Existence of density for the stochastic wave equation with space-time homogeneous Gaussian noise

Raluca M. Balan* Lluís Quer-Sardanyons† Jian Song‡

May 15, 2018

Abstract

In this article, we consider the stochastic wave equation on $\mathbb{R}_+ \times \mathbb{R}^d$, in spatial dimension $d = 1$ or $d = 2$, driven by a linear multiplicative space-time homogeneous Gaussian noise whose temporal and spatial covariance structure are given by locally integrable functions γ (in time) and f (in space), which are the Fourier transforms of tempered measures ν on $\mathbb{R}$, respectively μ on $\mathbb{R}^d$. Our main result shows that the law of the solution $u(t, x)$ of this equation is absolutely continuous with respect to the Lebesgue measure, provided that the spatial spectral measure μ satisfies an integrability condition which ensures that the sample paths of the solution are Hölder continuous.

MSC 2010: Primary 60H15; 60H07

Keywords: Gaussian noise; stochastic partial differential equations; Malliavin calculus

1 Introduction

In this article, we study the absolute continuity of the law of the solution to the stochastic wave equation with linear multiplicative noise, in spatial dimension $d = 1$ or $d = 2$:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) &= \Delta u(t,x) + u(t,x)\dot{W}(t,x), \quad t>0, x \in \mathbb{R}^d, \\ u(0,x) &= 1, \quad x \in \mathbb{R}^d, \\ \dfrac{\partial u}{\partial t}(0,x) &= 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{1}$$

*Department of Mathematics and Statistics, 585 King Edward Avenue, Ottawa, ON, K1N 6N5, Canada. E-mail address: rbalan@uottawa.ca. Research supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

†Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Catalonia, Spain. E-mail address: quer@mat.uab.cat. Research supported by the grant MTM2015-67802P.

‡Corresponding author. Department of Mathematics. University of Hong Kong. E-mail address: txjsong@hku.hk.

The noise W is given by a zero-mean Gaussian process $\{W(\varphi);\varphi\in\mathcal{D}(\mathbb{R}^{d+1})\}$ defined on a complete probability space $(\Omega,\mathcal{F},P)$, with covariance

$$\mathbb{E}[W(\varphi_1)W(\varphi_2)]=\int_{\mathbb{R}^2\times\mathbb{R}^{2d}}\gamma(t-s)f(x-y)\varphi_1(t,x)\varphi_2(s,y)dxdydtds=:J(\varphi_1,\varphi_2),$$

where $\gamma:\mathbb{R}\to[0,\infty]$ and $f:\mathbb{R}^d\to[0,\infty]$ are continuous, symmetric, locally integrable functions, such that

$$\gamma(t)<\infty \quad \text{if and only if} \quad t\neq 0;$$
$$f(x)<\infty \quad \text{if and only if} \quad x\neq 0.$$

Here $\mathcal{D}(\mathbb{R}^{d+1})$ is the space of C^∞-functions on $\mathbb{R}^{d+1}$ with compact support. We denote by $\mathcal{H}$ the completion of $\mathcal{D}(\mathbb{R}^{d+1})$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defined by $\langle\varphi_1,\varphi_2\rangle_{\mathcal{H}}=J(\varphi_1,\varphi_2)$.

We assume that f is non-negative-definite (in the sense of distributions), i.e.

$$\int_{\mathbb{R}^d}(\varphi*\widetilde{\varphi})(x)f(x)dx\geq 0, \quad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R}^d),$$

where $\widetilde{\varphi}(x)=\varphi(-x)$ and $\mathcal{S}(\mathbb{R}^d)$ is the space of rapidly decreasing C^∞-functions on $\mathbb{R}^d$. By the Bochner-Schwartz theorem, there exists a tempered measure μ on $\mathbb{R}^d$ such that $f=\mathcal{F}\mu$, where $\mathcal{F}\mu$ denotes the Fourier transform of μ in the space $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ of $\mathbb{C}$-valued tempered distributions on $\mathbb{R}^d$. We emphasize that this does not mean that $f(x)=\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\mu(d\xi)$ for all $x\in\mathbb{R}^d$, since μ may be an infinite measure. It means that

$$\int_{\mathbb{R}^d}f(x)\varphi(x)dx=\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\mu(d\xi) \quad \text{for all} \quad \varphi\in\mathcal{S}_{\mathbb{C}}(\mathbb{R}^d),$$

where $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$ is the space of $\mathbb{C}$-valued rapidly decreasing C^∞-functions on $\mathbb{R}^d$, and $\mathcal{F}\varphi(x)=\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\varphi(x)dx$ is the Fourier transform of φ. Here $\xi\cdot x$ denotes the scalar product in $\mathbb{R}^d$. Similarly, we assume that γ is non-negative-definite (in the sense of distributions), and so there exists a tempered measure ν on $\mathbb{R}$ such that $\gamma=\mathcal{F}\nu$ in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R})$.

We denote by G the fundamental solution of the wave equation on $\mathbb{R}^d$:

$$G(t,x)=\frac{1}{2}1_{\{|x|\leq t\}} \quad \text{if} \quad d=1 \qquad \text{and} \quad G(t,x)=\frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|\leq t\}} \quad \text{if} \quad d=2,$$

where $|\cdot|$ denotes the Euclidean norm if $d=2$. We recall from [2] the definition of the solution, refering to Section 3 below for the precise definition of the Skorohod integral.

Definition 1.1. We say that a process $u=\{u(t,x);t\geq 0,x\in\mathbb{R}^d\}$ with $u(0,x)=1$ for all $x\in\mathbb{R}^d$ is a (mild Skorohod) *solution* of equation (1) if u has a measurable modification (denoted also by u) such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\mathbb{E}|u(t,x)|^2<\infty$ for all $T>0$, and for any $t>0$ and $x\in\mathbb{R}^d$, the following equality holds in $L^2(\Omega)$:

$$u(t,x)=1+\int_0^t\int_{\mathbb{R}^d}G(t-s,x-y)u(s,y)W(\delta s,\delta y), \tag{2}$$

where the stochastic integral is understood in the Skorohod sense, and the process $v^{(t,x)}=\{v^{(t,x)}(s,y)=1_{[0,t]}(s)G(t-s,x-y)u(s,y);s\geq 0,y\in\mathbb{R}^d\}$ is Skorohod integrable.

In [2], it was proved that equation (1) has a unique solution in any spatial dimension d, provided that the spatial spectral measure μ of the noise satisfies *Dalang's condition*:

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty. \tag{3}$$

Condition (3) was introduced simultaneously in articles [7] and [26], and plays a crucial role in the study of stochastic partial differential equations (SPDEs) with spatially homogeneous Gaussian noise (see for instance [7, 25, 23, 14] for a sample of relevant references). Owing to Remark 10(b) in [7], note that (3) holds automatically if $d = 1$, and if $d = 2$, (3) is equivalent to

$$\int_{0<|x|\leq 1} f(x)\log\left(\frac{1}{|x|}\right)dx < \infty.$$

The problem of absolute continuity of the law and smoothness of the density for the solution of an SPDE goes back to article [6], in which the authors studied the equation:

$$Lu(t,x) = \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)) \tag{4}$$

with space-time white noise W, smooth functions σ and b, and L the wave operator on $\mathbb{R}_+ \times I$, for an interval $I \subset \mathbb{R}$ which could be bounded, semi-bounded, or even $\mathbb{R}$, with Dirichlet boundary conditions when I has a finite endpoint. These authors showed that the *mild solution* (defined similarly to (2) using the Green function of the wave operator) coincides with the *weak solution* (defined using integration against test functions) and proved that this solution has a smooth density. In [24], [4] and [20], it was shown that the same property holds for equation (4) in which L is replaced by the heat operator on $\mathbb{R}_+ \times [0,1]$, with Dirichlet (respectively Neumann) boundary conditions. (In this case, the fact that the mild and weak solutions coincide was known from Walsh' lecture notes [32].)

In the recent years, several authors revisited the problem of existence and smoothness of density for the mild solution of an SPDE of form (4) on the entire space $\mathbb{R}^d$, with Lipschitz functions σ and b, driven by the more general Gaussian noise introduced in [8, 7]. This noise is spatially homogeneous (with spatial spectral measure μ as above), but white in time, i.e. blue has temporal covariance structure given formally by the Dirac distribution at 0. The function σ is assumed to satisfy the condition:

$$|\sigma(x)| \geq c > 0 \quad \text{for all } x \in \mathbb{R}. \tag{5}$$

We refer the reader to [19] for the wave equation in spatial dimension $d = 2$, [23] for the heat equation in any dimension d and the wave equation in dimension $d = 1, 2, 3$ (see also [18, 28, 29]), and [30, 31] for the wave equation in dimension $d \geq 4$.

In the case of the space-time Gaussian noise which is colored in time (i.e. has temporal covariance structure given by a non-negative-definite locally integrable function γ as above), all references related to the problem of existence and smoothness of the density of the law of the solution focus on the stochastic heat equation with *linear* multiplicative noise (i.e. $\sigma(x) = x$ and $b = 0$):

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x), \quad t > 0, x \in \mathbb{R}^d, \tag{6}$$

with initial condition $u(0,x) = u_0(x)$, where u_0 is a continuous bounded function. In this case, the noise is not a martingale in time, and the techniques of Itô stochastic integration cannot be used. This leads to several types of solution: the *mild Skorohod solution* defined similarly to (2) using the Skorohod integral involving the Green function G_h of the heat operator, the *mild Stratonovich solution* defined using a Stratonovich-type integral of the same term as in (2) involving G_h, and the *weak solution* defined using Stratonovich integration against test functions. According to some well-known criteria from Malliavin calculus, to show that $u(t,x)$ has a density it is enough to prove that

$$\|Du(t,x)\|_{\mathcal{H}} > 0 \quad \text{a.s.} \tag{7}$$

and this density is smooth if $u(t,x)$ is infintely differentiable in the Malliavin sense and

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^{-2p} < \infty \quad \text{for all} \quad p > 0. \tag{8}$$

In [13], it was proved that relation (8) holds for the weak solution of (6), if the covariance functions of the noise are given by $\gamma(t) = \rho_H(t) := H(2H-1)|t|^{2H-2}$ and $f(x) = \prod_{i=1}^{d} \rho_{H_i}(x_i)$ with parameters $H, H_1, \ldots, H_d \in (1/2, 1)$. This result was obtained using the Feynman-Kac (FK) representation of the weak solution, which holds only when the parameters of the noise satisfy the condition $2H + \sum_{i=1}^{d} H_i > d+1$. Under the same condition, it might be possible to prove that the mild Skorohod solution of equation (6) satisfies (8), using the FK formula of this solution given in [13]. In the case of general covariance functions γ and f, the authors of [11] established the FK formula for the mild Stratonivich solution to equation (6), assuming that $0 \leq \gamma(t) \leq C_\beta |t|^{-(1-\beta)}$ and

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta} \mu(d\xi) < \infty, \tag{9}$$

for some $\beta \in (0,1)$ and $C_\beta > 0$. Under this assumption, it may be possible to show that the mild Stratonivich solution satisfies relation (8). In the recent preprint [12], it was proved that the mild Skorohod solution $u(t,x)$ to equation (6) satisfies (8) by deriving estimates for the small ball probability $P(\|Du(t,x)\|_{\mathcal{H}} \leq a)$ as $a \to 0+$. This was proved using an FK formula for the regularization of the solution, and assuming that the noise is white in either space or time, or $c_1 t^{\alpha_0} \leq \gamma(t) \leq c_2 t^{-\alpha_0}$ for all $t \in \mathbb{R}$, for some $c_0 > 0$ and $\alpha_0 \in [0,1)$, and f satisfies the scaling property $f(cx) \leq c^{-\alpha} f(x)$ for all $c > 0$ and $x \in \mathbb{R}^d$, for some $\alpha \in (0,2)$. In addition, these authors assume that $\gamma = \gamma_0 * \gamma_0$ and $f = f_0 * f_0$ for some functions γ_0 and f_0.

In the present article, we will prove the absolute continuity of the law of the solution to the wave equation (1) with space-time homogeneous Gaussian noise. Surprisingly, we are able to do this assuming only that the spatial spectral measure μ satisfies (9), without any additional constraints on γ. For the wave equation, it is not known if there is a FK formula for the solution. Our method for proving (7) is similar to the one used in [23] for the white noise in time. Since $\sigma(x) = x$ fails to satisfy condition (5), we will prove (7) by localizing on the event $\Omega_m = \{|u(t,x)| > m^{-1}\}$ and let $m \to \infty$. A closer look at the inner product in $\mathcal{H}$ reveals that it is enough to prove that (see Corollary 2.4 below):

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{r,z} u(t,x) D_{r,z'} u(t,x) f(z-z') dz dz' dr > 0 \quad \text{a.s. on} \quad \Omega_m. \tag{10}$$

For this, we will show that the process $\{D_{r,z}u(t,x); r \in [0,t], z \in \mathbb{R}^d\}$ has a jointly measurable modification and satisfies a certain integral equation involving a Hilbert-space valued Skorohod integral. As shown in Section 3 below, the proofs of these facts contain some of the main technical difficulties encountered throughout the article, which are mainly due to the time-space correlation of the underlying noise.

We suppose that the following assumption holds, which is important for describing the space $\mathcal{H}$ and its inner product, as shown by Theorem 2.3 below.

Assumption A. $\mu(d\xi) = (2\pi)^{-d}g(\xi)d\xi$, $\nu(d\tau) = (2\pi)^{-1}h(\tau)d\tau$ and $1/(hg)1_{\{hg>0\}}$ is a slow growth (or tempered) function, i.e. there exists an integer $k \geq 1$ such that

$$\int_{\{hg>0\}} \left(\frac{1}{1+\tau^2+|\xi|^2}\right)^k \frac{1}{h(\tau)g(\xi)} d\tau d\xi < \infty.$$

Our basic example is $\gamma(t) = H(2H-1)|t|^{2H-2}$ with $1/2 < H < 1$ and f is the Riesz kernel of order α, i.e. $f(x) = |x|^{-d+\alpha}$ with $0 < \alpha < d$. In this case, $h(\tau) = |\tau|^{2H-1}$ and $g(\xi) = c_{\alpha,d}|\xi|^{-\alpha}$ for some constant $c_{\alpha,d} > 0$ depending on α and d. Assumption A is satisfied for this example.

The following theorem is the main result of the present article.

Theorem 1.2. *If Assumption A holds and the spatial spectral density μ satisfies* (9) *for some $\beta \in (0,1)$, then the restriction of the law of the random variable $u(t,x)1_{\{u(t,x)\neq 0\}}$ to the set $\mathbb{R}\backslash\{0\}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}\backslash\{0\}$.*

Condition (9) is used to ensure the Hölder continuity of the sample paths of the solution, which was established in [2] and is an essential component of our proof. The same idea was used in [4] to get a suitable lower bound for the dominant term of the Malliavin matrix of the solution. These authors also used the Hölder continuity of the sample paths of the solution to prove that the law of the solution has a smooth density. Moreover, in [9], it was proved that (9) ensures the Hölder continuity of the sample paths of the solution of a general SPDE with white noise in time, and that this condition is optimal when the spatial covariance function of the noise is given by the Riez kernel. We believe that this condition is also optimal for the colored noise in time, and it may not be possible to obtain the absolute continuity of the solution assuming only Dalang's condition (3).

This article is organized as follows. In Section 2, we characterize the space $\mathcal{H}$ and its inner product. In Section 3, we review some basic elements of Malliavin calculus, we prove that the solution to equation (1) is infinitely differentiable in the Malliavin sense, and examine some properties of its Malliavin derivative. In Section 4, we give the proof of Theorem 1.2. In Appendix A, we discuss a Parseval-type identity, while in Appendix B we give a criterion for the existence of a measurable modification of a random field. Both these results are used in the present article.

2 Characterization of the space $\mathcal{H}$

In this section, we provide an alternative definition of the inner product in $\mathcal{H}$ in terms of the Fourier transform in the space variable, and we give a characterization of the space

$\mathcal{H}$ (which is due essentialy to [5]). This characterization plays an important role in the present article, because it allows us to focus on (10), instead of (7).

Note that, by the definition of the measure μ, for any $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi_1(x)\overline{\varphi_2(y)}dxdy = \int_{\mathbb{R}^d} \mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\varphi_2(\xi)}\mu(d\xi). \tag{11}$$

Similarly, for any $\phi_1, \phi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \gamma(t-s)\phi_1(t)\overline{\phi_2(s)}dtds = \int_{\mathbb{R}} \mathcal{F}\phi_1(\tau)\overline{\mathcal{F}\phi_2(\tau)}\nu(d\tau). \tag{12}$$

Building upon a remarkable result borrowed from [17], in Appendix A, we show that relation (11) holds for any functions $\varphi_1, \varphi_2 \in L^1_{\mathbb{C}}(\mathbb{R}^d)$ whose absolute values have "finite energy" with respect to the kernel f. This fact is used in the proof of Lemma 3.1 below.

By Lemma 2.1 of [2], for any $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^{d+1})$,

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^{d+1}} \mathcal{F}\varphi_1(\tau,\xi)\overline{\mathcal{F}\varphi_2(\tau,\xi)}\nu(d\tau)\mu(d\xi), \tag{13}$$

where $\mathcal{F}$ denotes the Fourier transform in both variables t and x.

Similarly to [16, 5], we consider the space:

$$\mathcal{U} = \{S \in \mathcal{S}'(\mathbb{R}^{d+1}); \mathcal{F}S \text{ is a (measurable) function and } \int_{\mathbb{R}^{d+1}} |\mathcal{F}S(\tau,\xi)|^2 \nu(d\tau)\mu(d\xi) < \infty\}.$$

The space $\mathcal{U}$ is endowed with the inner product

$$\langle S_1, S_2 \rangle_{\mathcal{U}} := \int_{\mathbb{R}^{d+1}} \mathcal{F}S_1(\tau,\xi)\overline{\mathcal{F}S_2(\tau,\xi)}\nu(d\tau)\mu(d\xi).$$

By (13), $\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{U}}$ for any $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^{d+1})$. We denote $\|S\|^2_{\mathcal{U}} = \langle S, S \rangle_{\mathcal{U}}$. Note that if $S \in \mathcal{S}'(\mathbb{R}^{d+1})$ is such that $\mathcal{F}S$ is a function, then

$$\mathcal{F}S(-\tau,-\xi) = \overline{\mathcal{F}S(\tau,\xi)} \quad \text{for almost all } (\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d. \tag{14}$$

The proof of this fact is very similar to that of Lemma 3.1 of [16]. This property and the symmetry of the measures ν and μ imply that $\langle\cdot,\cdot\rangle_{\mathcal{U}}$ is a well-defined $\mathbb{R}$-valued inner product, provided that we identify two elements S_1 and S_2 such that $\|S_1 - S_2\|_{\mathcal{U}} = 0$.

We let $L^2_{\mathbb{C}}(\nu\times\mu)$ be the space of functions $v : \mathbb{R}^{d+1} \to \mathbb{C}$ such that

$$\int_{\mathbb{R}^{d+1}} |v(\tau,\xi)|^2 \nu(d\tau)\mu(d\xi) < \infty,$$

and $\widetilde{L}^2_{\mathbb{C}}(\nu\times\mu)$ be the subset of $L^2_{\mathbb{C}}(\nu\times\mu)$ consisting of functions v such that $v(-\tau,-\xi) = \overline{v(\tau,\xi)}$ for almost all $(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^d$.

The next result generalizes Theorem 3.2 of [16] to higher dimensions. Assumption A is not needed for this result.

Theorem 2.1. *The space $\mathcal{D}(\mathbb{R}^{d+1})$ is dense in $\mathcal{U}$, with respect to $\langle\cdot,\cdot\rangle_{\mathcal{U}}$, and hence $\mathcal{U}$ is included in $\mathcal{H}$. Moreover, $\langle S_1, S_2\rangle_{\mathcal{U}} = \langle S_1, S_2\rangle_{\mathcal{H}}$ for any $S_1, S_2 \in \mathcal{U}$.*

Proof. We only need to prove that $\mathcal{D}(\mathbb{R}^{d+1})$ is dense in $\mathcal{U}$, with respect to $\langle\cdot,\cdot\rangle_{\mathcal{U}}$, i.e. for any $S \in \mathcal{U}$, there exists a sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R}^{d+1})$ such that $\|\varphi_n - S\|_{\mathcal{U}} \to 0$ as $n \to \infty$. For this, it suffices to prove that

$$\mathcal{F}\big(\mathcal{D}(\mathbb{R}^{d+1})\big) \quad \text{is dense in } \widetilde{L}^2_{\mathbb{C}}(\nu\times\mu), \tag{15}$$

where $\mathcal{F}\big(\mathcal{D}(\mathbb{R}^{d+1})\big)$ is the image of $\mathcal{D}(\mathbb{R}^{d+1})$ under the Fourier transform, and $\widetilde{L}^2_{\mathbb{C}}(\nu\times\mu)$ is endowed with the topology of $L^2_{\mathbb{C}}(\nu\times\mu)$. (To see this, let $S\in\mathcal{U}$ be arbitrary. Since $\mathcal{F}S \in \widetilde{L}^2_{\mathbb{C}}(\nu\times\mu)$, there exists a sequence $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathbb{R}^{d+1})$ such that $\|\mathcal{F}\varphi_n - \mathcal{F}S\|_{L^2_{\mathbb{C}}(\nu\times\mu)} = \|\varphi_n - S\|_{\mathcal{U}} \to 0$ as $n\to\infty$.)

It remains prove (15). First, we claim that

$$\mathcal{F}\big(\mathcal{D}_{\mathbb{C}}(\mathbb{R}^{d+1})\big) \quad \text{is dense in } L^2_{\mathbb{C}}(\nu\times\mu). \tag{16}$$

Indeed, this is an extension of Theorem 4.1 in [15], which is proved as follows. First, $\mathcal{F}\big(\mathcal{D}_{\mathbb{C}}(\mathbb{R}^{d+1})\big)$ is dense in $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{d+1})$, because the Fourier transform defines an homeomorphism from $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{d+1})$ onto itself. Secondly, using that ν and μ are tempered measures, one obtains that $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{d+1}) \subset L^2_{\mathbb{C}}(\nu\times\mu)$ and that convergence in $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{d+1})$ implies convergence in $L^2_{\mathbb{C}}(\nu\times\mu)$. These two facts imply the third one, namely $\mathcal{F}\big(\mathcal{D}_{\mathbb{C}}(\mathbb{R}^{d+1})\big)$ is dense in $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{d+1})$ with respect to the topology of $L^2_{\mathbb{C}}(\nu\times\mu)$. Finally, the conclusion follows by observing that $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^{d+1})$ is dense in $L^2_{\mathbb{C}}(\nu\times\mu)$, since $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^{d+1})$ is so. This proves (16).

We now prove (15). Let $v \in \widetilde{L}^2_{\mathbb{C}}(\nu\times\mu)$ be arbitrary. By (16), there exists a sequence $(\varphi_n)_{n\geq1}$ in $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^{d+1})$ such that $\mathcal{F}\varphi_n \to v$ in $L^2_{\mathbb{C}}(\nu\times\mu)$. Note that this implies that the following limits hold in $L^2(\nu\times\mu)$:

$$\lim_{n\to\infty} \mathrm{Re}(\mathcal{F}\varphi_n) = \mathrm{Re}(v) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Im}(\mathcal{F}\varphi_n) = \mathrm{Im}(v).$$

Let us construct a sequence $(\psi_n)_{n\geq1}$ in $\mathcal{D}(\mathbb{R}^{d+1})$ such that $\mathcal{F}\psi_n \to v$ in $L^2_{\mathbb{C}}(\nu\times\mu)$. For this, we will use the following notation. Namely, for any $\mathbb{C}$-valued function κ, $\mathrm{e}(\kappa) := \frac{1}{2}(\kappa+\widetilde{\kappa})$ and $\mathrm{o}(\kappa) := \frac{1}{2}(\kappa-\widetilde{\kappa})$ denote the even and odd parts of κ, respectively.

Since v belongs to $\widetilde{L}^2_{\mathbb{C}}(\nu\times\mu)$, we have that $\mathrm{Re}(v)$ is a even function and $\mathrm{Im}(v)$ is an odd function. Here, we say that a function g is *even* if $g(-x) = g(x)$ for almost all $x \in \mathbb{R}^{d+1}$ and is *odd* if $g(-x) = -g(x)$ for almost all $x\in\mathbb{R}^{d+1}$. These properties, together with the symmetry of the measures ν and μ, imply that the following limits hold in $L^2(\nu\times\mu)$:

$$\lim_{n\to\infty} \mathrm{e}(\mathrm{Re}(\mathcal{F}\varphi_n)) = \mathrm{Re}(v) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{o}(\mathrm{Im}(\mathcal{F}\varphi_n)) = \mathrm{Im}(v).$$

Define $\psi_n := \mathcal{F}^{-1}\big(\mathrm{e}(\mathrm{Re}(\mathcal{F}\varphi_n)) + i\mathrm{o}(\mathrm{Im}(\mathcal{F}\varphi_n))\big)$. Then, ψ_n is a real function, belongs to $\mathcal{D}(\mathbb{R}^{d+1})$ and, by construction, $\mathcal{F}\psi_n \to v$ in $L^2_{\mathbb{C}}(\nu\times\mu)$. □

The following result not only generalizes Theorem 3.4 of [16] to higher dimensions, but also specifies the necessary and sufficient condition for the completeness of $\mathcal{U}$. This result is an immediate consequence of Theorem 3.5.(2) of [5], applied to the space $\mathbb{R}^{d+1}$ and the measure $F = \nu\times\mu$.

Theorem 2.2. *Assume that* $\mu(\xi) = (2\pi)^{-d} g(\xi) d\xi$ *and* $\nu(d\tau) = (2\pi)^{-1} h(\tau) d\tau$. *Then* $\mathcal{U}$ *is complete if and only if for any function* $\varphi \in L^2_{\mathbb{C}}(\nu \times \mu)$ *there exists an integer* $k \geq 1$ *such that*

$$\int_{\{h>0,g>0\}} \left(\frac{1}{1+\tau^2+|\xi|^2} \right)^k |\varphi(\tau,\xi)| \, d\tau d\xi < \infty.$$

In particular, $\mathcal{U}$ *is a complete if* $1/(hg) 1_{\{hg>0\}}$ *is a slow growth function.*

Combining Theorems 2.1 and 2.2, we obtain the following result, which can be viewed as a generalization of Theorem 3.5 of [16] to higher dimensions.

Theorem 2.3. *If Assumption A holds, then* $\mathcal{H}$ *coincides with the space* $\mathcal{U}$. *Moreover, for any* $S_1, S_2 \in \mathcal{H}$,

$$\langle S_1, S_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^{d+1}} \mathcal{F}S_1(\tau,\xi) \overline{\mathcal{F}S_2(\tau,\xi)} \nu(d\tau)\mu(d\xi).$$

Corollary 2.4. *If Assumption A holds and* S *is a measurable function on* $\mathbb{R}_+ \times \mathbb{R}^d$ *such that* $S \in \mathcal{H}$ *and*

$$I := \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S(t,x) f(x-y) S(t,y) dx dy dt > 0, \tag{17}$$

then $\|S\|_{\mathcal{H}} > 0$.

Proof. Suppose that $\|S\|_{\mathcal{H}} = 0$. By Theorem 2.3, $\mathcal{F}S(\tau,\xi) = 0$ for almost all $(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^d$. Hence $S(t,x) = 0$ for almost all $t > 0$ and $x \in \mathbb{R}^d$, i.e. there exists a Borel set $N \subset \mathbb{R}_+ \times \mathbb{R}^d$ with $\lambda_{d+1}(N) = 0$ such that $S(t,x) = 0$ for all $(t,x) \notin N$, where λ_{d+1} denotes the Lebesgue measure on $\mathbb{R}^{d+1}$. Therefore,

$$I = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_M(t,x,y) S(t,x) f(x-y) S(t,y) dx dy dt,$$

where $M = \{(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d; (t,x) \in N, (t,y) \in N\}$.

Let $N_t = \{x \in \mathbb{R}^d; (t,x) \in N\}$ be the section of the set N at point $t > 0$. By Fubini's theorem, $\lambda_{d+1}(N) = \int_0^\infty \lambda_d(N_t) dt$. Since $\lambda_{d+1}(N) = 0$, we infer that $\lambda_d(N_t) = 0$ for almost all $t > 0$. Note that the section of the set M at point t is $M_t = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d; (t,x,y) \in M\} = N_t \times N_t$, and its Lebesque measure is $\lambda_{2d}(M_t) = \lambda_d^2(N_t) = 0$ for almost all $t > 0$. By applying Fubini's theorem again, we infer that $\lambda_{2d+1}(M) = \int_0^\infty \lambda_{2d}(M_t) dt = 0$. This shows that $I = 0$, which contradicts (17). □

3 Malliavin Derivative of the Solution

In this section, we examine some properties of the Malliavin derivative $Du(t,x)$ of the solution $u(t,x)$ to equation (1). In particular, we show that $Du(t,x)$ satisfies a certain integral equation, which is slightly different than the equation given by Theorem 6.7 of [1].

We recall some basic elements of Malliavin calculus. We refer the reader to [22] for more details. Any random variable $F \in L^2(\Omega)$ which is measurable with respect to the

σ-field generated by $\{W(\varphi); \varphi \in \mathcal{H}\}$ admits the representation $F = \sum_{n\geq 0} J_n F$ where $J_n F$ is the projection of F on the n-th Wiener chaos space $\mathcal{H}_n$ for $n \geq 1$, and $J_0 F = \mathbb{E}(F)$. We denote by I_n the multiple integral of order n with respect to W, which is a linear continuous operator from $\mathcal{H}^{\otimes n}$ onto $\mathcal{H}_n$.

Let $\mathcal{S}$ be the class of "smooth" random variables, i.e. random variables of the form $F = f(W(\varphi_1), \ldots, W(\varphi_n))$, where $n \geq 1$, $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$ and f is in the set $C_b^\infty(\mathbb{R}^n)$ of bounded infinitely differentiable functions on $\mathbb{R}^n$ whose partial derivatives are bounded. The Malliavin derivative of a random variable F of this form is the $\mathcal{H}$-valued random variable given by:

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_n))\varphi_i.$$

We let $\mathbb{D}^{1,2}$ be the completion of $\mathcal{S}$ with respect to the norm $\|F\|_{1,2} = \left(\mathbb{E}|F|^2 + \mathbb{E}\|DF\|_{\mathcal{H}}^2\right)^{1/2}$.

Similarly, the iterated derivative $D^k F$ can be defined as a $\mathcal{H}^{\otimes k}$-valued random variable, for any natural number $k \geq 1$. For any $p > 1$, let $\mathbb{D}^{k,p}$ be the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p} = \Big(\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}\|D^j F\|_{\mathcal{H}^{\otimes j}}^p\Big)^{1/p}.$$

It can be proved that: (see p. 28 of [22])

$$F \in \mathbb{D}^{k,2} \quad \text{if and only if} \quad \sum_{n\geq 1} n^k \mathbb{E}|J_n F|^2 < \infty.$$

For $p > 1$, it can be shown that

$$F \in \mathbb{D}^{k,p} \quad \text{if} \quad \sum_{n\geq 1} n^{k/2}(p-1)^{n/2}(\mathbb{E}|J_n F|^2)^{1/2} < \infty \tag{18}$$

(see p. 28 of [1]).

The divergence operator δ is defined as the adjoint of the operator D. The domain of δ, denoted by Dom δ, is the set of $u \in L^2(\Omega; \mathcal{H})$ such that

$$|\mathbb{E}\langle DF, u\rangle_{\mathcal{H}}| \leq c(\mathbb{E}|F|^2)^{1/2}, \quad \forall F \in \mathbb{D}^{1,2}, \tag{19}$$

where c is a constant depending on u. If $u \in$ Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\langle DF, u\rangle_{\mathcal{H}}, \quad \forall F \in \mathbb{D}^{1,2}. \tag{20}$$

If $u \in$ Dom δ, we will use the notation

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) W(\delta t, \delta x),$$

even if u is not a function in (t,x), and we say that $\delta(u)$ is the Skorohod integral of u with respect to W.

We return now to the solution u of equation (1). From [2], we know that $u(t,x)$ has the Wiener chaos expansion

$$u(t,x) = 1 + \sum_{n\geq 1} I_n(f_n(\cdot,t,x)),$$

where I_n is the multiple Wiener integral of order n with respect to W, and

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x) = G(t-t_n,x-x_n)\ldots G(t_2-t_1,x_2-x_1)1_{\{0<t_1<\ldots<t_n<t\}}.$$

It follows that

$$\mathbb{E}|u(t,x)|^2 = \sum_{n\geq 0} \frac{1}{n!}\alpha_n(t),$$

where $\alpha_n(t) = (n!)^2\|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}$ and $\widetilde{f}_n(\cdot,t,x)$ is the symmetrization of $f_n(\cdot,t,x)$.

In Section 6 of [1], it was proved that if $f(x) = |x|^{-\alpha}$ with $\alpha \in (0,d)$ and $\gamma(t) = H(2H-1)|t|^{2H-2}$ with $1/2 < H < 1$, then for any integer $k \geq 1$ and for any $p > 1$,

$$u(t,x) \in \mathbb{D}^{k,p}.$$

We will now extend this result to general functions f and γ.

We begin with a maximum principle, which is a refinement of Lemma 4.2 of [2], specific to the cases $d=1$ and $d=2$. Its proof is based on a Parseval-type identity given in Appendix A.

Lemma 3.1. *If G is the fundamental solution of the wave equation in spatial dimension $d=1$ or $d=2$, then*

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2\mu(d\xi) = \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi)|^2\mu(d\xi).$$

Proof. We denote by $\mathcal{E}_f(\varphi)$ the energy of φ with respect to f given by (56) (Appendix A). Note that $G(t,\cdot)$ is a non-negative integrable function with $\int_{\mathbb{R}^d} G(t,x)dx = t$ and

$$\mathcal{E}_f(G(t,\cdot)) = \int_{\mathbb{R}^d} \big(G(t,\cdot) * G(t,\cdot)\big)(x)f(x)dx = \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi)|^2\mu(d\xi) < \infty.$$

For any $\eta \in \mathbb{R}^d$, we let $G_\eta(t,x) = e^{-ix\cdot\eta}G(t,x)$. Then $G_\eta(t,\cdot) \in L^1_{\mathbb{C}}(\mathbb{R}^d)$ and

$$\begin{aligned}\mathcal{E}_f(|G_\eta(t,\cdot)|) &= \int_{\mathbb{R}^d} \big(|G_\eta(t,\cdot)| * |G_\eta(t,\cdot)|\big)(x)f(x)dx \\ &\leq \int_{\mathbb{R}^d} \big(G(t,\cdot) * G(t,\cdot)\big)(x)f(x)dx = \mathcal{E}_f(G(t,\cdot)) < \infty.\end{aligned}$$

Moreover, $\mathcal{F}G_\eta(t,\cdot)(\xi) = \mathcal{F}G(t,\cdot)(\xi+\eta)$ for any $\xi \in \mathbb{R}^d$. It follows that

$$\begin{aligned}0 &\leq \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2\mu(d\xi) = \int_{\mathbb{R}^d} |\mathcal{F}G_\eta(t,\cdot)(\xi)|^2\mu(d\xi) \\ &= \int_{\mathbb{R}^d} (G_\eta(t,\cdot) * G_\eta(t,\cdot))(x)f(x)dx = \left|\int_{\mathbb{R}^d} (G_\eta(t,\cdot) * G_\eta(t,\cdot))(x)f(x)dx\right| \\ &\leq \int_{\mathbb{R}^d} \big(|G_\eta(t,\cdot)| * |G_\eta(t,\cdot)|\big)(x)f(x)dx \leq \mathcal{E}_f(G(t,\cdot)),\end{aligned}$$

where for the second equality above, we used Lemma A.2 (Appendix A). □

The next result gives a stronger form of relation (4.10) of [2] (with a simplified proof). Its proof is based on Lemma 3.1.

Lemma 3.2. *For any $k \geq 0$,*

$$\sum_{n\geq 0} \frac{n^k}{n!} \alpha_n(t) < \infty.$$

Proof. We will borrow notations from [2, Theorem 4.4]. Recall that

$$\alpha_n(t) = \int_{[0,t]^{2n}} \prod_{j=1}^{n} \gamma(t_j - s_j) \psi_n(\mathbf{t}, \mathbf{s}) d\mathbf{t} d\mathbf{s}, \tag{21}$$

where

$$\psi_n(\mathbf{t}, \mathbf{s}) \;\; = \;\; \int_{\mathbb{R}^{nd}} \mathcal{F} g_{\mathbf{t}}^{(n)}(\cdot, t, x)(\xi_1, \dots, \xi_n) \overline{\mathcal{F} g_{\mathbf{s}}^{(n)}(\cdot, t, x)(\xi_1, \dots, \xi_n)} \mu(d\xi_1) \dots \mu(d\xi_n)$$

and

$$g_{\mathbf{t}}^{(n)}(\cdot, t, x) = n! \widetilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x). \tag{22}$$

If the permutation ρ of $\{1, \dots, n\}$ is chosen such that $t_{\rho(1)} < \dots < t_{\rho(n)}$, then

$$\begin{aligned} \mathcal{F} g_{\mathbf{t}}^{(n)}(\xi_1, \dots, \xi_n) \;\; &= \;\; e^{-i \sum_{j=1}^{n} \xi_j \cdot x} \overline{\mathcal{F} G(t_{\rho(2)} - t_{\rho(1)}, \cdot)(\xi_{\rho(1)})} \; \overline{\mathcal{F} G(t_{\rho(3)} - t_{\rho(2)}, \cdot)(\xi_{\rho(1)} + \xi_{\rho(2)})} \\ & \quad \dots \overline{\mathcal{F} G(t - t_{\rho(n)}, \cdot)(\xi_{\rho(1)} + \dots + \xi_{\rho(n)})} \end{aligned} \tag{23}$$

By relation (4.15) of [2],

$$\alpha_n(t) \leq \Gamma_t^n \int_{[0,t]^n} \psi_n(\mathbf{t}, \mathbf{t}) d\mathbf{t}.$$

To estimate $\psi_n(\mathbf{t}, \mathbf{t})$ we use the first inequality in (4.16) of [2]. We denote $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1, \dots, n$ and $t_{\rho(n+1)} = t$. We have:

$$\begin{aligned} \psi_n(\mathbf{t}, \mathbf{t}) \;\; &\leq \;\; \prod_{j=1}^{n} \left(\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F} G(u_j, \cdot)(\xi_j + \eta)|^2 \mu(d\xi_j) \right) \\ &\leq \;\; \prod_{j=1}^{n} \int_{\mathbb{R}^d} \frac{\sin^2(u_j |\xi_j|)}{|\xi_j|^2} \mu(d\xi_j), \end{aligned}$$

where for the last equality we used Lemma 3.1. Therefore,

$$\alpha_n(t) \leq \Gamma_t^n n! \int_{\{0 < t_1 < \dots < t_n < t\}} \int_{\mathbb{R}^{nd}} \frac{\sin^2((t_2 - t_1)|\xi_1|)}{|\xi_1|^2} \dots \frac{\sin^2((t - t_n)|\xi_n|)}{|\xi_n|^2} \mu(dt_1) \dots \mu(dt_n) d\mathbf{t}.$$

From Lemma 2.2 of [3], we know that for any $\beta > 0$ and $\xi \in \mathbb{R}^d$,

$$I_\beta^w(\xi) := \int_0^\infty e^{-\beta t} \frac{\sin^2(t|\xi|)}{|\xi|^2} dt = \frac{2}{\beta} \cdot \frac{1}{\beta^2 + 4|\xi|^2}.$$

Using the change of variables $t_2 - t_1 = u_1, \dots t - t_n = u_n$, we obtain:

$$\int_{\{0<t_1<\ldots<t_n<t\}} \frac{\sin^2((t_2-t_1)|\xi_1|)}{|\xi_1|^2} \cdots \frac{\sin^2((t-t_n)|\xi_n|)}{|\xi_n|^2} dt_1 \dots dt_n$$
$$= \int_{\{(u_1,\ldots,u_n)\in[0,t]^n; \sum_{j=1}^n u_j<t\}} e^{-Mu_1} \frac{\sin^2(u_1|\xi_1|)}{|\xi_1|^2} \cdots e^{-Mu_n} \frac{\sin^2(u_n|\xi_n|)}{|\xi_n|^2} e^{M(u_1+\ldots+u_n)} du_1 \dots du_n$$
$$\leq e^{Mt} \prod_{j=1}^n \left(\int_0^t e^{-Mu_j} \frac{\sin^2(u_j|\xi_j|)}{|\xi_j|^2} du_j \right) \leq e^{Mt} \prod_{j=1}^n I_M^w(\xi_j) = e^{Mt} \left(\frac{2}{M}\right)^n \prod_{j=1}^n \frac{1}{M^2+4|\xi_j|^2}.$$

It follows that

$$\alpha_n(t) \leq e^{Mt} n! \left(\frac{2\Gamma_t}{M}\right)^n \left(\int_{\mathbb{R}^d} \frac{1}{M^2+4|\xi|^2} \mu(d\xi)\right)^n = e^{Mt} n! \left(\frac{2\Gamma_t}{M} K_M\right)^n, \qquad (24)$$

where $K_M = \int_{\mathbb{R}^d} \frac{1}{M^2+4|\xi|^2} \mu(d\xi)$. By the dominated convergence theorem and (3), $K_M \to 0$ as $M \to \infty$. Hence, using the fact that $n \leq e^n$ for any $n \geq 0$, we have:

$$\sum_{n\geq 0} \frac{n^k}{n!} \alpha_n(t) \leq e^{Mt} \sum_{n\geq 0} n^k \left(\frac{2\Gamma_t}{M} K_M\right)^n \leq e^{Mt} \sum_{n\geq 0} \left(e^k \frac{2\Gamma_t}{M} K_M\right)^n < \infty,$$

if we choose M sufficiently large. □

Lemma 3.3. *Let u be the solution of equation* (1). *Then $u(t,x) \in \mathbb{D}^{k,p}$ for any $k \geq 1$ and $p > 1$.*

Proof. We apply (18) to the variable $F = u(t,x)$. Let $J_n(t,x) := J_n u(t,x) = I_n(f_n(\cdot,t,x))$. Then $\mathbb{E}|J_n(t,x)|^2 = (n!)^2 \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} = \frac{1}{n!}\alpha_n(t)$. By (24), we have:

$$\sum_{n\geq 1} n^{k/2}(p-1)^{n/2} \left(\frac{1}{n!}\alpha_n(t)\right)^{1/2} \leq e^{Mt/2} \sum_{n\geq 1} e^{nk/2}(p-1)^{n/2} \left(\frac{2\Gamma_t}{M} K_M\right)^{n/2} < \infty,$$

if we choose M large enough. □

We begin now to examine the Malliavin derivatives of first and second order of the variable $u(t,x)$. First, note that for any $r > 0$ and $z \in \mathbb{R}^d$, we have:

$$D_{r,z} u(t,x) = \sum_{n\geq 1} n I_{n-1}(f_n(\cdot,r,z,t,x)) = 1_{\{r<t\}} G(t-r,x-z) \sum_{n\geq 1} n I_{n-1}(f_{n-1}(\cdot,r,z)) \quad (25)$$

We recall the following definitions.

Definition 3.4. A random field $\{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is *measurable* if the map $(\omega,t,x) \mapsto X(\omega,t,x)$ is measurable with respect to $\mathcal{F} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^d)$.

Definition 3.5. The random fields $\{X(t,x); t \geq 0, x \in \mathbb{R}^d\}$ and $\{Y(t,x); t \geq 0, x \in \mathbb{R}^d\}$ are *modifications* of each other if $X(t,x) = Y(t,x)$ a.s. for all $t \geq 0$ and $x \in \mathbb{R}^d$.

Theorem 3.6. *The process $\{D_{r,z}u(t,x); r \in [0,t], z \in \mathbb{R}^d\}$ has a measurable modification.*

Proof. By (25), $D_{r,z}u(t,x) = G(t-r, x-z)F(r,z)$, where

$$F(r,z) = \sum_{n \geq 1} n I_{n-1}(f_{n-1}(\cdot, r, z)) = \sum_{n \geq 0} (n+1) J_n(r,z),$$

and $J_n(r,z) = I_n(f_n(\cdot, r, z))$. Since the map $(r,z) \mapsto G(t-r, x-z)$ is measurable on $[0,t] \times \mathbb{R}^d$, it suffices to show that the process $F = \{F(r,z); r \in [0,t], z \in \mathbb{R}^d\}$ has a measurable modification. This will follow by Proposition B.1.a) (Appendix B), once we show that F is stochastically continuous. In fact, we will show that F is $L^p(\Omega)$-continuous, for any $p \geq 2$. For this, we use the same argument as in the proof of Theorem 7.1 of [2]. We denote by $\| \cdot \|_p$ the $L^p(\Omega)$-norm.

Using the equivalence of norms $\| \cdot \|_p$ for random variables in the same Wiener chaos space (see last line of p.62 of [22]) and (24), for any $r \in [0,t]$ and $z \in \mathbb{R}^d$, we have:

$$\begin{aligned} \|J_n(r,z)\|_p &\leq (p-1)^{n/2} \|J_n(r,z)\|_2 = (p-1)^{n/2} \left(\frac{1}{n!} \alpha_n(r) \right)^{1/2} \\ &\leq (p-1)^{n/2} e^{Mr/2} \left(\frac{2\Gamma_r}{M} K_M \right)^{n/2}. \end{aligned}$$

Using the fact that $n+1 \leq e^n$ and choosing M sufficiently large, it follows that

$$\sum_{n \geq 0} (n+1) \sup_{(r,z) \in [0,t] \times \mathbb{R}^d} \|J_n(r,z)\|_p \leq e^{Mt/2} \sum_{n \geq 0} e^n (p-1)^{n/2} \left(\frac{2\Gamma_t}{M} K_M \right)^{n/2} < \infty.$$

Hence, the sequence $\{F_n(r,z) = \sum_{k=0}^n (k+1) J_k(r,z); n \geq 1\}$ converges to $F(r,z)$ in $L^p(\Omega)$, uniformly in $(r,z) \in [0,t] \times \mathbb{R}^d$. Note that F_n is $L^p(\Omega)$-continuous for any n, since J_n is $L^p(\Omega)$-continuous for any n (by Lemma 7.1 of [2]). Therefore, F is $L^p(\Omega)$-continuous. □

Remark 3.7. For the remaining part of the article, we fix $t > 0$ and $x \in \mathbb{R}^d$, and we work with the measurable modification given by Theorem 3.6, which will be denoted also by $\{D_{r,z}u(t,x); r \in [0,t], z \in \mathbb{R}^d\}$.

By equation (25),

$$\mathbb{E}|D_{r,z}u(t,x)|^2 = A(r) G^2(t-r, x-z) 1_{\{r<t\}}, \tag{26}$$

where

$$A(t) := \sum_{n \geq 1} n^2 \mathbb{E}|I_{n-1}(f_{n-1}(\cdot, t, x))|^2 = \sum_{n \geq 0} \frac{(n+1)^2}{n!} \alpha_n(t). \tag{27}$$

Note that $A(t)$ is finite by Lemma 3.2, and is non-decreasing in t.

Similarly,

$$D^2_{(\theta,w),(r,z)} u(t,x) = 1_{\{\theta<r<t\}} G(t-r, x-z) G(r-\theta, z-w) \sum_{n \geq 2} n(n-1) I_{n-2}(f_{n-2}(\cdot, \theta, w))$$

and

$$\mathbb{E}|D^2_{(\theta,w),(r,z)}u(t,x)|^2 = B(\theta)G^2(t-r,x-z)G^2(r-\theta,z-w)1_{\{\theta<r<t\}}, \tag{28}$$

where

$$B(t) = \sum_{n\geq 2} n^2(n-1)^2\mathbb{E}|I_{n-2}(f_{n-2}(\cdot,t,x))|^2 = \sum_{n\geq 0} \frac{(n+2)^2(n+1)^2}{n!}\alpha_n(t).$$

Note that $B(t)$ is finite by Lemma 3.2, and is non-decreasing in t.

Let $(u_n)_{n\geq 0}$ be the sequence of Picard iterations given by: $u_0(t,x)=1$ and

$$u_n(t,x) = 1 + \sum_{k=1}^{n} I_k(f_k(\cdot,t,x)).$$

Then

$$D_{r,z}u_n(t,x) = 1_{\{r<t\}}G(t-r,x-z)\sum_{k=1}^{n} kI_{k-1}(f_k(\cdot,r,z))$$

and

$$\mathbb{E}|D_{r,z}u_n(t,x)|^2 = A_n(r)G^2(t-r,x-z)1_{\{r<t\}}, \tag{29}$$

where

$$A_n(t) := \sum_{k=1}^{n} k^2\mathbb{E}|I_{k-1}(f_{k-1}(\cdot,t,x))|^2 = \sum_{k=0}^{n-1} \frac{(k+1)^2}{k!}\alpha_k(t).$$

Also,

$$D^2_{(\theta,w),(r,z)}u_n(t,x) = 1_{\{\theta<r<t\}}G(t-r,x-z)G(r-\theta,z-w)\sum_{k=2}^{n} k(k-1)I_{k-2}(f_k(\cdot,\theta,w))$$

and

$$\mathbb{E}|D^2_{(\theta,w),(r,z)}u_n(t,x)|^2 = B_n(\theta)G^2(t-r,x-z)G^2(r-\theta,z-w)1_{\{\theta<r<t\}}, \tag{30}$$

where

$$B_n(t) := \sum_{k=2}^{n} k^2(k-1)^2\mathbb{E}|I_{k-2}(f_{k-2}(\cdot,t,x))|^2 = \sum_{k=0}^{n-2} \frac{(k+2)^2(k+1)^2}{k!}\alpha_k(t).$$

Let $\mathcal{P}_0$ be the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to the inner product

$$\langle g,h\rangle_0 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x)f(x-x')h(x')dxdx'. \tag{31}$$

The space $\mathcal{P}_0$ may contain distributions. But $\mathcal{P}_0$ includes the space $|\mathcal{P}_0|$ of all measurable functions $g:\mathbb{R}^d\to\mathbb{R}$ such that $\|g\|_+ := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|g(x)||g(x')|f(x-x')dxdx' < \infty$. Note that $\|g\|_0 \leq \|g\|_+$.

By (26) and Cauchy-Schwarz inequality, for any $r \in [0,t]$, $s \in [0,t]$ and $y \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}\|D_{r,\cdot}u(s,y)\|_+^2 &= \mathbb{E}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |D_{r,z}u(s,y)||D_{r,z'}u(s,y)|f(z-z')dzdz' \\ &\le A(r)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s-r,y-z)G(s-r,y-z')f(z-z')dzdz' \\ &= A(r)\,\|G(s-r,\cdot)\|_0^2.\end{aligned}$$

We use the fact that

$$\|G(s,\cdot)\|_0^2 = \int_{\mathbb{R}^d} \frac{\sin^2(s|\xi|)}{|\xi|^2}\mu(d\xi) \le C_s^* \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) =: C_s, \tag{32}$$

with $C_s^* = 2(1+s^2)$. From this, we deduce that

$$\mathbb{E}\|D_{r,\cdot}u(s,y)\|_0^2 \le C_s A(r) \quad \text{for all } s \in [0,t], y \in \mathbb{R}^d. \tag{33}$$

Note that this argument also shows $D_{r,\cdot}u(t,x)(\omega) \in |\mathcal{P}_0| \subset \mathcal{P}_0$, for almost all $\omega \in \Omega$.

Similarly, based on (29), we obtain:

$$\|D_{r,\cdot}u_n(s,y)\|_0^2 \le C_s A_n(r). \tag{34}$$

By Theorem 6.7 of [1], we know that Malliavin derivative Du of the solution satisfies the following equation in $L^2(\Omega;\mathcal{H})$:

$$D.u(t,x) = G(t-\cdot,x-\cdot)u(\cdot) + \int_0^t\int_{\mathbb{R}^d} U(s,y)W(\delta^* s,\delta^* y) \tag{35}$$

where $U = U^{(t,x)} \in L^2(\Omega;\mathcal{H}\otimes\mathcal{H})$ belongs to the domain of the δ^* operator (with $\mathcal{A} = \mathcal{H}$) and is given by:

$$U((r',z'),(s,y)) = G(t-s,x-y)D_{r',z'}u(s,y).$$

In what follows we will show that $D_{r,\cdot}u$ satisfies an equation similar to (35), but in $L^2(\Omega;\mathcal{P}_0)$ for fixed $r \in [0,t]$.

Let $\overline{\delta}$ be the $\mathcal{P}_0$-valued Skorohod integral defined in Section 6 of [1] with $\mathcal{A} = \mathcal{P}_0$. By Proposition 6.2 of [1], $\mathbb{D}^{1,2}(\mathcal{H}\otimes\mathcal{P}_0) \subset \text{Dom}\ \overline{\delta}$.

We fix $t>0$, $x \in \mathbb{R}^d$ and $r \in [0,t]$. For any $s \in [0,t]$, $y \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$, we let

$$\begin{aligned}K^{(r)}\big((s,y),z\big) &= 1_{[0,t]}(s)G(t-s,x-y)D_{r,z}u(s,y) \\ K_n^{(r)}\big((s,y),z\big) &= 1_{[0,t]}(s)G(t-s,x-y)D_{r,z}u_n(s,y).\end{aligned}$$

We denote

$$\phi(t) := \int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(s,y)G(s',y')\gamma(s-s')f(y-y')dydy'dsds' = \alpha_1(t), \tag{36}$$

where we recall that $\alpha_1(t)$ has been defined in (21).

The following result is similar to Lemma 6.5 of [1].

Lemma 3.8. *For any $r \in [0,t]$, $K^{(r)} \in \mathbb{D}^{1,2}(\mathcal{H}\otimes\mathcal{P}_0)$ and $K_n^{(r)} \in \mathbb{D}^{1,2}(\mathcal{H}\otimes\mathcal{P}_0)$.*

Proof. Recall that

$$\|K^{(r)}\|^2_{\mathbb{D}^{1,2}(\mathcal{H}\otimes\mathcal{P}_0)} = \mathbb{E}\|K^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{P}_0} + \mathbb{E}\|DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0}.$$

It suffices to show that both terms above are finite. Using Cauchy-Schwarz inequality and (33), we have:

$$\begin{aligned}
&\mathbb{E}\|K^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{P}_0}\\
&= \mathbb{E}\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y')\langle D_{r,\cdot}u(s,y)D_{r,\cdot}u(s',y')\rangle_0\\
&\qquad\qquad \times \gamma(s-s')f(y-y')dydy'dsds'\\
&\le E\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y')\Big(\mathbb{E}\|D_{r,\cdot}u(s,y)\|_0^2\Big)^{1/2}\Big(\mathbb{E}\|D_{r,\cdot}u(s',y')\|_0^2\Big)^{1/2}\\
&\qquad\qquad \times \gamma(s-s')f(y-y')dydy'dsds'\\
&\le C_tA(r)\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y'\gamma(s-s')f(y-y')dydy'dsds'\\
&= C_tA(r)\phi(t) < \infty.
\end{aligned}$$

A similar calculation shows that $\mathbb{E}\|K_n^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{P}_0} < C_tA_n(r)\phi(t)$, using (34).

Secondly,

$$\begin{aligned}
&\mathbb{E}\|DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0}\\
&= \int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y')\gamma(s-s')f(y-y')\\
&\times \int_{\mathbb{R}^{2d}}\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} \mathbb{E}[D^2_{(\theta,w),(r,z)}u(s,y), D^2_{(\theta',w'),(r,z')}u(s',y')]\,\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\\
&\qquad \times f(z-z')dzdz'dydy'dsds'.
\end{aligned}$$

Using Cauchy-Schwarz inequality and (28), we obtain that

$$\begin{aligned}
&|\mathbb{E}[D^2_{(\theta,w),(r,z)}u(s,y), D^2_{(\theta',w'),(r,z')}u(s',y')]|\\
&\le \Big(\mathbb{E}|D^2_{(\theta,w),(r,z)}u(s,y)|^2\Big)^{1/2}\Big(\mathbb{E}|D^2_{(\theta',w'),(r,z')}u(s',y')|^2\Big)^{1/2}\\
&\le B(r)G(s-r,y-z)G(r-\theta,z-w)G(s'-r,y'-z')\\
&\qquad \times G(r-\theta',z'-w')1_{\{\theta<r<s\}}1_{\{\theta'<r<s'\}}.
\end{aligned}$$

Note that, by Cauchy-Schwarz inequality, for any $z, z' \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(r-\theta,z-w)1_{\{\theta<r\}}G(r-\theta',z'-w')1_{\{\theta'<r\}}\gamma(\theta-\theta')f(w-w')d\theta d\theta' dwdw'\\
&\le \sup_{\bar z\in\mathbb{R}^d}\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(r-\theta,\bar z-w)G(r-\theta',\bar z-w')\gamma(\theta-\theta')f(w-w')d\theta d\theta' dwdw'\\
&= \int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(\theta,w)G(\theta',w')\gamma(\theta-\theta')f(w-w')d\theta d\theta' dwdw' = \phi(r).
\end{aligned}$$

Hence,

$$\mathbb{E}\|DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0} \leq B(r)\phi(r)\int_{[0,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y')\gamma(s-s')f(y-y')$$
$$1_{\{r<s\}}1_{\{r<s'\}}\left(\int_{\mathbb{R}^{2d}} G(s-r,y-z)G(s'-r,y'-z')f(z-z')dzdz'\right)dydy'dsds'.$$

By the Cauchy-Schwarz inequality, we see that for any $s,s'\in[0,t]$,

$$\int_{\mathbb{R}^{2d}} G(s-r,y-z)G(s'-r,y'-z')f(z-z')dzdz' \leq \|G(s-r,y-\cdot)\|_0\|G(s'-r,y'-\cdot)\|_0,$$

which is bounded by C_t due to (32). Hence,

$$\mathbb{E}\|DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0} \leq B(r)\phi(r)C_t\phi(t) < \infty.$$

A similar argument based on (30) shows that $\mathbb{E}\|DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0} \leq B_n(r)\phi(r)C_t\phi(t)$. □

We use the following more suggestive notation:

$$\overline{\delta}(K^{(r)}) = \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)D_{r,\cdot}u(s,y)W(\overline{\delta}s,\overline{\delta}y)$$

$$\overline{\delta}(K^{(r)}_n) = \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)D_{r,\cdot}u_n(s,y)W(\overline{\delta}s,\overline{\delta}y).$$

By Lemma 3.8, these stochastic integrals are well-defined.

The following result establishes a recursive relation involving the Malliavin derivative of the Picard iteration scheme (see Proposition 6.6 of [1] for a similar result).

Proposition 3.9. *For any $r\in[0,t]$, the following equality holds in $L^2(\Omega;\mathcal{P}_0)$:*

$$D_{r,\cdot}u_n(t,x) = G(t-r,x-\cdot)u_{n-1}(r,\cdot) + \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)D_{r,\cdot}u_{n-1}(s,y)W(\overline{\delta}s,\overline{\delta}y).$$

Proof. For any $r\in[0,t]$ and $z\in\mathbb{R}^d$, we denote

$$V_n(r,z) = \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)D_{r,z}u_n(s,y)W(\delta s,\delta y),$$

where the stochastic integral is the usual Skorohod integral. By *Step 2* of the proof of Proposition 6.6 of [1], we know that for any $r\in[0,t]$ and $z\in\mathbb{R}^d$,

$$D_{r,z}u_n(t,x) = G(t-r,x-z)u_{n-1}(r,z) + V_{n-1}(r,z).$$

The conclusion will follow once we prove that $\overline{\delta}(K^{(r)}_{n-1})$ coincides in $L^2(\Omega;\mathcal{P}_0)$ with $V_{n-1}(r,\cdot)$. By duality, it suffices to prove that for any $F\in\mathbb{D}^{1,2}(\mathcal{P}_0)$,

$$\mathbb{E}\langle F, V_{n-1}(r,\cdot)\rangle_0 = \mathbb{E}\langle DF, K^{(r)}_{n-1}\rangle_{\mathcal{H}\otimes\mathcal{P}_0}.$$

Without loss of generality, we may assume that $F = F_0\varphi$, where F_0 is a "smooth" random variable and $\varphi \in \mathcal{P}_0$. Then $DF = DF_0 \otimes \varphi$ and

$$\begin{aligned}\mathbb{E}\langle DF, K^{(r)}_{n-1}\rangle_{\mathcal{H}\otimes\mathcal{P}_0} &= \mathbb{E}\int_{[0,t]^2}\int_{\mathbb{R}^{4d}} G(t-s,x-y)D_{r,z}u_{n-1}(s,y)(D_{s',y'}F_0)\varphi(z') \\ &\qquad \gamma(s-s')f(y-y')f(z-z')dzdz'dydy'dsds' \\ &= \int_{\mathbb{R}^{2d}} f(z-z')\varphi(z')E\langle DF_0, G(t-\cdot,x-\cdot)D_{r,z}u_{n-1}\rangle_{\mathcal{H}}dzdz'.\end{aligned}$$

By the duality between δ and D,

$$\mathbb{E}\langle DF_0, G(t-\cdot,x-\cdot)D_{r,z}u_{n-1}\rangle_{\mathcal{H}} = \mathbb{E}[\delta(G(t-\cdot,x-\cdot)D_{r,z}u_{n-1})F_0] = \mathbb{E}[V_{n-1}(r,z)F_0].$$

Hence,

$$\mathbb{E}\langle DF, K^{(r)}_{n-1}\rangle_{\mathcal{H}\otimes\mathcal{P}_0} = \mathbb{E}\int_{\mathbb{R}^{2d}} f(z-z')V_{n-1}(r,z)F_0\varphi(z')dzdz' = \mathbb{E}\langle F, V_{n-1}(r,\cdot)\rangle_0.$$

□

In the following theorem, we prove that the Malliavin derivative $Du(t,x)$ satisfies an equation in the space $\mathcal{P}_0$.

Theorem 3.10. *For any $r \in [0,t]$, the following equality holds in $L^2(\Omega;\mathcal{P}_0)$:*

$$D_{r,\cdot}u(t,x) = G(t-r,x-\cdot)u(r,\cdot) + \int_0^t\int_{\mathbb{R}^d} G(t-s,x-y)D_{r,\cdot}u(s,y)W(\overline{\delta}s,\overline{\delta}y). \tag{37}$$

Proof. Recall that the stochastic integral on the right-hand side of (37) is $\overline{\delta}(K^{(r)})$. By duality, it suffices to prove that for any $F \in \mathbb{D}^{1,2}(\mathcal{P}_0)$,

$$\mathbb{E}\langle D_{r,\cdot}u(t,x) - G(t-r,x-\cdot)u(r,\cdot), F\rangle_0 = \mathbb{E}\langle DF, K^{(r)}\rangle_{\mathcal{H}\otimes\mathcal{P}_0}. \tag{38}$$

By Proposition 3.9, we know that for any $F \in \mathbb{D}^{1,2}(\mathcal{P}_0)$,

$$\mathbb{E}\langle D_{r,\cdot}u_n(t,x) - G(t-r,x-\cdot)u_{n-1}(r,\cdot), F\rangle_0 = \mathbb{E}\langle DF, K^{(r)}_{n-1}\rangle_{\mathcal{H}\otimes\mathcal{P}_0}. \tag{39}$$

Relation (39) is obtained by taking $n \to \infty$ in (38). We justify this below.

On the right-hand side, using duality and Cauchy-Schwarz inequality, we have:

$$\begin{aligned}\mathbb{E}\langle DF, K^{(r)}_{n-1} - K^{(r)}\rangle_{\mathcal{H}\otimes\mathcal{P}_0} &= \mathbb{E}\langle\overline{\delta}(K^{(r)}_{n-1} - K^{(r)}), F\rangle_0 \\ &\leq \big(\mathbb{E}\|\overline{\delta}(K^{(r)}_{n-1} - K^{(r)})\|_0^2\big)^{1/2}\big(\mathbb{E}\|F\|_0^2\big)^{1/2}.\end{aligned}$$

By Proposition 6.2 of [1],

$$\mathbb{E}\|\overline{\delta}(K^{(r)}_{n-1} - K^{(r)})\|_0^2 \leq \mathbb{E}\|K^{(r)}_{n-1} - K^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{P}_0} + \mathbb{E}\|DK^{(r)}_{n-1} - DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0}.$$

As in the proof of Lemma 3.8, it can be shown that

$$\mathbb{E}\|K^{(r)}_{n-1} - K^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{P}_0} \leq C_t\phi(t)\big(A(r) - A_{n-1}(r)\big) \to 0 \quad \text{as } n \to \infty.$$

and

$$\mathbb{E}\|DK^{(r)}_{n-1} - DK^{(r)}\|^2_{\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{P}_0} \leq C_t\phi(r)\phi(t)\big(B(r) - B_{n-1}(r)\big) \to 0 \quad \text{as } n \to \infty.$$

We now treat the left-hand side of (39), which is a difference of two terms. For the first term, by the Cauchy-Schwarz inequality,

$$\mathbb{E}\langle D_{r,\cdot}u_n(t,x) - D_{r,\cdot}u(t,x), F\rangle_0 \leq \Big(\mathbb{E}\|D_{r,\cdot}u_n(t,x) - D_{r,\cdot}u(t,x)\|_0^2\Big)^{1/2}\Big(\mathbb{E}\|F\|_0^2\Big)^{1/2}.$$

Similarly to (33), it can be shown that

$$\mathbb{E}\|D_{r,\cdot}u_n(t,x) - D_{r,\cdot}u(t,x)\|_0^2 \leq C_t\big(A(r) - A_n(r)\big) \to 0 \quad \text{as } n \to \infty.$$

For the second term, we use again Cauchy-Schwarz inequality, and we observe that

$$\mathbb{E}\|G(t-r, x-\cdot)(u_{n-1}(r,\cdot) - u(r,\cdot))\|_0^2 = \int_{\mathbb{R}^{2d}} G(t-r, x-z)G(t-r', x-z') \\ \times \mathbb{E}[(u_{n-1}(r,z) - u(r,z))(u_{n-1}(r,z') - u(r,z'))]f(z-z')dzdz' \to 0,$$

as $n \to \infty$. This follows by the Cauchy-Schwarz inequality and the dominated convergence theorem, using the fact that

$$\sup_{(s,y)\in[0,t]\times\mathbb{R}^d} \mathbb{E}|u_n(s,y) - u(s,y)| \to 0,$$

which was shown in the proof of Theorem 7.1 of [2]. □

Remark 3.11. In what follows, we will use the following important consequence of Theorem 3.10. If we denote by $B(r,\cdot)$ the right-hand side of (37), then obviously

$$\mathbb{E}\int_0^t \|D_{r,\cdot}u(t,x) - B(r,\cdot)\|_0^2 dr = 0.$$

Hence, there exists a measurable set $N \subset \Omega \times [0,t]$ with $(P \times \lambda)(N) = 0$ (where λ is the Lebesgue measure on $\mathbb{R}$), such that for all $(\omega, r) \in (\Omega \times [0,t])\backslash N$

$$\|D_{r,\cdot}u(t,x)(\omega)\|_0 = \|B(r,\cdot)(\omega)\|_0. \tag{40}$$

Remark 3.12. In particular, from (37), we deduce that the Malliavin derivative Du satisfies the following equation in $L^2\big(\Omega; L^2([0,t];\mathcal{P}_0)\big)$:

$$D.u(t,x) = G(t-\cdot, x-\cdot)u(\cdot) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y)D.u(s,y)W(\overline{\delta}s, \overline{\delta}y).$$

This equation is different than equation (35), which holds in $L^2(\Omega;\mathcal{H})$ and contains the operator δ^*. We note in passing that the space $L^2([0,t];\mathcal{P}_0)$ is included in $\mathcal{H}$. This can be seen using the same argument as on page 281 of [22].

4 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2, which is based on Corollary 2.4 and Lemma 4.3 below.

We begin with a general result about absolute continuity of the law of a random variable, which corresponds to the remark right after the proof of Theorem 2.1.1 in [22]. We include the proof for the sake of completeness.

Lemma 4.1. *If F is a random variable such that $F \in \mathbb{D}^{2,p}$ for some $p > 1$, then the measure $(\|DF\|_{\mathcal{H}}^2 \cdot \mathbb{P}) \circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. In particular, for any Borel set $B \subset \mathbb{R}$ with Lebesgue measure zero,*

$$\mathbb{P}(F \in B, \|DF\|_{\mathcal{H}} > 0) = 0. \tag{41}$$

Proof. Let $\varphi \in C_b^\infty(\mathbb{R})$ be arbitrary. By the chain rule for the Malliavin derivative (see Proposition 1.2.3 of [22]), $\varphi(F) \in \mathbb{D}^{1,2}$ and $D(\varphi(F)) = \varphi'(F)DF$. Taking the scalar product with DF, we obtain that

$$\langle D(\varphi(F)), DF\rangle_{\mathcal{H}} = \varphi'(F)\|DF\|_{\mathcal{H}}^2.$$

Note that $F \in \mathbb{D}^{2,p}$ implies that $DF \in \mathbb{D}^{1,p}(\mathcal{H})$. (Recall that $\mathbb{D}^{1,p}(\mathcal{H})$ is the completion of the space $\mathcal{S}(\mathcal{H})$ of "smooth" $\mathcal{H}$-valued random variables F with respect to the norm $\|F\|_{\mathbb{D}^{1,p}(\mathcal{H})} = \left(\mathbb{E}\|F\|_{\mathcal{H}}^p + \mathbb{E}\|DF\|_{\mathcal{H}\otimes\mathcal{H}}^p\right)^{1/p}$.) Since $\mathbb{D}^{1,2}(\mathcal{H}) \subset \operatorname{Dom}\delta$, it follows that $DF \in \operatorname{Dom}\delta$. By definition (19) of $\operatorname{Dom}\delta$, we have

$$\left|\mathbb{E}\big(\|DF\|_{\mathcal{H}}^2\varphi'(F)\big)\right| = \left|\mathbb{E}\big(\langle D\varphi(F), DF\rangle_{\mathcal{H}}\big)\right| \le c\big(\mathbb{E}|\varphi(F)|^2\big)^{\frac{1}{2}} \le c\|\varphi\|_\infty.$$

We have thus proved that, for all $\varphi \in C_b^\infty(\mathbb{R})$,

$$\left|\int_{\mathbb{R}} \varphi'(x)\big((\|DF\|_{\mathcal{H}}^2 \cdot \mathbb{P}) \circ F^{-1}\big)(dx)\right| \le c\|\varphi\|_\infty.$$

By Lemma 2.1.1 of [22], the measure $\mu = (\|DF\|_{\mathcal{H}}^2 \cdot P) \circ F^{-1}$ is absolutely continuous with respect to the Lebesgue measure.

To prove the last statement, let $B \subset \mathbb{R}$ be a Borel set with Lebesgue measure zero. Then $\int_{F^{-1}(B)} \|DF\|_{\mathcal{H}}^2 dP = \mu(B) = 0$, which implies (41). □

We continue with a result about existence of density of a truncated random variable. First note that if Γ is a Borel set in $\mathbb{R}$ such that $0 \notin \Gamma$, then the law of the truncated variable $G = F1_{\{F\in\Gamma\}}$ is a probability measure $\mathbb{P}_G$ on $\Gamma \cup \{0\}$ given by $\mathbb{P}_G(A) = \mathbb{P}(F \in A)$ for any Borel set $A \subset \Gamma$, and $\mathbb{P}_G(\{0\}) = \mathbb{P}(G = 0) = \mathbb{P}(F \notin \Gamma)$.

Lemma 4.2. *Let Γ be a Borel set in $\mathbb{R}$ such that $0 \notin \Gamma$. Let $F \in \mathbb{D}^{2,p}$ for some $p > 1$ be such that*

$$\|DF\|_{\mathcal{H}} > 0 \quad a.s. \text{ on} \quad \{F \in \Gamma\}. \tag{42}$$

Then, the restriction of the law of the variable $G = F1_{\{F\in\Gamma\}}$ to the set Γ is absolutely continuous with respect to the Lebesgue measure on Γ.

Proof. Let $B \subset \mathbb{R}$ be a Borel set with zero Lebesgue measure. By (42), we have

$$\mathbb{P}(F \in \Gamma, \|DF\|_{\mathcal{H}} = 0) = 0. \tag{43}$$

By (41) and (43), we obtain:

$$\mathbb{P}(F \in B, F \in \Gamma) = \mathbb{P}(F \in B, F \in \Gamma, \|DF\|_{\mathcal{H}} > 0) + \mathbb{P}(F \in B, F \in \Gamma, \|DF\|_{\mathcal{H}} = 0) = 0.$$

□

Lemma 4.3. *Let $(\Gamma_m)_{m\geq 1}$ be a sequence of open sets in $\mathbb{R}$ such that $0 \notin \Gamma_m$ and $\Gamma_m \subset \Gamma_{m+1}$, for all $m \geq 1$. Let $\Gamma = \cup_{m\geq 1}\Gamma_m$. Let $F \in \mathbb{D}^{2,p}$ for some $p > 1$ be such that, for all $m \geq 1$,*

$$\|DF\|_{\mathcal{H}} > 0 \quad a.s.\ on \quad \{F \in \Gamma_m\}.$$

Then, the restriction of the law of the variable $F1_{\{F\in\Gamma\}}$ to the set Γ is absolutely continuous with respect to the Lebesgue measure on Γ.

Proof. Let $B \subset \mathbb{R}$ be a Borel set with zero Lebesgue measure. By Lemma 4.2, $\mathbb{P}(F \in B, F \in \Gamma_m) = 0$ for all $m \geq 1$. By taking $m \to \infty$, we infer that $\mathbb{P}(F \in B, F \in \Gamma) = 0$. □

Recall that $u(t, x) \in \mathbb{D}^{1,2}$ and, by (26), we have, for all $(r, z) \in [0, t] \times \mathbb{R}^d$,

$$\mathbb{E}|D_{r,z}u(t, x)|^2 \leq A(t)\, G^2(t - r, x - z), \tag{44}$$

where $A(t)$ is the constant given by (27).

Proof of Theorem 1.2: We will apply Lemma 4.3 in the case $F = u(t, x)$ and $\Gamma_m = \{v \in \mathbb{R}; |v| > \frac{1}{m}\}$. For any $m \geq 1$, let $\Omega_m := \{|u(t, x)| > \frac{1}{m}\} \cap \widetilde{\Omega}$, where $\widetilde{\Omega}$ is an event of probability 1 which will be defined below. By Lemma 3.3, $u(t, x) \in \mathbb{D}^{2,p}$ for any $p > 1$. We will prove that

$$\|Du(t, x)\|_{\mathcal{H}} > 0 \quad \text{a.s. on } \Omega_m.$$

In view of Corollary 2.4, it is enough to prove that

$$\int_0^t \|D_{r\cdot}u(t, x)\|_0^2\, dr > 0 \quad \text{a.s. on } \Omega_m. \tag{45}$$

By Remark 3.7, the map $(r, z) \mapsto D_{r,z}u(t, x)(\omega)$ is measurable on $\mathbb{R}_+ \times \mathbb{R}^d$, for any $\omega \in \Omega$.

We use Remark 3.11. By Fubini's theorem, $(P \times \lambda)(N) = \int_\Omega \lambda(N_\omega)P(d\omega)$, where $N_\omega := \{r \in [0, t]; (\omega, r) \in N\}$ is the section of N at point $\omega \in \Omega$. Since $(P \times \lambda)(N) = 0$, $\lambda(N_\omega) = 0$ for P-almost all $\omega \in \Omega$. Say this happens on the event $\widetilde{\Omega}$ of probability 1.

Let $A = \big(\Omega \times [0, t]\big)\backslash N$. Note that the section of A at $\omega \in \Omega$ is the set

$$A_\omega := \{r \in [0, t]; (\omega, r) \in A\} = \{r \in [0, t]; (\omega, r) \notin N\} = [0, t] - N_\omega.$$

Fix $\omega \in \widetilde{\Omega}$. For any $r \in A_\omega$, equality (40) holds. Using the inequality $\|a+b\|_0^2 \geq \frac{1}{2}\|a\|_0^2 - \|b\|_0^2$, we obtain:

$$\begin{aligned}\|D_{r\cdot}u(t,x)\|_0^2 &= \left\| G(t-r,x-\cdot)u(r,\cdot) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r\cdot}u(s,y)W(\overline{\delta}s,\overline{\delta}y)\right\|_0^2 \\ &\geq \frac{1}{2}\|G(t-r,x-\cdot)u(r,\cdot)\|_0^2 - \left\|\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r\cdot}u(s,y)W(\overline{\delta}s,\overline{\delta}y)\right\|_0^2.\end{aligned}$$

Let $\delta \in (0,1)$ be arbitrary. Taking the integral with respect to r on A_ω (and using the fact that $\lambda(N_\omega)=0$), we obtain that on $\widetilde{\Omega}$,

$$\begin{aligned}\int_0^t \|D_{r\cdot}u(t,x)\|_0^2\, dr &\geq \int_{t-\delta}^t \|D_{r\cdot}u(t,x)\|_0^2\, dr \\ &\geq \frac{1}{2}\int_{t-\delta}^t \|G(t-r,x-\cdot)u(r,\cdot)\|_0^2\, dr - I(\delta), \qquad (46)\end{aligned}$$

where

$$\begin{aligned}I(\delta) &= \int_{t-\delta}^t \left\|\int_0^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r\cdot}u(s,y)W(\overline{\delta}s,\overline{\delta}y)\right\|_0^2 dr \\ &= \int_{t-\delta}^t \left\|\int_{t-\delta}^t \int_{\mathbb{R}^d} G(t-s,x-y)D_{r\cdot}u(s,y)W(\overline{\delta}s,\overline{\delta}y)\right\|_0^2 dr.\end{aligned}$$

The second equality is due to the fact that $D_{r\cdot}u(s,y)=0$ if $r>s$, and so we must have $s \geq r \geq t-\delta$. On the event Ω_m,

$$\begin{aligned}&\int_{t-\delta}^t \|G(t-r,x-\cdot)u(r,\cdot)\|_0^2\, dr \\ &\geq \frac{1}{m^2}\int_{t-\delta}^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-r,x-z)G(t-r,x-z')f(z-z')dzdz'dr - J(\delta) \\ &= \frac{1}{m^2}\int_0^\delta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(r,z)G(r,z')f(z-z')dzdz'dr - J(\delta),\end{aligned}$$

where

$$J(\delta) = \int_{t-\delta}^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-r,x-z)G(t-r,x-z')\big(u(t,x)^2 - u(r,z)u(r,z')\big)f(z-z')dzdz'dr.$$

Set

$$\psi(\delta) := \int_0^\delta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(r,z)G(r,z')f(z-z')dzdz'dr.$$

Note that, by Dalang's condition, $\psi(\delta) < \infty$ and $\psi(\delta)$ converges to zero as $\delta \to 0$.

By (46), we have that

$$\int_0^t \|D_{r,\cdot}u(t,x)\|_0^2\, dr \geq \frac{1}{2m^2}\psi(\delta) - \frac{1}{2}J(\delta) - I(\delta). \qquad (47)$$

Let us now estimate the first moment of $J(\delta)$ and $I(\delta)$, respectively. To start with, we have

$$\mathbb{E}[|J(\delta)|] \leq \int_{t-\delta}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-r, x-z)G(t-r, x-z')f(z-z') \\ \times \mathbb{E}\big[|u(t,x)^2 - u(r,z)u(r,z')|\big] dzdz'dr, \tag{48}$$

where we observe that

$$\mathbb{E}\big[|u(t,x)^2 - u(r,z)u(r,z')|\big] \leq \big(\mathbb{E}|u(r,z)|^2\big)^{\frac{1}{2}}\big(\mathbb{E}|u(r,z') - u(t,x)|^2\big)^{\frac{1}{2}} \\ + \big(\mathbb{E}|u(t,x)|^2\big)^{\frac{1}{2}}\big(\mathbb{E}|u(r,z) - u(t,x)|^2\big)^{\frac{1}{2}}.$$

By Theorem 7.1 of [2], we have that

$$\sup_{(s,y)\in[0,t]\times\mathbb{R}^d} \mathbb{E}(|u(s,y)|^2) < \infty.$$

Moreover, by Theorem 8.3 in [2], since we are assuming condition (9) the process $\{u(r,z);\ r \in [0,t], z \in \mathbb{R}^d\}$ has a modification with Hölder-continuous paths of order $(1-\beta)-\varepsilon$ for any $\varepsilon > 0$. Thus, taking into account that $r \in [t-\delta, t]$ and the fundamental solution $G(t-r, x-z)$ contains the indicator function $1_{\{|x-z|<t-r\}}$ (resp. $G(t-r, x-z')$ contains $1_{\{|x-z'|<t-r\}}$), we have

$$\mathbb{E}\big[|u(t,x)^2 - u(r,z)u(r,z')|\big] \leq C'_t\big(|t-r|^{1-\beta} + |z'-x|^{1-\beta}\big)^{\frac{1}{2}} + \big(|t-r|^{1-\beta} + |z-x|^{1-\beta}\big)^{\frac{1}{2}} \\ \leq C'_t \delta^{\frac{1-\beta}{2}},$$

where $C'_t > 0$ is a constant depending on t. Plugging this estimate in (48), we obtain that

$$\mathbb{E}[|J(\delta)|] \leq C'_t\, \delta^{\frac{1-\beta}{2}} \psi(\delta). \tag{49}$$

Now we proceed to bound $\mathbb{E}[|I(\delta)|]$. Indeed, we have that

$$\mathbb{E}[I(\delta)] = \int_{t-\delta}^{t} \mathbb{E}\left\| \int_{t-\delta}^{t} \int_{\mathbb{R}^d} G(t-s, x-y) D_{r,\cdot} u(s,y) W(\overline{\delta} s, \overline{\delta} y) \right\|_0^2 dr.$$

At this point, we apply Proposition 6.2 of [1] to get that

$$\mathbb{E}[I(\delta)] \leq I_1(\delta) + I_2(\delta),$$

where the latter two terms are the following:

$$I_1(\delta) = \mathbb{E} \int_{t-\delta}^{t} \int_{[t-\delta,t]^2} \int_{\mathbb{R}^{2d}} G(t-s, x-y)G(t-s', x-y')\gamma(s-s')f(y-y') \\ \times \langle D_{r\cdot} u(s,y), D_{r\cdot} u(s',y') \rangle_0\, dydy'dsds'dr,$$

and

$$
\begin{aligned}
I_2(\delta) = \mathbb{E} \int_{t-\delta}^{t} \int_{[t-\delta,t]^2} \int_{\mathbb{R}^{2d}} \int_{[0,t]^2} \int_{\mathbb{R}^{2d}} & G(t-s, x-y) G(t-s', x-y') \gamma(s-s') f(y-y') \\
& \times \langle D^2_{(\theta,w),(r,\cdot)} u(s,y), D^2_{(\theta',w'),(r,\cdot)} u(s',y') \rangle_0 \, \gamma(\theta-\theta') f(w-w') \\
& \times dw dw' d\theta d\theta' dy dy' ds ds' dr.
\end{aligned}
$$

In the above terms, we have used the inner product $\langle \cdot, \cdot \rangle_0$ given by (31).

In order to deal with $I_1(\delta)$, we observe that, by Cauchy-Schwarz inequality and (44),

$$
\begin{aligned}
& \int_{t-\delta}^{t} \mathbb{E} \langle D_{r\cdot} u(s,y), D_{r\cdot} u(s',y') \rangle_0 \, dr \\
& \leq \int_{t-\delta}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\mathbb{E} |D_{r,z} u(s,y)|^2 \right)^{\frac{1}{2}} \left(\mathbb{E} |D_{r,z'} u(s',y')|^2 \right)^{\frac{1}{2}} f(z-z') dz dz' dr \\
& \leq A(t) \int_{t-\delta}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(s-r, y-z) G(s'-r, y'-z) f(z-z') dz dz' dr \\
& = A(t) \int_{t-\delta}^{t} \langle G(s-r, y-\cdot), G(s'-r, y'-\cdot) \rangle_0 \, dr \\
& \leq A(t) \left(\int_{t-\delta}^{t} \| G(s-r, y-\cdot) \|_0^2 \, dr \right)^{\frac{1}{2}} \left(\int_{t-\delta}^{t} \| G(s'-r, y'-\cdot) \|_0^2 \, dr \right)^{\frac{1}{2}} \\
& \leq A(t) \sup_{(s,y)s \in [t-\delta,t] \times \mathbb{R}^d} \int_{t-\delta}^{t} \| G(s-r, y-\cdot) \|_0^2 \, dr.
\end{aligned}
$$

Note that the above supremun equals to

$$
\begin{aligned}
& \sup_{s \in [t-\delta,t]} \int_{t-\delta}^{s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(s-r, z) G(s-r, z') f(z-z') dz dz' dr \\
& \quad = \sup_{s \in [t-\delta,t]} \int_{0}^{s-(t-\delta)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(r,z) G(r,z') f(z-z') dz dz' dr \\
& \quad = \int_0^{\delta} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(r,z) G(r,z') f(z-z') dz dz' dr \\
& \quad = \psi(\delta).
\end{aligned} \tag{50}
$$

Thus, we have proved that

$$
\int_{t-\delta}^{t} \mathbb{E} \langle D_{r,\cdot} u(s,y), D_{r,\cdot} u(s',y') \rangle_0 \, dr \leq A(t) \psi(\delta),
$$

for all $s, s' \in [t-\delta, t]$ and all $y, y' \in \mathbb{R}^d$. This implies that

$$
\begin{aligned}
& I_1(\delta) \\
& \leq A(t) \psi(\delta) \int_{t-\delta}^{t} \int_{t-\delta}^{t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-y) G(t-s', x-y') \gamma(s-s') f(y-y') ds ds' dy dy' \\
& = A(t) \psi(\delta) \phi(\delta),
\end{aligned} \tag{51}
$$

with $\phi(t)$ given by (36). Observe that $\phi(\delta)$ is well-defined and converges to zero as $\delta \to 0$.

As far as the term $I_2(\delta)$ is concerned, one follows analogous arguments as in $I_1(\delta)$. More precisely, we first observe that, using Cauchy-Schwarz inequality and (28)

$$\begin{aligned}
&I_2(\delta)\\
&\leq B(t)\int_{[t-\delta,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y')\gamma(s-s')f(y-y')\\
&\quad\times\int_{t-\delta}^{s}\int_{\mathbb{R}^{2d}} G(s-r,y-z)G(s'-r,y'-z')f(z-z') \qquad (52)\\
&\quad\times\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(r-\theta,z-w)G(r-\theta',z'-w')\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\\
&\quad\times dzdz'drdydy'dsds'. \qquad (53)
\end{aligned}$$

Note that, by Cauchy-Schwarz inequality for any $t-\delta<r<s<t$ and $z,z'\in\mathbb{R}^d$,

$$\begin{aligned}
&\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(r-\theta,z-w)G(r-\theta',z'-w')\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\\
&\leq\left(\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(r-\theta,z-w)G(r-\theta',z-w')\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\right)^{1/2}\\
&\quad\left(\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(r-\theta,z'-w)G(r-\theta',z'-w')\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\right)^{1/2}\\
&\leq\sup_{(\bar r,\bar z)\in[t-\delta,s]\times\mathbb{R}^d}\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(\overline{r}-\theta,\overline{z}-w)G(\overline{r}-\theta',\overline{z}-w')\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\\
&=\sup_{\overline{r}\in[t-\delta,s]}\int_{[0,r]^2}\int_{\mathbb{R}^{2d}} G(\overline{\theta},\overline{w})G(\overline{\theta}',\overline{w}')\gamma(\overline{\theta}-\overline{\theta}'f(\overline{w}-\overline{w}')d\overline{w}d\overline{w}'d\overline{\theta}d\overline{\theta}'\\
&=\int_{[0,s]^2}\int_{\mathbb{R}^{2d}} G(\theta,w)G(\theta',w')\gamma(\theta-\theta')f(w-w')dwdw'd\theta d\theta'\\
&=\phi(s)\leq\phi(t),
\end{aligned}$$

where the first equality follows using the change of variables $\overline{\theta}=\overline{r}-\theta,\overline{\theta}'=\overline{r}-\theta',\overline{w}=\overline{z}-w,\overline{w}'=\overline{z}-w'$. Plugging the latter estimate in (53), we end up with

$$\begin{aligned}
&I_2(\delta)\\
&\leq B(t)\phi(t)\int_{[t-\delta,t]^2}\int_{\mathbb{R}^{2d}} G(t-s,x-y)G(t-s',x-y')\gamma(s-s')f(y-y')dsds'dydy'\\
&\quad\times\sup_{(s,y)\in[t-\delta,t]\times\mathbb{R}^d}\int_{t-\delta}^{s}\int_{\mathbb{R}^{2d}} G(s-r,y-z)G(s'-r,y'-z')f(z-z')dzdz'dr\\
&=B(t)\phi(t)\psi(\delta)\phi(\delta),
\end{aligned}$$

where for the last equality we used (50). Thus, we have proved that

$$I_2(\delta)\leq B(t)\phi(t)\psi(\delta)\phi(\delta). \qquad (54)$$

Using (51) and (54), we obtain:

$$\mathbb{E}[I(\delta)] \leq \Big(A(t) + B(t)\phi(t)\Big)\psi(\delta)\phi(\delta) =: C''_t\psi(\delta)\phi(\delta). \tag{55}$$

Taking into account the latter estimate and the one obtained for $\mathbb{E}[|J(\delta|]$ (see (49)), we will be able to conclude the proof, as follows. By (47) and Markov's inequality, for any $n \geq 1$, we have

$$\begin{aligned}\mathbb{P}\left(\left\{\int_0^t \|D_{r\cdot}u(t,x)\|_0^2\,dr < \frac{1}{n}\right\} \cap \Omega_m\right) &\leq \mathbb{P}\left(I(\delta) + \frac{1}{2}J(\delta) > \frac{1}{2m^2}\psi(\delta) - \frac{1}{n}\right)\\ &\leq \left(\frac{1}{2m^2}\psi(\delta) - \frac{1}{n}\right)^{-1}\left(\mathbb{E}[I(\delta)] + \frac{1}{2}\mathbb{E}[|J(\delta)|]\right)\\ &\leq \left(\frac{1}{2m^2}\psi(\delta) - \frac{1}{n}\right)^{-1}\psi(\delta)\Big(C''_t\phi(\delta) + \frac{1}{2}C'_t\delta^{\frac{1-\beta}{2}}\Big).\end{aligned}$$

Taking $n \to \infty$, one gets

$$\mathbb{P}\left(\left\{\int_0^t \|D_r u(t,x)\|_0^2\,dr = 0\right\} \cap \Omega_m\right) \leq \left(\frac{1}{2m^2}\psi(\delta)\right)^{-1}\psi(\delta)\Big(C''_t\phi(\delta) + \frac{1}{2}C'_t\delta^{\frac{1-\beta}{2}}\Big)$$

Next, we take $\delta \to 0$. Since $\phi(\delta) \to 0$ as $\delta \to 0$ and $\beta < 1$, we obtain:

$$\mathbb{P}\left(\left\{\int_0^t \|D_r u(t,x)\|_0^2\,dr = 0\right\} \cap \Omega_m\right) = 0.$$

This concludes the proof of (45), and the proof of Theorem 1.2.

A A Parseval-type identity

In this section, we give the Parseval-type identity which is used in the proof of Lemma A.2. We begin by recalling a remarkable result of [17], and comment on a small correction of a related result from the same paper.

Let $f : \mathbb{R}^d \to [0,\infty]$ be a kernel of *positive type*, i.e. f is locally integrable and its Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ is a function g which is non-negative almost everywhere. In addition, we suppose that f is continuous, symmetric and $f(x) < \infty$ if and only if $x \neq 0$.

By Lemma 5.6 of [17], for any Borel probability measure μ on $\mathbb{R}^d$, we have:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\mu(dx)\mu(dy) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\mathcal{F}\mu(\xi)|^2 g(\xi)d\xi =: \mathcal{E}_f(\mu).$$

In particular, if $\mu(dx) = \varphi(x)dx$ where φ is a density function on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi(x)\varphi(y)dxdy = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 g(\xi)d\xi =: \mathcal{E}_f(\varphi). \tag{56}$$

It follows that relation (56) holds for any *non-negative* function $\varphi \in L^1(\mathbb{R}^d)$.

Relation (5.37) of [17] says that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\mu(dx)\nu(dy) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}\mu(\xi)\overline{\mathcal{F}\nu(\xi)}g(\xi)d\xi \tag{57}$$

for any Borel probability measures μ and ν on $\mathbb{R}^d$. When $\nu=\delta_0$ and $\mu=\delta_x$ for $x\in\mathbb{R}$ arbitrary, this relation becomes

$$f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\xi\cdot x}g(\xi)d\xi, \quad x\in\mathbb{R},$$

which is not true if g is not integrable. The problem is caused by the fact that in the proof of (5.37), on the right hand side of (5.39), we may have $\infty-\infty$.

But (57) *does hold* for any Borel probability measures μ and ν on $\mathbb{R}^d$ such that $\mathcal{E}_f(\mu)<\infty$ and $\mathcal{E}_f(\nu)<\infty$. In particular,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi(x)\psi(y)dxdy = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}g(\xi)d\xi =: \mathcal{E}_f(\varphi,\psi) \tag{58}$$

for any density functions φ and ψ on $\mathbb{R}^d$ with $\mathcal{E}_f(\varphi)<\infty$ and $\mathcal{E}_f(\psi)<\infty$.

It follows that relation (58) holds for any *non-negative* functions $\varphi,\psi\in L^1(\mathbb{R}^d)$ with $\mathcal{E}_f(\varphi)<\infty$ and $\mathcal{E}_f(\psi)<\infty$. (To see this, we write (58) for the density functions $\varphi/\|\varphi\|_1$ and $\psi/\|\psi\|_1$ and then we multiply by $\|\varphi\|_1\|\psi\|_1$.) Moreover, in this case $|\mathcal{E}_f(\varphi,\psi)|<\infty$ since by the Cauchy-Schwarz inequality

$$\left|\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}g(\xi)d\xi\right| \leq \left(\int_{\mathbb{R}^d}|\mathcal{F}\varphi(\xi)|^2 g(\xi)d\xi\right)^{1/2}\left(\int_{\mathbb{R}^d}|\mathcal{F}\psi(\xi)|^2 g(\xi)d\xi\right)^{1/2}$$

In fact, we have the following more general result.

Lemma A.1. *Relation* (58) *holds for any functions* $\varphi,\psi\in L^1(\mathbb{R}^d)$ *with* $\mathcal{E}_f(|\varphi|)<\infty$ *and* $\mathcal{E}_f(|\psi|)<\infty$*, and in this case* $|\mathcal{E}_f(\varphi,\psi)|<\infty$*.*

Proof: We write $\varphi=\varphi^+-\varphi^-$, where $\varphi^+=\max(\varphi,0)$ and $\varphi^-=\max(-\varphi,0)$. Similarly, $\psi=\psi^+-\psi^-$, where $\psi^+=\max(\psi,0)$ and $\psi^-=\max(-\psi,0)$. Then $|\varphi|=\varphi^++\varphi^-$. Since $\varphi^+(x)\leq|\varphi(x)|$ for any $x\in\mathbb{R}$, we have

$$\mathcal{E}_f(\varphi^+)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi^+(x)\varphi^+(y)dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)|\varphi(x)||\varphi(y)|dxdy=\mathcal{E}_f(|\varphi|)<\infty.$$

Similarly, we obtain that $\mathcal{E}_f(\varphi^-)<\infty$, $\mathcal{E}_f(\psi^+)<\infty$ and $\mathcal{E}_f(\psi^-)<\infty$.

Hence relation (58) holds for the pairs (φ^+,ψ^+), (φ^-,ψ^-), (φ^+,ψ^-) and (φ^-,ψ^+):

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi^+(x)\psi^+(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi^+(\xi)\overline{\mathcal{F}\psi^+(\xi)}g(\xi)d\xi\\
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi^-(x)\psi^-(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi^-(\xi)\overline{\mathcal{F}\psi^-(\xi)}g(\xi)d\xi\\
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi^+(x)\psi^-(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi^+(\xi)\overline{\mathcal{F}\psi^-(\xi)}g(\xi)d\xi\\
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi^-(x)\psi^+(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi^-(\xi)\overline{\mathcal{F}\psi^+(\xi)}g(\xi)d\xi
\end{aligned}$$

and all the integrals appearing above are finite. We take the sum of the first two relations above and from this, we subtract the sum of the last two. We obtain

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)(\varphi^+(x)-\varphi^-(x))(\psi^+(y)-\psi^-(y))dxdy =$$

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(\mathcal{F}\varphi^+(\xi)-\mathcal{F}\varphi^-(\xi))\overline{(\mathcal{F}\psi^+(\xi)-\mathcal{F}\psi^-(\xi))}g(\xi)d\xi,$$

which is exactly relation (58). Finally, we note that

$$|\mathcal{E}_f(\varphi,\psi)| \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)|\varphi(x)||\psi(y)|dxdy = \mathcal{E}_f(|\varphi|,|\psi|) \leq \mathcal{E}_f(|\varphi|)^{1/2}\mathcal{E}_f(|\psi|)^{1/2} < \infty.$$

□

For complex-valued functions, we have the following result.

Lemma A.2. *For any functions $\varphi,\psi \in L^1_{\mathbb{C}}(\mathbb{R}^d)$ with $\mathcal{E}_f(|\varphi|)<\infty$ and $\mathcal{E}_f(|\psi|)<\infty$, we have:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi(x)\overline{\psi(y)}dxdy = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}g(\xi)d\xi =: \mathcal{E}_f(\varphi,\psi), \tag{59}$$

and in this case, $|\mathcal{E}_f(\varphi,\psi)|<\infty$.

Proof: We write $\varphi=\varphi_1+i\varphi_2$ and $\psi=\psi_1+i\psi_2$, where $\varphi_1,\varphi_2,\psi_1,\psi_2 \in L^1(\mathbb{R}^d)$. Note that $|\varphi|^2=|\varphi_1|^2+|\varphi_2|^2$. It follows that $|\varphi_1(x)| \leq |\varphi(x)|$ for all $x \in \mathbb{R}$, and hence

$$\mathcal{E}_f(|\varphi_1|) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)|\varphi_1(x)||\varphi_1(y)|dxdy \leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)|\varphi(x)||\varphi(y)|dxdy = \mathcal{E}_f(|\varphi|) < \infty.$$

Similarly, $\mathcal{E}_f(|\varphi_2|)<\infty$, $\mathcal{E}_f(|\psi_1|)<\infty$ and $\mathcal{E}_f(|\psi_2|)<\infty$.

Note that

$$\begin{aligned}\varphi(x)\overline{\psi(y)} &= [\varphi_1(x)+i\varphi_2(x)][\psi_1(y)-i\psi_2(y)] \\ &= [\varphi_1(x)\psi_1(y)+\varphi_2(y)\psi_2(y)] + i[\varphi_2(x)\psi_1(y)-\varphi_1(y)\psi_2(y)]\end{aligned}$$

and

$$\begin{aligned}\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)} &= [\mathcal{F}\varphi_1(\xi)+i\mathcal{F}\varphi_2(\xi)][\overline{\mathcal{F}\psi_1(\xi)}-i\overline{\mathcal{F}\psi_2(\xi)}] \\ &= \mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\psi_1(\xi)}+\mathcal{F}\varphi_2(\xi)\overline{\mathcal{F}\psi_2(\xi)}+i[\mathcal{F}\varphi_2(\xi)\overline{\mathcal{F}\psi_1(\xi)}-\mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\psi_2(\xi)}].\end{aligned}$$

We apply Lemma A.1 to the pairs of functions (φ_1,ψ_1), (φ_2,ψ_2), (φ_2,ψ_1), (φ_1,ψ_2):

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi_1(x)\psi_1(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\psi_1(\xi)}g(\xi)d\xi \\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi_2(x)\psi_2(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi_2(\xi)\overline{\mathcal{F}\psi_2(\xi)}g(\xi)d\xi \\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi_2(x)\psi_1(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi_2(\xi)\overline{\mathcal{F}\psi_1(\xi)}g(\xi)d\xi \\ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\varphi_1(x)\psi_2(y)dxdy &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\psi_2(\xi)}g(\xi)d\xi,\end{aligned}$$

where all the integrals above are finite. We take the sum of the first two relations and then we add the difference between the third and fourth relations multiplied by i. We obtain:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)\{\varphi_1(x)\psi_1(y)+\varphi_2(x)\psi_2(y)-i[\varphi_2(x)\psi_1(y)+\varphi_1(x)\psi_2(y)]\}dxdy =$$

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\{\mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\psi_1(\xi)}+\mathcal{F}\varphi_2(\xi)\overline{\mathcal{F}\psi_2(\xi)}+i[\mathcal{F}\varphi_2(\xi)\overline{\mathcal{F}\psi_1(\xi)}-\mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\psi_2(\xi)}]\}g(\xi)d\xi,$$

which is exactly relation (59). The fact that $|\mathcal{E}_f(\varphi,\psi)|<\infty$ follows as in Lemma A.1. □

B Existence of measurable modifications

In this section, we prove a result about existence of measurable modifications of random fields, which is used frequently in the literature on SPDEs using Walsh' approach [32]. Parts a) and b) of this result are extensions to random fields of Theorem 30 in Chapter IV of [10], respectively Proposition 3.21 of [27]. We include the proof since we could not find a direct reference in the literature. Part a) of this result is used in the proof of Theorem 3.6. Part b) of this result is not needed in the present article, but we include it here since its proof requires only a minor modification of the proof of part a).

A random field is a collection $X=\{X(t,x); t\geq 0, x\in\mathbb{R}^d\}$ of random variables defined on the same probability space $(\Omega,\mathcal{F},P)$. We say that X is *stochastically continuous* if it is continuous in probability, i.e. $X(t_n,x_n)\stackrel{P}{\to}X(t,x)$ if $t_n\to t$ and $x_n\to x$. X is *predictable* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if the map $(\omega,t,x)\mapsto X(\omega,t,x)$ is measurable with respect to the *predictable σ-field* on $\Omega\times\mathbb{R}_+\times\mathbb{R}^d$, which is the σ-field generated by elementary processes. (An *elementary process* is a process of the form $Y(\omega,t,x)=Y_0(\omega)1_{(a,b]}(t)1_A(x)$, where Y_0 is $\mathcal{F}_a$-measurable, $0<a<b$ and $A\in\mathcal{B}(\mathbb{R}^d)$ is a bounded set.) X is *adapted* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ if $X(t,x)$ is $\mathcal{F}_t$-measurable, for any $t\geq 0$ and $x\in\mathbb{R}^d$.

Proposition B.1. *a) Any stochastically continuous random field has a measurable modification. b) Any stochastically continuous random field which is adapted with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ has a predictable modification, with respect to the same filtration.*

Proof. a) Let $X=\{X(t,x); t\geq 0, x\in\mathbb{R}^d\}$ be a stochastically continuous random field defined on a probability space $(\Omega,\mathcal{F},P)$. Let $(E_m)_{m\geq 1}$ be an increasing sequence of compact sets in $\mathbb{R}^d$ such that $\cup_m E_m=\mathbb{R}^d$. Fix $m\geq 1$ and let $I=[0,m]\times E_m$. Since X is stochastically continuous, it is uniformly stochastically continuous on I, i.e. for any $\varepsilon>0$ there exists $\delta_\varepsilon>0$ such that for any $(t,x),(s,y)\in I$ with $|t-s|^2+|x-y|^2\leq\delta_m^2$,

$$\mathbb{P}(|X(t,x)-X(s,y)|>2^{-m})\leq 2^{-m}.$$

Let $0=t_0^{(m)}<t_1^{(m)}<\ldots<t_{n_m}^{(m)}=m$ be a partition of $[0,m]$ into subintervals of length smaller than δ_m and $(U_l^{(m)})_{l=1,\ldots,K_m}$ be a partition of E_m into Borel sets of diameter smaller than δ_m. (The diameter of a set S is defined as $\sup\{|x-y|; x,y\in S\}$.) Let $x_l^{(m)}\in U_l^{(m)}$ be arbitrary. For any $t\in(t_k^{(m)},t_{k+1}^{(m)}]$ and $x\in U_l^{(m)}$,

$$\mathbb{P}(|X(t_k^{(m)},x_l^{(m)})-X(t,x)|>2^{-m})\leq 2^{-m}. \quad (60)$$

Define

$$X_m(\omega,t,x)=\sum_{k=0}^{n_m-1}\sum_{l=1}^{K_m}X(\omega,t_k^{(m)},x_l^{(m)})1_{(t_k^{(m)},t_{k+1}^{(m)}]}(t)1_{U_l^{(m)}}(x).$$

Note that X_m is a measurable process, since $X(t_k^{(m)},x_l^{(m)})$ is $\mathcal{F}$-measurable.

Let $A=\{(\omega,t,x)\in\Omega\times\mathbb{R}_+\times\mathbb{R}^d;(X_m(\omega,t,x))_{m\geq 0}\text{ converges}\}$. Then $A\in\mathcal{F}\times\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(\mathbb{R}^d)$ and the process $\widetilde{X}$ defined by

$$\widetilde{X}(\omega,t,x)=1_A(\omega,t,x)\lim_{m\to\infty}X_m(\omega,t,x)$$

is measurable.

We now show that $\widetilde{X}$ is a modification of X. Let $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ be arbitrary. Then $(t,x)\in I_m$ for m large enough, and $t\in(t_k^{(m)},t_{k+1}^{(m)}]$ for some $k=k_m$ and $x\in U_l^{(m)}$ for some $l=l_m$. Let

$$A_m=\{|X(t_k^{(m)},x_l^{(m)})-X(t,x)|>2^{-m}\}.$$

By (60), $\sum_m\mathbb{P}(A_m)<\infty$. By the Borel-Cantelli Lemma, $\mathbb{P}(\Omega_{t,x})=1$ where $\Omega_{t,x}=\liminf_m A_m^c=\cup_{m_0}\cap_{m\geq m_0}A_m^c$. Let $\omega\in\Omega_{t,x}$. Then there exists $m_0=m_0(\omega)$ such that

$$|X(\omega,t_k^{(m)},x_l^{(m)})-X(\omega,t,x)|\leq 2^{-m}\quad\forall m\geq m_0(\omega).$$

By the definition of X_m, it follows that

$$|X_m(\omega,t,x)-X(\omega,t,x)|\leq 2^{-m}\quad\forall m\geq m_0(\omega).$$

From this, we infer that $\lim_{m\to\infty}X_m(\omega,t,x)=X(\omega,t,x)$, i.e. $(\omega,t,x)\in A$. This shows that $\widetilde{X}(\omega,t,x)=X(\omega,t,x)$ for any $\omega\in\Omega_{t,x}$. Since $\mathbb{P}(\Omega_{t,x})=1$, we infer that $\mathbb{P}(\widetilde{X}(t,x)=X(t,x))=1$, as required.

b) Assume, in addition, that X is adapted. Then X_m is predictable, being a linear combination of elementary processes. Hence A and $\widetilde{X}$ are predictable. $\square$

Acknowledgement

The first author is grateful to Mohammud Foondun who asked her (in a personal communication) about the existence of a predictable modification of a random field. The answer to his question is given by Proposition B.1.b).

References

[1] Balan, R. M. (2012). The stochastic wave equation with multiplicative fractional noise: a Malliavin calculus approach. *Potential Anal.* **36**, 1-34.

[2] Balan, R. M. and Song, J. (2017). Hyperbolic Anderson Model with space-time homogeneous Gaussian noise. *Latin Amer. J. Probab. Math. Stat.* **14**, 799-849.

[3] Balan, R. M. and Song, J. (2017). Second order Lyapunov exponents for parabolic and hyperbolic Anderson models. Preprint available on arxiv:1704.02411.

[4] Bally, V. and Pardoux, E. (1998). Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.* **9**, 27-64.

[5] Basse-O'Connor, A., Graversen, S.-E. and Pedersen, J. (2012). Multiparameter processes with stationary increments. Spectral representation and intregration. *Electr. J. Probab.* **17**, paper no. 74, 21 pages.

[6] Carmona, R. and Nualart, D. (1988). Random non-linear wave equations: smoothness of the solutions. *Probab. Th. Rel. Fields* **79**, 469-508.

[7] Dalang R.C. (1999) Extending martingale measure stochastic integral with applications to spatially homogeneous spde's. Electr. J. Probab. **4**.

[8] Dalang, R. C. and Frangos, N. (1998). The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26**, 187-212.

[9] Dalang, R. C. and Sanz-Solé, M. (2005). Regularity of the sample paths of a class of second-order spde's. *J. Funct. Anal.* **227**, 304-337.

[10] Dellacherie, C. and Meyer, P.A. (1975). *Probabilités et Potentiel.* Hermann, Paris.

[11] Hu, Y., Huang, J., Nualart, D. and Tindel, S. (2015). Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electr. J. Probab.* **20**, no. 55, 1-50.

[12] Hu, Y. and Le, K. (2018). Asumptotics of the density of parabolic Anderson random fields. Preprint available on arxiv:1801.03386

[13] Hu, Y., Nualart, D. and Song, J. (2011). Feynman-Kac formula for heat equation driven by fractional white noise. *Ann. Probab.* **39**, 291-326.

[14] Huang, J., Lê, K. and Nualart, D. (2017). Large time asymptotics for the Parabolic Anderson model driven by spatially correlated noise. *Ann. Inst. Henri Poincaré* **53**, 1305-1340.

[15] Itô, K. (1954) Stationary random distributions. *Mem. Coll. Sci. Univ. Kyoto Ser. A Math.* **28**, 209-223.

[16] Jolis, M. (2010). The Wiener integral with respect to second-order processes with stationary increments. *J. Math. Anal. Appl.* **366**, 607-620.

[17] Khoshnevisan, D. and Xiao, Y. (2009). Harmonic analysis of additive Lévy processes. *Probab. Th. Rel. Fields* **145**, 459-515.

[18] Márquez-Carreras, D., Mellouk, M., Sarrà, M. (2001). On stochastic partial differential equations with spatially correlated noise: smoothness of the law. *Stoch. Proc. Appl.* **93**, 269-284.

[19] Millet, A. and Sanz-Solé, M. (1999). A stochastic wave equation in two space dimension: smoothness of the law. *Ann. Probab.* **27**, 803-844.

[20] Mueller, C. and Nualart, D. (2008). Regularity of the density for the stochastic heat equation. *Electr. J. Probab.* **13**, Paper no. 74, 2248-2258.

[21] Nualart, D. (1998). Analysis on Wiener space and anticipative stochastic calculus. *Lecture Notes Math.* **1690**. Ecole d'ete de probabilités de Saint Flour XXV-1995, P. Bernard Ed. 123-227.

[22] Nualart, D. (2006). *Malliavin Calculus and Related Topics*. Second Edition. Springer.

[23] Nualart, D. and Quer-Sardanyons, L. (2007). Existence and smoothness of the density for spatially homogeneous SPDEs. *Potential Anal.* **27**, 281-299.

[24] Pardoux, E. and Zhang, T. (1993). Absolute continuity for the law of the solution of a parabolic SPDE. *J. of Funct. Anal.* **112**, 447-458.

[25] Peszat, S. (2002). The Cauchy problem for a nonlinear stochastic wave equation in any dimension. *J. Evol. Equ.* **2**, 383-394.

[26] Peszat, S. and Zabczyk, J. (2000). Nonlinear stochastic wave and heat equations. *Probab. Th. Rel. Fields* **116**, 421-443.

[27] Peszat, S. and Zabczyk, J. (2007). *Stochastic partial differential equations with Lévy noise*. Cambridge University Press.

[28] Quer-Sardanyons, L., Sanz-Solé, M. (2004). Absolute continuity of the law of the solution to the 3-dimensional stochastic wave equation. *J. Funct. Anal.* **206**, No.1, 1-32.

[29] Quer-Sardanyons, L., Sanz-Solé, M. (2004) A stochastic wave equation in dimension 3: Smoothness of the law. *Bernoulli* **10**, No.1, 165-186.

[30] Sanz-Solé, M. and Süß, A. (2013). The stochastic wave equation in high dimensions: Malliavin differentiability and absolute continuity. *Electr. J. Probab.* **18**, no. 64, 1-28

[31] Sanz-Solé, M. and Süß, A. (2015). Absolute continuity for SPDEs with irregular fundamental solution. *Electr. Comm. Probab.* **20**, no.14, 1-11.

[32] Walsh, J. B. (1986). An introduction to stochastic partial differential equations. Ecole d'Eté de Probabilités de Saint-Flour XIV. *Lecture Notes in Math.* **1180**, 265-439, Springer-Verlag.