

# Counterexample to the Vector Generalization of Costa's EPI, and Partial Resolution

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**Abstract**—We give a counterexample to the vector generalization of Costa's entropy power inequality (EPI) due to Liu, Liu, Poor and Shamai. In particular, the claimed inequality can fail if the matrix-valued parameter in the convex combination does not commute with the covariance of the additive Gaussian noise. Conversely, the inequality holds if these two matrices commute.

**Index Terms**—Entropy power, Costa's EPI

## I. INTRODUCTION AND MAIN RESULT

For a random vector  $X$  with density on  $\mathbb{R}^n$ , let  $h(X)$  denote its differential entropy. Let  $Z \sim N(0, \Sigma_Z)$  be a Gaussian vector in  $\mathbb{R}^n$  independent of  $X$ , and let  $A$  be a (real symmetric) positive semidefinite  $n \times n$  matrix satisfying  $A \preceq I$  with respect to the positive semidefinite ordering, where  $I$  denotes the identity matrix. In [1, Theorem 1], Liu, Liu, Poor and Shamai claim the following generalization of Costa's EPI<sup>1</sup> [2]:

$$e^{\frac{2}{n}h(X+A^{1/2}Z)} \geq |I - A|^{1/n} e^{\frac{2}{n}h(X)} + |A|^{1/n} e^{\frac{2}{n}h(X+Z)}. \quad (1)$$

Liu et al. apply (1) in their investigation of the secrecy capacity region of the degraded vector Gaussian broadcast channel with layered confidential messages.

The purpose of this note is to demonstrate that (1) can fail for  $n \geq 2$ , and also to offer a partial resolution. Toward the first goal, consider  $n = 2$  and let us define

$$\Sigma_X = \begin{pmatrix} 200 & 100 \\ 100 & 51 \end{pmatrix}, \Sigma_Z = \begin{pmatrix} 200 & 0 \\ 0 & 1 \end{pmatrix}, A^{1/2} = \frac{1}{20} \begin{pmatrix} 10 & 5 \\ 5 & 17 \end{pmatrix}. \quad (2)$$

Taking  $X \sim N(0, \Sigma_X)$  and  $Z \sim N(0, \Sigma_Z)$  to be independent Gaussian vectors, we have

$$\frac{1}{2\pi e} e^{\frac{2}{n}h(X+A^{1/2}Z)} = |\Sigma_X + A^{1/2}\Sigma_Z A^{1/2}|^{1/2} \approx 19.53.$$

On the other hand,

$$\begin{aligned} & \frac{1}{2\pi e} \left( |I - A|^{1/n} e^{\frac{2}{n}h(X)} + |A|^{1/n} e^{\frac{2}{n}h(X+Z)} \right) \\ &= |I - A|^{1/2} |\Sigma_X|^{1/2} + |A|^{1/2} |\Sigma_X + \Sigma_Z|^{1/2} \approx 40.28. \end{aligned}$$

Thus, a contradiction to (1) is obtained. We remark that there is nothing particularly unique about this counterexample, except that the matrices were chosen to violate (1) by a significant margin.

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<sup>1</sup>Entropies are taken to be base  $e$  throughout. For a positive semidefinite matrix  $M$ , we write  $M^{1/2}$  to denote the unique positive semidefinite matrix such  $M = M^{1/2}M^{1/2}$ .

Evidently, further assumptions are needed in order for (1) to hold. To give a simple resolution, we note that it suffices for the matrices  $A$  and  $\Sigma_Z$  to commute.

**Theorem 1.** *Let  $X$  be a random vector with density on  $\mathbb{R}^n$  whose entropy exists in the usual Lebesgue sense, and let  $Z \sim N(0, \Sigma_Z)$  be a Gaussian vector in  $\mathbb{R}^n$  independent of  $X$ . If  $A \preceq I$  is positive semidefinite and commutes with  $\Sigma_Z$ , then*

$$e^{\frac{2}{n}h(X+A^{1/2}Z)} \geq |I - A|^{1/n} e^{\frac{2}{n}h(X)} + |A|^{1/n} e^{\frac{2}{n}h(X+Z)}.$$

*Proof.* For brevity, we refer the reader to the original proof of Theorem 1 in [1], and only point out where the argument needs to be corrected. To this end, Liu et al.'s proof contains an incorrect application of the AM-GM inequality in the form [1, Eq. (28)]:

$$\begin{aligned} & |\Sigma_Z^{-1} \text{Cov}(Z|D_\gamma X + Z)(I - D_\gamma^{-2})|^{1/n} \\ & \leq \frac{1}{n} \text{Tr}(\Sigma_Z^{-1} \text{Cov}(Z|D_\gamma X + Z)(I - D_\gamma^{-2})), \quad (3) \end{aligned}$$

where  $D_\gamma := (I + \gamma(A - I))^{1/2}$ , and  $\gamma \in [0, 1]$  parameterizes a path of perturbation. Indeed, a product of positive semidefinite matrices is not necessarily positive semidefinite, which can lead to failure of the AM-GM inequality in the form (3). For example, returning to the counterexample above where  $X \sim N(0, \Sigma_X)$  and matrices are chosen according to (2), the eigenvalues of  $\Sigma_Z^{-1} \text{Cov}(Z|D_\gamma X + Z)(I - D_\gamma^{-2})$  can be approximately computed as  $\{-0.0053, -0.7273\}$  for  $\gamma = 0.5$ , in violation of (3).

However, if  $A$  and  $\Sigma_Z$  commute, then so do  $\Sigma_Z^{-1/2}$  and  $(I - D_\gamma^{-2})^{1/2}$  since real symmetric matrices commute if and only if they are simultaneously diagonalizable by some orthogonal matrix  $U$ . Hence,

$$\begin{aligned} & \text{Tr}(\Sigma_Z^{-1} \text{Cov}(Z|D_\gamma X + Z)(I - D_\gamma^{-2})) \\ &= \text{Tr}((I - D_\gamma^{-2})^{1/2} \Sigma_Z^{-1/2} \text{Cov}(Z|D_\gamma X + Z) \Sigma_Z^{-1/2} (I - D_\gamma^{-2})^{1/2}). \end{aligned}$$

The argument of the second trace term is clearly positive semidefinite, and therefore (3) holds for all  $\gamma \in [0, 1]$  under the additional assumption that  $A$  and  $\Sigma_Z$  commute, thereby repairing Liu et al.'s proof.  $\square$

## II. REMARKS

The critical application of inequality (1) by Liu et al. (see [1, p. 1877]) assumes only that  $A$  and  $\Sigma_Z$  are diagonal matrices, so the conclusions of [1] appear to be unaffected, aside from [1, Theorem 1]. In fact, Ekrem and Ulukus [3] obtained the same secrecy capacity results using a different argument. Nevertheless, reference [1] has been cited numerous times

in the literature, so other published results may be affected to varying degrees. As one example, a computation in [4] similarly overlooks non-commutativity of the matrices  $A$  and  $\Sigma_Z$ , leading to the incorrect conclusion that [1, Theorem 1] is a corollary of [4, Theorem 3]. Despite the error, the validity of [4, Theorem 3] is not impacted, and the corrected computation, showing that [4, Theorem 3] implies Theorem 1 above, can be found in [5, Section III.A]. Other works may be more seriously affected, but we do not attempt to give an accounting of consequences here.

In closing, we remark that the additional assumption that  $A$  and  $\Sigma_Z$  commute is a relatively strong one. It can easily be seen, using the simultaneous diagonalization property of  $A$  and  $\Sigma_Z$  by a common orthogonal matrix, that Theorem 1 has a completely equivalent statement where  $Z \sim N(0, I)$  and  $A$  is restricted to be a diagonal matrix with diagonal entries  $0 \leq a_i \leq 1$ ,  $i = 1, \dots, n$ . In this sense, Theorem 1 should be viewed as a modest extension of Costa's original 1985 result which assumed that the parameters  $a_i$  were identical for all  $i$ . Indeed, as pointed out to the authors by an anonymous referee, a standard information-theoretic argument may be used to establish Theorem 1 as a corollary of Costa's original inequality, without appealing to the perturbation framework used in [1]. This argument has been included in the appendix.

#### REFERENCES

- [1] R. Liu, T. Liu, H.V. Poor, and S. Shamai. A vector generalization of Costa's entropy-power inequality with applications. *Information Theory, IEEE Transactions on*, 56(4):1865–1879, April 2010.
- [2] M. Costa. A new entropy power inequality. *Information Theory, IEEE Transactions on*, 31(6): pp. 751–760, November 1985.
- [3] E. Ekrem and S. Ulukus. The secrecy capacity region of the Gaussian MIMO multi-receiver wiretap channel. *Information Theory, IEEE Transactions on*, 57(4): 2083–2114, April 2011.
- [4] T. A. Courtade. Strengthening the entropy power inequality. In *Information Theory (ISIT), 2016 IEEE International Symposium on*, pages 2294–2298. IEEE, 2016.
- [5] T. A. Courtade. A strong entropy power inequality. In *Information Theory, IEEE Transactions on*, 64(4): 2173–2192, April 2018.

#### APPENDIX

What follows is a proof of Theorem 1 using Costa's entropy power inequality [2]. In particular, we shall prove Theorem 1 in the equivalent setting noted in the closing statement above, where  $Z \sim N(0, I)$  and  $A$  is a diagonal matrix with diagonal entries  $0 \leq a_i \leq 1$ ,  $i = 1, \dots, n$ . The argument below was provided by an anonymous referee.

Toward this end, for a random vector  $X$  with density on  $\mathbb{R}^n$ , define random vectors  $Y_1, Y_2$  via

$$Y_1 := X + A^{1/2}Z_1, \text{ and } Y_2 := Y_1 + (I - A)^{1/2}Z_2,$$

where  $Z_1, Z_2$  are independent copies of  $Z \sim N(0, I)$ , also independent of  $X$ . Since  $A$  is assumed diagonal with entries  $a_1, \dots, a_n$ , Theorem 1 may thus be written as

$$\prod_{i=1}^n (1 - a_i)^{1/n} e^{\frac{2}{n}(h(X) - h(Y_1))} + \prod_{i=1}^n a_i^{1/n} e^{\frac{2}{n}(h(Y_2) - h(Y_1))} \leq 1. \quad (4)$$

We consider the exponential terms separately. By the Csiszár sum identity,

$$\begin{aligned} h(X) - h(Y_1) &= \sum_{i=1}^n (h(X_i | X^{i-1}, Y_{1,i+1}^n) - h(Y_{1,i} | X^{i-1}, Y_{1,i+1}^n)) \\ &= \sum_{i=1}^n (h(X_i | V_i) - h(Y_{1,i} | V_i)), \end{aligned}$$

where  $V_i := (X^{i-1}, Y_{1,i+1}^n)$ . Similarly,

$$\begin{aligned} h(Y_2) - h(Y_1) &= \sum_{i=1}^n (h(Y_{2,i} | Y_2^{i-1}, Y_{1,i+1}^n) - h(Y_{1,i} | Y_2^{i-1}, Y_{1,i+1}^n)) \\ &\leq \sum_{i=1}^n (h(Y_{2,i} | V_i) - h(Y_{1,i} | V_i)), \end{aligned}$$

where the last inequality follows from

$$I(X^{i-1}; Y_{2,i} | Y_2^{i-1}, Y_{1,i+1}^n) \leq I(X^{i-1}; Y_{1,i} | Y_2^{i-1}, Y_{1,i+1}^n),$$

which holds by Markovity induced by construction of  $Y_1, Y_2$ . Hence, (4) holds if

$$\begin{aligned} e^{\frac{1}{n} \sum_{i=1}^n \log(1 - a_i) + 2h(X_i | V_i) - 2h(Y_{1,i} | V_i)} \\ + e^{\frac{1}{n} \sum_{i=1}^n \log(a_i) + 2h(Y_{2,i} | V_i) - 2h(Y_{1,i} | V_i)} \leq 1. \end{aligned}$$

By convexity of  $x \mapsto e^x$ , it is sufficient to show that

$$(1 - a_i) e^{2h(X_i | V_i) - 2h(Y_{1,i} | V_i)} + a_i e^{2h(Y_{2,i} | V_i) - 2h(Y_{1,i} | V_i)} \leq 1$$

for each  $i$ . However, this is just the conditional form of Costa's scalar inequality for concavity of entropy power, which holds again by convexity of  $x \mapsto e^x$ , and the easily verified fact that  $V_i \rightarrow X_i \rightarrow Y_{1,i} \rightarrow Y_{2,i}$ .

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