\omega_j)}{8\sqrt{e_2-e_1}(\wp(2\omega T)-\wp(\pm 2\omega U+\omega_j)}\\
&= -\frac{1}{4\alpha_j}\frac{\alpha_j^3\sqrt{t(t-1)(t-x)}\mp\alpha_j^2\beta_j\sqrt{u(u-1)(u-x)}\pm(u(u-1)(u-x))^{3/2}}{u(u-1)(u-x)-\alpha_j^2\gamma_j},\\
B_{j,\pm} &= \frac{\wp(2\omega(\pm U+T))-e_j}{\wp(2\omega U)-e_j}=\frac{(\sqrt{t(t-1)(t-x)}\mp\sqrt{u(u-1)(u-x)})^2-\gamma_j(t-u)^2}{\alpha_j(t-u)^2},\\
\alpha_j &= u-\frac{e_j-e_1}{e_2-e_1},\ \beta_j=u+\frac{2e_j+e_1}{e_2-e_1},\ \gamma_j=t+u+\frac{e_j+2e_1}{e_2-e_1}\cdot
\end{aligned}\tag{56}$$

3. Third order matrix Lax pair of Harnad

Finally, it is worth saying a few words on a nice third order matrix Lax pair[21], associated to a three-degree of freedom Hamiltonian system. This Lax pair, which has one Fuchsian singularity and one nonFuchsian, admits a dual, second order matrix Lax pair defined in[21] (Eq. (3.55), (3.61)), which has the structure (8) and whose residues can be found in[14] (Eqs. (65)–(68)). Its identification to the present Lax pair (42) could define a rational representation of the Hamilton variables (q_j,p_j), $j=1,2,3$, instead of the algebraic ones obtained by identification to the Lax pair of Jimbo and Miwa. This could also help to solve the factorization problem of the three-wave system mentioned in[14] (§7.2).

V. QUANTUM CORRESPONDENCE

In 1994 Suleimanov[42] noticed remarkable properties for P_{VI}, which are inherited by all the lower Painlevé equations under the confluence.

1. There exists a linear PDE of the parabolic type

$$\left[x(x-1)(\partial_x+g_{\rm h}(x))-t(t-1)(t-x)\partial_t^2+v(x,t)\right]\psi_{\rm h}=0,\tag{57}$$

   (the subscript h refers to the heat equation), in which the potential v is a rational function, and $g_{\rm h}$ an irrelevant arbitrary function, both of them independent of the nonlinear field u. This generalized heat equation, which is defined up to the multiplication of $\psi_{\rm h}$ by an arbitrary function of (x,t,θ_j), can be normalized in different ways, either affine in the four θ_j^2's[42], or affine in $(\theta_0,\,\theta_1,\,\theta_x,\,\theta_\infty^2-(\theta_0+\theta_1+\theta_x-1)^2)$[13,30].

2. There exists a representation of P_{VI} by a classical Hamiltonian $H(q,p,x)$ with $q=u$, different from the one of Malmquist[28], and there exists a quantization $q\to t$, $p\to\partial_t$, $H(q,p,x)\to H(t,\partial_t,x)$ (with an appropriate ordering of the noncommuting operators t, ∂_t) allowing the identification of the generalized heat equation (57) to the time-dependent Schrödinger equation of quantum mechanics,

$$\left[x(x-1)(\partial_x+g_{\rm h}(x))-H(t,\partial_t,x)\right]\psi_{\rm h}=0.\tag{58}$$

3. This "quantum correspondence" extends[47,48] to the representation of $\mathrm{P_{VI}}$ in elliptic coordinates (X,T). The heat equation (57) then takes the form[48] (Eq. (5.21))

$$\left[-a(\partial_X - G_{\mathrm{H}}(X)) + \frac{1}{2}(\partial_T^2 + V(T,X,\{\theta_j^2 - 1/4\}))\right]\Psi_{\mathrm{H}} = 0, \tag{59}$$

and the potential V is deduced from the elliptic potential V in (5) by shifting each θ_j^2 by $-1/4$.

However, as pointed out in[47] (p. 4), the quantum correspondence in rational coordinates (58) relies, at least for $\mathrm{P_{VI}}$, $\mathrm{P_{V}}$, $\mathrm{P_{IV}}$ and $\mathrm{P_{III}}$ (see details in[42,43]), on a skilful, unexplained, choice of the ordering of the products of t and ∂_t. Indeed, while the Hamiltonian (5) in elliptic coordinates is the sum of a "kinetic energy" $P^2/2$ and a "potential energy" $V(Q,X)$, this is no more the case in rational coordinates (u,x,t). Moreover, if one uses a nonoptimal matrix Lax pair, the derivation of the heat equation in elliptic coordinates makes the computations rather involved[48].

It is therefore necessary to give a direct, deterministic derivation of all these results. Let us do that only starting from the matrix Lax pair provided by the moving frame of Bonnet surfaces.

First step. Since the heat equation is scalar, one must first derive a scalar Lax pair from the matrix one. Each of the four off-diagonal elements M_{12}, M_{21} of (40) and (42) possesses a single zero $t = f(u',u,x)$ (provided one sets $a = \pm\theta_\infty$ in (40)), and each of these four zeroes obeys a $\mathrm{P_{VI}}$ equation. These contiguous $\mathrm{P_{VI}}$ are linked by birational transformations as sketched in[26] (Eq. (4.4)). The simplest of these elements is

$$(40):\ \ M_{12} = \frac{t-u}{t(t-1)}\cdot \tag{60}$$

The elimination of anyone of the two components of the moving frame, whether in (40) or in (42), therefore generates the unique apparent singularity ($t = u$ in the above example, $t =$ another $\mathrm{P_{VI}}$ function in the three other cases) which the scalar Lax pair must possess[37] (p. 219).

If one denotes the two components of (40) and (42) as, respectively, $\psi_{j,\mathrm{q}}$ (q like quadratic in the θ_j's), $\psi_{j,\mathrm{m}}$ (m like meromorphic in θ_∞), then the scalar wave vector

$$\psi_{\mathrm{d}} = \sqrt{\frac{t(t-1)}{t-u}}\psi_{1,\mathrm{q}}, \tag{61}$$

obeys the scalar Lax pair (6) of R. Fuchs[17],

$$\left(\partial_t^2 + \frac{S}{2}\right)\psi_{\mathrm{d}} = 0,\ \left(\partial_x + C\partial_t - \frac{C_t}{2} + g_{\mathrm{d}}(x,u,u')\right)\psi_{\mathrm{d}} = 0, \tag{62}$$

in which the arbitrary function g_{d}, which depends on $x, u(x), u'(x)$ but not on t, will be later used to cancel various terms independent of t.

The coefficient C is independent of the four θ_j's and the coefficient S (the Schwarzian), which has five double poles in t (hence the notation ψ_{d}) has two nice properties which we will preserve throughout this section: (i) it is an even function of the four θ_j's; (ii) as noticed by Garnier[18] (p. 51), it displays a remarkable symmetry between u and t. Indeed, if one introduces the potential function

$$V_{\rm G}(z,s)=\frac{1}{4}\left[-3z+(\theta_\infty^2-s)(z-x)+(\theta_0^2-s)\left(\frac{x}{z}-1\right)\right.$$
$$\left.+(\theta_1^2-s)\left(-\frac{x-1}{z-1}+1\right)+(\theta_x^2-s)\left(\frac{x(x-1)}{z-x}+2x-1\right)\right], \tag{63}$$

the dependence of S on the θ_j's is only through the difference $V_{\rm G}(u,s)-V_{\rm G}(t,s)$ in which the shift s is unity,

$$C=-\frac{t(t-1)(u-x)}{x(x-1)(t-u)}, \tag{64}$$

$$\frac{S}{2}=-\frac{3/4}{(t-u)^2}-\frac{\beta_1 u'+\beta_0}{t(t-1)(t-u)}-\frac{[(\beta_1 u')^2-\beta_0^2](u-x)}{u(u-1)t(t-1)(t-x)}$$
$$+\frac{1}{t(t-1)(t-x)}\left[V_{\rm G}(u,1)-V_{\rm G}(t,1)\right], \tag{65}$$

$$\beta_1=-\frac{x(x-1)}{2(u-x)},\quad \beta_0=-u+\frac{1}{2}\cdot \tag{66}$$

Remark. In the correspondence matrix-scalar, defined by (41) and

$$\psi_{1,{\rm q}}=\sqrt{\frac{t-u}{t(t-1)}}\psi_{\rm d},\ \psi_{2,{\rm q}}=\left(\frac{x(x-1)}{u-x}\left(\partial_x-\Theta_x\frac{1}{2(t-x)}\right)+\frac{a}{2}\right)\psi_{1,{\rm q}}, \tag{67}$$

only $\psi_{1,{\rm q}}$ has a simple dependence on $\psi_{\rm d}$, and the correspondence between the second component $\psi_{2,{\rm q}}$ and $\psi_{\rm d}$ is indeed quite difficult[11,26] to find simply by some good guess.

Second step. This is the elimination of u between the two equations of the scalar Lax pair (62). It is realized[13] by the linear combination $x(x-1)(\partial_x+\dots)-t(t-1)(t-x)(\partial_t^2+\dots)$ of the two scalar equations (62), followed by a change of the wave function $\psi_{\rm d}$ and a suitable choice of the arbitrary function $g_{\rm d}(x,u,u')$. This change is essentially $\psi_{\rm d}=(t-u)^{-1/2}\psi_{\rm h}$ but, because of the freedom $\psi_{\rm h}\to f(x,t)\psi_{\rm h}$ with f independent of u, it is more convenient to define it as

$$\psi_{\rm d}=(t-u)^{-1/2}t^{k_0/2}(t-1)^{k_1/2}(t-x)^{k_x+1/2}\psi_{\rm h}, \tag{68}$$

with k_0, k_1, k_x adjustable constants independent of the θ_j's. The transformed Lax pair thus retains the parity in the θ_j's.

The crucial question at this point is to find the Hamiltonian $H(q,p,x)$ whose quantization $H(t,\partial_t,x)$ is unambiguous and succeeds to describe the resulting scalar heat equation (58) for the $\psi_{\rm h}$ defined in (68). Indeed, the Hamiltonian description of P_{VI} (at least in rational coordinates) is not unique. The three Hamiltonians we are aware of (Malmquist[28], Suleimanov[42], Tsegel'nik[44]) have different properties: polynomial in q (Malmquist), even functions of p (Malmquist, Suleimanov), even functions of the four θ_j's (Suleimanov, Tsegel'nik), but none of these properties is relevant, and this is a fourth property which dictates the Hamiltonian.

Indeed, in order to prove the Painlevé property of P_{VI}, Painlevé built[33] (p. 26)[34] (Eq. (3)) four rational functions of u', u, x having as only movable singularities two movable simple poles of residue unity (reached when $u \sim \pm x(x-1)\theta_\infty^{-1}(x-x_{0,\pm})^{-1}$ in the expression below, and similarly for the three other rational functions),

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,P}} &= \frac{x(x-1)u'^2}{2u(u-1)(u-x)} - \frac{u'}{u-x} \\
&+\frac{1}{2x(x-1)}\left[\theta_\infty^2\left(\frac{1}{2}-u\right)+\theta_0^2\left(\frac{1}{2}-\frac{x}{u}\right)\right. \\
&\qquad \left. +\theta_1^2\left(\frac{x-1}{u-1}-\frac{1}{2}\right)+(\theta_x^2-1)\left(\frac{1}{2}-x-\frac{x(x-1)}{u-x}\right)-x+1\right],
\end{aligned}
\tag{69}
$$

and this is the Hamiltonian associated to this tau-function which correctly defines the quantum correspondence. The Hamiltonians of Malmquist and Suleimanov are associated to a different tau-function, which is the one built by Chazy[9] (expression t page 341) and whose logarithmic derivative has only one (instead of two) movable simple pole of residue unity (reached when $u \sim x(x-1)\theta_\infty^{-1}(x-x_0)^{-1}$).

The Hamiltonian[44] associated to (69),

$$
\left\{
\begin{aligned}
& H_{\mathrm{VI,T}}(q,p,x) = \frac{1}{a_{\mathrm{T}}x(x-1)}\left[q(q-1)(q-x)a_{\mathrm{T}}^2p^2+q(q-1)a_{\mathrm{T}}p\right. \\
&\qquad +\frac{1}{4}\left((\theta_\infty^2-1)\left(\frac{1}{2}-q\right)+\theta_0^2\left(\frac{1}{2}-\frac{x}{q}\right)\right. \\
&\qquad\quad \left.\left.+\theta_1^2\left(-\frac{1}{2}+\frac{x-1}{u-1}\right)+\theta_x^2\left(\frac{1}{2}-x-\frac{x(x-1)}{u-x}\right)+2x-1\right)\right], \\
& q=u,\ p=\frac{1}{2a_{\mathrm{T}}}\left(\frac{x(x-1)u'}{u(u-1)(u-x)}-\frac{1}{q-x}\right),
\end{aligned}
\right.
\tag{70}
$$

(a_{T} being a constant of normalization), only differs from (69) by an additive term which reflects the singling out of one singular point among four,

$$
\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,P}} - 2a_{\mathrm{T}}H_{\mathrm{VI,T}}(q,p,x)+\frac{\mathrm{d}}{\mathrm{d}x}\log(q-x)=0. \tag{71}
$$

Then, if one defines the quantum Hamiltonian as

$$
\forall\psi:\ H_{\mathrm{VI,T}}(t,\partial_t,x)\psi = a_{\mathrm{T}}\frac{t(t-1)(t-x)}{a_{\mathrm{T}}^2x(x-1)}\partial_t^2\psi+\frac{t(t-1)}{a_{\mathrm{T}}x(x-1)}\partial_t\psi+(\partial_t^0\ \text{terms})\psi, \tag{72}
$$

and chooses the above adjustable parameters as $k_0=0, k_1=0, k_x=1$, the scalar Lax pair for the ψ_{h} defined in (68) can be written as

$$
\left\{
\begin{aligned}
& \left[\partial_x - a_{\mathrm{T}}H_{\mathrm{VI,T}}(t,\partial_t,x,\theta_j^2+s_j)+g_{\mathrm{h}}(x)\right]\psi_{\mathrm{h}}=0, \\
& \left[\partial_x - \frac{t(t-1)(u-x)}{x(x-1)(t-u)}\partial_t - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,P}}+g_{\mathrm{h}}(x)-\frac{3}{4x}-\frac{3}{4(x-1)}\right. \\
& \qquad \left. + \frac{x(x-1)u'+u(u-1)(u-x)\left(\dfrac{1}{u}+\dfrac{1}{u-1}-\dfrac{2}{u-x}\right)}{2x(x-1)(t-u)}\right]\psi_{\mathrm{h}}=0, \\
& g_{\mathrm{d}}-g_{\mathrm{h}} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log\tau_{\mathrm{VI,P}}-\frac{3}{4x}-\frac{3}{4(x-1)}, \\
& s_\infty=1,\ s_0=s_1=s_x=-1,
\end{aligned}
\right.
\tag{73}
$$

and the quantum correspondence is realized by taking g_{h} as the arbitrary function. The reason why only three θ_j^2 display the same shift is a consequence of the necessity to single out one of the four singular points in order to define a tau-function.

Third step. The conversion of this Lax pair to elliptic coordinates (U,X,T) is performed in a systematic way following the guidelines and the formulae of Halphen recalled in Appendix A. In order that the heat equation in elliptic coordinates takes the normalized form (59) (i.e. without ∂_T term), the wave function Ψ_{H} must be

$$\left\{\begin{aligned} &\Psi_{\mathrm{H}}=(t(t-1))^{-1/4}(t-x)^{3/4}e^F\psi_{\mathrm{h}},\\ &\mathrm{d}F=\frac{W(t,x)}{2(t(t-1)(t-x))^{1/2}}\mathrm{d}t+\frac{G_{\mathrm{H}}(X)}{x(x-1)}\mathrm{d}x\\ &\quad+\frac{-2(t(t-1)(t-x))^{1/2}W(t,x)-(t-x)W^2(t,x)-(t-x)^2}{4x(x-1)(t-x)}\mathrm{d}x, \end{aligned}\right. \tag{74}$$

in which $G_{\mathrm{H}}(X)$ is an arbitrary function. As expected, the link between the wave function Ψ_{H} of the heat equation in elliptic coordinates and the wave function $\psi_{1,\mathrm{q}}$ of the first canonical form of the matrix Lax pair of $\mathrm{P_{VI}}$ does not involve the apparent singularity $t=u$,

$$\Psi_{\mathrm{H}}=(t(t-1))^{1/4}(t-x)^{-3/4}e^F\psi_{1,\mathrm{q}}. \tag{75}$$

In the elliptic coordinates, the Lax pair (73) then becomes

$$\left\{\begin{aligned} &\left[2a_X\pi^2\partial_X-\frac{1}{2}G_{\mathrm{H}}(X)+\frac{1}{2}\partial_T^2-\frac{1}{2}(2\omega)^2\sum_{j=\infty,0,1,x}\left(\theta_j^2-\frac{1}{4}\right)\wp(2\omega T+\omega_j)\right]\Psi_{\mathrm{H}}=0,\\ &\left[a_X\pi^2\partial_X+\omega\left[\frac{1}{2}\frac{\wp'(2\omega T)}{\wp(2\omega T)-\wp(2\omega U)}+\zeta(2\omega T)-2\eta T\right]\partial_T+(e_2-e_1)\omega^2G_{\mathrm{H}}(X)\right.\\ &-(e_2-e_1)\omega^2[\theta_\infty^2\wp(2\omega U)+\theta_0^2\wp(2\omega U+\omega_1)+\theta_1^2\wp(2\omega U+\omega_2)+(\theta_x^2-1)\wp(2\omega U+\omega_3)]\\ &+\frac{(2\omega)^2(e_3-e_2)^2(e_1-e_3)^2}{\wp'^2(2\omega U)}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+\frac{2\omega^2(e_3-e_2)(e_1-e_3)(\wp(2\omega T)-e_3)}{(\wp(2\omega T)-\wp(2\omega U))(\wp(2\omega U)-e_3)}\frac{\mathrm{d}u}{\mathrm{d}x}\\ &-\omega^2(\zeta(2\omega T)-2\eta T)^2-\omega^2\frac{(\zeta(2\omega T)-2\eta T)\wp'(2\omega T)}{\wp(2\omega T)-\wp(2\omega U)}-\omega(\eta+e_3\omega)\\ &\left.-\omega^2\frac{(\wp(2\omega T)-e_3)^2-(e_3-e_1)(e_3-e_2)}{\wp(2\omega T)-\wp(2\omega U)}\right]\Psi_{\mathrm{H}}=0, \end{aligned}\right. \tag{76}$$

in which $\mathrm{d}u/\mathrm{d}x$ should be replaced by the expression (A14).

The reduction $\partial_X=0$ of the remarkable parabolic PDE $(76)_1$ is a generalization to nonconstant half-periods of the equation introduced by Darboux[16],

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}T^2}-\lambda-(2\omega)^2\sum_{j=\infty,0,1,x}n_j(n_j+1)\wp(2\omega T+\omega_j)\right]\Psi=0, \tag{77}$$

integrated for arbitrary complex λ and integer n_j's by de Sparre[40], and rediscovered by Treibich and Verdier more than one century later. Its writing in rational coordinates is identical to the Heun equation[23]. For a modern account on this Darboux ODE, see[38,45].

Remark. The shifts $-1/4$ in $(76)_1$ are a direct consequence of the identity $n_j(n_j+1) = (n_j+1/2)^2 - 1/4$.

The confluence to the lower Painlevé equations of the results presented in this section can be found in Appendix C.

VI. GENERALIZED BONNET SURFACES

Given the previous results, it is natural to ask whether there exist analytic surfaces represented by the full $\mathrm{P_{VI}}$, which would therefore generalize Bonnet surfaces. The answer is indeed positive.

The matrix Lax pair (40) can be converted back to the moving frame (15) of some surface by solving the six scalar equations

$$\mathbb{U}P^{-1}\mathrm{d}z + \mathbb{V}P^{-1}\mathrm{d}\bar{z} = L\mathrm{d}x + M\mathrm{d}t,\ P = \begin{pmatrix} G & 0 \\ 0 & 1/G \end{pmatrix}, \tag{78}$$

for the five unknowns e^v, H, Q, $\overline{Q}$, G^2 (indeed, c scales out because of (18)), thus defining an extrapolation of the Bonnet moving frame (25) to arbitrary values of the four θ_j's. If one defines f_{jk} and g_{jk} by

$$L_{jk}\mathrm{d}x + M_{jk}\mathrm{d}t = f_{jk}(x,t)\mathrm{d}z + g_{jk}(x,t)\mathrm{d}\bar{z}, \tag{79}$$

this system of six scalar equations,

$$\begin{cases} Qe^{-v/2}G^{-2} = -4c_{\mathrm{z}}(t-x),\ \overline{Q}e^{-v/2}G^2 = g_{21}, \\ (H+c)e^{v/2}G^2 = 2f_{21},\ (H-c)e^{v/2}G^{-2} = 8c_{\mathrm{z}}(u-x), \\ \mathrm{d}\log(e^vG^4) = -f_{11}\mathrm{d}z + 2c_{\mathrm{z}}\left[a(u-x) - \Theta_x \dfrac{x(x-1)}{t-x}\right]\mathrm{d}\bar{z}, \end{cases} \tag{80}$$

is equivalent to

$$\begin{cases} G^4 = -4\dfrac{f_{21}}{4c_{\mathrm{z}}(u-x)}\dfrac{H-c}{H+c},\ e^v = -\dfrac{16c_{\mathrm{z}}(u-x)f_{21}}{H^2-c^2}, \\ Q = -\dfrac{8c_{\mathrm{z}}(t-x)f_{21}}{H+c},\ \overline{Q} = -\dfrac{8c_{\mathrm{z}}(u-x)g_{21}}{H-c}, \\ \mathrm{d}H = (H-c)A\mathrm{d}z + (H+c)B\mathrm{d}\bar{z}, \\ A(x,t) = -2f_{11} + 4x(x-1)\dfrac{u'-1}{u-x}, \\ B(x,t) = 4c_{\mathrm{z}}\left[\Theta_x\dfrac{x(x-1)}{t-x} - a(u-x) + x(x-1)\partial_x\log(f_{21})\right], \end{cases} \tag{81}$$

and the condition $\mathrm{d}^2H = 0$,

$$\begin{aligned} &(H-c)\left[AB - 4c_{\mathrm{z}}x(x-1)A_x\right] \\ &-(H+c)\left[AB - 4c_{\mathrm{z}}t(t-1)B_t - 4c_{\mathrm{z}}x(x-1)B_x\right] = 0, \end{aligned} \tag{82}$$

admits three solutions

$$\begin{cases} \text{(A) } c \ne 0,\ (\theta_\infty,\Theta_x) \ne (-1,-1):\ \dfrac{H}{c} = \text{ the unique solution of (82)}, \\ \text{(B) } c = 0,\ \theta_\infty = \Theta_x \ne -1,\ \theta_1^2 = \theta_0^2:\ H = \text{ any integral of } (81)_3, \\ \text{(C) } c \text{ arbitrary},\ (\theta_\infty,\Theta_x) = (-1,-1):\ H = \text{ any integral of } (81)_3. \end{cases} \tag{83}$$

The generic solution (A)

$$
\begin{aligned}
c \neq 0,\ (\theta_\infty, \Theta_x) \neq (-1,-1):\ & \frac{H}{c} = \frac{(\theta_\infty - \Theta_x)P_5 + (\theta_\infty + \Theta_x - 2)P_3}{(\theta_\infty - \Theta_x)P_4 + (\theta_\infty + \Theta_x - 2)P_5},\\
& \frac{H-c}{H+c} = \frac{P_5}{x(x-1)P_3 \frac{\mathrm{d}}{\mathrm{d}x}\log \tau_{\mathrm{VI,M}}},
\end{aligned}
\tag{84}
$$

where the P_n's, all different, are polynomials of u', u, x, t, θ_∞, θ_0^2, θ_1^2, Θ_x of degree n in u', defines an extrapolation of Bonnet surfaces to more general surfaces in $\mathbb{R}^3(c)$ represented by the generic $\mathrm{P_{VI}}$, in which, as a consequence of (18), the nonzero value of c is arbitrary and independent of the θ_j's. Its Bonnet limit is, by construction,

$$
\lim_{\theta_\infty \to 0, \Theta_x \to 0} \frac{H}{c} = \frac{8 c_{\mathrm{z}} x(x-1)\frac{\mathrm{d}}{\mathrm{d}x}\log \tau_{\mathrm{VI,M}}}{c_{\mathrm{z}}(\theta_1^2 - \theta_0^2)},
\tag{85}
$$

but for arbitrary θ_j's the value of H is different from $8c_{\mathrm{z}}x(x-1)\frac{\mathrm{d}}{\mathrm{d}x}\log \tau_{\mathrm{VI,M}}$ and does depend on t.

As to the two nongeneric solutions (B) and (C), in which $\mathrm{d}^2 H$ is identically zero, they are not essentially different from the solution of Bonnet. For instance, in the third solution (C), two θ_j's are the same as those of Bonnet and the two others are shifted by ± 1, therefore this third solution is the Schlesinger transform[19,39] of the Bonnet solution, and we leave it to the interested reader to establish the explicit expressions for H.

Remark. One can similarly define an extrapolation of harmonic inverse mean curvature surfaces to the full $\mathrm{P_{VI}}$. It is sufficient to first transpose the moving frame and therefore, instead of solving (78), to solve

$$
{}^t\mathbb{U}P^{-1}\mathrm{d}z + {}^t\mathbb{V}P^{-1}\mathrm{d}\bar{z} = L\mathrm{d}x + M\mathrm{d}t,\ P = \begin{pmatrix} G & 0 \\ 0 & 1/G \end{pmatrix}.
\tag{86}
$$

VII. CONCLUSION

The present results are threefold: (i) the natural Lax pair of $\mathrm{P_{VI}}$; (ii) a rigorous derivation of the quantum correspondence of $\mathrm{P_{VI}}$; (iii) an extension of Bonnet surfaces to two more parameters, thus matching the completeness of $\mathrm{P_{VI}}$ and the completeness of Gauss-Codazzi equations.

In future work, we plan two directions of research.

(i) To find a geometric characterization of these extended Bonnet surfaces.

(ii) To similarly improve the discrete matrix Lax pair of q-$\mathrm{P_{VI}}$ introduced by Jimbo and Sakai[25], i.e. to remove its meromorphic dependence on a fixed parameter of this q-$\mathrm{P_{VI}}$.

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Appendix A: Conversion between rational and elliptic coordinates

This Appendix has two guidelines. The first one is to define the Weierstrass functions $\wp$, ζ, σ as functions of three arguments (z, g_2, g_3), not two arguments $(z|\tau)$ by setting one period to unity. The reason is that $\wp(z, g_2, g_3)$ is a homogeneous function, thus making all formulae homogeneous and therefore easy to check. Our second guideline is to never deal with partial derivatives, always with differentials, in order to avoid thinking about which depends on what. The reference is, naturally, the first volume of Halphen[20] (t. I Chap. IX–X).

The ratio of the two periods 2ω, $2\omega'$ and the discriminant are respectively denoted Ω and Δ[20] (t. I p. 321),

$$\Omega = i\pi\frac{\omega'}{\omega},\ \Delta = g_2^3 - 27g_3^2. \tag{A1}$$

The transformation (2)–(3) between rational and elliptic coordinates is equivalently defined as

$$(u, 0, 1, x) = (\wp(2\omega U), e_1, e_2, e_3),\ \Omega = i\pi\frac{\omega'}{\omega}\cdot \tag{A2}$$

In order to obtain (4), it is sufficient to establish the differentials of u, x, $\mathrm{d}u/\mathrm{d}x$ in terms of $\mathrm{d}U$, $\mathrm{d}\Omega$, $\mathrm{d}(\mathrm{d}U/\mathrm{d}\Omega)$, then to substitute these values in (1).

Since x, u and Ω, U have no dimension, it is convenient to replace the triplet $(2\omega U, g_2, g_3)$ of the arguments of $\wp$ by a triplet containing two dimensionless variables, for instance (U, Ω, Δ), whose Jacobian

$$\frac{D(2\omega U, g_2, g_3)}{D(U, \Omega, \Delta)} = 2\omega\frac{2\omega^2}{9\pi^2} \tag{A3}$$

never vanishes. The differential of a dimensionless variable will therefore have no contribution of $\mathrm{d}\Delta$, without the need for assuming the period 2ω to be unity.

If one denotes (z, g_2, g_3) the three arguments of $\wp$, ζ, σ, and L the dimension of the first argument z, the various variables have the dimensions

$$\begin{gathered}[z] = [\omega] = [\sigma] = L,\ [\wp] = [e_\alpha] = L^{-2},\ [\zeta] = [\eta] = L^{-1},\\ [g_2] = L^{-4},\ [g_3] = L^{-6},\ [\Delta] = L^{-12}.\end{gathered} \tag{A4}$$

Abbreviating $\wp(z, g_2, g_3)$ (resp. $\zeta(z, g_2, g_3)$, $\sigma(z, g_2, g_3)$) as $\wp$ (resp. ζ, σ), and denoting by a quote $'$ the derivative with respect to the first argument, the only necessary formulae, apart (A1), are the following[20] (t. I Chap. IX–X)[51],

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3),\ \zeta' = -\wp,\ \sigma' = \sigma\zeta, \tag{A5}$$

$$\begin{aligned}\mathrm{d}\sigma = \sigma\zeta\mathrm{d}z &+ \frac{1}{\Delta}\left(-\frac{9}{4}g_3\sigma'' + \frac{1}{4}g_2^2 z\sigma' - \left(\frac{1}{4}g_2^2 + \frac{3}{16}g_2 g_3 z^2\right)\sigma\right)\mathrm{d}g_2\\ &+ \frac{1}{\Delta}\left(\frac{3}{2}g_2\sigma'' - \frac{9}{2}g_3 z\sigma' + \left(\frac{9}{2}g_3 + \frac{1}{8}g_2^2 z^2\right)\sigma\right)\mathrm{d}g_3,\end{aligned} \tag{A6}$$

$$\zeta(\omega, g_2, g_3) = \eta,\ \zeta(\omega', g_2, g_3) = \eta',\ \eta\omega' - \eta'\omega = i\frac{\pi}{2}, \tag{A7}$$

$$\mathrm{d}\omega = \frac{1}{\Delta}\left(-\frac{1}{4}g_2^2\omega + \frac{9}{2}g_3\eta\right)\mathrm{d}g_2 + \frac{1}{\Delta}\left(\frac{9}{2}g_3\omega - 3g_2\eta\right)\mathrm{d}g_3, \tag{A8}$$

$$\mathrm{d}\eta = \frac{1}{\Delta}\left(-\frac{3}{8}g_2 g_3\omega + \frac{1}{4}g_2^2\eta\right)\mathrm{d}g_2 + \frac{1}{\Delta}\left(\frac{1}{4}g_2^2\omega - \frac{9}{2}g_3\eta\right)\mathrm{d}g_3. \tag{A9}$$

In particular, the expressions of $\mathrm{d}\zeta$ and $\mathrm{d}\wp$ result from $\mathrm{d}\sigma$ by action of the operator $'$ (we hope that no confusion occurs with ω'), which commutes with the operator d. To this list one should add the transformed of (A8)–(A9) under $(\omega, \eta) \to (\omega', \eta')$.

From the above formulae, one deduces the quite useful formula,

$$\mathrm{d}e_\alpha = \frac{e_\alpha \mathrm{d}g_2 + \mathrm{d}g_3}{12 e_\alpha^2 - g_2}, \tag{A10}$$

together with the formulae of the change of variables

$$\mathrm{d}g_2 = \frac{1}{\Delta}\left(-12 g_3 \frac{\mathrm{d}\Omega}{\pi^2} + \frac{g_2}{3}\mathrm{d}\Delta\right),\ \mathrm{d}g_3 = \frac{1}{\Delta}\left(-\frac{2}{3} g_2^2 \frac{\mathrm{d}\Omega}{\pi^2} + \frac{g_3}{2}\mathrm{d}\Delta\right), \tag{A11}$$

$$\mathrm{d}\omega = 2\eta\omega^2 \frac{\mathrm{d}\Omega}{\pi^2} - \frac{\omega}{12\Delta}\mathrm{d}\Delta,\ \mathrm{d}\eta = -g_2\omega^3 \frac{\mathrm{d}\Omega}{6\pi^2} + \frac{\eta}{12\Delta}\mathrm{d}\Delta. \tag{A12}$$

1. From rational to elliptic coordinates

Given the definitions (3) and now taking $z = 2\omega U$, the differentials $\mathrm{d}x$ and $\mathrm{d}u$ are then linear forms of $\mathrm{d}U$, $\mathrm{d}\Omega$ independent of $\mathrm{d}\Delta$,

$$\left\{ \begin{aligned} \mathrm{d}x &= -\frac{4\omega^2 (e_3 - e_1)(e_3 - e_2)}{\pi^2 (e_2 - e_1)} \mathrm{d}\Omega, \\ \mathrm{d}u &= \frac{2\omega\wp'}{(e_2 - e_1)}\mathrm{d}U + \left[(\zeta - 2\eta U)\wp' + 2(\wp - e_1)(\wp - e_2)\right] \frac{2\omega^2}{\pi^2 (e_2 - e_1)} \mathrm{d}\Omega, \end{aligned} \right. \tag{A13}$$

hence the value of $\mathrm{d}u/\mathrm{d}x$,

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\omega(e_3 - e_1)(e_3 - e_2)} \left[\pi^2 \wp' \frac{\mathrm{d}U}{\mathrm{d}\Omega} + \left((\zeta - 2\eta U)\wp' + 2(\wp - e_1)(\wp - e_2)\right)\omega\right]. \tag{A14}$$

The differential of $\mathrm{d}u/\mathrm{d}x$ is similarly expressed as a linear form of $\mathrm{d}U$, $\mathrm{d}\Omega$, $\mathrm{d}(\mathrm{d}U/\mathrm{d}\Omega)$, and one obtains

$$\begin{aligned} \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = \frac{e_2 - e_1}{8\omega^3 (e_3 - e_1)^2 (e_3 - e_2)^2} & \left[\pi^4 \wp' \frac{\mathrm{d}^2 U}{\mathrm{d}\Omega^2} + 2\pi^4 \omega \wp'' \left(\frac{\mathrm{d}U}{\mathrm{d}\Omega}\right)^2 \right. \\ & -\pi^2 \omega^2 \left\{12 \wp\wp' + 4(\zeta - 2\eta U)\wp''\right\} \frac{\mathrm{d}U}{\mathrm{d}\Omega} \\ & \left. +\omega^3 \left(2(\zeta - 2\eta U)^2 \wp'' + 12(\zeta - 2\eta U)\wp\wp' + 3\wp'^2\right)\right]. \end{aligned} \tag{A15}$$

The transformed equation of (1) then results from the substitution of the expressions x, u, $\mathrm{d}u/\mathrm{d}x$, $\mathrm{d}^2u/\mathrm{d}x^2$ respectively defined by (3), (A14), (A15),

$$\begin{aligned} & \frac{\pi^2}{(2\omega)^3} \frac{\mathrm{d}^2 U}{\mathrm{d}\Omega^2} - \theta_\infty^2 \wp' \\ & \quad - \theta_0^2 \frac{(e_3 - e_1)(e_1 - e_2)\wp'}{(\wp - e_1)^2} - \theta_1^2 \frac{(e_1 - e_2)(e_2 - e_3)\wp'}{(\wp - e_2)^2} - \theta_x^2 \frac{(e_2 - e_3)(e_3 - e_1)\wp'}{(\wp - e_3)^2} = 0. \end{aligned} \tag{A16}$$

Finally, the addition formula

$$\forall x_1, x_2 :\ \wp(x_1 + x_2) + \wp(x_1) + \wp(x_2) = \frac{1}{4}\left(\frac{\wp'(x_1) - \wp'(x_2)}{\wp(x_1) - \wp(x_2)}\right)^2, \tag{A17}$$

applied to the choice $x_2 = \omega_\alpha$, $\wp(x_2) = e_\alpha$ reduces to

$$\forall z: \ \wp(z+\omega_\alpha) - e_\alpha = \frac{(e_\alpha - e_\beta)(e_\alpha - e_\gamma)}{\wp(z) - e_\alpha}, \tag{A18}$$

whose z-derivatives are the last three terms of (A16).

The usual normalization (4) is achieved by taking the independent variable X to be an adequate multiple of Ω, cf. Eq. (2).

2. From elliptic to rational coordinates

Because of homogeneity, any function of the elliptic coordinates (U, Ω, T) can be written as the product of a function of (u, x, t) by a monomial of ω (or $(e_2 - e_1)$ since $(e_2 - e_1)\omega^2$ has no dimension). For instance, Eq. (3) implies

$$\begin{aligned} & \frac{\wp(2\omega U)}{u - \dfrac{x+1}{3}} = \frac{\wp(2\omega U + \omega_1)}{\dfrac{x}{u} - \dfrac{x+1}{3}} = \frac{\wp(2\omega U + \omega_2)}{-\dfrac{x-1}{u-1} - \dfrac{x-2}{3}} = \frac{\wp(2\omega U + \omega_3)}{-\dfrac{x(x-1)}{u-x} + \dfrac{2x-1}{3}} \\ & = \frac{3e_1}{-(x+1)} = \frac{3e_2}{-(x-2)} = \frac{3e_3}{2x-1} = \frac{e_2 - e_1}{1} = \frac{e_3 - e_2}{x-1} = \frac{e_1 - e_3}{-x}, \end{aligned} \tag{A19}$$

and similarly by changing (u, U) to (t, T).

The expressions of $\mathrm{d}U$, $\mathrm{d}T$, $\mathrm{d}\Omega$ in terms of $\mathrm{d}u$, $\mathrm{d}t$, $\mathrm{d}x$ result from the differentials (A13) and their transform under $(u, U) \to (t, T)$,

$$\left\{ \begin{aligned} \mathrm{d}U &= \frac{(e_2 - e_1)}{2\omega\wp'(2\omega U)}\mathrm{d}u - \frac{(e_2 - e_1)\left[(\zeta(2\omega U) - 2\eta U)\wp'(2\omega U) + 2(\wp(2\omega U) - e_1)(\wp(2\omega U) - e_2)\right]}{2(e_3 - e_1)(e_3 - e_2)\omega\wp'(2\omega U)}\mathrm{d}x, \\ \mathrm{d}T &= \frac{(e_2 - e_1)}{2\omega\wp'(2\omega T)}\mathrm{d}t - \frac{(e_2 - e_1)\left[(\zeta(2\omega T) - 2\eta T)\wp'(2\omega T) + 2(\wp(2\omega T) - e_1)(\wp(2\omega T) - e_2)\right]}{2(e_3 - e_1)(e_3 - e_2)\omega\wp'(2\omega T)}\mathrm{d}x, \\ \mathrm{d}\Omega &= -\frac{\pi^2(e_2 - e_1)}{4\omega^2(e_3 - e_1)(e_3 - e_2)}\mathrm{d}x. \end{aligned} \right. \tag{A20}$$

After elimination of e_3, $\wp(2\omega U)$, $\wp(2\omega T)$, $\wp'(2\omega U)$, $\wp'(2\omega T)$ with the definitions (3), the one-form $\pi^2 \mathrm{d}U \mathrm{d}T / \mathrm{d}\Omega$, defined by

$$\left\{ \begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}\Omega} &= -\frac{\omega(e_2 - e_1)^{1/2}}{\pi^2\sqrt{u(u-1)(u-x)}}\left[x(x-1)\frac{\mathrm{d}u}{\mathrm{d}x} - u(u-1) - \sqrt{u(u-1)(u-x)}W(u,x)\right], \\ \mathrm{d}T &= \frac{(e_2 - e_1)^{-1/2}}{4\omega\sqrt{t(t-1)(t-x)}}\left[x(x-1)\mathrm{d}t - t(t-1)\mathrm{d}x - \sqrt{t(t-1)(t-x)}W(t,x)\mathrm{d}x\right]\frac{1}{x(x-1)}, \\ \mathrm{d}\Omega &= -\frac{\pi^2}{4\omega^2(e_2 - e_1)x(x-1)}\mathrm{d}x, \end{aligned} \right. \tag{A21}$$

depends algebraically on u, t, x, $\mathrm{d}u$, $\mathrm{d}t$, $\mathrm{d}x$ and on a dimensionless function W defined by

$$W(u,x) = \frac{\zeta(2\omega U) - 2\eta U}{\sqrt{e_2 - e_1}}, \ W(t,x) = \frac{\zeta(2\omega T) - 2\eta T}{\sqrt{e_2 - e_1}}, \tag{A22}$$

which obeys a closed differential system with coefficients depending only on u, t, x, $\mathrm{d}u$, $\mathrm{d}t$, $\mathrm{d}x$,

$$\begin{aligned}
\mathrm{d}W(z,x) &= \frac{W(z,x)}{2x(x-1)} V_3(x)\mathrm{d}x - \frac{\sqrt{z(z-1)(z-x)}}{2x(x-1)}\mathrm{d}x + [z-x+V_3(x)]\,\frac{z(z-1)\mathrm{d}x - x(x-1)\mathrm{d}z}{2x(x-1)\sqrt{z(z-1)(z-x)}},\\
\mathrm{d}V_3(x) &= \left[\frac{1}{2} + \frac{V_3^2(x)}{2x(x-1)}\right]\mathrm{d}x,\ V_3(x) = \frac{\eta + e_3\omega}{(e_2-e_1)\omega}.
\end{aligned} \tag{A23}$$

This system integrates with the complete elliptic integrals,

$$\begin{cases}
V_3 = -2x(x-1)(\log\psi)',\ \psi = c_1 K(\sqrt{x}) + c_2 K_{\mathrm{C}}(\sqrt{x}),\\
K(k) = \displaystyle\int_0^1 \frac{\mathrm{d}\lambda}{\sqrt{(1-\lambda^2)(1-k^2\lambda^2)}},\ K_{\mathrm{C}}(k) = \displaystyle\int_0^1 \frac{\mathrm{d}\lambda}{\sqrt{(1-\lambda^2)(1-(k^2-1)\lambda^2)}}.
\end{cases} \tag{A24}$$

Appendix B: The solution of Bonnet to his problem

Problem[7] (§11 pp. 72–73). Given a surface in $\mathbb{R}^3$, to find all surfaces which are applicable[52] on that surface and possess the same two principal radii of curvature.

Solution. Using conformal coordinates, Bonnet gave a complete solution to his problem in $\mathbb{R}^3$[7] (§11–12 pp 72–92). What we present here is the (easy to perform, see[10]) extrapolation to $\mathbb{R}^3(c)$ of his solution, using the method of Bonnet and the usual notation for the Gauss and Codazzi equations.

Since a surface in $\mathbb{R}^3(c)$ is characterized by $(\upsilon,\ H,\ Q,\ \overline{Q})$, the problem is equivalent to: given a solution $(\upsilon,\ H,\ |Q|^2)$ of the Gauss-Codazzi equations (16), to determine all the values of $e^{i\omega} = Q/|Q| = |Q|/\overline{Q}$.

By elimination of $|Q|$ between the two Codazzi equations $(16)_{2,3}$, the variable $e^{i\omega}$ obeys a second degree algebraic equation whose coefficients only depend on υ, H, $|Q|$[7] (§11 p. 75)

$$H_{\bar z}(\log\alpha)_{\bar z} e^{i\omega} + H_z(\log\beta)_z e^{-i\omega} - 4e^{-\upsilon}|Q|\upsilon_{z\bar z} + e^{\upsilon}|Q|^{-2}H_z H_{\bar z} = 0, \tag{B1}$$

$$\alpha = e^{\upsilon}|Q|^{-2}H_{\bar z},\ \ \beta = e^{\upsilon}|Q|^{-2}H_z. \tag{B2}$$

Therefore the discussion splits into three cases $(H_{\bar z}, H_z) = (0,0),\ (0,\neq 0),\ (\neq 0,\neq 0)$, the third case splits into two cases $(\alpha_{\bar z},\beta_z) = (0,0),\ (\neq 0,\neq 0)$, and finally the case $(H_{\bar z}, H_z, \alpha_{\bar z}, \beta_z) = (\neq 0, \neq 0, 0, 0)$ splits into two cases.

Totally, there are five solutions, summarized in Table I.

TABLE I. The different types of analytic surfaces which solve the Bonnet problem[7] (§11 and 12). The page numbers refer to Bonnet[7]. CMC is short for constant mean curvature surfaces.

	Characterization	Applicable surfaces	Comment	Pages
1	$H_{\bar z}=0, H_z=0$	CMC (two arb f + one PDE)	sine-Gordon or Liouville	76–78
2	$H_{\bar z}=0, H_z\neq 0$	one cone (two arb f)	not real[7]	78–81
3	$\substack{H_{\bar z}H_z\neq 0 \\ h'+h^2-c^2=0}$	Dual to minimal (two arb f)	not real[8]	82–84
4	$\substack{H_{\bar z}H_z\neq 0 \\ h'+h^2-c^2\neq 0}$	Bonnet surfaces (6-param)	Painlevé VI	84–85
5	$\alpha_{\bar z}\beta_z\neq 0$	One surface (Bonnet pair)	3 PDEs p 90	85–92

These five types of applicable surfaces sharing the same first fundamental form and mean curvature were obtained by Bonnet as follows.

1. Type 1 (constant mean curvature surfaces). Defined by

$$H_z = H_{\bar z} = 0, \tag{B3}$$

this solution is characterized by

$$\left\{\begin{array}{l} \upsilon_{z\bar z} + \dfrac{h^2-c^2}{2}e^{\upsilon} - 2g_1^2(z)g_2^2(\bar z)e^{-\upsilon} = 0, \\ H = h,\ Q = g_1^2(z),\ \overline{Q} = g_2^2(\bar z), \end{array}\right. \tag{B4}$$

in which h is the constant mean curvature and g_1, g_2 are two nonzero integration functions.

If $h^2 = c^2$, the Liouville equation for e^υ integrates as

$$h^2 = c^2 : \; e^\upsilon = -(g_1(z) g_2(\bar{z}))^2 \frac{(g_3(z) + g_4(\bar{z}))^2}{g_3'(z) g_4'(\bar{z})}, \tag{B5}$$

with g_3 and g_4 arbitrary, and the relations (B4) and (B5) are identical to the so-called Weierstrass representation of minimal surfaces (Weierstrass 1863),

$$H = c, \; Q = -\eta^2(z)\psi'(z), \; e^\upsilon = \left(1 + |\psi|^2\right)^2 |\eta|^4, \tag{B6}$$

with the correspondence

$$g_1^2 = -\eta^2\psi', \; g_2^2 = -\bar{\eta}^2\bar{\psi}', \; \frac{g_3}{\psi} = g_4\bar{\psi} = \text{ arbitrary constant.} \tag{B7}$$

If $h^2 \neq c^2$, a conformal transformation (17) with $G_1' = \lambda g_1$, $G_2' = \lambda^{-1} g_2$ and λ constant maps the PDE for υ to the sine-Gordon equation

$$\upsilon_{Z\bar{Z}} + \frac{h^2 - c^2}{2} e^\upsilon - 2e^{-\upsilon} = 0, \tag{B8}$$

and the reduced moving frame equations depend on λ which is then a spectral parameter.

2. Type 2 (single complex cone). Defined by

$$H_{\bar{z}} = 0, \; (e^\upsilon |Q|^{-2} H_z)_{\bar{z}} = 0, \; \mathrm{d}H \neq 0, \tag{B9}$$

this surface is characterized by

$$\begin{cases} e^\upsilon = 2i \dfrac{g_3'(z) g_4'(\bar{z})}{c \cos g_3(z)}, \; H = c \sin g_3(z), \\ Q = i(g_3'(z))^2 g_4(\bar{z}), \; \overline{Q} = -i(g_4'(z))^2 / g_4(\bar{z}), \end{cases} \tag{B10}$$

and it is not real[7] (p. 81).

3. Types 3 and 4. The defining relations

$$e^\upsilon |Q|^{-2} H_{\bar{z}} = g_1(z) \neq 0, \; e^\upsilon |Q|^{-2} H_z = g_2(\bar{z}) \neq 0, \tag{B11}$$

imply that H and $e^\upsilon |Q|^{-2}$ only depend on one variable, whose differential is $(1/2)(g_1 \mathrm{d}z + g_2 \mathrm{d}\bar{z})$. After the conformal transformation (17) with $G_1' = 2/g_1$, $G_2' = 2/g_2$ followed by the elimination of υ, the variables $H = h(\xi)$ and $Q(z, \bar{z})$ obey the coupled system

$$\begin{cases} (\log h')'' + 2h' - 8|Q|^2 \left(\dfrac{h' + h^2 - c^2}{h'} \right) = 0, \; \xi = \dfrac{z + \bar{z}}{2}, \\ (\log |Q|)_{z\bar{z}} - |Q|^2 = 0, \end{cases} \tag{B12}$$

therefore two subcases arise, depending on whether h obeys or not the Riccati ODE defined by canceling the coefficient of $|Q|^2$ in $(\mathrm{B12})_1$.

In both cases, the remaining equations are

$$e^v = 4\frac{|Q|^2}{h'(\xi)},\ Q_{\bar z} = \overline{Q}_z = |Q|^2. \tag{B13}$$

4. Type 3 (Surfaces dual to minimal surfaces). If h obeys the Riccati ODE, the general solution

$$\begin{cases} e^v = 4\dfrac{h_1'(z)h_2'(\bar z)}{(h_1(z)+h_2(\bar z))^2 c^2 \cosh^2 c\Re(z-z_0)},\ H = c\tanh c\Re(z-z_0), \\ Q = -\dfrac{h_1'(z)}{h_1(z)+h_2(\bar z)},\ \overline{Q} = -\dfrac{h_2'(\bar z)}{h_1(z)+h_2(\bar z)}, \end{cases} \tag{B14}$$

depends on one arbitrary constant $\Re(z_0)$ and two arbitrary functions of one variable. These analytic surfaces are not real[8] (p. 57), and at least when c is zero there exists a conformal transformation[46],[5] (Remark 4.3.1 p. 68) mapping them to minimal surfaces in $\mathbb{R}^3$.

5. Type 4 (Bonnet surfaces). They are characterized by

$$\begin{aligned} & e^v|Q|^{-2}H_{\bar z} = g_1(z) \neq 0,\ e^v|Q|^{-2}H_z = g_2(\bar z) \neq 0, \\ & \frac{2\mathrm{d}H}{g_1\mathrm{d}z + g_2\mathrm{d}\bar z} + H^2 - c^2 \neq 0. \end{aligned} \tag{B15}$$

Since $|Q|^2$ is defined by $(\mathrm{B12})_1$, it only depends on $\Re(z)$ and the equation $(\mathrm{B12})_2$ is an ODE for $|Q|$ which integrates as

$$|Q| = \varepsilon\frac{2c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}\Re(z-z_0)},\ \varepsilon^2 = 1, \tag{B16}$$

in which $\Re(z_0)$ and c_{z} are arbitrary, while the Codazzi equations (B13) integrate as

$$\begin{cases} Q = 2c_{\mathrm{z}}\coth 2c_{\mathrm{z}}(z-z_0) - 2c_{\mathrm{z}}\coth 4c_{\mathrm{z}}\Re(z-z_0) = \dfrac{\sinh 2c_{\mathrm{z}}(\bar z-\bar z_0)}{\sinh 2c_{\mathrm{z}}(z-z_0)}\dfrac{2c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}\Re(z-z_0)}, \\ \overline{Q} = 2c_{\mathrm{z}}\coth 2c_{\mathrm{z}}(\bar z-\bar z_0) - 2c_{\mathrm{z}}\coth 4c_{\mathrm{z}}\Re(z-z_0) = \dfrac{\sinh 2c_{\mathrm{z}}(z-z_0)}{\sinh 2c_{\mathrm{z}}(\bar z-\bar z_0)}\dfrac{2c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}\Re(z-z_0)}, \end{cases} \tag{B17}$$

with z_0 and $\bar z_0$ arbitrary. As a consequence, $\omega = \arg Q$ is characterized by the nice relation[7] (Eq. (53) p. 85),

$$\tan\frac{\omega}{2} = i\frac{\tanh c_{\mathrm{z}}(z-z_0-\varepsilon(\bar z-\bar z_0))}{\tanh c_{\mathrm{z}}(z-z_0+\varepsilon(\bar z-\bar z_0))},\ \varepsilon^2 = 1. \tag{B18}$$

As to $H(z,\bar z) = h(\xi)$, with $\xi = \Re(z)$, it obeys the third order ODE[7] (Eq. (52) p. 84),

$$(\log h')'' + 2h' - 2\left(\frac{4c_{\mathrm{z}}}{\sinh 4c_{\mathrm{z}}(\xi-\xi_0)}\right)^2\left(\frac{h'+h^2-c^2}{h'}\right) = 0, \tag{B19}$$

which admits the first integral ([22] (p. 48) for $c = 0$,[5] for arbitrary c)

$$K = \left(\frac{h''}{h'} + 8c_z \coth 4c_z(\xi - \xi_0) \right)^2 + 8 \left[\left(\frac{4c_z}{\sinh 4c_z(\xi - \xi_0)} \right)^2 \frac{h^2 - c^2}{h'} + h' + 8c_z \coth 4c_z(\xi - \xi_0) h \right]. \tag{B20}$$

The general solution[4] of this ODE (which Bonnet could not obtain for obvious chronological reasons) is a Hamiltonian of either a codimension-two P_{VI} ($c_z \neq 0$) or a codimension-three P_V ($c_z = 0$). This defines a family of analytic surfaces, called Bonnet surfaces, which, in addition to the fixed parameter c, depend on six arbitrary movable constants (the two origins of z and $\bar{z}$, the first integrals c_z and K, and the two constants of integration of the ODE (B20)). Their main property is to be applicable on a surface of revolution but to never be a surface of revolution[7] (Eq. (53) p. 85). The real surfaces defined by these analytic surfaces have been determined by É. Cartan[8], they require c_z^2 to be real and consist of three disjoint classes denoted A, B, C corresponding respectively to c_z^2 negative, positive, zero.

Remark 1. Bonnet surfaces are characterized by the local condition

$$|Q|^2 (\log |Q|)_{z\bar{z}} - g_1(z) g_2(\bar{z})/4 = 0, \tag{B21}$$

i.e., after elimination of g_1 and g_2, by the global conditions[10]

$$(\log Q)_{z\bar{z}} - (\log Q)_{\bar{z}} (\log \overline{Q})_z = 0, \ (\log \overline{Q})_{z\bar{z}} - (\log \overline{Q})_z (\log Q)_{\bar{z}} = 0. \tag{B22}$$

Remark 2. Since a Liouville PDE such as (B21) is equivalent to a d'Alembert PDE $\varphi_{z\bar{z}} = 0$ (i.e. φ harmonic), many geometers like to characterize Bonnet surfaces by the condition that some function (e.g. $Q - \overline{Q}$ in Eq. (B17), or $1/Q$ in[22]) be harmonic, but one should keep in mind that such a condition is only local.

6. Type 5 (Bonnet pairs). They are characterized by

$$(e^u |Q|^{-2} H_{\bar{z}})_{\bar{z}} \neq 0, \ (e^u |Q|^{-2} H_z)_z \neq 0. \tag{B23}$$

Since there are exactly two applicable surfaces, the proof of Bonnet can be simplified as follows[5] (§4.8.1). Denoting $Q_j, j = 1, 2$ the two solutions, the difference of the two sets of Gauss-Codazzi equations,

$$Q_1 \overline{Q}_1 - Q_2 \overline{Q}_2 = 0, \ (Q_1 - Q_2)_{\bar{z}} = 0, \ (\overline{Q}_1 - \overline{Q}_2)_z = 0, \tag{B24}$$

integrates as

$$Q_1 = \frac{q}{g_2} + g_1, \ Q_2 = \frac{q}{g_2} - g_1, \ \overline{Q}_1 = -\frac{q}{g_1} + g_2, \ \overline{Q}_2 = -\frac{q}{g_1} - g_2, \tag{B25}$$

in which $q(z,\bar{z})$, $g_1(z)$ and $g_2(\bar{z})$ are integration functions. Then, to the half-sum of the two sets of Gauss-Codazzi equations

$$\left\{\begin{array}{l} v_{z\bar{z}} + \dfrac{H^2-c^2}{2}e^v + 2\left(\dfrac{q^2}{g_1 g_2} - g_1 g_2\right)e^{-v} = 0, \\ \left(\dfrac{q}{g_2}\right)_{\bar{z}} - \dfrac{1}{2}e^v H_z = 0,\ \left(\dfrac{q}{-g_1}\right)_z - \dfrac{1}{2}e^v H_{\bar{z}} = 0, \end{array}\right. \tag{B26}$$

one applies the conformal transformation (17), completed by

$$\forall G(z):\ (z,q) \to \left(G(z), |G'(z)|^2 q\right). \tag{B27}$$

The system resulting from the choice $G'^2 = g_1$, $\overline{G}'^2 = g_2$[7] (p. 90)

$$\left\{\begin{array}{l} v_{z\bar{z}} + \dfrac{1}{2}(H^2-c^2)e^v - 2(1-q^2)e^{-v} = 0, \\ q_{\bar{z}} - \dfrac{1}{2}H_z e^v = 0,\ -q_z - \dfrac{1}{2}H_{\bar{z}}e^v = 0, \end{array}\right. \tag{B28}$$

is a particular Gauss-Codazzi system in $\mathbb{R}^3(c)$, with

$$Q = 1+q,\ \overline{Q} = 1-q. \tag{B29}$$

A member $Q = Q_j$ of a Bonnet pair is characterized by the local condition

$$\frac{Q}{g_1} + \frac{\overline{Q}}{g_2} + (-1)^j 2 = 0, \tag{B30}$$

i.e., after elimination of g_1 and g_2, by the two global conditions

$$\left(\frac{\left(\frac{\overline{Q}_z}{Q\overline{Q}}\right)_{\bar{z}}}{\left(\log\frac{Q}{\overline{Q}}\right)_{z\bar{z}}}\right)_{\bar{z}} = 0 \text{ and c.c..} \tag{B31}$$

This terminates the solution given by Bonnet to his problem.

The five types of surfaces which solve the Bonnet problem can also be characterized by the following global conditions which only involve Q and $\overline{Q}$,

$$\mathbb{R}^3(c):\ \left\{\begin{array}{l} 1.\ Q_{\bar{z}} = 0,\ \overline{Q}_z = 0, \\ 2.\ (\log Q)_{z\bar{z}} = 0,\ (\log\overline{Q})_{z\bar{z}} = 0, \\ 3.\ (Q_{\bar{z}}/|Q|^2)_z = 0,\ (\overline{Q}_z/|Q|^2)_{\bar{z}} = 0, \\ 4.\ (\log Q)_{z\bar{z}} - (\log Q)_{\bar{z}}(\log\overline{Q})_z = 0,\ (\log\overline{Q})_{z\bar{z}} - (\log\overline{Q})_z(\log Q)_{\bar{z}} = 0, \\ 5.\ \left(\dfrac{\left(\frac{\overline{Q}_z}{Q\overline{Q}}\right)_{\bar{z}}}{\left(\log\frac{Q}{\overline{Q}}\right)_{z\bar{z}}}\right)_{\bar{z}} = 0 \text{ and c.c..} \end{array}\right. \tag{B32}$$

Appendix C: Confluence to the lower Painlevé equations

Following Garnier[18], we define the lower $\mathrm{P_n}$ as four-parameter equations derived from $\mathrm{P_{VI}}(u,x,\alpha,\beta,\gamma,\delta)$ by the classical confluence of poles.

Definition in rational coordinates[18],

$$
\begin{aligned}
\mathrm{P_{VI}}\ &:\ u''=\frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]u'^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]u'\\
&\qquad +\frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{u^2}+\gamma\frac{x-1}{(u-1)^2}+\delta\frac{x(x-1)}{(u-x)^2}\right],\\
\mathrm{P_{V}}\ &:\ u''=\left[\frac{1}{2u}+\frac{1}{u-1}\right]u'^2-\frac{u'}{x}+\frac{(u-1)^2}{x^2}\left[\alpha u+\frac{\beta}{u}\right]+\gamma\frac{u}{x}+\delta\frac{u(u+1)}{u-1},\\
\mathrm{P_{III}}\ &:\ u''=\frac{u'^2}{u}-\frac{u'}{x}+\frac{\alpha u^2+\gamma u^3}{4x^2}+\frac{\beta}{4x}+\frac{\delta}{4u},\\
\mathrm{P_{IV}}'\ &:\ u''=\frac{u'^2}{2u}+\gamma\left(\frac{3}{2}u^3+4xu^2+2x^2u\right)+4\delta(u^2+xu)-2\alpha u+\frac{\beta}{u},\\
\mathrm{P_{II}}'\ &:\ u''=\delta(2u^3+xu)+\gamma(6u^2+x)+\beta u+\alpha,\\
\mathrm{J}\ &:\ u''=2\delta u^3+6\gamma u^2+\beta u+\alpha.
\end{aligned}
$$

The added equation J (like "Jacobi") is the autonomous limit of $\mathrm{P_{II}}'$, which is itself the synthesis of $\mathrm{P_{II}}$ and $\mathrm{P_{I}}$ made by Garnier.

Transformation between rational and elliptic or degenerate elliptic coordinates[2],

$$
\begin{aligned}
\mathrm{P_{VI}}\ &: x=\frac{e_3-e_1}{e_2-e_1},\ u=\frac{\wp(2\omega U,g_2,g_3)-e_1}{e_2-e_1},\ t=\frac{\wp(2\omega T,g_2,g_3)-e_1}{e_2-e_1},\\
\mathrm{P_{V}}\ &: x=e^{2X},\ u=\coth^2 U,\ t=\coth^2 T,\\
\mathrm{P_{III}}\ &: x=e^{2X},\ u=e^{X}e^{2U},\ t=e^{X}e^{2T},\\
\mathrm{P_{IV}}'\ &: x=X,\ u=U^2,\ t=T^2,\\
\mathrm{P_{II}}'\ &: x=X,\ u=U,\ t=T.
\end{aligned}
$$

$\mathrm{P_n}$ in elliptic or degenerate elliptic coordinates[2],

$$
\begin{aligned}
\mathrm{P_{VI}}\ &: \frac{\mathrm{d}^2U}{\mathrm{d}X^2}=\frac{(2\omega)^3}{\pi^2 a_{\mathrm{X}}^2}\sum_{j=\infty,0,1,x}\theta_j^2\wp'(2\omega U+\omega_j,g_2,g_3),\\
\mathrm{P_{V}}\ &: \frac{\mathrm{d}^2U}{\mathrm{d}X^2}=-2\alpha\frac{\cosh U}{\sinh^3 U}-2\beta\frac{\sinh U}{\cosh^3 U}-2\gamma e^{2X}\sinh(2U)-\frac{1}{2}\delta e^{4X}\sinh(4U),\\
\mathrm{P_{III}}\ &: \frac{\mathrm{d}^2U}{\mathrm{d}X^2}=\frac{1}{2}e^{X}(\alpha e^{2U}+\beta e^{-2U})+\frac{1}{2}e^{2X}(\gamma e^{4U}+\delta e^{-4U}),\\
\mathrm{P_{IV}}'\ &: \frac{\mathrm{d}^2U}{\mathrm{d}X^2}=-\alpha U+\frac{\beta}{2U^3}+\gamma\left(\frac{3}{4}U^5+2XU^3+X^2U\right)+2\delta(U^3+XU),\\
\mathrm{P_{II}}'\ &: \frac{\mathrm{d}^2U}{\mathrm{d}X^2}=\delta(2U^3+XU)+\gamma(6U^2+X)+\beta U+\alpha.
\end{aligned}
$$

1. Matrix Lax pairs holomorphic in the four parameters

The confluence preserves the unique zero $t=u$ of M_{12} and the invertibility of M_∞ under one nonvanishing condition.

$\mathrm{P_{VI}}$ see (40).

$$\mathrm{P_V}\begin{cases} L=-\dfrac{M_1}{t-1}-\dfrac{u-1}{x}M_\infty,\ M=-\dfrac{M_\infty}{t-1}+\dfrac{xM_1}{(t-1)^2}+\left(\dfrac{1}{t}-\dfrac{1}{t-1}\right)M_0,\\ M_\infty=\dfrac{1}{2}\begin{pmatrix}0&-2\\-\alpha&0\end{pmatrix},\\ M_0=\dfrac{1}{4u(u-1)^2}\begin{pmatrix}-2u(u-1)r_5 & 4u^2(u-1)^2\\ -r_5^2-2\beta(u-1)^2 & 2u(u-1)r_5\end{pmatrix},\\ M_1=2d\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\dfrac{dr_5+(\gamma-d)(u-1)}{2(u-1)^2}\begin{pmatrix}0&0\\1&0\end{pmatrix},\\ r_5=x(u'-du),\\ -4\det M_\infty=2\alpha,\ -4\det M_0=-2\beta,\ -4\det M_1=-2\delta=d^2;\end{cases} \tag{C1}$$

$$\mathrm{P_{III}}\begin{cases} L=-\dfrac{L_0}{t}-\dfrac{u}{x}M_\infty,\ M=\dfrac{xL_0}{t^2}+\dfrac{M_0}{t}-M_\infty,\ M_\infty=\dfrac{1}{8}\begin{pmatrix}0&-4\\-\gamma&0\end{pmatrix},\\ M_0=\dfrac{2r_3}{4u}\begin{pmatrix}1&0\\0&-1\end{pmatrix}-\dfrac{u}{2}\begin{pmatrix}0&1\\0&0\end{pmatrix}+\dfrac{\gamma u+2\alpha}{8}\begin{pmatrix}0&0\\1&0\end{pmatrix},\\ L_0=-\dfrac{d}{4}\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\dfrac{-2dr_3+(\beta+2d)u}{4u^2}\begin{pmatrix}0&0\\1&0\end{pmatrix},\\ r_3=xu'+\dfrac{d}{2}x,\ -4\det M_\infty=\dfrac{\gamma}{4},\ -4\det L_0=-\dfrac{\delta}{2}=\dfrac{d^2}{4};\end{cases} \tag{C2}$$

$$\mathrm{P_{IV}}'\begin{cases} L=2(t+u)M_\infty+2M_0,\ M=tM_\infty+M_0+\dfrac{M_{-1}}{t},\ M_\infty=\dfrac{1}{4}\begin{pmatrix}-c&0\\2\delta&c\end{pmatrix},\\ M_0=\dfrac{1}{4}\begin{pmatrix}-2cx & 2\\ cr_4+2\delta(u+2x)-2(\alpha+c) & 2cx\end{pmatrix},\ M_{-1}=\dfrac{1}{8u}\begin{pmatrix}2ur_4 & -4u^2\\ r_4^2+2\beta & -2ur_4\end{pmatrix},\\ r_4=u'+cu^2+2cxu,\ -4\det M_\infty=\dfrac{\gamma}{4}=\dfrac{c^2}{4},\ -4\det M_{-1}=\dfrac{\beta}{2};\end{cases} \tag{C3}$$

$$\mathrm{P_{II}}'\begin{cases} L=\dfrac{t+u}{2}M_\infty+\dfrac{M_1}{2},\ M=t^2M_\infty+tM_1+M_0,\\ M_\infty=\begin{pmatrix}-d&0\\2\gamma&d\end{pmatrix},\ M_1=\begin{pmatrix}0&2\\ dr_2+\frac{\beta}{2}+2\gamma u & 0\end{pmatrix},\\ M_0=\begin{pmatrix}u'+du^2 & -2u\\ d(2ur_2+1/2)+\alpha+\beta u/2+\gamma(2u^2+x) & 2-(u'+du^2)\end{pmatrix},\\ r_2=u'+d\left(u^2+\dfrac{x}{2}\right),\ -4\det M_\infty=4\delta=4d^2.\end{cases} \tag{C4}$$

For $\mathrm{P_V}$, the two Fuchsian singularities of M are naturally put at $t=\infty$ and $t=0$ by the confluence. The choice $t=0,1$ for their location made in Ref.[24] ((C.38)) breaks the symmetry between t and u, resulting in distorted values of the invariants $\det M_j$.

2. Matrix Lax pairs symmetric with respect to the diagonal

They are generated from (42) by the confluence. Alternatively, they are obtained from those in section C 1 by action of the transition matrix P displayed in the first line of each entry. They have a meromorphic dependence in one of the four parameters. From $\mathrm{P_{IV}}'$ down, M_∞ is no more diagonal and all elements are rational.

$\mathrm{P_{VI}}$ see (42).

$$\mathrm{P_V}\begin{cases} \alpha=\dfrac{\theta_\infty^2}{2}\neq 0:\ P=\begin{pmatrix} 2 & 2 \\ a-\theta_\infty & a+\theta_\infty\end{pmatrix}\begin{pmatrix} g^{-1/2} & 0 \\ 0 & g^{1/2}\end{pmatrix}, \\ L=-\dfrac{M_1}{t-1},\ M=-\dfrac{M_\infty}{t-1}+\dfrac{xM_1}{(t-1)^2}+\left(\dfrac{1}{t}-\dfrac{1}{t-1}\right)M_0, \\ M_\infty=\dfrac{\theta_\infty}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \\ M_0=\dfrac{r_5^2+2\beta u^2(u-1)^2}{4\theta_\infty u(u-1)^2}\begin{pmatrix} 1 & g \\ -1/g & -1\end{pmatrix}-\dfrac{r_5}{2(u-1)}\begin{pmatrix} 0 & g \\ 1/g & 0\end{pmatrix} \\ \qquad +\dfrac{\theta_\infty u}{4}\begin{pmatrix} -1 & g \\ -1/g & 1\end{pmatrix}, \\ M_1=\dfrac{1}{2\theta_\infty(u-1)^2}\left[-dr_5-(\gamma-d)(u-1)\right]\begin{pmatrix} 1 & -g \\ 1/g & -1\end{pmatrix}+\dfrac{d}{2}\begin{pmatrix} 0 & g \\ 1/g & 0\end{pmatrix}, \\ r_5=x(u'-du),\ \dfrac{g'}{g}=\theta_\infty\dfrac{u-1}{x},\ \delta=-\dfrac{d^2}{2}, \\ -4\det M_\infty=2\alpha=\theta_\infty^2,\ -4\det M_0=-2\beta,\ -4\det M_1=d^2; \end{cases} \tag{C5}$$

$$\mathrm{P_{III}}\begin{cases} \gamma=c^2\neq 0:\ P=\begin{pmatrix} 2 & 2 \\ -c & c\end{pmatrix}\begin{pmatrix} g^{-1/2} & 0 \\ 0 & g^{1/2}\end{pmatrix}, \\ L=-\dfrac{L_0}{t},\ M=\dfrac{xL_0}{t^2}+\dfrac{M_0}{t}-M_\infty,\ M_\infty=\dfrac{c}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}, \\ L_0=\dfrac{2dr_3-(\beta+2d)u}{4cu^2}\begin{pmatrix} 1 & g \\ -1/g & -1\end{pmatrix}-\dfrac{d}{4}\begin{pmatrix} 0 & g \\ 1/g & 0\end{pmatrix}, \\ M_0=\dfrac{r_3}{2u}\begin{pmatrix} 0 & g \\ 1/g & 0\end{pmatrix}-\dfrac{\alpha}{4c}\begin{pmatrix} 1 & g \\ -1/g & -1\end{pmatrix}-\dfrac{cu}{4}\begin{pmatrix} 0 & g \\ -1/g & 0\end{pmatrix}, \\ r_3=xu'+\dfrac{d}{2}x,\ \dfrac{g'}{g}=c\dfrac{u}{2x},\ \gamma=c^2,\ \delta=-d^2, \\ -4\det M_\infty=c^2/4,\ -4\det L_0=d^2/4; \end{cases} \tag{C6}$$

$$\mathrm{P_{IV}}'\begin{cases} \gamma=c^2\neq 0:\ P=\begin{pmatrix} 2 & 2 \\ c & -c\end{pmatrix},\ L=2(t+u)M_\infty+2M_0,\ M=tM_\infty+M_0+\dfrac{M_{-1}}{t}, \\ M_\infty=\dfrac{1}{4c}\begin{pmatrix} 2\delta & 2\delta-c^2 \\ -2\delta-c^2 & -2\delta\end{pmatrix}, \\ M_0=\dfrac{cr_4+2\delta(u+2x)-2\alpha+2c}{4c}\begin{pmatrix} 1 & 1 \\ -1 & -1\end{pmatrix}-\dfrac{cx}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix},\ +\dfrac{c}{8}\begin{pmatrix} 1 & -1 \\ 1 & -1\end{pmatrix} \\ M_{-1}=\dfrac{r_4^2+2\beta}{8cu}\begin{pmatrix} 1 & 1 \\ -1 & -1\end{pmatrix}+\dfrac{r_4}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}-\dfrac{cu}{8}\begin{pmatrix} 1 & -1 \\ 1 & -1\end{pmatrix}, \\ r_4=u'+cu^2+2cxu,\ \gamma=c^2,\ -4\det M_\infty=\dfrac{c^2}{4},\ -4\det M_{-1}=-\dfrac{\beta}{2}; \end{cases} \tag{C7}$$

$$
\mathrm{P_{II}}' \begin{cases}
\delta = d^2 \neq 0: \ P = \begin{pmatrix} 2 & 2 \\ d & -d \end{pmatrix}, \ L = \dfrac{t+u}{2} M_\infty + \dfrac{M_1}{2}, \ M = t^2 M_\infty + t M_1 + M_0, \\
M_\infty = \dfrac{1}{d} \begin{pmatrix} 2\gamma & 2\gamma - d^2 \\ -2\gamma - d^2 & -2\gamma \end{pmatrix}, \\
M_1 = -\dfrac{d}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \dfrac{2dr_2 + 4\gamma u + \beta}{2d} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \\
M_0 = -\dfrac{du}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \dfrac{2dur_2 + 4\gamma u^2 + \beta u + 2\gamma x + 2\alpha + d}{2d} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\
\qquad + \dfrac{2r_2 - dx}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
r_2 = u' + d\left(u^2 + \dfrac{x}{2}\right), \ -4 \det M_\infty = 4\delta = 4d^2.
\end{cases} \tag{C8}
$$

3. Quantum correspondence

First step. The classical Hamiltonians are generated by the confluence acting on (70). As explained in the text, these Hamiltonians for $\mathrm{P_{VI}}$, $\mathrm{P_V}$ and $\mathrm{P_{III}}$ are different from those in Ref.[42], they only coincide at the $\mathrm{P_{IV}}'$ and $\mathrm{P_{II}}'$ levels because of the absence of a p term.

Second step. One defines the scalar Lax pairs (62) of the four-parameter $\mathrm{P_n}$'s by their two coefficients (S, C), see Ref.[18] (pp. 49, 52), reproduced in Ref.[15] (p. 211), and one changes the scalar wave vector from ψ_d to ψ_h.

Third step. The quantum Hamiltonians are defined from the classical ones as the confluence of the quantization rule (72). If one chooses the normalization constant a_T adequately for each $\mathrm{P_n}$, the quantization rule (72) is the same for every $\mathrm{P_n}$,

$$
\forall \mathrm{P_n}, \ \forall f(q,x), \ \forall k = 1,2: \ f(q,x) p^k \psi_\mathrm{h} \to f(t,x) a_\mathrm{T}^{-k} \partial_t^k \psi_\mathrm{h}. \tag{C9}
$$

The result is as follows.

Equations (62) (scalar Lax pair), (C9) (quantization) and $q = u$ are common to all $\mathrm{P_n}$.

$$
\mathrm{P_V} \begin{cases}
V(z) = \alpha z + \dfrac{\beta}{2} z^2 + \gamma(2z^3 + zx) + \dfrac{\delta}{2}(z^4 + z^2 x), \\
\dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \left[\dfrac{1}{2u} + \dfrac{1}{u-1}\right] u'^2 + \dfrac{u'}{x} - \dfrac{u(u-1)^2}{x} \dfrac{\partial V(u)}{\partial u} = 0, \\
S = -\dfrac{3}{2(t-u)^2} + 2\dfrac{xu' + (u-1)(2u-1)}{(u-1)t(t-1)(t-u)} - \dfrac{1}{2tu}\left(\dfrac{xu'}{(t-1)(u-1)}\right)^2 \\
\qquad + \dfrac{x}{t(t-1)^2}(V(u) - V(t)) + \dfrac{1}{2}\left(\dfrac{2t-1}{t(t-1)}\right)^2, \ C = -\dfrac{t(t-1)(u-1)}{x(t-u)}, \\
H(q,p,x,\alpha,\beta,\gamma,\delta) = \dfrac{q(q-1)}{x}((q-1)a_\mathrm{T} p^2 + p) - \dfrac{1}{2a_\mathrm{T}} V(q), \\
\qquad p = \dfrac{1}{2a_\mathrm{T}}\left(\dfrac{xu'}{u(u-1)^2} - \dfrac{1}{u-1}\right), \\
g_\mathrm{d} - g_\mathrm{h} = -a_\mathrm{T} H(q,p,x,\alpha,\beta,\gamma,\delta) + \dfrac{xu' - (u-1)}{2x(u-1)}, \ \psi_\mathrm{d} = \psi_\mathrm{h}(t-u)^{-1/2}(t-1), \\
\left[\partial_x + g_\mathrm{h}(x) - a_\mathrm{T} H(t, \partial_t, x, \alpha + 1/2, \beta + 1/2, \gamma, \delta)\right] \psi_\mathrm{h} = 0, \\
\left[\partial_x + g_\mathrm{h}(x) - a_\mathrm{T} H(q,p,x,\alpha,\beta,\gamma,\delta) + C\partial_t + \dfrac{(xu' - (u-1))(t-1)}{2x(u-1)(t-u)}\right] \psi_\mathrm{h} = 0,
\end{cases} \tag{C10}
$$

$$
\mathrm{P_{III}} \begin{cases}
V(z) = \frac{1}{16}\left(2\alpha\frac{z}{x} - 2\frac{\beta}{z} + \gamma\frac{z^2}{x} - \delta\frac{x}{z^2}\right), \; \left(\frac{x\mathrm{d}u}{u\mathrm{d}x}\right)' - 2u\frac{\partial V(u)}{\partial u} = 0, \\
S = -\frac{3}{2(t-u)^2} + 2\frac{xu'+u}{tu(t-u)} - \frac{1}{2}\left(\frac{xu'}{tu}\right)^2 + \frac{2x}{t^2}\left(V(u) - V(t)\right), \; C = -\frac{tu}{x(t-u)}, \\
H(q,p,x,\alpha,\beta,\gamma,\delta) = \frac{q^2}{x}a_{\mathrm{T}}p^2 + \frac{q}{x}p - \frac{1}{a_{\mathrm{T}}}V(q) + \frac{1}{4a_{\mathrm{T}}x}, \; p = \frac{1}{2a_{\mathrm{T}}}\left(\frac{xu'}{u^2} - \frac{1}{u}\right), \\
g_{\mathrm{d}} - g_{\mathrm{h}} = -a_{\mathrm{T}}H(q,p,x,\alpha,\beta,\gamma,\delta) + \frac{u'}{2u}, \; \psi_{\mathrm{d}} = \psi_{\mathrm{h}}(t-u)^{-1/2}tx^{-1/2}, \\
\left[\partial_x + g_{\mathrm{h}}(x) - a_{\mathrm{T}}H(t,\partial_t,x,\alpha,\beta,\gamma,\delta)\right]\psi_{\mathrm{h}} = 0, \\
\left[\partial_x + g_{\mathrm{h}}(x) - a_{\mathrm{T}}H(q,p,x,\alpha,\beta,\gamma,\delta) + C\partial_t + \frac{(xu'-u)t}{2xu(t-u)}\right]\psi_{\mathrm{h}} = 0,
\end{cases} \tag{C11}
$$

$$
\mathrm{P_{IV}}' \begin{cases}
V(z) = -2\alpha z - \frac{\beta}{z} + \gamma(\frac{z^3}{2} + 2xz^2 + 2x^2 z) + \delta(2z^2 + 4xz), \\
\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{u'^2}{2u} - u\frac{\partial V(u)}{\partial u} = 0, \\
S = -\frac{3}{2(t-u)^2} + 2\frac{u'+2}{2t(t-u)} - \frac{u'^2}{8tu} + \frac{1}{2t^2} + \frac{1}{4t}\left(V(u) - V(t)\right), \; C = -\frac{2t}{t-u}, \\
H(q,p,x,\alpha,\beta,\gamma,\delta) = qa_{\mathrm{T}}p^2 - \frac{1}{4a_{\mathrm{T}}}V(q), \; p = \frac{u'}{4a_{\mathrm{T}}u}, \\
g_{\mathrm{d}} - g_{\mathrm{h}} = -a_{\mathrm{T}}H(q,p,x,\alpha,\beta,\gamma,\delta), \; \psi_{\mathrm{d}} = \psi_{\mathrm{h}}(t-u)^{-1/2}, \\
\left[\partial_x + g_{\mathrm{h}}(x) - a_{\mathrm{T}}H(t,\partial_t,x,\alpha,\beta,\gamma,\delta)\right]\psi_{\mathrm{h}} = 0, \\
\left[\partial_x + g_{\mathrm{h}}(x) - a_{\mathrm{T}}H(q,p,x,\alpha,\beta,\gamma,\delta) + C\partial_t + \frac{u'+2}{2(t-u)}\right]\psi_{\mathrm{h}} = 0,
\end{cases} \tag{C12}
$$

$$
\mathrm{P_{II}}' \begin{cases}
V(z) = \alpha z + \frac{\beta}{2}z^2 + \gamma(2z^3 + zx) + \frac{\delta}{2}(z^4 + z^2 x), \; \frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\partial V(u)}{\partial u} = 0, \\
S = -\frac{3}{2(t-u)^2} + 2\frac{u'}{t-u} - 2u'^2 + 4V(u) - 4V(t), \; C = -\frac{1}{2(t-u)}, \\
H(q,p,x,\alpha,\beta,\gamma,\delta) = a_{\mathrm{T}}\frac{p^2}{2} - \frac{1}{a_{\mathrm{T}}}V(q), \; p = \frac{u'}{a_{\mathrm{T}}}, \\
g_{\mathrm{d}} - g_{\mathrm{h}} = -a_{\mathrm{T}}H(q,p,x,\alpha,\beta,\gamma,\delta), \; \psi_{\mathrm{d}} = \psi_{\mathrm{h}}(t-u)^{-1/2}, \\
\left[\partial_x + g_{\mathrm{h}}(x) - a_{\mathrm{T}}H(t,\partial_t,x,\alpha,\beta,\gamma,\delta)\right]\psi_{\mathrm{h}} = 0, \\
\left[\partial_x + g_{\mathrm{h}}(x) - a_{\mathrm{T}}H(q,p,x,\alpha,\beta,\gamma,\delta) + C\partial_t + \frac{u'}{2(t-u)}\right]\psi_{\mathrm{h}} = 0.
\end{cases} \tag{C13}
$$

4. Generalized heat equations and associated Lax pairs

The confluence of the relations (73) yields for each $\mathrm{P_n}$ a scalar Lax pair made of a generalized heat equation and a first order PDE.

If one denotes $H_{\mathrm{n}}(q,p,x,\alpha,\beta,\gamma,\delta)$ the above classical Hamiltonians, the generalized heat equations are (we omit the $g_{\mathrm{h}}(x)$ terms),

$$
\forall \mathrm{P_n}: \; \left[\partial_x - a_{\mathrm{T}}H_{\mathrm{n}}(t,\partial_t,x,\alpha+s_\alpha,\beta+s_\beta,\gamma+s_\gamma,\delta+s_\delta)\right]\psi_{\mathrm{h}} = 0, \tag{C14}
$$

in which the shifts s_* of the parameters are nonzero only for $\mathrm{P_{VI}}$ and $\mathrm{P_V}$,

$$(s_\alpha, s_\beta, s_\gamma, s_\delta) = \begin{cases} (1/2, 1/2, -1/2, 1/2), \ \mathrm{P_{VI}} \\ (1/2, 1/2, 0, 0), \ \mathrm{P_V} \\ (0, 0, 0, 0), \ \mathrm{P_{III}}, \mathrm{P_{IV}}', \mathrm{P_{II}}'. \end{cases} \tag{C15}$$

The second half of the Lax pairs is as follows,

$$\mathrm{P_V} \begin{cases} \Big[\partial_x - \dfrac{t(t-1)(u-1)}{x(t-u)}\partial_t + \dfrac{xu' - u + 1}{2x(t-u)} - \dfrac{xu'^2}{4u(u-1)^2} + \dfrac{xu'}{2x(u-1)} \\ \quad + \alpha\left(\dfrac{u}{2x} - \dfrac{1}{4x}\right) + \beta\left(\dfrac{1}{4x} - \dfrac{1}{2xu}\right) + \gamma\left(\dfrac{1}{4} - \dfrac{u}{2(u-1)}\right) - \delta\dfrac{xu}{2(u-1)^2}\Big]\psi_\mathrm{h} = 0, \end{cases} \tag{C16}$$

$$\mathrm{P_{III}} \left[\partial_x - \frac{tu}{x(t-u)}\partial_t + \frac{xu' - u}{2(t-u)} - \frac{xu'^2}{4u^2} + \frac{u'}{2u} - \frac{1}{2x} + \frac{1}{8}\left(\alpha\frac{u}{x} - \frac{\beta}{u} + \gamma\frac{u^2}{2x} - \delta\frac{x}{2u^2}\right)\right]\psi_\mathrm{h} = 0, \tag{C17}$$

$$\mathrm{P_{IV}}' \begin{cases} \Big[\partial_x - \dfrac{2t}{t-u}\partial_t + \dfrac{u' + 2}{2(t-u)} - \dfrac{u'^2}{8u} \\ \quad - \dfrac{1}{4}\left(2\alpha u + \dfrac{\beta}{u} - \gamma\dfrac{u^3 + 4xu^2 + 4x^2u}{2} - \delta(2u^2 + 4xu)\right)\Big]\psi_\mathrm{h} = 0, \end{cases} \tag{C18}$$

$$\mathrm{P_{II}}' \left[\partial_x - \frac{2}{t-u}\partial_t + \frac{u'}{2(t-u)} - \frac{u'^2}{2} + \alpha u + \beta\frac{u^2}{2} + \gamma(2u^3 + xu) + \delta\frac{u^4 + xu^2}{2}\right]\psi_\mathrm{h} = 0. \tag{C19}$$

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[50]The Codazzi equations were in fact first written in 1853 by Karl M. Peterson, a Latvian student, before Mainardi (1856) and Codazzi (1868), see the historical notes by Phillips[35].
[51]Erratum. In formula 18.6.23 of Abramowitz and Stegun[1], the last g_2 should be g_3.
[52]This means that both surfaces have the same first fundamental form.