Proof of the Goldberg-Seymour Conjecture on Edge-Colorings of Multigraphs

Guantao Chen[a] Guangming Jing[a] Wenan Zang[b*†]

[a] Department of Mathematics and Statistics, Georgia State University
Atlanta, GA 30303, USA

[b] Department of Mathematics, The University of Hong Kong
Hong Kong, China

Abstract

Given a multigraph $G = (V, E)$, the *edge-coloring problem* (ECP) is to color the edges of G with the minimum number of colors so that no two adjacent edges have the same color. This problem can be naturally formulated as an integer program, and its linear programming relaxation is called the *fractional edge-coloring problem* (FECP). In the literature, the optimal value of ECP (resp. FECP) is called the *chromatic index* (resp. *fractional chromatic index*) of G, denoted by $\chi'(G)$ (resp. $\chi^*(G)$). Let $\Delta(G)$ be the maximum degree of G and let

$$\Gamma(G) = \max\left\{\frac{2|E(U)|}{|U|-1} :\ U \subseteq V,\ |U| \geq 3 \ \text{and} \ \text{odd}\right\},$$

where $E(U)$ is the set of all edges of G with both ends in U. Clearly, $\max\{\Delta(G), \lceil\Gamma(G)\rceil\}$ is a lower bound for $\chi'(G)$. As shown by Seymour, $\chi^*(G) = \max\{\Delta(G), \Gamma(G)\}$. In the 1970s Goldberg and Seymour independently conjectured that $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil\Gamma(G)\rceil\}$. Over the past four decades this conjecture, a cornerstone in modern edge-coloring, has been a subject of extensive research, and has stimulated a significant body of work. In this paper we present a proof of this conjecture. Our result implies that, first, there are only two possible values for $\chi'(G)$, so an analogue to Vizing's theorem on edge-colorings of simple graphs, a fundamental result in graph theory, holds for multigraphs; second, although it is NP-hard in general to determine $\chi'(G)$, we can approximate it within one of its true value, and find it exactly in polynomial time when $\Gamma(G) > \Delta(G)$; third, every multigraph G satisfies $\chi'(G) - \chi^*(G) \leq 1$, so FECP has a fascinating integer rounding property.

*E-mail addresses: gchen@gsu.edu (G. Chen), gjing1@gsu.edu (G. Jing), wzang@maths.hku.hk (W. Zang).

†The third author was supported in part by the Research Grants Council of Hong Kong.

1 Introduction

Given a multigraph $G = (V, E)$, the *edge-coloring problem* (ECP) is to color the edges of G with the minimum number of colors so that no two adjacent edges have the same color. Its optimal value is called the *chromatic index* of G, denoted by $\chi'(G)$. In addition to its great theoretical interest, ECP arises in a variety of applications, so it has attracted tremendous research efforts in several fields, such as combinatorial optimization, theoretical computer science, and graph theory. Holyer [15] proved that it is NP-hard in general to determine $\chi'(G)$, even when restricted to a simple cubic graph, so there is no efficient algorithm for solving ECP exactly unless $NP = P$, and hence the focus of extensive research has been on near-optimal solutions to ECP or good estimates of $\chi'(G)$.

Let $\Delta(G)$ be the maximum degree of G. Clearly, $\chi'(G) \geq \Delta(G)$. There are two classical upper bounds on the chromatic index: the first of these, $\chi'(G) \leq \lfloor \frac{3\Delta(G)}{2} \rfloor$, was established by Shannon [35] in 1949, and the second, $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum multiplicity of edges in G, was proved independently by Vizing [38] and Gupta [11] in the 1960s. This second result is widely known as Vizing's theorem, which is particularly appealing when applied to a simple graph G, because it reveals that $\chi'(G)$ can take only two possible values: $\Delta(G)$ and $\Delta(G)+1$. Nevertheless, in the presence of multiple edges, the gap between $\chi'(G)$ and these three bounds can be arbitrarily large. Therefore it is desirable to find some other graph theoretic parameters connected to the chromatic index.

Observe that each color class in an edge-coloring of G is a matching, so it contains at most $(|U| - 1)/2$ edges in $E(U)$ for any $U \subseteq V$ with $|U|$ odd, where $E(U)$ is the set of all edges of G with both ends in U. Hence *the density of* G, defined by

$$\Gamma(G) = \max\Big\{\frac{2|E(U)|}{|U| - 1} : \ U \subseteq V, \ |U| \geq 3 \ \text{and} \ \text{odd}\Big\},$$

provides another lower bound for $\chi'(G)$. It follows that $\chi'(G) \geq \max\{\Delta(G), \Gamma(G)\}$.

To facilitate better understanding of the parameter $\max\{\Delta(G), \Gamma(G)\}$, let A be the edge-matching incidence matrix of G. Then ECP can be naturally formulated as an integer program, whose linear programming (LP) relaxation is exactly as given below:

$$\begin{array}{ll} \text{Minimize} & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \mathbf{1} \\ & \ \ \boldsymbol{x} \geq \mathbf{0}. \end{array}$$

This linear program is called the *fractional edge-coloring problem* (FECP), and its optimal value is called the *fractional chromatic index* of G, denoted by $\chi^*(G)$. As shown by Seymour [34] using Edmonds' matching polytope theorem [7], it is always true that $\chi^*(G) = \max\{\Delta(G), \Gamma(G)\}$. Thus the preceding inequality actually states that $\chi'(G) \geq \chi^*(G)$.

As $\chi'(G)$ is integer-valued, we further obtain $\chi'(G) \geq \max\{\Delta(G), \lceil\Gamma(G)\rceil\}$. How good is this bound? In the 1970s Goldberg [9] and Seymour [34] independently made the following conjecture.

Conjecture 1.1. *Every multigraph G satisfies* $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil\Gamma(G)\rceil\}$.

Let r be a positive integer. A multigraph $G = (V, E)$ is called an *r-graph* if G is regular of degree r and for every $X \subseteq V$ with $|X|$ odd, the number of edges between X and $V - X$ is at least r. Note that if G is an r-graph, then $|V(G)|$ is even and $\Gamma(G) \le r$. Seymour [34] also proposed the following weaker version of Conjecture 1.1, which amounts to saying that $\chi'(G) \le \max\{\Delta(G), \lceil\Gamma(G)\rceil\} + 1$ for any multigraph G.

Conjecture 1.2. *Every r-graph G satisfies $\chi'(G) \le r + 1$.*

The following conjecture was posed by Gupta [11] in 1967 and can be deduced from Conjecture 1.1, as verified by Scheide [30].

Conjecture 1.3. *Let G be a multigraph with $\mu(G) = \mu$, such that $\Delta(G)$ cannot be expressed in the form $2p\mu - q$, where $q \ge 0$ and $p > \lfloor(q + 1)/2\rfloor$. Then $\chi'(G) \le \Delta(G) + \mu(G) - 1$.*

A multigraph G is called *critical* if $\chi'(H) < \chi'(G)$ for any proper subgraph H of G. In edge-coloring theory, critical multigraphs are of special interest, because they have much more structural properties than arbitrary multigraphs. The following two conjectures are due to Jakobsen [16, 17] and were proved by Andersen [1] to be weaker than Conjecture 1.1.

Conjecture 1.4. *Let G be a critical multigraph with $\chi'(G) \ge \Delta(G) + 2$. Then G contains an odd number of vertices.*

Conjecture 1.5. *Let G be a critical multigraph with $\chi'(G) > \frac{m\Delta(G)+(m-3)}{m-1}$ for an odd integer $m \ge 3$. Then G has at most $m - 2$ vertices.*

Motivated by Conjecture 1.1, Hochbaum, Nishizeki, and Shmoys [14] formulated the following conjecture concerning the approximability of ECP.

Conjecture 1.6. *There is a polynomial-time algorithm that colors the edges of any multigraph G using at most $\chi'(G) + 1$ colors.*

Over the past four decades Conjecture 1.1 has been a subject of extensive research, and has stimulated a significant body of work, with contributions from many researchers; see McDonald [23] for a survey on this conjecture and Stiebitz *et al.* [36] for a comprehensive account of edge-colorings.

Several approximate results state that $\chi'(G) \le \max\{\Delta(G) + \rho(G), \lceil\Gamma(G)\rceil\}$, where $\rho(G)$ is a positive number depending on G. Asymptotically, Kahn [19] showed that $\rho(G) = o(\Delta(G))$. Scheide [31] and Chen, Yu, and Zang [5] independently proved that $\rho(G) \le \sqrt{\Delta(G)/2}$. Chen *et al.* [3] improved this to $\rho(G) \le \sqrt[3]{\Delta(G)/2}$. Recently, Chen and Jing [4] further strengthened this as $\rho(G) \le \sqrt[3]{\Delta(G)/4}$.

There is another family of results, asserting that $\chi'(G) \le \max\{\frac{m\Delta(G)+(m-3)}{m-1}, \lceil\Gamma(G)\rceil\}$, for increasing values of m. Such results have been obtained by Andersen [1] and Goldberg [9] for $m = 5$, Andersen [1] for $m = 7$, Goldberg [10] and Hochbaum, Nishizeki, and Shmoys [14] for $m = 9$, Nishizeki and Kashiwagi [25] and Tashkinov [37] for $m = 11$, Favrholdt, Stiebitz, and Toft [8] for $m = 13$, Scheide [31] for $m = 15$, Chen *et al.* [3] for $m = 23$, and Chen and Jing [4] for $m = 39$. It is worthwhile pointing out that, when $\Delta(G) \le 39$, the validity of Conjecture 1.1 follows instantly from Chen and Jing's result [4], because $\frac{39\Delta(G)+36}{38} < \Delta(G) + 2$.

Haxell and McDonald [13] obtained a different sort of approximation to Conjecture 1.1: $\chi'(G) \le \max\{\Delta(G) + 2\sqrt{\mu(G)\log\Delta(G)},\ \lceil\Gamma(G)\rceil\}$. Another way to obtain approximations for Conjecture 1.1 is to incorporate the order n of G (that is, number of vertices) into a bound. In this direction, Plantholt [28] proved that $\chi'(G) \le \max\{\Delta(G),\ \lceil\Gamma(G)\rceil + 1 + \sqrt{n\log(n/6)}\}$ for any multigraph G with ever order $n \ge 572$. In [29], he established an improved result that is applicable to all multigraphs.

Marcotte [22] showed that $\chi'(G) = \max\{\Delta(G),\ \lceil\Gamma(G)\rceil\}$ for any multigraph G with no K_5^--minor, thereby confirming Conjecture 1.1 for this multigraph class. Recently, Haxell, Krivelevich, and Kronenberg [12] established Conjecture 1.1 for random multigraphs.

The purpose of this paper is to present a proof of the Goldberg-Seymour conjecture.

Theorem 1.1. *Every multigraph G satisfies $\chi'(G) \le \max\{\Delta(G)+1,\ \lceil\Gamma(G)\rceil\}$.*

As stated before, Conjectures 1.2-1.5 are all weaker than the Goldberg-Seymour conjecture, so the truth of them follows from Theorem 1.1 as corollaries.

Theorem 1.2. *Every r-graph G satisfies $\chi'(G) \le r + 1$.*

Theorem 1.3. *Let G be a multigraph with $\mu(G) = \mu$, such that $\Delta(G)$ cannot be expressed in the form $2p\mu - q$, where $q \ge 0$ and $p > \lfloor (q+1)/2 \rfloor$. Then $\chi'(G) \le \Delta(G) + \mu(G) - 1$.*

Theorem 1.4. *Let G be a critical multigraph with $\chi'(G) \ge \Delta(G) + 2$. Then G contains an odd number of vertices.*

Theorem 1.5. *Let G be a critical multigraph with $\chi'(G) > \frac{m\Delta(G)+(m-3)}{m-1}$ for an odd integer $m \ge 3$. Then G has at most $m - 2$ vertices.*

We have seen that FECP is intimately tied to ECP. For any multigraph G, the fractional chromatic index $\chi^*(G) = \max\{\Delta(G),\ \Gamma(G)\}$ can be determined in polynomial time by combining the Padberg-Rao separation algorithm for b-matching polyhedra [26] (see also [21, 27]) with binary search. In [6], Chen, Zang, and Zhao designed a combinatorial polynomial-time algorithm for finding the density $\Gamma(G)$ of any multigraph G, thereby resolving a problem posed in both Stiebitz *et al.* [36] and Jensen and Toft [18]. Nemhauser and Park [24] observed that FECP can be solved in polynomial time by an ellipsoid algorithm, because the separation problem of its LP dual is exactly the maximum-weight matching problem (see also Schrijver [33], Theorem 28.6 on page 477). In [6], Chen, Zang, and Zhao devised a combinatorial polynomial-time algorithm for FECP as well.

We believe that our proof of Theorem 1.1 can be adapted to yield a polynomial-time algorithm for finding an edge-coloring of any multigraph G with at most $\max\{\Delta(G)+1,\ \lceil\Gamma(G)\rceil\}$ colors, and we are working on the design of this algorithm. A successful implementation would lead to an affirmative answer to Conjecture 1.6 as well.

Some remarks may help to put Theorem 1.1 in proper perspective.

First, by Theorem 1.1, there are only two possible values for the chromatic index of a multigraph G: $\max\{\Delta(G),\ \lceil\Gamma(G)\rceil\}$ and $\max\{\Delta(G)+1,\ \lceil\Gamma(G)\rceil\}$. Thus an analogue to Vizing's theorem on edge-colorings of simple graphs, a fundamental result in graph theory, holds for multigraphs.

Second, Theorem 1.1 exhibits a dichotomy on edge-coloring: While Holyer's theorem [15] tells us that it is NP-hard to determine $\chi'(G)$, we can approximate it within one of its true value, because $\max\{\Delta(G)+1, \lceil\Gamma(G)\rceil\} - \chi'(G) \le 1$. Furthermore, if $\Gamma(G) > \Delta(G)$, then $\chi'(G) = \lceil\Gamma(G)\rceil$, so it can be found in polynomial time [6, 26].

Third, by Theorem 1.1 and aforementioned Seymour's theorem, every multigraph $G = (V, E)$ satisfies $\chi'(G) - \chi^*(G) \le 1$, which can be naturally extended to the weighted case. Let $w(e)$ be a nonnegative integral weight on each edge $e \in E$ and let $\boldsymbol{w} = (w(e) : e \in E)$. The *chromatic index* of $(G, \boldsymbol{w})$, denoted by $\chi'_w(G)$, is the minimum number of matchings in G such that each edge e is covered exactly $w(e)$ times by these matchings, and the *fractional chromatic index* of $(G, \boldsymbol{w})$, denoted by $\chi^*_w(G)$, is the optimal value of the following linear program:

$$\begin{array}{ll} \text{Minimize} & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{w} \\ & \boldsymbol{x} \ge \mathbf{0}, \end{array}$$

where A is again the edge-matching incidence matrix of G. Clearly, $\chi'_w(G)$ is the optimal value of the corresponding integer program. Let G_w be obtained from G by replacing each edge e with $w(e)$ parallel edges between the same ends. It is then routine to check that $\chi'_w(G) = \chi'(G_w)$ and $\chi^*_w(G) = \chi^*(G_w)$. So the inequality $\chi'_w(G) - \chi^*_w(G) \le 1$ holds for all nonnegative integral weight functions $\boldsymbol{w}$, and hence FECP has a fascinating integer rounding property (see Schrijver [32, 33]).

So far the most powerful and sophisticated technique for multigraph edge-coloring is the method of Tashkinov trees [37], which generalizes the earlier methods of Vizing fans [38] and Kierstead paths [20]. (These methods are named after the authors who invented them, respectively.) Most approximate results described above Theorem 1.1 were obtained by using the method of Tashkinov trees. As remarked by McDonald [23], the Goldberg-Seymour conjecture and ideas culminating in this method are two cornerstones in modern edge-coloring. Nevertheless, this method suffers some theoretical limitation when applied to prove the conjecture; the reader is referred to Asplund and McDonald [2] for detailed information. Despite various attempts to extend the Tashkinov trees (see, for instance, [3, 4, 5, 31, 36]), the difficulty encountered by the method remains unresolved. Even worse, new problem emerges: it becomes very difficult to preserve the structure of an extended Tashkinov tree under Kempe changes (the most useful tool in edge-coloring theory). In this paper we introduce a new type of extended Tashkinov trees and develop an effective control mechanism over Kempe changes, which can overcome all the aforementioned difficulties.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic concepts and techniques of edge-coloring theory, and exhibit some important properties of stable colorings. In Section 3, we define the extended Tashkinov trees to be employed in subsequent proof, and give an outline of our proof strategy. In Section 4, we establish some auxiliary results concerning the extended Tashkinov trees and stable colorings, which ensure that this type of trees is preserved under some restricted Kempe changes. In Section 5, we develop an effective control mechanism over Kempe changes, the so-called good hierarchy of an extended Tashkinov tree; our proof relies heavily on this novel recoloring technique. In Section 6, we derive some properties satisfied by the good hierarchies introduced in the preceding section. In Section 7, we present the last step of our proof based on these good hierarchies.

2 Preliminaries

This section presents some basic definitions, terminology, and notations used in our paper, along with some important properties and results.

2.1 Terminology and Notations

Let $G = (V, E)$ be a multigraph. For each $X \subseteq V$, let $G[X]$ denote the subgraph of G induced by X, and let $G - X$ denote $G[V - X]$; we write $G - x$ for $G - X$ if $X = \{x\}$. Moreover, we use $\partial(X)$ to denote the set of all edges with precisely one end in X, and write $\partial(x)$ for $\partial(X)$ if $X = \{x\}$. For each pair $x, y \in V$, let $E(x, y)$ denote the set of all edges between x and y. As it is no longer appropriate to represent an edge f between x and y by xy in a multigraph, we write $f \in E(x, y)$ instead. For each subgraph H of G, let $V(H)$ and $E(H)$ denote the vertex set and edge set of H, respectively, let $|H| = |V(H)|$, and let $G[H] = G[V(H)]$ and $\partial(H) = \partial(V(H))$.

Let e be an edge of G. A *tree sequence* with respect to G and e is a sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with $p \geq 1$, consisting of distinct edges $e_1, e_2, \ldots, e_p$ and distinct vertices $y_0, y_1, \ldots, y_p$, such that $e_1 = e$ and each edge e_j with $1 \leq j \leq p$ is between y_j and some y_i with $0 \leq i < j$. Given a tree sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$, we can naturally associate a linear order $\prec$ with its vertices, such that $y_i \prec y_j$ if $i < j$. We write $y_i \preceq y_j$ if $i \leq j$. This linear order will be used repeatedly in subsequent sections. For each vertex y_j of T with $j \geq 1$, let $T(y_j)$ denote $(y_0, e_1, y_1, \ldots, e_j, y_j)$. Clearly, $T(y_j)$ is also a tree sequence with respect to G and e. We call $T(y_j)$ the *segment* of T induced by y_j. Let T_1 and T_2 be two tree sequences with respect to G and e. We write $T_2 - T_1$ for $E[T_2] - E[T_1]$, write $T_1 \subseteq T_2$ if T_1 is a segment of T_2, and write $T_1 \subset T_2$ if T_1 is a proper segment of T_2; that is, $T_1 \subseteq T_2$ and $T_1 \neq T_2$.

A *k-edge-coloring* of G is an assignment of k colors, $1, 2, \ldots, k$, to the edges of G so that no two adjacent edges have the same color. By definition, the chromatic index $\chi'(G)$ of G is the minimum k for which G has a k-edge-coloring. We use $[k]$ to denote the color set $\{1, 2, \ldots, k\}$, and use $\mathcal{C}^k(G)$ to denote the set of all k-edge-colorings of G. Note that every k-edge-coloring of G is a mapping from E to $[k]$.

Let φ be a k-edge-coloring of G. For each $\alpha \in [k]$, the edge set $E_{\varphi,\alpha} = \{e \in E : \varphi(e) = \alpha\}$ is called a *color class*, which is a matching in G. For any two distinct colors α and β in $[k]$, let H be the spanning subgraph of G with $E(H) = E_{\varphi,\alpha} \cup E_{\varphi,\beta}$. Then each component of H is either a path or an even cycle; we refer to such a component as an (α, β)-*chain* with respect to φ, and also call it an (α, β)-*path* (resp. (α, β)-*cycle*) if it is a path (resp. cycle). We use $P_v(\alpha, \beta, \varphi)$ to denote the unique (α, β)-chain containing each vertex v. Clearly, for any two distinct vertices u and v, either $P_u(\alpha, \beta, \varphi)$ and $P_v(\alpha, \beta, \varphi)$ are identical or are vertex-disjoint. Let C be an (α, β)-chain with respect to φ, and let φ' be the k-edge-coloring arising from φ by interchanging α and β on C. We say that φ' is obtained from φ by *recoloring* C, and write $\varphi' = \varphi/C$. This operation is called a *Kempe change*.

Let F be an edge subset of G. As usual, $G - F$ stands for the multigraph obtained from G by deleting all edges in F; we write $G - f$ for $G - F$ if $F = \{f\}$. Let $\pi \in \mathcal{C}^k(G - F)$. For each $K \subseteq E$, define $\pi(K) = \cup_{e \in K - F}\, \pi(e)$. For each $v \in V$, define

$$\pi(v) = \pi(\partial(v)) \text{ and } \overline{\pi}(v) = [k] - \pi(v).$$

We call $\pi(v)$ the set of colors *present* at v and call $\overline{\pi}(v)$ the set of colors *missing* at v. For each $X \subseteq V$, define

$$\overline{\pi}(X) = \cup_{v \in X} \overline{\pi}(v).$$

We call X *elementary* with respect to π if $\overline{\pi}(u) \cap \overline{\pi}(v) = \emptyset$ for any two distinct vertices $u, v \in X$. We call X *closed* with respect to π if $\pi(\partial(X)) \cap \overline{\pi}(X) = \emptyset$; that is, no missing coloring of X appears on the edges in $\partial(X)$. Furthermore, we call X *strongly closed* with respect to π if X is closed with respect to π and $\pi(e) \neq \pi(f)$ for any two distinct colored edges $e, f \in \partial(X)$. For each subgraph H of G, write $\overline{\pi}(H)$ for $\overline{\pi}(V(H))$, and write $\pi\langle H \rangle$ for $\pi(E(H))$. Moreover, define

$$\partial_{\pi,\alpha}(H) = \{e \in \partial(H) : \pi(e) = \alpha\},$$

and define

$$I[\partial_{\pi,\alpha}(H)] = \{v \in V(H) : v \text{ is incident with an edge in } \partial_{\pi,\alpha}(H)\}.$$

For an edge $e \in \partial(H)$, we call its end in (resp. outside) H the *in-end* (resp. *out-end*) relative to H. Thus $I[\partial_{\pi,\alpha}(H)]$ consists of all in-ends (relative to H) of edges in $\partial_{\pi,\alpha}(H)$. A color α is called a *defective color* of H with respect to π if $|\partial_{\pi,\alpha}(H)| \geq 2$. A color $\alpha \in \overline{\pi}(H)$ is called *closed* in H under π if $\partial_{\pi,\alpha}(H) = \emptyset$. For convenience, we say that H is *closed* (resp. *strongly closed*) with respect to π if $V(H)$ is closed (resp. strongly closed) with respect to π. Let α and β be two colors that are not assigned to $\partial(H)$ under π. We use $\pi/(G - H, \alpha, \beta)$ to denote the coloring π' obtained from π by interchanging α and β in $G - V(H)$; that is, for any edge f in $G - V(H)$, if $\pi(f) = \alpha$ then $\pi'(f) = \beta$, and if $\pi(f) = \beta$ then $\pi'(f) = \alpha$. Obviously, $\pi' \in \mathcal{C}^k(G - F)$.

2.2 Elementary Multigraphs

Let $G = (V, E)$ be a multigraph. We call G an *elementary multigraph* if $\chi'(G) = \lceil \Gamma(G) \rceil$. With this notion, Conjecture 1.1 can be rephrased as follows.

Conjecture 2.1. *Every multigraph G with $\chi'(G) \geq \Delta(G) + 2$ is elementary.*

Recall that G is critical if $\chi'(H) < \chi'(G)$ for any proper subgraph H of G. As pointed out by Stiebitz *et al.* [36] (see page 7), to prove Conjecture 2.1, it suffices to consider critical multigraphs. To see this, let G be an arbitrary multigraph with $\chi'(G) \geq \Delta(G) + 2$. Then G contains a critical multigraph H with $\chi'(H) = \chi'(G)$, which implies that $\chi'(H) \geq \Delta(H) + 2$. Note that if H is elementary, then so is G, because $\lceil \Gamma(G) \rceil \leq \chi'(G) = \chi'(H) = \lceil \Gamma(H) \rceil \leq \lceil \Gamma(G) \rceil$. Thus both inequalities hold with equalities, and hence $\chi'(G) = \lceil \Gamma(G) \rceil$.

To prove Conjecture 1.1, we shall actually establish the following statement.

Theorem 2.1. *Every critical multigraph G with $\chi'(G) \geq \Delta(G) + 2$ is elementary.*

In our proof we shall appeal to the following theorem, which reveals some intimate connection between elementary multigraphs and elementary sets. This result is implicitly contained in Andersen [1] and Goldberg [10], and explicitly proved in Stiebitz *et al.* [36] (see Theorem 1.4 on page 8).

Theorem 2.2. *Let $G = (V, E)$ be a multigraph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. If G is critical, then the following conditions are equivalent:*

(i) *G is an elementary multigraph.*

(ii) *For each edge* $e \in E$ *and each coloring* $\varphi \in \mathcal{C}^k(G-e)$*, the vertex set* V *is elementary with respect to* φ*.*

(iii) *There exists an edge* $e \in E$ *and a coloring* $\varphi \in \mathcal{C}^k(G-e)$*, such that the vertex set* V *is elementary with respect to* φ*.*

(iv) *There exists an edge* $e \in E$*, a coloring* $\varphi \in \mathcal{C}^k(G-e)$*, and a subset* X *of* V*, such that* X *contains both ends of* e*, and* X *is elementary as well as strongly closed with respect to* φ*.*

2.3 Stable Colorings

In this subsection, we assume that T is a tree sequence with respect to a multigraph $G = (V, E)$ and an edge e, C is a subset of $[k]$, and φ is a coloring in $\mathcal{C}^k(G - e)$, where $k \geq \Delta(G) + 1$.

A coloring $\pi \in \mathcal{C}^k(G - e)$ is called a (T, C, φ)-*stable coloring* if the following two conditions are satisfied:

(i) $\pi(f) = \varphi(f)$ for any $f \in E$ incident to T with $\varphi(f) \in \overline{\varphi}(T) \cup C$; and

(ii) $\overline{\pi}(v) = \overline{\varphi}(v)$ for any $v \in V(T)$.

By convention, $\pi(e) = \varphi(e) = \emptyset$. From the definition we see that if $\varphi\langle T\rangle \subseteq \overline{\varphi}(T) \cup C$, then $\pi(f) = \varphi(f)$ for all edges f on T; this special type of stable colorings will be our major concern.

In our proof we shall perform a sequence of Kempe changes so that the resulting colorings are stable in some sense. The following lemma gives an equivalent definition of stable colorings.

Lemma 2.3. *A coloring* $\pi \in \mathcal{C}^k(G - e)$ *is* (T, C, φ)*-stable iff the following two conditions are satisfied:*

(i') $\pi(f) = \varphi(f)$ *for any* $f \in E$ *incident to* T *with* $\varphi(f) \in \overline{\varphi}(T) \cup C$ *or* $\pi(f) \in \overline{\varphi}(T) \cup C$*; and*

(ii) $\overline{\pi}(v) = \overline{\varphi}(v)$ *for any* $v \in V(T)$*.*

Proof. Note that condition (ii) described here is exactly the same as given in the definition and that (i') implies (i), so the "if" part is trivial. To establish the "only if" part, let $f \in E$ be an arbitrary edge incident to T with $\pi(f) \in \overline{\varphi}(T) \cup C$. We claim that $\varphi(f) = \pi(f)$, for otherwise, let $v \in V(T)$ be an end of f. By (ii), we have $\overline{\pi}(v) = \overline{\varphi}(v)$. So $\pi(v) = \varphi(v)$ and hence there exists an edge $g \in \partial(v) - \{f\}$ with $\varphi(g) = \pi(f)$. It follows that $\varphi(g) \in \overline{\varphi}(T) \cup C$. By (i), we obtain $\pi(g) = \varphi(g)$, which implies $\pi(f) = \pi(g)$, contradicting the hypothesis that $\pi \in \mathcal{C}^k(G - e)$. Our claim asserts that $\varphi(f) = \pi(f)$ for any $f \in E$ incident to T with $\pi(f) \in \overline{\varphi}(T) \cup C$. Combining this with (i), we see that (i') holds. ∎

Let us derive some properties satisfied by stable colorings.

Lemma 2.4. *Being* $(T, C, \cdot)$*-stable is an equivalence relation on* $\mathcal{C}^k(G - e)$*.*

Proof. From Lemma 2.3 it is clear that being $(T, C, \cdot)$-stable is reflexive, symmetric, and transitive. So it defines an equivalence relation on $\mathcal{C}^k(G - e)$. ∎

Let P be a path in G whose edges are colored alternately by α and β in φ, with $|P| \geq 2$, and let u and v be the ends of P with $v \in V(T)$. We say that P is a T-*exit path* with respect to φ if $V(T) \cap V(P) = \{v\}$ and $\overline{\varphi}(u) \cap \{\alpha, \beta\} \neq \emptyset$; in this case, v is called a $(T, \varphi, \{\alpha, \beta\})$-*exit* and P is also called a $(T, \varphi, \{\alpha, \beta\})$-*exit path*. Note that possibly $\overline{\varphi}(v) \cap \{\alpha, \beta\} = \emptyset$.

Let $f \in E(u, v)$ be an edge in $\partial(T)$ with $v \in V(T)$. We say that f is $T \vee C$-*nonextendable* with respect to φ if there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring π and a color $\alpha \in \overline{\pi}(v)$, such that v is a $(T, \pi, \{\alpha, \varphi(f)\})$-exit. Otherwise, we say that f is $T \vee C$-*extendable* with respect to φ.

Lemma 2.5. *Suppose T is closed with respect to φ, and $f \in E(u, v)$ is an edge in $\partial(T)$ with $v \in V(T)$. If there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring π, such that $\overline{\pi}(u) \cap \overline{\pi}(T) \neq \emptyset$, then f is $T \vee C$-nonextendable with respect to φ.*

Proof. Let $\alpha \in \overline{\pi}(u) \cap \overline{\pi}(T)$ and $\beta \in \overline{\pi}(v)$. By the definition of stable colorings, we have $\alpha \in \overline{\varphi}(T)$ and $\beta \in \overline{\varphi}(v)$. Since both α and β are closed in T under φ, they are also closed in T under π by Lemma 2.3. Define $\pi' = \pi/(G - T, \alpha, \beta)$. Clearly, π' is a $(T, C \cup \{\varphi(f)\}, \pi)$-stable coloring. By Lemma 2.4, π' is also a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring. Since $P_v(\beta, \varphi(f), \pi')$ consists of a single edge f, it is a T-exit path with respect to π'. Hence f is $T \vee C$-nonextendable with respect to φ. ∎

Lemma 2.6. *Suppose T is closed with respect to φ, and $f \in E(u, v)$ is an edge in $\partial(T)$ with $v \in V(T)$. If f is $T \vee C$-nonextendable with respect to φ, then for any $\alpha \in \overline{\varphi}(v)$ there exists a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring π, such that v is a $(T, \pi, \{\alpha, \varphi(f)\})$-exit.*

Proof. Since f is $T \vee C$-nonextendable, by definition, there exist a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring φ' and a color $\beta \in \overline{\varphi}(v)$, such that v is a $(T, \varphi', \{\beta, \varphi(f)\})$-exit. Since both α and β are closed in T under φ, they are also closed in T under φ' by Lemma 2.3. Define $\pi = \varphi'/(G - T, \alpha, \beta)$. Clearly, π is a $(T, C \cup \{\varphi(f)\}, \varphi')$-stable coloring. By Lemma 2.4, π is also a $(T, C \cup \{\varphi(f)\}, \varphi)$-stable coloring. Note that $P_v(\alpha, \varphi(f), \pi) = P_v(\beta, \varphi(f), \varphi')$, so $P_v(\alpha, \varphi(f), \pi)$ is a T-exit path with respect to π, and hence v is a $(T, \pi, \{\alpha, \varphi(f)\})$-exit. ∎

Lemma 2.7. *Suppose T is closed but not strongly closed with respect to φ, with $|V(T)|$ odd, and suppose π is a (T, C, φ)-stable coloring. Then T is also closed but not strongly closed with respect to π.*

Proof. Let $X = V(T)$ and let t be the size of the set $[k] - \overline{\varphi}(X)$. Since π is a (T, C, φ)-stable coloring, from Lemma 2.3 we deduce that T is closed with respect to π and that $\overline{\pi}(X) = \overline{\varphi}(X)$ (so $[k] - \overline{\pi}(X)$ is also of size t). By hypotheses, $|V(T)|$ is odd and T is not strongly closed with respect to φ. Thus under the coloring φ each color in $[k] - \overline{\varphi}(X)$ is assigned to at least one edge in $\partial(T)$, and some color in $[k] - \overline{\varphi}(X)$ is assigned to at least two edges in $\partial(T)$. It follows that $|\partial(T)| \geq t + 1$. Note that under the coloring π only colors in $[k] - \overline{\pi}(X)$ can be assigned to edges in $\partial(T)$, so some of these colors is used at least twice by the Pigeonhole Principle. Hence T is not strongly closed with respect to π. ∎

2.4 Tashkinov Trees

A multigraph G is called k-*critical* if it is critical and $\chi'(G) = k + 1$. Throughout this paper, by a k-*triple* we mean a k-critical multigraph $G = (V, E)$, where $k \geq \Delta(G) + 1$, together with an uncolored edge $e \in E$ and a coloring $\varphi \in \mathcal{C}^k(G - e)$; we denote it by (G, e, φ).

Let (G, e, φ) be a k-triple. A *Tashkinov tree* with respect to e and φ is a tree sequence $T = (y_0, e_1, y_1, \ldots, e_p, y_p)$ with respect to G and e, such that for each edge e_j with $2 \le j \le p$, there is a vertex y_i with $0 \le i < j$ satisfying $\varphi(e_j) \in \overline{\varphi}(y_i)$.

The following theorem is due to Tashkinov [37]; its proof can also be found in Stiebitz *et al.* [36] (see Theorem 5.1 on page 116).

Theorem 2.8. *Let (G, e, φ) be a k-triple and let T be a Tashkinov tree with respect to e and φ. Then $V(T)$ is elementary with respect to φ.*

Let $G = (V, E)$ be a critical multigraph G with $\chi'(G) \ge \Delta(G) + 2$. For each edge $e \in E$ and each coloring $\varphi \in \mathcal{C}^k(G - e)$, there is a Tashkinov tree T with respect to e and φ. The *Tashkinov order* of G, denoted by $t(G)$, is the largest number of vertices contained in such a Tashkinov tree. Scheide [31] (see Proposition 4.5) has established the following result, which will be employed in our proof.

Theorem 2.9. *Let G be a critical multigraph G with $\chi'(G) \ge \Delta(G) + 2$. If $t(G) < 11$, then G is an elementary multigraph.*

The method of Tashkinov trees consists of modifying a given partial edge-coloring with sequences of Kempe changes and resulting extensions (that is, coloring an edge e with a color α, which is missing at both ends of e). When applied to prove Conjecture 1.1, the crux of this method is to capture the density $\Gamma(G)$ by exploring a sufficiently large Tashkinov tree (see Theorem 2.8). However, this target may become unreachable when $\chi'(G)$ gets close to $\Delta(G)$, even if we allow for an unlimited number of Kempe changes; such an example has been found by Asplund and McDonald [2]. To circumvent this difficulty and to make this method work, we shall introduce a new type of extended Tashkinov trees in this paper by using the procedure described below.

Given a k-triple (G, e, φ) and a tree sequence T with respect to G and e, we may construct a tree sequence $T' = (T, e_1, y_1, \ldots, e_p, y_p)$ from T by recursively adding edges $e_1, e_2, \ldots, e_p$ and vertices $y_1, y_2, \ldots, y_p$ outside T, such that

- e_1 is incident to T and each edge e_i with $1 \le i \le p$ is between y_i and $V(T) \cup \{y_1, y_2, \ldots, y_{i-1}\}$;
- for each edge e_i with $1 \le i \le p$, there is a vertex x_i in $V(T) \cup \{y_1, y_2, \ldots, y_{i-1}\}$, satisfying $\varphi(e_i) \in \overline{\varphi}(x_i)$.

Such a procedure is referred to as *Tashkinov's augmentation algorithm* (TAA). We call T' a *closure* of T under φ if it cannot grow further by using TAA (equivalently, T' becomes closed). We point out that, although there might be several ways to construct a closure of T under φ, the vertex set of these closures is unique.

3 Extended Tashkinov Trees

The purpose of this section is to present extended Tashkinov trees to be used in our proof and to give an outline of our proof strategy.

Given a k-triple (G, e, φ), we first propose an algorithm for constructing a *Tashkinov series*, which is a series of tuples $(T_n, \varphi_{n-1}, S_{n-1}, F_{n-1}, \Theta_{n-1})$ for $n = 1, 2, \ldots$, where

- φ_{n-1} is the k-edge-coloring of $G - e$ exhibited in iteration $n - 1$,

• T_n is the tree sequence with respect to e and φ_{n-1} constructed in iteration $n-1$,

• S_{n-1} consists of the connecting colors used in iteration $n-1$ with $|S_{n-1}| \le 2$,

• F_{n-1} consists of the connecting edge used in iteration $n-1$ if $n \ge 2$ and $F_0 = \emptyset$, and

• $\Theta_{n-1} \in \{RE, SE, PE\}$ if $n \ge 2$, which stands for the extension type used in iteration $n-1$; we set $\Theta_0 = \emptyset$.

For ease of description, we make some preparations. Since each T_n is a tree sequence with respect to G and e, the linear order $\prec$ defined in Subsection 2.1 is valid for T_n. By $T_n + f_n$ we mean the tree sequence augmented from T_n by adding an edge f_n. By a *segment* of a cycle we mean a path contained in it.

Let D_{n-1} be a certain subset of $[k]$ and let π be a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring. We use $v_{\pi,\alpha}$ to denote the maximum vertex in $I[\partial_{\pi,\alpha}(T_n)]$ in the order $\prec$ for each defective color α of T_n with respect to π, and use v_π to denote the maximum vertex in the order $\prec$ among all these vertices $v_{\pi,\alpha}$. We reserve the symbol v_n for the maximum vertex in the order $\prec$ among all these vertices v_π, where π ranges over all $(T_n, D_{n-1}, \varphi_{n-1})$-stable colorings. We also reserve the symbol π_{n-1} for the corresponding π (that is, $v_n = v_{\pi_{n-1}}$), and reserve $f_n \in E(u_n, v_n)$ for an edge in $\partial(T_n)$ such that $\pi_{n-1}(f_n)$ is a defective color with respect to π_{n-1}. We call v_n the *maximum defective vertex* with respect to $(T_n, D_{n-1}, \varphi_{n-1})$.

(3.1) In our algorithm, there are three types of augmentations: revisiting extension (RE), series extension (SE), and parallel extension (PE). Each iteration $n\,(\ge 1)$ involves a special vertex v_n, which is called an *extension* vertex if $\Theta_n = SE$ and a *supporting* vertex if $\Theta_n = PE$.

Algorithm 3.1

Step 0. Let $\varphi_0 = \varphi$ and let T_1 be a closure of e under φ_0, which is a closed Tashkinov tree with respect to e and φ_0. Set $S_0 = F_0 = \Theta_0 = \emptyset$ and set $n = 1$.

Step 1. If T_n is strongly closed with respect to φ_{n-1}, stop. Else, if there exists a subscript $h \le n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that $\Theta_i = RE$ for all i with $h+1 \le i \le n-1$, if any, and such that some (γ_h, δ_h)-cycle with respect to φ_{n-1} contains both an edge $f_n \in \partial_{\varphi_{n-1},\gamma_h}(T_n)$ and a segment L connecting $V(T_h)$ and v_n with $V(L) \subseteq V(T_n)$, where v_n is the end of f_n in T_n, go to Step 2. Else, let $D_{n-1} = \cup_{i \le n-1} S_i - \overline{\varphi}_{n-1}(T_{n-1})$, where $T_0 = \emptyset$. Let v_n, π_{n-1}, and $f_n \in E(u_n, v_n)$ be as defined above the algorithm, and let $\delta_n = \pi_{n-1}(f_n)$. If for every $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring σ_{n-1}, we have $\overline{\sigma}_{n-1}(u_n) \cap \overline{\sigma}_{n-1}(T_n) = \emptyset$, go to Step 3. Else, go to Step 4.

Step 2. Let $\varphi_n = \varphi_{n-1}$, let T_{n+1} be a closure of $T_n + f_n$ under φ_n, and let $\delta_n = \delta_h$, $\gamma_n = \gamma_h$, $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = RE$. Set $n = n+1$, return to Step 1. (We call this augmentation a **revisiting extension** (RE), call f_n an *RE connecting edge*, and call δ_n and γ_n *connecting colors*. Note that v_n is neither called an extension vertex nor called a supporting vertex (see (3.1)).

Step 3. Let $\varphi_n = \pi_{n-1}$, let T_{n+1} be a closure of $T_n + f_n$ under φ_n, and let $S_n = \{\delta_n\}$, $F_n = \{f_n\}$, and $\Theta_n = SE$. Set $n = n+1$, return to Step 1. (We call this augmentation a **series extension** (SE), call f_n an *SE connecting edge*, call δ_n a *connecting color*, and call v_n an *extension vertex*.)

Step 4. Let A_{n-1} be the set of all iterations i with $1 \le i \le n-1$ such that $\Theta_i = PE$ and $v_i = v_n$.

Let γ_n be a color in $\overline{\pi}_{n-1}(v_n)\cap(\cup_{i\in A_{n-1}}S_i)$ if $A_{n-1}\neq\emptyset$ (see (3.5) below), and a color in $\overline{\pi}_{n-1}(v_n)$ otherwise. By Lemmas 2.5 and 2.6, there exists a $(T_n, D_{n-1}\cup\{\delta_n\}, \pi_{n-1})$-stable coloring π'_{n-1}, such that v_n is a $(T_n, \pi'_{n-1}, \{\gamma_n, \delta_n\})$-exit. Let $\varphi_n = \pi'_{n-1}/P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$, $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = PE$. Let T_{n+1} be a closure of T_n under φ_n. Set $n = n+1$, return to Step 1. (We call this augmentation a **parallel extension** (PE), call f_n a *PE connecting edge*, call δ_n and γ_n *connecting colors*, and call v_n a *supporting vertex*. Note that f_n is not necessarily contained in T_{n+1}.)

Throughout the remainder of this paper, we reserve all symbols used for the same usage as in this algorithm. In particular, $D_n = \cup_{i\leq n}S_i - \overline{\varphi}_n(T_n)$ (see Step 1) for $n\geq 0$. So $D_0=\emptyset$.

To help understand the algorithm better, let us make a few remarks and offer some simple observations.

(3.2) In our proof we shall restrict our attention to the case when $|T_n|$ is odd (as we shall see). Suppose T_n is not strongly closed with respect to φ_{n-1} (see Step 1). Then, by Lemma 2.7, T_n is closed but not strongly closed with respect to any $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring. Thus v_n, π_{n-1}, and f_n involved in Step 1 are all well defined. It follows that at least one of RE, SE and PE applies to each iteration, and hence the algorithm terminates only when T_n is strongly closed with respect to φ_{n-1}, which contains the case when $V(T_n)=V(G)$.

(3.3) As described in the algorithm, revisiting extension (RE) has priority over both series and parallel extensions (SE and PE). If $\Theta_n = RE$, then from Algorithm 3.1 we see that the (γ_h, δ_h)-cycle with respect to φ_{n-1} displayed in Step 1 must contain at least one edge in $G[T_h]$, at least two boundary edges of T_h colored with γ_h, and at least two boundary edges of T_n colored with γ_h, because δ_h is a missing color in T_h under both φ_h and φ_{n-1}.

(3.4) It is clear that δ_n is a defective color of T_n with respect to φ_n when $\Theta_n = SE$ or PE (as $|\partial_{\pi_{n-1},\delta_n}(T_n)|\geq 3$ when $|T_n|$ is odd), while γ_n is a defective color of T_n with respect to φ_n when $\Theta_n = RE$. Moreover, D_{n-1} is the set of all connecting colors in $\cup_{h\leq n-1}S_h$ that are not missing in T_{n-1} with respect to φ_{n-1}.

(3.5) As we shall prove in Lemma 3.3, if $A_{n-1}\neq\emptyset$ in Step 4, then $\overline{\pi}_{n-1}(v_n)\cap(\cup_{i\in A_{n-1}}S_i) = \overline{\varphi}_{n-1}(v_n)\cap(\cup_{i\in A_{n-1}}S_i)$ contains precisely one color, so γ_n can be selected in a unique way. This property will play an important role in our proof.

Lemma 3.2. *For $n\geq 1$, the following statements hold:*

(i) $\overline{\varphi}_{n-1}(T_n)\cup D_{n-1}\subseteq\overline{\varphi}_n(T_n)\cup D_n\subseteq\overline{\varphi}_n(T_{n+1})\cup D_n$.

(ii) For any edge f incident to T_n, if $\varphi_{n-1}(f)\in\overline{\varphi}_{n-1}(T_n)\cup D_{n-1}$, then $\varphi_n(f)=\varphi_{n-1}(f)$, unless $\Theta_n = PE$ and $f=f_n$. So $\varphi_n(f)\in\overline{\varphi}_n(T_n)\cup D_n$ provided that $\varphi_{n-1}(f)\in\overline{\varphi}_{n-1}(T_n)\cup D_{n-1}$.

(iii) $\varphi_{n-1}\langle T_n\rangle\subseteq\overline{\varphi}_{n-1}(T_n)\cup D_{n-1}$ and $\varphi_n\langle T_n\rangle\subseteq\overline{\varphi}_n(T_n)\cup D_n$. So $\sigma_n(f)=\varphi_n(f)$ for any (T_n, D_n, φ_n)-stable coloring σ_n and any edge f on T_n.

(iv) If $\Theta_n = PE$, then $\partial_{\varphi_n,\gamma_n}(T_n)=\{f_n\}$, and edges in $\partial_{\varphi_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n)-v_n)$. Furthermore, each color in $\overline{\varphi}_n(T_n)-\{\delta_n\}$ is closed in T_n under φ_n.

Proof. By definition, $D_{n-1}=\cup_{i\leq n-1}S_i-\overline{\varphi}_{n-1}(T_{n-1})$. So $\overline{\varphi}_{n-1}(T_n)\cup D_{n-1}=\overline{\varphi}_{n-1}(T_n)\cup[\cup_{i\leq n-1}S_i-\overline{\varphi}_{n-1}(T_{n-1})]$. Since $\overline{\varphi}_{n-1}(T_{n-1})\subseteq\overline{\varphi}_{n-1}(T_n)$, we obtain

(1) $\overline{\varphi}_{n-1}(T_n)\cup D_{n-1}=\overline{\varphi}_{n-1}(T_n)\cup(\cup_{i\leq n-1}S_i)$.

Similarly, we can prove that

(2) $\overline{\varphi}_n(T_n) \cup D_n = \overline{\varphi}_n(T_n) \cup (\cup_{i\le n} S_i)$.

(i) For any $\alpha \in \overline{\varphi}_{n-1}(T_n)$, from Algorithm 3.1 and definition of stable colorings we see that $\alpha \in \overline{\varphi}_n(T_n)$, unless $\Theta_n = PE$ and $\alpha = \gamma_n$; in this exceptional case, $\alpha \in S_n$. So $\overline{\varphi}_{n-1}(T_n) \subseteq \overline{\varphi}_n(T_n) \cup S_n$ and hence $\overline{\varphi}_{n-1}(T_n) \cup (\cup_{i\le n-1} S_i) \subseteq \overline{\varphi}_n(T_n) \cup (\cup_{i\le n} S_i)$. It follows from (1) and (2) that $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$. Clearly, $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T_{n+1}) \cup D_n$.

(ii) Let f be an edge incident to T_n with $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. If $\Theta_n = RE$, then $\varphi_n = \varphi_{n-1}$ by Step 1 of Algorithm 3.1, which implies $\varphi_n(f) = \varphi_{n-1}(f)$. So we may assume that $\Theta_n \neq RE$. Let π_{n-1} be the $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring as specified in Step 1 of Algorithm 3.1. By the definition of stable colorings, we obtain $\pi_{n-1}(f) = \varphi_{n-1}(f)$. If $\Theta_n = SE$, then $\varphi_n(f) = \pi_{n-1}(f)$ by Step 3 of Algorithm 3.1. Hence $\varphi_n(f) = \varphi_{n-1}(f)$. It remains to consider the case when $\Theta_n = PE$. Let π'_{n-1} be the $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring as specified in Step 4 of Algorithm 3.1. By Lemma 2.4, π'_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. Hence $\pi'_{n-1}(f) = \varphi_{n-1}(f)$. Since $\varphi_n = \pi'_{n-1}/P_{v_n}(\delta_n, \gamma_n, \pi'_{n-1})$ and $P_{v_n}(\delta_n, \gamma_n, \pi'_{n-1})$ contains only one edge f_n incident to T_n (see Step 4 of Algorithm 3.1), we have $\varphi_n(f) = \pi'_{n-1}(f)$, unless $f = f_n$. It follows that $\varphi_n(f) = \varphi_{n-1}(f)$, unless $f = f_n$; in this exceptional case, $\varphi_{n-1}(f) = \delta_n$ and $\varphi_n(f) = \gamma_n \in S_n$. Hence $\varphi_n(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \cup S_n \subseteq \overline{\varphi}_n(T_n) \cup D_n \cup S_n = \overline{\varphi}_n(T_n) \cup D_n$ by (i) and (2), as desired.

(iii) Let us first prove the statement $\varphi_{n-1}\langle T_n\rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$ by induction on n. As the statement holds trivially when $n = 1$, we proceed to the induction step and assume that the statement has been established for $n-1$; that is,

(3) $\varphi_{n-2}\langle T_{n-1}\rangle \subseteq \overline{\varphi}_{n-2}(T_{n-1}) \cup D_{n-2}$.

By (3) and (ii) (with $n-1$ in place of n), for each edge f on T_{n-1} we have $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_{n-1}) \cup D_{n-1} \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. For each edge $f \in T_n - T_{n-1}$, from Algorithm 3.1 and TAA we see that $\varphi_{n-1}(f) \in D_{n-1}$ if f is a connecting edge and $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n)$ otherwise. Combining these observations, we obtain $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. Hence $\varphi_{n-1}\langle T_n\rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$, which together with (ii) implies $\varphi_n\langle T_n\rangle \subseteq \overline{\varphi}_n(T_n) \cup D_n$.

It follows that for any edge f on T_n, we have $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$. Thus $\sigma_n(f) = \varphi_n(f)$ for any (T_n, D_n, φ_n)-stable coloring σ_n.

(iv) From the definitions of π_{n-1} and stable colorings, we see that edges in $\partial_{\pi_{n-1},\delta_n}(T_n)$ are all incident to $V(T_n(v_n))$, and each color in $\overline{\pi}_{n-1}(T_n)$ is closed in T_n under π_{n-1}. So, by the definitions of π'_{n-1} and stable colorings, edges in $\partial_{\pi'_{n-1},\delta_n}(T_n)$ are all incident to $V(T_n(v_n))$, and each color in $\overline{\pi}'_{n-1}(T_n)$ is closed in T_n under π'_{n-1}. Thus the desired statements follow instantly from the definition of φ_n in Step 4. ∎

Lemma 3.3. *Let u be a vertex of T_n and let B_{n-1} be the set of all iterations j with $1 \le j \le n-1$, such that $\Theta_j = PE$ and $v_j = u$. Suppose $B_{n-1} = \{i_1, i_2, \ldots, i_p\}$, where $1 \le i_1 < i_2 < \ldots < i_p \le n-1$. Then the following statements hold:*

(i) $\overline{\varphi}_{n-1}(u) \cap (\cup_{j\in B_{n-1}} S_j) = \overline{\varphi}_{i_p}(u) \cap (\cup_{j\in B_{n-1}} S_j) = \{\delta_{i_p}\}$;

(ii) $\gamma_{i_2} = \delta_{i_1}$, $\gamma_{i_3} = \delta_{i_2}$, $\ldots$, $\gamma_{i_p} = \delta_{i_{p-1}}$; *and*

(iii) $\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\} = \overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}$. *So* $\overline{\varphi}_{i_1-1}(u) = (\overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}) \cup \{\gamma_{i_1}\}$ *and* $\overline{\varphi}_{i_p}(u) = (\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\}) \cup \{\delta_{i_p}\}$.

Proof. By the definition of B_{n-1}, for any iteration j with $i_p+1 \le j \le n-1$, if $v_j = u$, then $\Theta_j = RE$ or SE. So $\overline{\varphi}_{n-1}(u) = \overline{\varphi}_{i_p}(u)$ by Algorithm 3.1 and the definition of stable colorings. Thus, to prove (i), it suffices to show that $\overline{\varphi}_{i_p}(u) \cap (\cup_{j\in B_{n-1}} S_j) = \{\delta_{i_p}\}$.

Set $C_h = \{i_1, i_2, \ldots, i_h\}$ for $1 \le h \le p$. Then $C_p = B_{n-1}$ and hence (i) is equivalent to saying that

(i') $\overline{\varphi}_{i_p}(u) \cap (\cup_{j\in C_p} S_j) = \{\delta_{i_p}\}$.

Let us prove statements (i'), (ii), and (iii) simultaneously by induction on p.

From Step 4 and the definition of stable colorings, we see that $\gamma_{i_1} \in \overline{\pi}_{i_1-1}(u) = \overline{\varphi}_{i_1-1}(u)$, $\delta_{i_1} \notin \overline{\pi}_{i_1-1}(u) = \overline{\varphi}_{i_1-1}(u)$, and $\overline{\varphi}_{i_1}(u)$ is obtained from $\overline{\varphi}_{i_1-1}(u)$ by replacing γ_{i_1} with δ_{i_1}. So $\overline{\varphi}_{i_1}(u) \cap (\cup_{j\in C_1} S_j) = \overline{\varphi}_{i_1}(u) \cap \{\gamma_{i_1}, \delta_{i_1}\} = \{\delta_{i_1}\}$ and $\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\} = \overline{\varphi}_{i_1}(u) - \{\delta_{i_1}\}$. Thus both (i') and (iii) hold for $p = 1$. For (ii), there is nothing to prove now.

Suppose we have established these statements for $p-1$. Let us proceed to the induction step for p.

By the induction hypotheses on (i') and (iii), we obtain the following two equalities:

(1) $\overline{\varphi}_{i_{p-1}}(u) \cap (\cup_{j\in C_{p-1}} S_j) = \{\delta_{i_{p-1}}\}$ and

(2) $\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\} = \overline{\varphi}_{i_{p-1}}(u) - \{\delta_{i_{p-1}}\}$.

From the definition of B_{n-1}, we see that for any iteration j with $i_{p-1}+1 \le j \le i_p - 1$, if $v_j = u$, then $\Theta_j = RE$ or SE. Thus, by Algorithm 3.1 and the definition of stable colorings, we obtain $\overline{\varphi}_{i_{p-1}}(u) = \overline{\varphi}_{i_p-1}(u) = \overline{\pi}_{i_p-1}(u)$. According to Step 4 and using (1),

(3) $\gamma_{i_p} = \delta_{i_{p-1}} \in \overline{\varphi}_{i_{p-1}}(u)$, $\delta_{i_p} \notin \overline{\varphi}_{i_{p-1}}(u)$, and $\overline{\varphi}_{i_p}(u)$ is obtained from $\overline{\varphi}_{i_{p-1}}(u)$ by replacing γ_{i_p} with δ_{i_p}.

Clearly, (3) implies (ii) and the following equality:

(4) $\overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\} = \overline{\varphi}_{i_{p-1}}(u) - \{\delta_{i_{p-1}}\}$.

It follows from (2) and (4) that $\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\} = \overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}$, thereby proving (iii).

By (1), we have $(\overline{\varphi}_{i_{p-1}}(u) - \{\delta_{i_{p-1}}\}) \cap (\cup_{j\in C_{p-1}} S_j) = \emptyset$. So $(\overline{\varphi}_{i_{p-1}}(u) - \{\delta_{i_{p-1}}\}) \cap (\cup_{j\in C_p} S_j) = (\overline{\varphi}_{i_{p-1}}(u) - \{\delta_{i_{p-1}}\}) \cap S_{i_p} = (\overline{\varphi}_{i_{p-1}}(u) - \{\gamma_{i_p}\}) \cap \{\gamma_{i_p}, \delta_{i_p}\} = \emptyset$, where last two equalities follow from (3). Combining this observation with (4) yields $\overline{\varphi}_{i_p}(u) \cap (\cup_{j\in C_p} S_j) = [(\overline{\varphi}_{i_{p-1}}(u) - \{\delta_{i_{p-1}}\}) \cup \{\delta_{i_p}\}] \cap (\cup_{j\in C_p} S_j) = \{\delta_{i_p}\} \cap (\cup_{j\in C_p} S_j) = \{\delta_{i_p}\}$. Hence (i') is established.

Since $\gamma_{i_1} \in \overline{\varphi}_{i_1-1}(u)$ and $\delta_{i_p} \in \overline{\varphi}_{i_p}(u)$, from the equality $\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\} = \overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}$, we immediately deduce that $\overline{\varphi}_{i_1-1}(u) = (\overline{\varphi}_{i_p}(u) - \{\delta_{i_p}\}) \cup \{\gamma_{i_1}\}$ and $\overline{\varphi}_{i_p}(u) = (\overline{\varphi}_{i_1-1}(u) - \{\gamma_{i_1}\}) \cup \{\delta_{i_p}\}$. ∎

Lemma 3.4. *$|D_n| \le n$.*

Proof. Recall that $D_n = \cup_{i\le n} S_i - \overline{\varphi}_n(T_n)$. For $1 \le i \le n$, by Algorithm 3.1, we have $S_i = \{\delta_i\}$ if $\Theta_i = SE$ and $S_i = \{\delta_i, \gamma_i\}$ otherwise.

To establish the desired inequality, we apply induction on n. Trivially, the statement holds when $n = 0, 1$. So we proceed to the induction step, and assume that $|D_{n-1}| \le n-1$ for some $n \ge 2$.

If $\Theta_n = RE$, then $\varphi_n = \varphi_{n-1}$ and $S_n = S_{n-1}$ by Step 2 in Algorithm 3.1. So $D_n \subseteq D_{n-1}$ and hence $|D_n| \le n-1$.

If $\Theta_n = SE$, then $S_n = \{\delta_n\}$ and $\overline{\varphi}_n(T_n) = \overline{\varphi}_{n-1}(T_n)$ by Step 3 in Algorithm 3.1 and the definition of stable colorings. It follows that $D_n \subseteq D_{n-1} \cup \{\delta_n\}$. Hence $|D_n| \le |D_{n-1}| + 1 \le n$.

It remains to consider the case when $\Theta_n = PE$. By Step 4 in Algorithm 3.1 and the definition of stable colorings, we obtain $\delta_n \notin \overline{\varphi}_{n-1}(T_n)$ and $(\overline{\varphi}_{n-1}(T_n) - \{\gamma_n\}) \cup \{\delta_n\} \subseteq \overline{\varphi}_n(T_n)$. So

$$\begin{aligned} D_n &= \cup_{i \le n} S_i - \overline{\varphi}_n(T_n) \\ &\subseteq \cup_{i \le n-1} S_i \cup \{\delta_n, \gamma_n\} - [(\overline{\varphi}_{n-1}(T_n) - \{\gamma_n\}) \cup \{\delta_n\}] \\ &\subseteq \cup_{i \le n-1} S_i \cup \{\gamma_n\} - (\overline{\varphi}_{n-1}(T_n) - \{\gamma_n\}) \\ &\subseteq [\cup_{i \le n-1} S_i - \overline{\varphi}_{n-1}(T_n)] \cup \{\gamma_n\} \\ &\subseteq D_{n-1} \cup \{\gamma_n\}. \end{aligned}$$

Hence $|D_n| \le |D_{n-1}| + 1 \le n$. ∎

Lemma 3.5. *Suppose $\Theta_n = PE$ (see Step 4). Let σ_n be a (T_n, D_n, φ_n)-stable coloring and let $\sigma_{n-1} = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. If $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$, then σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable and hence is $(T_n, D_{n-1}, \varphi_{n-1})$-stable.*

Proof. Let π'_{n-1} be as specified in Step 4 of Algorithm 3.1. Recall that

(1) π'_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable.

By definition, $\varphi_n = \pi'_{n-1} / P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$. So

(2) $\pi'_{n-1} = \varphi_n / P_{v_n}(\gamma_n, \delta_n, \varphi_n)$.

We propose to show that

(3) σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi'_{n-1})$-stable.

To justify this, note that $\overline{\sigma}_n(v) = \overline{\varphi}_n(v)$ for all $v \in V(T_n)$, because σ_n is a (T_n, D_n, φ_n)-stable coloring. Thus, by the definition of σ_{n-1} and (2), we obtain

(4) $\overline{\sigma}_{n-1}(v) = \overline{\pi}'_{n-1}(v)$ for all $v \in V(T_n)$.

Let f be an edge incident to T_n with $\pi'_{n-1}(f) \in \overline{\pi}'_{n-1}(T_n) \cup D_{n-1} \cup \{\delta_n\}$. By (1), we have $\pi'_{n-1}(f) \in \overline{\pi}_{n-1}(T_n) \cup D_{n-1} \cup \{\delta_n\}$. Since π_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable, we further obtain $\pi'_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \cup \{\delta_n\}$. So $\pi'_{n-1}(f) \in \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i). From Step 4 we see that

(5) $\pi'_{n-1}(f) = \varphi_n(f)$ if $f \ne f_n$, $\pi'_{n-1}(f_n) = \delta_n$, and $\varphi_n(f_n) = \gamma_n$.

So $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$. Hence $\sigma_n(f) = \varphi_n(f)$, because σ_n is a (T_n, D_n, φ_n)-stable coloring. From the definition of σ_{n-1}, we deduce that $\sigma_{n-1}(f) = \sigma_n(f)$ if $f \ne f_n$ and $\sigma_{n-1}(f_n) = \delta_n$. Combining these observations with (5) yields

(6) $\sigma_{n-1}(f) = \pi'_{n-1}(f)$ for any edge f incident to T_n with $\pi'_{n-1}(f) \in \overline{\pi}'_{n-1}(T_n) \cup D_{n-1} \cup \{\delta_n\}$.

Thus (3) follows instantly from (4) and (6). Using (1), (3) and Lemma 2.4, we conclude that σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable. So σ_{n-1} is $(T_n, D_{n-1}, \pi_{n-1})$-stable. Since π_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable, from Lemma 2.4 it follows that σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. ∎

Let us now present a generalized version of Tashkinov trees to be used in our proof.

Definition 3.6. Let (G, e, φ) be a k-triple. A tree sequence T with respect to G and e is called an *extended Tashkinov tree* (ETT) if there exists a Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ constructed from (G, e, φ) by using Algorithm 3.1, such that $T_n \subset T \subseteq T_{n+1}$, where $T_0 = \emptyset$.

As introduced in Subsection 2.1, by $T_n \subset T \subseteq T_{n+1}$ we mean that T_n is a proper segment of T, and T is a segment of T_{n+1}.

Observe that the extended Tashkinov tree T has a built-in ladder-like structure. So we propose to call the sequence $T_1 \subset T_2 \subset \ldots \subset T_n \subset T$ the *ladder* of T, and call n the *rung number* of T and denote it by $r(T)$. Moreover, we call $(\varphi_0, \varphi_1, \ldots, \varphi_n)$ the *coloring sequence* of T, call φ_n the *generating coloring* of T, and call $\mathcal{T}$ the Tashkinov series *corresponding* to T.

In our proof we shall frequently work with stable colorings; the following concept will be used to keep track of the structures of ETTs.

Definition 3.7. Let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series constructed from a k-triple (G, e, φ) by using Algorithm 3.1. A coloring $\sigma_n \in \mathcal{C}^k(G-e)$ is called $\varphi_n \bmod T_n$ if there exists an ETT T^* with corresponding Tashkinov series $\mathcal{T}^* = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, satisfying $\sigma_0 \in \mathcal{C}^k(G-e)$ and the following conditions for all i with $1 \le i \le n$:

- $T_i^* = T_i$ and
- σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G-e)$, where $D_i = \cup_{h \le i} S_h - \overline{\varphi}_i(T_i)$.

We call T^* an ETT *corresponding* to (σ_n, T_n) (or simply *corresponding* to σ_n if no ambiguity arises).

Remark. Comparing $\mathcal{T}^*$ with $\mathcal{T}$, we see that T_{i+1}^* in $\mathcal{T}^*$ is obtained from T_i by using the same connecting edge, connecting color, and extension type as T_{i+1} in $\mathcal{T}$ for $1 \le i \le n$. Furthermore, $T_1 \subset T_2 \subset \ldots \subset T_n \subset T^*$ is the ladder of T^* and $r(T^*) = n$. Since σ_i is a (T_i, D_i, φ_i)-stable coloring, by Lemma 3.2(iii), we have $\sigma_i(f) = \varphi_i(f)$ for any edge f on T_i and $1 \le i \le n$; this fact will be used repeatedly in our paper.

To ensure that the structures of ETTs are preserved under taking stable colorings, we impose some restrictions on such trees.

Definition 3.8. Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. We say that T has the *maximum property* (MP) under $(\varphi_0, \varphi_1, \ldots, \varphi_n)$ (or simply under φ_n if no ambiguity arises), if $|T_1|$ is maximum over all Tashkinov trees T_1' with respect to an edge $e' \in E$ and a coloring $\varphi_0' \in \mathcal{C}^k(G-e')$, and $|T_{i+1}|$ is maximum over all (T_i, D_i, φ_i)-stable colorings for any i with $1 \le i \le n-1$; that is, $|T_{i+1}|$ is maximum over all tree sequences T_{i+1}', which is a closure of $T_i + f_i$ (resp. T_i) under a (T_i, D_i, φ_i)-stable coloring φ_i' if $\Theta_i = RE$ or SE (resp. if $\Theta_i = PE$), where f_i is the connecting edge in F_i.

At this point a natural question is to ask whether an ETT with sufficiently large size and satisfying the maximum property can be constructed to fulfill our needs. We shall demonstrate that it is indeed the case (see Lemma 3.11). The statement below follows instantly from the above two definitions and Lemma 2.4 (the details can also be found in the proof of Lemma 3.11).

Lemma 3.9. *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, let σ_n be a $\varphi_n \bmod T_n$ coloring, and let T^* be an ETT corresponding to (σ_n, T_n) (see Definition 3.7). If T satisfies MP under φ_n, then T^* satisfies MP under σ_n.* ∎

The importance of the maximum property is revealed by the following statement to be established: If T enjoys the maximum property under φ_n, then $V(T)$ is elementary with respect to φ_n; Theorem 2.1 follows from it and Theorem 2.2 as a corollary. We shall prove this statement by induction on the rung number $r(T)$. To carry out the induction step, we need several auxiliary results concerning ETTs with the maximum property. Thus what we are going to establish is a stronger version.

Let us define a few terms before presenting our theorem. For each $v \in V(T)$, we use $m(v)$ to denote the minimum subscript i such that $v \in V(T_i)$. Let α and β be two colors in $[k]$. We say that α and β are *T-interchangeable* under φ_n if there is at most one (α, β)-path with respect to φ_n intersecting T. When T is closed (that is, $T = T_{n+1}$), we also say that T has the *interchangeability property* with respect to φ_n if under any (T, D_n, φ_n)-stable coloring σ_n, any two colors α and β are T-interchangeable, provided that $\overline{\sigma}_n(T) \cap \{\alpha, \beta\} \neq \emptyset$ (equivalently $\overline{\varphi}_n(T) \cap \{\alpha, \beta\} \neq \emptyset$).

The undefined symbols and notations in the theorem below can all be found in Algorithm 3.1.

Theorem 3.10. *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. If T has the maximum property under φ_n, then the following statements hold:*

- *(i) $V(T)$ is elementary with respect to φ_n.*
- *(ii) T_{n+1} has the interchangeability property with respect to φ_n.*
- *(iii) For any $i \leq n$, if v_i is a supporting vertex and $m(v_i) = j$, then every (T_i, D_i, φ_i)-stable coloring σ_i is $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable. In particular, σ_i is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable. Furthermore, for any two distinct supporting vertices v_i and v_j with $i, j \leq n$, if $m(v_i) = m(v_j)$, then $S_i \cap S_j = \emptyset$.*
- *(iv) If $\Theta_n = PE$, then $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains precisely one vertex, v_n, from T_n for any (T_n, D_n, φ_n)-stable coloring σ_n.*
- *(v) For any (T_n, D_n, φ_n)-stable coloring σ_n and any defective color δ of T_n with respect to σ_n, if v is a vertex but not the smallest one (in the order $\prec$) in $I[\partial_{\sigma_n, \delta}(T_n)]$, then $v \preceq v_i$ for any supporting or extension vertex v_i with $i \geq m(v)$.*
- *(vi) Every (T_n, D_n, φ_n)-stable coloring σ_n is a φ_n mod T_n coloring. (So every ETT corresponding to (σ_n, T_n) (see Definition 3.7) satisfies MP under σ_n by Lemma 3.9.)*

Recall that in the definition of maximum property (see Definition 3.8), $|T_{n+1}|$ is not required to be maximum over all (T_n, D_n, φ_n)-stable colorings. This relaxation allows us to proceed by induction in our proof of Theorem 3.10. Now let us show that Theorem 2.1 can be deduced easily from this theorem.

Lemma 3.11. *Let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$ be a Tashkinov series constructed from a k-triple (G, e, φ) by using Algorithm 3.1. Suppose T_{n+1} has MP under φ_n. Then there exists a Tashkinov series $\mathcal{T}^* = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, satisfying $\sigma_0 \in \mathcal{C}^k(G-e)$, $|T_{n+1}^*| \geq |T_{n+1}|$, and the following conditions for $1 \leq i \leq n$:*

- *(i) $T_i^* = T_i$;*
- *(ii) σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G-e)$; and*
- *(iii) $|T_{i+1}^*|$ is maximum over all (T_i, D_i, σ_i)-stable colorings (see Definition 3.8).*

Furthermore, if T^*_{n+1} is not strongly closed with respect to σ_n, then there exists a Tashkinov series $\{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+2\}$, such that $T^*_{n+1} \subset T^*_{n+2}$ and T^*_{n+2} satisfies MP under σ_{n+1}.

Proof. Let μ be an arbitrary (T_n, D_n, φ_n)-stable coloring. Then μ is a φ_n mod T_n coloring by Theorem 3.10(vi) (with $T = T_{n+1}$). Thus Definition 3.7 guarantees the existence of an ETT, denoted by $T_{n+1}(\mu)$, corresponding to (μ, T_n), which is a closure of $T_n + f_n$ (resp. T_n) under μ if $\Theta_n = RE$ or SE (resp. if $\Theta_n = PE$). Let us reserve σ_n for a (T_n, D_n, φ_n)-stable coloring such that $T_{n+1}(\sigma_n)$ has the maximum number of vertices among all these $T_{n+1}(\mu)$'s, and let $T^*_{i+1} = T_{n+1}(\sigma_n)$. Then $|T^*_{n+1}| \ge |T_{n+1}|$. By Lemma 2.4, every (T_n, D_n, σ_n)-stable coloring is a (T_n, D_n, φ_n)-stable coloring. So $|T^*_{n+1}|$ is also maximum over all (T_n, D_n, σ_n)-stable colorings.

Since σ_n is a φ_n mod T_n coloring, by Definition 3.7, there exists a Tashkinov series $\mathcal{T}^* = \{(T^*_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ that satisfies conditions (i) and (ii) as described in the lemma. Using the same argument as employed in the preceding paragraph, we see that $|T^*_{i+1}|$ is maximum over all (T_i, D_i, σ_i)-stable colorings as well for $1 \le i \le n-1$.

Suppose T^*_{n+1} is not strongly closed with respect to σ_n. Then we can construct a new tuple $(T^*_{n+2}, \sigma_{n+1}, S_{n+1}, F_{n+1}, \Theta_{n+1})$ by using Algorithm 3.1. Clearly, $T^*_{n+1} \subset T^*_{n+2}$ and T^*_{n+2} satisfies MP under σ_{n+1}. ∎

Proof of Theorem 2.1. Let $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ be a Tashkinov series constructed from a k-triple (G, e, φ), such that

(a) T_{n+1} satisfies MP under φ_n;

(b) subject to (a), $|T_{n+1}|$ is maximum over all (T_n, D_n, φ_i)-stable colorings; and

(c) subject to (a) and (b), T_{n+1} contains as many vertices as possible.

By Lemma 3.11, such a Tashkinov series $\mathcal{T}$ exists, and T_{n+1} is strongly closed with respect to φ_n. By Theorem 3.10(i), $V(T_{n+1})$ is elementary with respect to φ_n. From Theorem 2.2(i) and (iv), we thus deduce that G is an elementary multigraph. ∎

The proof of Theorem 3.10 will take up the entire remainder of this paper.

4 Auxiliary Results

We prove Theorem 3.10 by induction on the rung number $r(T) = n$. The present section is devoted to a proof of statement (ii) in Theorem 3.10 in the base case and proofs of statements (iii)-(vi) in the general case.

For $n = 0$, statement (i) follows from Theorem 2.8, statements (iii)-(vi) hold trivially, and statement (ii) is a corollary of the following more general lemma.

Lemma 4.1. *Let (G, e, φ) be a k-triple, let T be a closed Tashkinov tree with respect to e and φ, and let α and β be two colors in $[k]$ with $\overline{\varphi}(T) \cap \{\alpha, \beta\} \ne \emptyset$. Then there is at most one (α, β)-path with respect to φ intersecting T.*

Proof. Assume the contrary: there are at least two (α, β)-paths Q_1 and Q_2 with respect to φ intersecting T. By Theorem 2.8, $V(T)$ is elementary with respect to φ. So T contains at most two vertices v with $\overline{\varphi}(v) \cap \{\alpha, \beta\} \ne \emptyset$, which in turn implies that at least two ends of Q_1 and

Q_2 are outside T. By hypothesis, T is closed with respect to φ. Hence precisely one of α and β, say α, is in $\overline{\varphi}(T)$. Thus we further deduce that at least three ends of Q_1 and Q_2 are outside T. Traversing Q_1 and Q_2 from these ends respectively, we can find at least three $(T, \varphi, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3. We call the tuple $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* and use $\mathcal{K}$ to denote the set of all such counterexamples.

With a slight abuse of notation, let $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$ be a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$. For $i = 1, 2, 3$, let a_i and b_i be the ends of P_i with $b_i \in V(T)$, and f_i be the edge of P_i incident to b_i. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2 \prec b_3$. Let $\gamma \in \overline{\varphi}(b_3)$ and let $\sigma_1 = \varphi/(G - T, \alpha, \gamma)$. Clearly, $\sigma_1 \in \mathcal{C}^k(G - e)$ and T is also a Tashkinov tree with respect to e and σ_1. Furthermore, f_i is colored by β under both φ and σ_1 for $i = 1, 2, 3$.

Consider $\sigma_2 = \sigma_1/P_{b_3}(\gamma, \beta, \sigma_1)$. Note that $\beta \in \overline{\sigma}_2(b_3)$. Let T' be obtained from $T(b_3)$ by adding f_1 and f_2 and let T'' be a closure of T' under σ_2. Obviously, both T' and T'' are Tashkinov trees with respect to e and σ_2. By Theorem 2.8, $V(T'')$ is elementary with respect to σ_2.

Observe that none of a_1, a_2, a_3 is contained in T'', for otherwise, let $a_i \in V(T'')$ for some i with $1 \le i \le 3$. Since $\{\beta, \gamma\} \cap \overline{\sigma}_2(a_i) \ne \emptyset$ and $\beta \in \overline{\sigma}_2(b_3)$, we obtain $\gamma \in \overline{\sigma}_2(a_i)$. Hence from TAA we see that P_1, P_2, P_3 are all entirely contained in $G[T'']$, which in turn implies $\gamma \in \overline{\sigma}_2(a_j)$ for $j = 1, 2, 3$. So $V(T'')$ is not elementary with respect to σ_2, a contradiction. Each P_i contains a subpath Q_i, which is a T_2-exit path with respect to σ_2. Since f_1 is not contained in Q_1, we obtain $|Q_1| + |Q_2| + |Q_3| < |P_1| + |P_2| + |P_3|$. Thus the existence of the counterexample $(\sigma_2, T'', \gamma, \beta, Q_1, Q_2, Q_3)$ violates the minimality assumption on $(\varphi, T, \alpha, \beta, P_1, P_2, P_3)$. ∎

So Theorem 3.10 is true in the base case. Suppose we have established that

(4.1) Theorem 3.10 holds for all ETTs with at most $n-1$ rungs and satisfying MP, for some $n \ge 1$.

Let us proceed to the induction step. We postpone the proof of Theorem 3.10(i) and (ii) to Section 7, and present a proof of Theorem 3.10(iii)-(vi) in this section. In our proof of the $(i+2)$th statement in Theorem 3.10 for $2 \le i \le 4$, we further assume that

(4.*i*) the jth statement in Theorem 3.10 holds for all ETTs with at most n rungs and satisfying MP, for all j with $3 \le j \le i+1$.

We break the proof of the induction step into a series of lemmas. The following lemma generalizes Lemma 3.2(ii), and will be used in the proofs of Theorem 3.10(iii) and (iv).

Lemma 4.2. *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. For any $1 \le s \le n$ and any edge f incident to T_s, if $\varphi_{s-1}(f) \in \overline{\varphi}_{s-1}(T_s) \cup D_{s-1}$, then $\varphi_t(f) = \varphi_{s-1}(f)$ for any t with $s \le t \le n$, unless $f = f_p \in F_p$ for some p with $s \le p \le t$ and $\Theta_p = PE$. In particular, if f is an edge in $G[T_s]$ with $\varphi_{s-1}(f) \in \overline{\varphi}_{s-1}(T_s) \cup D_{s-1}$, then $\varphi_t(f) = \varphi_{s-1}(f)$ for any t with $s \le t \le n$.*

Proof. By Lemma 3.2(i), we have $\overline{\varphi}_{i-1}(T_i) \cup D_{i-1} \subseteq \overline{\varphi}_i(T_{i+1}) \cup D_i$ for all $i \ge 1$. So to establish the first half, it suffices to prove the statement for $t = s$, which is exactly the same as Lemma 3.2(ii).

Note that if f is an edge in $G[T_s]$, then $f \notin \partial(T_p)$ for any p with $s \le p \le t$. Hence $f \ne f_p \in F_p$ for any p with $s \le p \le t$ and $\Theta_p = PE$. Thus the second half also holds. ∎

Lemma 4.3. *(Assuming (4.1)) Theorem 3.10(iii) holds for all ETTs with n rungs and satisfying MP; that is, for any $i \le n$, if v_i is a supporting vertex and $m(v_i) = j$, then every (T_i, D_i, φ_i)-stable coloring σ_i is $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable. In particular, σ_i is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable. Furthermore, for any two distinct supporting vertices v_i and v_j with $i, j \le n$, if $m(v_i) = m(v_j)$, then $S_i \cap S_j = \emptyset$.*

Proof. By hypothesis, T is an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, and T satisfies MP under φ_n.

We prove the first half by contradiction. Assume the contrary: there exists a subscript $i \le n$, such that v_i is a supporting vertex, $m(v_i) = j$, and some (T_i, D_i, φ_i)-stable coloring σ_i is not $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable. By definition, there exists an edge f incident to $T(v_i) - v_i$, with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T(v_i) - v_i) \cup D_{j-1}$, such that $\sigma_i(f) \ne \varphi_{j-1}(f)$, or there exists a vertex v of $T(v_i) - v_i$ such that $\overline{\sigma}_i(v) \ne \overline{\varphi}_{j-1}(v)$. In the former case, since $j \le i$, repeated application of Lemma 3.2(i) and (ii) yields $\overline{\varphi}_{j-1}(T(v_i) - v_i) \cup D_{j-1} \subseteq \overline{\varphi}_{j-1}(T_j) \cup D_{j-1} \subseteq \overline{\varphi}_{i-1}(T_i) \cup D_{i-1} \subseteq \overline{\varphi}_i(T_i) \cup D_i$ and $\varphi_i(f) \in \overline{\varphi}_i(T_i) \cup D_i$. Hence $\sigma_i(f) = \varphi_i(f)$, which implies $\varphi_i(f) \ne \varphi_{j-1}(f)$. In the latter case, since $\overline{\sigma}_i(v) = \overline{\varphi}_i(v)$, we have $\overline{\varphi}_i(v) \ne \overline{\varphi}_{j-1}(v)$. From Lemma 2.3 we deduce that φ_i is not $(T(v_i) - v_i, D_{j-1}, \varphi_{j-1})$-stable in either case.

Set $V_i^- = V(T(v_i) - v_i)$. Then there exists an edge f incident to V_i^- with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(V_i^-) \cup D_{j-1}$ such that $\varphi_{j-1}(f) \ne \varphi_i(f)$, or there exist a vertex $v \in V_i^-$ such that $\overline{\varphi}_{j-1}(v) \ne \overline{\varphi}_i(v)$. In either case, by Lemma 4.2 and Algorithm 3.1, there exists a supporting vertex $v_k \in V_i^-$ with $j \le k < i$. Thus $j \le i - 1$ and $v_k \prec v_i$.

Since $v_i \in V(T_j)$, we have $v_i \in V(T_{i-1})$. Let π_{i-1} be the $(T_i, D_{i-1}, \varphi_{i-1})$-stable coloring as specified in Steps 1 and 4 of Algorithm 3.1. Recall that $\delta_i = \pi_{i-1}(f_i)$. Since v_i is the maximum vertex in $I[\partial_{\pi_{i-1}, \delta_i}(T_i)]$, we see that δ_i is a defective color of T_{i-1} with respect to π_{i-1}, and v_i is not the smallest vertex in $I[\partial_{\pi_{i-1}, \delta_i}(T_{i-1})]$. As π_{i-1} is also $(T_{i-1}, D_{i-1}, \varphi_{i-1})$-stable, applying (4.1) and Theorem 3.10(v) to $v = v_i$ and π_{i-1}, we obtain $v_i \preceq v_k$; this contradiction establishes the first half of the assertion. Since $m(v_i) = j$, we have $v_i \notin V(T_{j-1})$. So T_{j-1} is entirely contained in $T(v_i) - v_i$, and hence σ_i is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable.

To establish the second half, let v_i and v_j be two distinct supporting vertices with $i < j \le n$ and $m(v_i) = m(v_j)$. We aim to show that $S_i \cap S_j = \emptyset$.

For $k = i, j$, let π_{k-1} be the $(T_k, D_{k-1}, \varphi_{k-1})$-stable coloring as specified in Steps 1 and 4 of Algorithm 3.1. Recall that $\delta_k = \pi_{k-1}(f_k)$ is a defective color of T_k with respect to π_{k-1}, and v_k is the maximum vertex in $I[\partial_{\pi_{k-1}, \delta_k}(T_k)]$. Let $r = m(v_i) = m(v_j)$. Since $r \le i < j$ and $v_j \in V(T_r)$, we have $v_j \in V(T_{j-1})$. As π_{j-1} is also $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable, applying Theorem 3.10(v) to π_{j-1}, T_{j-1} and $v = v_j$, we obtain $v_j \prec v_i$. By definition, $S_i = \{\delta_i, \gamma_i\}$. Observe that

(1) $\gamma_i \notin S_j$. Indeed, since $\gamma_i \in \overline{\varphi}_{i-1}(v_i)$ and $V(T_i)$ is elementary with respect to φ_{i-1} by (4.1) and Theorem 3.10(i), we have $\gamma_i \notin \overline{\varphi}_{i-1}(v_j)$. Let f be the edge incident to v_j with $\varphi_{i-1}(f) = \gamma_i$. Then f is an edge in $G[T_i]$, because T_i is closed with respect to φ_{i-1}. By Lemma 4.2, we have $\varphi_{j-1}(f) = \varphi_{i-1}(f) = \gamma_i$. So $\gamma_i \notin \overline{\varphi}_{j-1}(v_j)$ and $f \notin \partial(T_{j-1})$. Let π'_{j-1} be as specified in Step 4 in Algorithm 3.1. Since π'_{j-1} is $(T_j, D_{j-1}, \varphi_{j-1})$-stable, we have $\gamma_i \notin \overline{\pi}'_{j-1}(v_j)$ and $\pi'_{j-1}(f) = \gamma_i$, which implies $\gamma_i \notin S_j$.

(2) $\delta_i \notin S_j$. To justify this, note that $V(T_{i+1})$ is elementary with respect to φ_i by (4.1) and Theorem 3.10(i). Since $\delta_i \in \overline{\varphi}_i(v_i)$, we have $\delta_i \notin \overline{\varphi}_i(v_j)$. Let f be the edge incident to v_j with $\varphi_i(f) = \delta_i$. Since T_{i+1} is closed with respect to φ_i, edge f is contained in $G[T_{i+1}]$. Since $j > i$

and $\varphi_i(f) \in \overline{\varphi}_i(T_i) \cup D_i$, we have $\varphi_{j-1}(f) = \varphi_i(f) = \delta_i$ and $f \notin \partial(T_{j-1})$ by Lemma 4.2. Let π'_{j-1} be as specified in Step 4 of Algorithm 3.1. Then π'_{j-1} is $(T_j, D_{j-1}, \varphi_{j-1})$-stable. By definition, $\delta_i \notin \overline{\pi}'_{j-1}(v_j)$ and $\pi'_{j-1}(f) = \delta_i$. Hence $\delta_i \notin S_j$.

Combining (1) and (2), we conclude that $S_i \cap S_j = \emptyset$, as desired. ∎

The following lemma asserts that parallel extensions used in Algorithm 3.1 are preserved under taking certain stable colorings.

Lemma 4.4. *(Assuming (4.1) and (4.2)) Theorem 3.10(iv) holds for all ETTs with n rungs and satisfying MP; that is, if $\Theta_n = PE$, then $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains precisely one vertex, v_n, from T_n for any (T_n, D_n, φ_n)-stable coloring σ_n.*

Proof. Assume the contrary: $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains at least two vertices from T_n for some (T_n, D_n, φ_n)-stable coloring σ_n. Let $j = m(v_n)$. By applying a series of Kempe changes to σ_n, we shall construct a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring μ and an ETT T_j^* corresponding to (μ, T_{j-1}) with ladder $T_1 \subset T_2 \subset \ldots \subset T_{j-1} \subset T_j^*$, such that either $|T_j^*| > |T_j|$ or $V(T_j^*)$ is not elementary with respect to μ, which contradicts either the maximum property satisfied by T or the induction hypothesis (4.1) on Theorem 3.10(i).

Let L denote the set of all subscripts i with $j \le i \le n$, such that $\Theta_i = PE$ and $m(v_i) = j$, where v_i is the supporting vertex involved in iteration i. We partition L into disjoint subsets $L_1, L_2, \ldots, L_\kappa$, such that two subscripts $s, t \in L$ are in the same subset if and only if $v_s = v_t$. For $1 \le i \le \kappa$, write $L_i = \{i_1, i_2, \ldots, i_{c(i)}\}$, where $i_1 < i_2 < \ldots < i_{c(i)}$, and let w_i denote the common supporting vertex corresponding to L_i. Renaming subscripts if necessary, we may assume that $w_1 \prec w_2 \prec \ldots \prec w_\kappa$. For each L_i, define P_i to be the graph with vertex set $V(P_i) = \cup_{t \in L_i} S_t = \cup_{t \in L_i} \{\delta_t, \gamma_t\}$ and edge set $E(P_i) = \{\delta_t\gamma_t \ : \ t \in L_i\}$.

For each $t \in L$, we have $v_t \notin V(T_{j-1})$ because $m(v_t) = j$. It follows that $w_i \notin V(T_{j-1})$ for $1 \le i \le \kappa$. So each L_i consists of all subscripts t with $1 \le t \le n$, such that $\Theta_t = PE$ and $v_t = w_i$. By Lemma 3.3(ii) (with w_i and L_i in place of u and B_{n-1}, respectively), we obtain

(1) $\gamma_{i_2} = \delta_{i_1}$, $\gamma_{i_3} = \delta_{i_2}$, $\ldots$, $\gamma_{i_{c(i)}} = \delta_{i_{c(i)-1}}$. So P_i is the walk: $\gamma_{i_1} \to \delta_{i_1} = \gamma_{i_2} \to \delta_{i_2} = \gamma_{i_3} \to \cdots \to \delta_{i_{c(i)-1}} = \gamma_{i_{c(i)}} \to \delta_{i_{c(i)}}$, where $\gamma_{i_1} \in \overline{\varphi}_{i_1-1}(w_i)$ and $\delta_{i_{c(i)}} \in \overline{\varphi}_{i_{c(i)}}(w_i)$.

(2) $P_1, P_2, \ldots, P_\kappa$ are pairwise vertex-disjoint paths. In particular, for any $1 \le i \le \kappa$ and any $1 \le s < t \le c(i)$, we have $\gamma_{i_s} \ne \delta_{i_t}$.

To justify this, note that $S_p \cap S_q = \emptyset$ whenever p and q are contained in different L_i's by (4.2) and Theorem 3.10(iii). So $P_1, P_2, \ldots, P_\kappa$ are pairwise vertex-disjoint. It remains to prove that each P_i is a path.

Assume on the contrary that P_i contains a cycle. Then $\gamma_{i_s} = \delta_{i_t}$ for some subscripts s and t with $s < t$ by (1). Let $v \in V(T)$ be an arbitrary vertex with $v \prec w_i$. Since $\gamma_{i_s} \in \overline{\varphi}_{i_s-1}(w_i)$, we have $\gamma_{i_s} \notin \overline{\varphi}_{i_s-1}(v)$ by (4.1) and Theorem 3.10(i). Let f be the edge incident with v with $\varphi_{i_s-1}(f) = \gamma_{i_s}$. Since T_{i_s} is closed with respect to φ_{i_s-1}, edge f is contained $G[T_{i_s}]$. By Lemma 4.2, we have $\varphi_{i_t-1}(f) = \varphi_{i_s-1}(f) = \gamma_{i_s}$. From the definitions of π_{i_t-1} and π'_{i_t-1} in Step 4 of Algorithm 3.1, it follows that $v \notin I[\partial_{\varphi_{i_t-1}, \gamma_{i_s}}(T_{i_t})] = I[\partial_{\varphi_{i_t-1}, \delta_{i_t}}(T_{i_t})]$. Therefore w_i cannot be the supporting vertex of T_{i_t} with respect to φ_{i_t} and connecting color δ_{i_t} (see Algorithm 3.1); this contradiction proves (2).

(3) $v_n = w_1$.

Assume the contrary: $v_n \neq w_1$. Then $w_1 \prec v_n$. By (2), P_1 is a path and $\gamma_{1_1} \neq \delta_{1_{c(1)}}$. From Lemma 3.3(iii) (with w_i in place of u) we thus deduce that $\overline{\varphi}_{1_1-1}(w_1) = (\overline{\varphi}_{1_{c(1)}}(w_1) - \{\delta_{1_{c(1)}}\}) \cup \{\gamma_{1_1}\} \neq \overline{\varphi}_{1_{c(1)}}(w_1)$. Since L_1 consists of all subscripts t with $1 \leq t \leq n$, such that $v_t = w_1$ and $\Theta_t = PE$, by Algorithm 3.1 and the definition of stable colorings, $\overline{\varphi}_{j-1}(w_1) = \overline{\varphi}_{1_1-1}(w_1)$ and $\overline{\varphi}_n(w_1) = \overline{\varphi}_{1_{c(1)}}(w_1)$. Hence $\overline{\varphi}_{j-1}(w_1) \neq \overline{\varphi}_n(w_1)$. On the other hand, by (4.2) and Theorem 3.10(iii), φ_n is a $(T(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring, which implies $\overline{\varphi}_{j-1}(w_1) = \overline{\varphi}_n(w_1)$; this contradiction justifies (3).

For each t with $1 \leq t \leq n-1$ and $\Theta_t = PE$, let $\epsilon(t)$ be the smallest subscript $s > t$ such that $\Theta_s \neq RE$. This $\epsilon(t)$ is well defined and $\epsilon(t) \leq n$, as $\Theta_n = PE \neq RE$. Given a coloring φ and two colors α and β, recall that α and β are called T_t-interchangeable under φ if there is at most one (α, β)-path with respect to φ intersecting T_t; that is, all (α, β)-chains intersecting T_t are (α, β)-cycles except possibly one, which is an (α, β)-path. We say that α and β are T_t-*strongly interchangeable* (T_t-SI) under φ if for each vertex v in $T_t - v_t$, the chain $P_v(\alpha, \beta, \varphi)$ is an (α, β)-cycle which is fully contained in $G[T_{\epsilon(t)}]$ (equivalently, $V(P_v(\alpha, \beta, \varphi)) \subseteq V(T_{\epsilon(t)})$). Observe that if α and β are T_t-SI under φ, then they are T_t-interchangeable under φ. Furthermore, $P_{v_t}(\alpha, \beta, \varphi)$ contains only one vertex v_t from T_t, if it is a path.

Claim 4.1. *The coloring σ_n satisfies the following properties:*

(a1) σ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;

(a2) $\sigma_n(f) = \varphi_{j-1}(f)$ for all edges f in $G[T_j]$ with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T_j) \cup D_{j-1}$; in particular, this equality holds for all edges on T_j;

(a3) $\overline{\sigma}_n(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j) - \{w_1, w_2, \dots, w_\kappa\}$ and $\overline{\varphi}_{j-1}(w_i) = (\overline{\varphi}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\} = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for each $i = 1, 2, \dots, \kappa$; and

(a4) for any $t \in L - \{n\}$, the colors γ_t and δ_t are T_t-SI under σ_n

To justify this claim, note that $(a1)$ follows instantly from (4.2) and Theorem 3.10(iii).

$(a2)$ For each edge f in $G[T_j]$ with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T_j) \cup D_{j-1}$, by Lemma 4.2, we have $\varphi_n(f) = \varphi_{j-1}(f)$. Repeated application of Lemma 3.2(i) and (ii) implies that $\varphi_{n-1}(f) \in \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. By Lemma 3.2(ii), we further obtain $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, $\sigma_n(f) = \varphi_n(f)$, which implies $\sigma_n(f) = \varphi_{j-1}(f)$. By Lemma 3.2(iii), each edge f on T_j satisfies $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(T_j) \cup D_{j-1}$, so the equality $\sigma_n(f) = \varphi_{j-1}(f)$ holds for all edges f on T_j.

$(a3)$ Let v be a vertex in $V(T_j) - \{w_1, w_2, \dots, w_\kappa\}$. If v is contained in $T_j(v_n) - v_n$, then $\overline{\sigma}_n(v) = \overline{\varphi}_{j-1}(v)$ by $(a1)$. So we assume that v is outside $T_j(v_n) - v_n$. Note that v is not a supporting vertex during any iteration p with $j \leq p \leq n$ by the definition of L. Hence $\overline{\varphi}_n(v) = \overline{\varphi}_{j-1}(v)$ by Algorithm 3.1 and the definition of stable colorings. As σ_n is a (T_n, D_n, φ_n)-stable coloring, $\overline{\sigma}_n(v) = \overline{\varphi}_n(v) = \overline{\varphi}_{j-1}(v)$.

Since L_i consists of all subscripts t with $1 \leq t \leq n$, such that $v_t = w_i$ and $\Theta_t = PE$, we have $\overline{\varphi}_{j-1}(w_i) = \overline{\varphi}_{i_1-1}(w_i)$ and $\overline{\varphi}_n(w_i) = \overline{\varphi}_{i_{c(i)}}(w_i)$ by Algorithm 3.1 and the definition of stable colorings. Furthermore, $\overline{\varphi}_{i_1-1}(w_i) = (\overline{\varphi}_{i_{c(i)}}(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ by Lemma 3.3(iii) (with w_i in place of u). So $\overline{\varphi}_{j-1}(w_i) = (\overline{\varphi}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for $1 \leq i \leq \kappa$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, $\overline{\sigma}_n(w_i) = \overline{\varphi}_n(w_i)$. Hence $\overline{\varphi}_{j-1}(w_i) = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for $1 \leq i \leq \kappa$.

($a4$) By the induction hypothesis (4.1) on Theorem 3.10(ii), γ_t and δ_t are T_t-interchangeable under φ_t. Since $\Theta_t = PE$, $P_{v_t}(\gamma_t, \delta_t, \varphi_t)$ is a path containing only one vertex v_t from T_t by Algorithm 3.1. For each vertex v in $T_t - v_t$, observe that $P_v(\gamma_t, \delta_t, \varphi_t)$ is a (γ_t, δ_t)-cycle, for otherwise, $P_{v_t}(\gamma_t, \delta_t, \varphi_t)$ and $P_v(\gamma_t, \delta_t, \varphi_t)$ are two distinct (γ_t, δ_t)-paths intersecting T_t, a contradiction. Since RE has priority over PE and SE in Algorithm 3.1, the cycle $P_v(\gamma_t, \delta_t, \varphi_t)$ is fully contained in $G[T_{\epsilon(t)}]$, for otherwise, $\Theta_{\epsilon(t)} = RE$, contradicting the definition of $\epsilon(t)$. Hence γ_t and δ_t are T_t-SI under φ_t. By Lemma 4.2, we have $\varphi_t(f) = \varphi_n(f)$ for each edge f on $P_v(\gamma_t, \delta_t, \varphi_t)$, because $\gamma_t, \delta_t \in D_t$. It follows that γ_t and δ_t are T_t-SI under φ_n. Since σ_n is (T_n, D_n, φ_n)-stable, $\{\gamma_t, \delta_t\} = S_t \subseteq \overline{\varphi}_n(T_n) \cup D_n$, and $T_{\epsilon(t)} \subseteq T_n$, we deduce that γ_t and δ_t are also T_t-SI under σ_n. Hence Claim 4.1 is established.

Let us now apply the following algorithm to construct a new coloring from σ_n, which has the same missing color set as φ_{j-1} at each w_i with $i \geq 2$.

(A) Let $I = \emptyset$ and $\sigma = \sigma_n$. While $I \neq L - L_1$, do: let $i \geq 2$ be a subscript with $L_i - I \neq \emptyset$ and let t be the largest member of $L_i - I$. Set

$$\mathbf{A}(i,t): \qquad \sigma = \sigma / P_{w_i}(\gamma_t, \delta_t, \sigma) \quad \text{ and } \quad I = I \cup \{t\}.$$

Let us make some observations about this algorithm.

(4) Let I, i, t, σ be as specified in Algorithm (A) before performing the iteration $A(i,t)$. Then $P_{w_i}(\gamma_t, \delta_t, \sigma)$ is a path containing precisely one vertex w_i from T_t, with $\delta_t \in \overline{\sigma}(w_i)$. Furthermore, let $\sigma' = \sigma / P_{w_i}(\gamma_t, \delta_t, \sigma)$ and $I' = I \cup \{t\}$ denote the objects generated in the iteration $A(i,t)$. Then for any $s \in L - \{n\} - I'$, the colors γ_s and δ_s are T_s-SI under the coloring σ'.

To justify this, we apply induction on $|I|$. Note that $w_i = v_t$ for each iteration $A(i,t)$ by the definitions of L_i and w_i.

Let us first consider the case when $I = \emptyset$. Now t is the largest subscript $i_{c(i)}$ in L_i. By Algorithm 3.1 and the definition of stable colorings, we have $\delta_t \in \overline{\varphi}_t(w_i)$ and $\overline{\sigma}_n(w_i) = \overline{\varphi}_n(w_i) = \overline{\varphi}_t(w_i)$. So $\delta_t \in \overline{\sigma}_n(w_i)$. By ($a4$) in Claim 4.1,

(5) for any $s \in L - \{n\}$, the colors γ_s and δ_s are T_s-SI under σ_n.

In particular, (5) holds for $s = t$, so $P_{w_i}(\gamma_t, \delta_t, \sigma_n)$ is a path containing precisely one vertex w_i from T_t. For any $s \in L - \{n, t\}$, either $s \in L_h$ for some h with $h \neq i$ or $s \in L_i$ with $s < t$. In the former case, $S_s \cap S_t = \emptyset$ by (4.2) and Theorem 3.10(iii), so γ_s and δ_s are T_s-SI under σ' by (5). In the latter case, $v_s = v_t = w_i$ and $\epsilon(s) \leq t$. Furthermore, no color on any edge in $G[T_t]$ is changed under the Kempe change that transforms σ into σ'. So γ_s and δ_s are still T_s-SI under σ'.

So we proceed to the induction step. Let us show that (4) holds for a general I with $I \neq L - L_1$.

Let I, i, t, σ be as specified in Algorithm (A) before performing the iteration $A(i,t)$. By induction hypothesis,

(6) for any $s \in L - \{n\} - I$, the colors γ_s and δ_s are T_s-SI under the coloring σ.

It follows from (6) (with $s = t$) that γ_t and δ_t are T_t-SI under the coloring σ. So $P_{w_i}(\gamma_t, \delta_t, \sigma)$ contains precisely one vertex $v_t = w_i$ from T_t, if it is a path.

To prove that $P_{w_i}(\gamma_t, \delta_t, \sigma)$ is a path with $\delta_t \in \overline{\sigma}(w_i)$, we first assume that t is the largest subscript in L_i. By (4.2) and Theorem 3.10(iii), neither γ_t not δ_t has been used in any Kempe change before the iteration $A(i,t)$. By Algorithm 3.1, the definition of stable colorings, and the induction hypothesis, we have $\delta_t \in \overline{\varphi}_t(w_i)$ and $\overline{\sigma}(w_i) = \overline{\sigma}_n(w_i) = \overline{\varphi}_n(w_i) = \overline{\varphi}_t(w_i)$. So $\delta_t \in \overline{\sigma}(w_i)$. Next, we assume that t is not the largest subscript in L_i. Let $t = i_p$. Then i_{p+1} is the smallest element of L_i greater than i_p. So the last Kempe change involving w_i before iteration $A(i,t)$ was performed on a path of the form $P_{w_i}(\gamma_{i_{p+1}}, \delta_{i_{p+1}}, \cdot)$. By induction hypothesis, $\delta_{i_{p+1}}$ was a color missing at w_i before this Kempe change. Thus $\gamma_{i_{p+1}}$ becomes a missing color at w_i after this operation; it remains to be missing at w_i until the iteration $A(i,t)$ by (4.2) and Theorem 3.10(iii). Hence $\gamma_{i_{p+1}} \in \overline{\sigma}(w_i)$. By (1), we have $\delta_t = \delta_{i_p} = \gamma_{i_{p+1}}$. It follows that $\delta_t \in \overline{\sigma}(w_i)$. Therefore, $P_{w_i}(\gamma_t, \delta_t, \sigma)$ is a path containing precisely one vertex w_i from T_t, with $\delta_t \in \overline{\sigma}(w_i)$.

Let $I' = I \cup \{t\}$ and $\sigma' = \sigma / P_{w_i}(\gamma_t, \delta_t, \sigma)$. For each $s \in L - \{n\} - I'$, either $s \in L_h$ for some h with $h \neq i$ or $s \in L_i$ with $s < t$. In the former case, $S_s \cap S_t = \emptyset$ by (4.2) and Theorem 3.10(iii), so γ_s and δ_s are T_s-SI under σ' by (6). In the latter case, $v_s = v_t$ and $\epsilon(s) \le t$. Furthermore, no color on any edge in $G[T_t]$ is changed under the Kempe change that transforms σ into σ'. So γ_s and δ_s are still T_s-SI under σ' by (6). Hence (4) holds.

Claim 4.2. *Let ϱ_1 denote the coloring σ output by Algorithm (A). Then the following statements hold:*

- *(b1) ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;*
- *(b2) $\overline{\varrho}_1(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$, $\overline{\varrho}_1(v_n) = \overline{\varphi}_n(v_n)$, and $\varrho_1(f) = \sigma_n(f) = \varphi_{j-1}(f)$ for all edges f on T_j;*
- *(b3) for any edge $f \in E(G)$, if $\varrho_1(f) \neq \sigma_n(f)$, then f is not contained in $G[T_j]$ and $\sigma_n(f) \in \cup_{i \in L - L_1} S_i$; and*
- *(b4) for any $i \in L_1 - \{n\}$ (so $v_i = v_n$), the colors γ_i and δ_i are T_i-SI under ϱ_1.*

To justify this claim, recall from (4) that

(7) at each iteration $A(i,t)$ of Algorithm (A), the chain $P_{w_i}(\gamma_t, \delta_t, \sigma)$ is a path containing precisely one vertex $w_i = v_t$ from T_t, with $\delta_t \in \overline{\sigma}(w_i)$ and $i \ge 2$.

By (3) and the definitions of L and w_i's, we have

(8) $v_n = w_1 \prec w_i$ for all $i \ge 2$, and $T_j \subset T_t$ for each iteration $A(i,t)$ of Algorithm (A).

It follows from (7) and (8) that $\sigma(f) = \sigma_n(f)$ for all edges f incident to $T_j(v_n) - v_n$. So σ and hence ϱ_1 is a $(T_j(v_n) - v_n, D_{j-1}, \sigma_n)$-stable coloring. By (4.2) and Theorem 3.10(iii), σ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable. From Lemma 2.4 we deduce that ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable. So $(b1)$ holds.

By $(a3)$ in Claim 4.1, we have

(9) $\overline{\varphi}_{j-1}(w_i) = (\overline{\varphi}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\} = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$ for each vertex w_i with $i \ge 2$.

Recall that $S_p \cap S_q = \emptyset$ whenever p and q are contained in different L_i's by (4.2) and Theorem 3.10(iii), and that $P_1, P_2, \ldots, P_\kappa$ are pairwise vertex-disjoint paths by (2). After executing Algorithm (A), the direction of each P_i gets reversed (see (1)). Using Lemma 3.3(iii), we obtain $\overline{\varrho}_1(w_i) = (\overline{\sigma}_n(w_i) - \{\delta_{i_{c(i)}}\}) \cup \{\gamma_{i_1}\}$, so $\overline{\varrho}_1(w_i) = \overline{\varphi}_{j-1}(w_i)$ for $i \ge 2$ by (9). Combining this with $(a3)$ in Claim 4.1, we see that $\overline{\varrho}_1(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$. By (4), the path $P_{w_i}(\gamma_t, \delta_t, \sigma)$ involved in each iteration $A(i,t)$ of Algorithm (A) is disjoint from $v_n = w_1$. So $\overline{\varrho}_1(v_n) = \overline{\sigma}_n(v_n) = \overline{\varphi}_n(v_n)$. In view of (7) and (8), we get $\sigma(f) = \sigma_n(f)$ for all edges f on T_j at each iteration $A(i,t)$ of Algorithm (A). Hence $\varrho_1(f) = \sigma_n(f) = \varphi_{j-1}(f)$ for all edges f on T_j, where the second equality follows from $(a2)$ in Claim 4.1. Thus $(b2)$ is established.

Since the Kempe changes performed in Algorithm (A) only involve edges outside $G[T_j]$ and colors in $\cup_{h\in L-L_1} S_h$, we immediately get $(b3)$. Clearly, $(b4)$ follows from (4). This proves Claim 4.2.

Consider the coloring $\varrho_1 \in \mathcal{C}^k(G-e)$ described in Claim 4.2. Let T'_j be a closure of $T_j(v_n)$ under ϱ_1. By (4.1) and Theorem 3.10(i), $V(T_n)$ is elementary with respect to φ_{n-1}, so $|V(T_n)|$ is odd. From Step 4 in Algorithm 3.1, we see that $|\partial_{\pi'_{n-1},\delta_n}(T_n)| \ge 3$. Hence $|\partial_{\varphi_n,\delta_n}(T_n)| \ge 2$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, we have $|\partial_{\sigma_n,\delta_n}(T_n)| \ge 2$. By Lemma 3.2(iv), edges in $\partial_{\sigma_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. Furthermore, each color in $\overline{\sigma}_n(T_n) - \{\delta_n\}$ is closed in T_n under σ_n. It follows from $(b3)$ and TAA that $T'_j - T_n \ne \emptyset$ and at least one edge in $T'_j - T_n$ is colored by δ_n under ϱ_1. By $(b1)$, ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, so it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a φ_{j-1} mod T_{j-1} coloring by (4.1) and Theorem 3.10(vi). By $(b2)$, we have $\varrho_1(f) = \varphi_{j-1}(f)$ for any edge f on $T_j(v_n)$. Note that $T_j(v_n)$ under ϱ_1 is obtained from T_{j-1} by using the same connecting edge, connecting color, and extension type as T_j. By (4.1) and Theorem 3.10(vi), we obtain

(10) T'_j is an ETT corresponding to coloring ϱ_1 and satisfies MP under ϱ_1. So $V(T'_j)$ is elementary with respect to ϱ_1 by (4.1) and Theorem 3.10(i), because $j \le n$.

Depending on the intersection of $\overline{\varrho}_1(T'_j - v_n)$ and $\cup_{i\in L_1} S_i$, we consider two cases.

Case 1. $\overline{\varrho}_1(T'_j - v_n) \cap (\cup_{i\in L_1} S_i) \ne \emptyset$.

Let u be the minimum vertex (in the order $\prec$) in $T'_j - v_n$ such that $\overline{\varrho}_1(u) \cap (\cup_{i\in L_1} S_i) \ne \emptyset$. Clearly, $u \ne v_n$. By (10), $V(T'_j)$ is elementary with respect to ϱ_1. Since $\delta_n \in \overline{\varphi}_n(v_n) = \overline{\varrho}_1(v_n)$ by $(b2)$, we have $\delta_n \notin \overline{\varrho}_1(T'_j - v_n)$; in particular, $\delta_n \notin \overline{\varrho}_1(u)$. Recall that $L_1 = \{1_1, 1_2, \ldots, 1_{c(1)}\}$ and that $n = 1_{c(1)}$. Since $\delta_{1_{c(1)}} \notin \overline{\varrho}_1(u)$ and $\delta_{1_s} = \gamma_{1_{s+1}}$ for any $1_s \in L_1$ with $1_s < n$ (see (1)), the definition of u guarantees the existence of a minimum member r (as an integer) of L_1, such that $\gamma_r \in \overline{\varrho}_1(u)$. Note that $\gamma_r \in \cup_{i\in L_1} S_i$. Since $m(v_r) = j$, we have $r \ge j$. Let us show that

(11) $u \in V(T'_j) - V(T_r)$.

Otherwise, $u \in V(T_r)$. Since $\gamma_r \in \overline{\varrho}_1(u)$, we obtain $\gamma_r \in \overline{\sigma}_n(u)$, for otherwise, there exists an edge f incident with u such that $\varrho_1(f) \ne \sigma_n(f) = \gamma_r$. It follows from $(b3)$ that $\gamma_r \in \cup_{i\in L-L_1} S_i$, so $(\cup_{i\in L_1} S_i) \cap (\cup_{i\in L-L_1} S_i) \ne \emptyset$, contradicting Theorem 3.10(iii). Since σ_n is (T_n, D_n, φ_n)-stable, $\gamma_r \in \overline{\varphi}_n(u)$. On the other hand, by (4.1) and Theorem 3.10(i), $V(T_r)$ is elementary with respect to φ_{r-1}. From Step 4 in Algorithm 4.1, we see that $\gamma_r \in \overline{\varphi}_{r-1}(v_n)$ (as $v_r = v_n$), so $G[T_r]$ contains an edge f incident to u with $\varphi_{r-1}(f) = \gamma_r$. By Lemma 4.2, we obtain $\varphi_n(f) = \gamma_r$. Hence $\gamma_r \in \varphi_n(u)$; this contradiction justifies (11).

(12) $\overline{\varrho}_1(T'_j(u) - u) \cap (\cup_{i\in L_1} S_i - \{\delta_n\}) = \emptyset$.

By the minimality assumption on u, we have $\overline{\varrho}_1(T'_j(u) - \{v_n, u\}) \cap (\cup_{i\in L_1} S_i) = \emptyset$. Using Lemma 3.3(i), we obtain $\overline{\varphi}_n(v_n) \cap (\cup_{i\in L_1} S_i) = \{\delta_n\}$. It follows from $(b2)$ in Claim 4.2 that $\overline{\varrho}_1(v_n) \cap (\cup_{i\in L_1} S_i) = \{\delta_n\}$. Thus (12) holds.

Let r be the subscript as defined above (11). Then $r = 1_p$ for some $1 \le p \le c(1)$. By (1), we have $\gamma_r = \gamma_{1_p} = \delta_{1_{p-1}}$. Let $L_1^* = \{1_1, 1_2, \ldots, 1_{p-1}\}$. Since $1_{p-1} < 1_p = r \le n$, we have $n \notin L_1^*$. Observe that

(13) $\overline{\varrho}_1(v_n) \cap (\cup_{i\in L_1^*} S_i) = \emptyset$.

Indeed, by ($b2$) in Claim 4.2 and Lemma 3.3(i), we obtain $\overline{\varrho}_1(v_n) = \overline{\varphi}_n(v_n)$ and $\overline{\varphi}_n(v_n) \cap (\cup_{i\in L_1} S_i) = \{\delta_n\}$. As $n \notin L_1^*$, from (1) and (2) we see that $\delta_n \notin \cup_{i\in L_1^*} S_i$. So $\overline{\varphi}_n(v_n) \cap (\cup_{i\in L_1^*} S_i) = \emptyset$. Hence (13) follows.

We construct a new coloring from ϱ_1 by using the following algorithm.

(B) Let $I = \emptyset$ and $\varrho = \varrho_1$. While $I \neq L_1^*$, do: let t be the largest member of $L_1^* - I$ and set

$$\mathbf{B}(t): \qquad \varrho = \varrho / P_u(\gamma_t, \delta_t, \varrho) \quad \text{ and } \quad I = I \cup \{t\}.$$

Let us exhibit some properties satisfied by this algorithm.

(14) Let I, t, ϱ be as specified in Algorithm (B) before performing the iteration $B(t)$. Then $\delta_t \in \overline{\varrho}(u)$, and $P_u(\gamma_t, \delta_t, \varrho)$ is a path containing at most one vertex v_n from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho)$. Furthermore, let $\varrho' = \varrho/P_u(\gamma_t, \delta_t, \varrho)$ and $I' = I \cup \{t\}$ denote the objects generated in the iteration $B(t)$. Then for any $s \in L_1^* - I'$, the colors γ_s and δ_s are T_s-SI under the coloring ϱ'.

To justify this, we apply induction on $|I|$. Note that $v_n = v_t$ for each iteration $B(t)$ by the definition of L_1.

Let us first consider the case when $I = \emptyset$. Now t is the largest member of L_1^* (that is, $t = 1_{p-1}$). So $\delta_t = \delta_{1_{p-1}} = \gamma_{1_p} = \gamma_r \in \overline{\varrho}(u)$. By ($b4$) in Claim 4.2,

(15) for any $s \in L_1 - \{n\}$, the colors γ_s and δ_s are T_s-SI under ϱ_1.

In particular, (15) holds for $s = t$, so $P_u(\gamma_t, \delta_t, \varrho)$ is a path containing at most one vertex $v_n = v_t$ from T_t. From (13) we see that v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho)$. For any $s \in L_1^* - \{t\}$, we have $s < t$. So $\epsilon(s) \le t$. Since no color on any edge in $G[T_t]$ is changed under the Kempe change that transforms ϱ into ϱ', the colors γ_s and δ_s are still T_s-SI under ϱ' by (15).

So we proceed to the induction step. Let us show that (14) holds for a general I with $I \neq L_1^*$.

Let I, t, ϱ be as specified in Algorithm (B) before performing the iteration $B(t)$. Let $t = i_q$. Then i_{q+1} is the smallest element of L_1^* greater than i_q. So in the iteration $B(i_{q+1})$, the Kempe change was performed on a path of the form $P_u(\gamma_{i_{q+1}}, \delta_{i_{q+1}}, \cdot)$. By induction hypothesis, $\delta_{i_{q+1}}$ was a color missing at u before the iteration $B(i_{q+1})$. So $\gamma_{i_{q+1}}$ becomes a missing color at u after this operation. Hence $\gamma_{i_{q+1}} \in \overline{\varrho}(u)$. By (1), we have $\delta_t = \delta_{i_q} = \gamma_{i_{q+1}}$. Thus $\delta_t \in \overline{\varrho}(u)$. By induction hypothesis,

(16) for any $s \in L_1^* - I$, the colors γ_s and δ_s are T_s-SI under ϱ.

In particular, (16) holds for $s = t$, so $P_u(\gamma_t, \delta_t, \varrho)$ is a path containing at most one vertex $v_n = v_t$ from T_t. Since none of the path involved in previous Kempe changes terminates at v_n, by (13) we have $\overline{\varrho}(v_n) \cap (\cup_{i\in L_1^*} S_i) = \emptyset$. It follows that v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho)$.

Let $I' = I \cup \{t\}$ and $\varrho' = \varrho/P_u(\gamma_t, \delta_t, \varrho)$. For each $s \in L_1^* - I'$, we have $s < t$. So $\epsilon(s) \le t$. Since no color on any edge in $G[T_t]$ is changed under the Kempe change that transforms ϱ into ϱ', the colors γ_s and δ_s are still T_s-SI under σ'. Hence (14) holds.

Claim 4.3. *Let ϱ_2 denote the coloring ϱ output by Algorithm (B). Then the following statements hold:*

(c1) ϱ_2 *is* $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$*-stable;*

(c2) $\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ *for all* $v \in V(T_j \cup T_j'(u) - u)$ *and* $\varrho_2(f) = \varrho_1(f)$ *for all* $f \in E(T_j \cup T_j'(u))$*;*

(c3) $\gamma_{1_1} \in \overline{\varrho}_2(u)$.

To justify this claim, recall from (14) that

(17) at each iteration $B(t)$, the path $P_u(\gamma_t, \delta_t, \varrho)$ contains at most one vertex v_n from T_t, but v_n is not an end of $P_u(\gamma_t, \delta_t, \varrho)$.

Since $T_j \subseteq T_t$, we have $\varrho(f) = \varrho_1(f)$ (and hence $\varrho_2(f) = \varrho_1(f)$) for each edge f incident to $T_j(v_n) - v_n$ by (17). It follows that ϱ_2 is a $(T_j(v_n) - v_n, D_{j-1}, \varrho_1)$-stable coloring. By ($b1$) in Claim 4.2, ϱ_1 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. From Lemma 2.4 we see that ($c1$) holds.

Similarly, from (17) we deduce that $\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ for all $v \in V(T_j)$ and $\varrho_2(f) = \varrho_1(f)$ for all $f \in E(T_j)$. In view of (12), it is clear that $T'_j(u)$ does not contain the other end of $P_u(\gamma_t, \delta_t, \varrho)$ at each iteration $B(t)$. So $\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ for each $v \in V(T'_j(u) - u)$. By (1), (2) and (12), we also have $\overline{\varrho}_1(T'_j(u) - u) \cap (\cup_{i \in L_1^*} S_i) = \emptyset$. Since T'_j is a closure of $T_j(v_n)$ under ϱ_1, from TAA we deduce that $\varrho_1\langle T'_j(u) - T_j(v_n)\rangle \cap (\cup_{i \in L_1^*} S_i) = \emptyset$. It follows that $\varrho(f) = \varrho_1(f)$ for all edges f in $T'_j(u) - T_j(v_n)$ at each iteration $B(t)$. So $\varrho_2(f) = \varrho_1(f)$ for all edges f in $T'_j(u) - T_j(v_n)$ and hence ($c2$) holds.

By (14), we have $\delta_t \in \overline{\varrho}(u)$ before each iteration $B(t)$. So γ_t becomes a missing color at u after performing iteration $B(t)$. It follows that $\gamma_{1_1} \in \overline{\varrho}_2(u)$. Hence ($c3$) and therefore Claim 4.3 is established.

By ($c1$) in Claim 4.3, ϱ_2 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, so it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a φ_{j-1} mod T_{j-1} coloring by (4.1) and Theorem 3.10(vi). By ($b2$) and ($c2$), we have $\varrho_2(f) = \varphi_{j-1}(f)$ for each edge f on $T_j(v_n)$. Let T_j^1 be a closure of $T_j(v_n)$ under ϱ_2. Then T_j^1 is an ETT corresponding to the coloring ϱ_2 and satisfies the maximum property under ϱ_2 by (4.1) and Theorem 3.10(vi), because it is obtained from T_{j-1} by using the same connecting edge, connecting color, and extension type as T_j. In view of ($b2$) and ($c2$), we have

- $\overline{\varrho}_2(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$;
- $\varrho_2(f) = \varphi_{j-1}(f)$ for all $f \in E(T_j)$;
- $\overline{\varrho}_2(v) = \overline{\varrho}_1(v)$ for all $v \in V(T'_j(u) - u)$;
- $\varrho_2(f) = \varrho_1(f)$ for all $f \in E(T'_j(u))$; and
- $\overline{\varrho}_2(v_n) = \overline{\varphi}_n(v_n)$.

Using ($c3$) and Lemma 3.3(iii), we obtain $\gamma_{1_1} \in \overline{\varrho}_2(u)$ and $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) \subseteq \overline{\varphi}_{1_{c(1)}}(v_n) \cup \{\gamma_{1_1}\} = \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\} = \overline{\varrho}_2(v_n) \cup \{\gamma_{1_1}\}$. Therefore, $V(T_j \cup T'_j(u)) \subseteq V(T_j^1)$ by TAA, which contradicts the maximum property satisfied by T under φ_n, because $u \notin V(T_j)$.

Case 2. $\overline{\varrho}_1(T'_j - v_n) \cap (\cup_{i \in L_1} S_i) = \emptyset$.

Recall that $L_1 = \{1_1, 1_2, ..., 1_{c(1)}\}$. Set $S' = \cup_{i \in L_1} S_i$. Let us make some simple observations about T_j and T'_j.

(18) $\overline{\varrho}_1(T'_j) \cap S' = \overline{\varrho}_1(v_n) \cap S' = \{\delta_n\}$ and $\varrho_1\langle T'_j - T_j(v_n)\rangle \cap S' = \{\delta_n\}$.

To justify this, note that $V(T'_j)$ is elementary with respect to ϱ_1 by (10) and that $\overline{\varrho}_1(v_n) = \overline{\varphi}_n(v_n)$ by ($b2$). By Lemma 3.3(i), we have $\overline{\varphi}_n(v_n) \cap S' = \{\delta_n\}$. So $\overline{\varrho}_1(v_n) \cap S' = \{\delta_n\}$ and hence $\delta_n \notin \overline{\varrho}_1(T'_j - v_n)$. By the hypothesis of the present case, we obtain $\overline{\varrho}_1(T'_j) \cap S' = \overline{\varrho}_1(v_n) \cap S' = \{\delta_n\}$. Since T'_j is a closure of $T_j(v_n)$ under ϱ_1, from TAA and the paragraph above (10), we deduce that $\varrho_1\langle T'_j - T_j(v_n)\rangle \cap S' = \{\delta_n\}$. Hence (18) holds.

(19) $\delta_n \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$ and $\delta_n \notin \varrho_1\langle T_j - T_j(v_n)\rangle$.

Assume on the contrary that $\delta_n \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$. Then $\delta_n \in \overline{\varphi}_{j-1}(T_j - V(T_j(v_n)))$

by ($b2$) in Claim 4.2. Since $V(T_j)$ is elementary with respect to φ_{j-1} by (4.1) and Theorem 3.10(vi), we have $\delta_n \notin \overline{\varphi}_{j-1}(v_n)$. So $G[T_j]$ contains an edge f incident to v_n colored by δ_n under φ_{j-1}. By Lemma 4.2, $\varphi_n(f) = \varphi_{j-1}(f) = \delta_n$. Hence $\delta_n \in \varphi_n(v_n)$; this contradiction proves that $\delta_n \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$.

By (3) and Lemma 3.3(iii), we have $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) = (\overline{\varphi}_{1_{c(1)}}(v_n) - \{\delta_{1_{c(1)}}\}) \cup \{\gamma_{1_1}\} = (\overline{\varphi}_n(v_n) - \{\delta_n\}) \cup \{\gamma_{1_1}\}$. So $\delta_n \notin \overline{\varphi}_{j-1}(v_n)$ (see (2)). By (18), we obtain $\delta_n \notin \overline{\varrho}_1(T_j(v_n) - v_n)$, which together with ($b2$) implies $\delta_n \notin \overline{\varphi}_{j-1}(T_j(v_n) - v_n)$. Hence $\delta_n \notin \overline{\varphi}_{j-1}(T_j(v_n))$. As $\delta_n \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$, we further conclude that $\delta_n \notin \overline{\varphi}_{j-1}(T_j)$ by ($b2$). Hence no edge in $T_j - T_j(v_n)$ is colored by δ_n under φ_{j-1}, because T_j is a closure of $T_j(v_n)$ under φ_{j-1} by TAA. It follows from ($b2$) that $\delta_n \notin \varrho_1\langle T_j - T_j(v_n)\rangle$. So (19) is justified.

By Lemma 3.2(iv), $\partial_{\varphi_n,\gamma_n}(T_n) = \{f_n\}$, and edges in $\partial_{\varphi_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. These two properties remain valid if we replace φ_n by σ_n, because σ_n is (T_n, D_n, φ_n)-stable. Thus, by ($b3$) in Claim 4.2, they also hold true if we replace φ_n by ϱ_1. Since T'_j is a closure of $T_j(v_n)$ under ϱ_1 and $\delta_n \in \overline{\varphi}_n(v_n) = \overline{\varrho}_1(v_n)$ by ($b2$), from TAA we see that no boundary edge of $T_n \cup T'_j$ is colored by δ_n under ϱ_1. So $\partial_{\varrho_1,\gamma_n}(T_n) = \{f_n\}$ and $\partial_{\varrho_1,\delta_n}(T_n \cup T'_j) = \emptyset$.

At the beginning of our proof, we assume that $P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ contains at least two vertices from T_n. Let P denote $P_{v_n}(\gamma_n, \delta_n, \varrho_1)$. Then $P = P_{v_n}(\gamma_n, \delta_n, \sigma_n)$ by ($b3$) and hence $P \cap T_n \neq \{v_n\}$. Since $\partial_{\varrho_1,\gamma_n}(T_n) = \{f_n\}$ and $\partial_{\varrho_1,\delta_n}(T_n \cup T'_j) = \emptyset$, from the hypothesis of the present case, we deduce that the other end x of P is outside $T_n \cup T'_j$. Furthermore, P contains a subpath $P[w, x]$, which is a $T_n \cup T'_j$-exit path with respect to ϱ_1. Note that w is contained in $T'_j - V(T_n)$, because the edge incident with w on $P[w, x]$ is colored by γ_n and $\partial_{\varrho_1,\gamma_n}(T_n) = \{f_n\}$. Let $\beta \in \overline{\varrho}_1(w)$. By the hypothesis of the present case, we have

(20) $\beta \notin S'$.

Possibly $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$; in this situation, let z be the first vertex in $T_j - V(T_j(v_n))$ in the order $\prec$ such that $\beta \in \overline{\varrho}_1(z)$.

Claim 4.4. *There exists a coloring $\varrho_3 \in \mathcal{C}^k(G - e)$ with the following properties:*

(d1) *ϱ_3 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable;*

(d2) *if $\beta \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$, then $\overline{\varrho}_3(v) = \overline{\varrho}_1(v)$ for all $v \in V(T_j \cup T'_j(w) - w)$ and $\varrho_3(f) = \varrho_1(f)$ for all $f \in E(T_j \cup T'_j(w))$;*

(d3) *if $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$, then $\overline{\varrho}_3(v) = \overline{\varrho}_1(v)$ for all $v \in V(T_j(z) \cup T'_j(w)) - \{w, z\}$ and $\varrho_3(f) = \varrho_1(f)$ for all $f \in E(T_j(z) \cup T'_j(w))$. Furthermore, $\delta_n \in \overline{\varrho}_3(z)$; and*

(d4) *$\gamma_{1_1} \in \overline{\varrho}_3(w)$.*

(Assuming Claim 4.4) By ($d1$) in Claim 4.4, ϱ_3 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. So it is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a φ_{j-1} mod T_{j-1} coloring by (4.1) and Theorem 3.10(vi). By ($b2$), ($d2$) and ($d3$), we have $\varrho_3(f) = \varrho_1(f) = \varphi_{j-1}(f)$ for each edge f on $T_j(v_n)$. Let T_j^2 be a closure of $T'_j(w)$ under ϱ_3. Then T_j^2 is an ETT corresponding to the coloring ϱ_3 and satisfies MP under ϱ_3 by (4.1) and Theorem 3.10(vi), because it is obtained from T_{j-1} by using the same connecting edge, connecting color, and extension type as T_j. By (4.1) and Theorem 3.10(i), $V(T_j^2)$ is elementary with respect to ϱ_3. By ($d4$), we have $\gamma_{1_1} \in \overline{\varrho}_3(w)$. By Lemma 3.3(iii), we obtain $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) \subseteq \overline{\varphi}_{1_{c(1)}}(v_n) \cup \{\gamma_{1_1}\} = \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\}$. So $\overline{\varphi}_{j-1}(v_n) \subseteq \overline{\varrho}_3(v_n) \cup \{\gamma_{1_1}\}$ by ($b2$), ($d2$) and ($d3$).

When $\beta \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$, by ($b2$) and ($d2$) we have

- $\overline{\varrho}_3(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j - v_n)$;
- $\varrho_3(f) = \varphi_{j-1}(f)$ for all $f \in E(T_j)$;
- $\overline{\varrho}_3(v) = \overline{\varrho}_1(v)$ for all $v \in V(T'_j(w) - w)$; and
- $\varrho_3(f) = \varrho_1(f)$ for all $f \in E(T'_j(w))$.

From TAA we see that $V(T_j \cup T'_j(w)) \subseteq V(T^2_j)$, which contradicts the maximum property satisfied by T.

When $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$, by ($b2$) and ($d3$) we get

- $\overline{\varrho}_3(v) = \overline{\varphi}_{j-1}(v)$ for all $v \in V(T_j(z) - \{z, v_n\})$;
- $\varrho_3(f) = \varphi_{j-1}(f)$ for all $f \in E(T_j(z))$;
- $\overline{\varrho}_3(v) = \overline{\varrho}_1(v)$ for all $v \in V(T'_j(w) - w)$; and
- $\varrho_3(f) = \varrho_1(f)$ for all $f \in E(T'_j(w))$.

From TAA we conclude that $V(T_j(z) \cup T'_j(w)) \subseteq V(T^2_j)$. As $\delta_n \in \overline{\varrho}_3(z) \cap \overline{\varrho}_3(v_n)$, $V(T^2_j)$ is not elementary with respect to ϱ_3, a contradiction again.

To prove Claim 4.4, we consider the coloring $\varrho_0 = \varrho_1/(G - T'_j, \beta, \delta_n)$. Since T'_j is closed with respect to ϱ_1 and $\{v_n, w\} \subseteq V(T'_j)$, no boundary edge of T'_j is colored by β or δ_n under ϱ_1. So ϱ_0 is $(T'_j, D_{j-1}, \varrho_1)$-stable and hence is $(T_j(v_n) - v_n, D_{j-1}, \varrho_1)$-stable. Clearly, $P_w(\gamma_n, \beta, \varrho_0) = P_w(\gamma_n, \delta_n, \varrho_1)$. Define $\mu_0 = \varrho_0/P_w(\gamma_n, \beta, \varrho_0)$.

Claim 4.5. *The coloring μ_0 satisfies the following properties:*

(e1) μ_0 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring;

(e2) if $\beta \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$, then $\overline{\mu}_0(v) = \overline{\varrho}_1(v)$ for all $v \in V(T_j \cup T'_j(w) - w)$ and $\mu_0(f) = \varrho_1(f)$ for all $f \in E(T_j \cup T'_j(w))$;

(e3) if $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$, then $\overline{\mu}_0(v) = \overline{\varrho}_1(v)$ for all $v \in V(T_j(z) \cup T'_j(w)) - \{w, z\}$ and $\mu_0(f) = \varrho_1(f)$ for all $f \in E(T_j(z) \cup T'_j(w))$. Furthermore, $\delta_n \in \overline{\varrho}_3(z)$;

(e4) $\gamma_n = \delta_{1_{c(1)-1}} \in \overline{\mu}_0(w)$ and $\beta \notin \overline{\mu}_0(w)$;

(e5) for any $t \in L_1 - \{n\}$, the colors γ_t and δ_t are T_t-SI under μ_0; and

(e6) $\overline{\mu}_0(T'_j - w) \cap S' = \overline{\mu}_0(v_n) \cap S' = \{\delta_n\}$ and $\mu_0\langle T'_j - T_j(v_n)\rangle \cap S' = \{\delta_n\}$.

To justify this, recall that ϱ_1 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable by ($b1$). By the definitions of ϱ_0 and μ_0, the transformation from ϱ_1 to μ_0 only changes colors on some edges disjoint from $V(T_j(v_n))$. So ($e1$) holds. Statement ($e4$) follows instantly from the definition of μ_0. Note that $\delta_n, \beta \notin \cup_{t \in L_1 - \{n\}} S_t$ by (1), (2) and (20), and that $T_{\epsilon(t)} \subseteq T_n$ for each $t \in L_1 - \{n\}$. Furthermore, $P_w(\gamma_n, \beta, \varrho_0)$ is disjoint from $V(T_n)$. So ($e5$) can be deduced from ($b4$) immediately. Using (18) and the definitions of ϱ_0 and μ_0, we obtain ($e6$).

By (10), $V(T'_j)$ is elementary with respect to ϱ_1. Since $\beta \in \overline{\varrho}_1(w)$, we have $\beta \notin \overline{\varrho}_1(T'_j - w)$. By ($b2$), we obtain $\beta \notin \overline{\varphi}_{j-1}(T_j(v_n) - v_n)$ and $\beta \notin \overline{\varphi}_n(v_n)$. From Lemma 3.3(iii) we deduce that $\overline{\varphi}_{j-1}(v_n) = \overline{\varphi}_{1_1-1}(v_n) \subseteq \overline{\varphi}_{1_{c(1)}}(v_n) \cup \{\gamma_{1_1}\} = \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\}$. Since $\beta \neq \gamma_{1_1}$ by (20), we get $\beta \notin \overline{\varphi}_{j-1}(v_n)$. Hence

(21) $\beta \notin \overline{\varphi}_{j-1}(T_j(v_n))$.

Suppose $\beta \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$. Then $\beta \notin \overline{\varphi}_{j-1}(T_j)$ by ($b2$) and (21). Thus $\beta \notin \varphi_{j-1}\langle T_j - T_j(v_n)\rangle$, because T_j is obtained from $T_j(v_n)$ by TAA under φ_{j-1}. By ($b2$) and (19), we obtain

$\beta \notin \varrho_1\langle T_j - T_j(v_n)\rangle$, $\delta_n \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$, and $\delta_n \notin \varrho_1\langle T_j - T_j(v_n)\rangle$. From the definitions of ϱ_0 and μ_0, we see that $(e2)$ holds.

Suppose $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$. Recall that z is the first vertex in $T_j - V(T_j(v_n))$ in the order $\prec$ with $\beta \in \overline{\varrho}_1(z)$. By $(b2)$ and (21), we get $\beta \in \overline{\varphi}_{j-1}(z)$ and $\beta \notin \overline{\varphi}_{j-1}(T_j(z) - z)$. Since T_j is obtained from $T_j(v_n)$ by TAA under φ_{j-1}, we have $\beta \notin \varphi_{j-1}\langle T_j(z) - T_j(v_n)\rangle$. It follows from $(b2)$ that $\beta \notin \varrho_1\langle T_j(z) - T_j(v_n)\rangle$. By (19), we obtain $\delta_n \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$ and $\delta_n \notin \varrho_1\langle T_j - T_j(v_n)\rangle$. From the definition of ϱ_0 and μ_0, we see that $(e3)$ holds. So Claim 4.5 is established.

Let $L_1^* = L_1 - \{n\}$. We construct a new coloring from μ_0 by using the following algorithm.

(C) Let $I = \emptyset$ and $\mu = \mu_0$. While $I \neq L_1^*$, do: let t be the largest member in $L_1^* - I$ and set

$$\mathbf{C}(\text{t}): \qquad \mu = \mu/P_w(\gamma_t, \delta_t, \mu) \quad \text{ and } \quad I = I \cup \{t\}.$$

Let ϱ_3 denote the coloring μ output by Algorithm (C). We aim to show that ϱ_3 is as described in Claim 4.4; our proof is based on the following statement.

(22) Let I, t, μ be as specified in Algorithm (C) before performing the iteration $C(t)$. Then $\delta_t \in \overline{\mu}(w)$, and $P_w(\gamma_t, \delta_t, \mu)$ is a path containing at most one vertex v_n from T_t, but v_n is not an end of $P_w(\gamma_t, \delta_t, \mu)$. Furthermore, let $\mu' = \mu/P_w(\gamma_t, \delta_t, \varrho)$ and $I' = I \cup \{t\}$ denote the objects generated in the iteration $C(t)$. Then for any $s \in L_1^* - I'$, the colors γ_s and δ_s are T_s-SI under the coloring μ'.

To justify this, we apply induction on $|I|$. Let us first consider the case when $I = \emptyset$. Now t is the largest member of L_1^* (that is, $t = 1_{c(1)-1}$). By $(e4)$ in Claim 4.5, we have $\delta_t = \delta_{1_{c(1)-1}} = \gamma_n \in \overline{\mu}_0(w)$. By $(e5)$, we obtain

(23) for any $s \in L_1^*$, the colors γ_s and δ_s are T_s-SI under μ_0.

In particular, (23) holds for $s = t$, so $P_w(\gamma_t, \delta_t, \mu_0)$ is a path containing at most one vertex $v_n = v_t$ from T_t. By (1), (2) and $(e6)$, we obtain

(24) $\overline{\mu}_0(v_n) \cap (\cup_{i \in L_1^*} S_i) = \emptyset$.

From (24) we deduce that v_n is not an end of $P_w(\gamma_t, \delta_t, \mu_0)$. For any $s \in L_1^* - \{t\}$, we have $s < t$. So $\epsilon(s) \leq t$. Since no color on any edge in $G[T_t]$ is changed under the Kempe change that transforms $\mu = \mu_0$ into μ', the colors γ_s and δ_s are still T_s-SI under μ' by (23).

So we proceed to the induction step. Let us show that (22) holds for a general I with $I \neq L_1^*$.

Let I, t, μ be as specified in Algorithm (C) before performing the iteration $C(t)$. Let $t = i_q$. Then i_{q+1} is the smallest element of L_1^* greater than i_q. So in the iteration $C(i_{q+1})$, the Kempe change was performed on a path of the form $P_w(\gamma_{i_{q+1}}, \delta_{i_{q+1}}, \cdot)$. By induction hypothesis, $\delta_{i_{q+1}}$ was a color missing at w before the iteration $B(i_{q+1})$. So $\gamma_{i_{q+1}}$ becomes a missing color at w after this operation. Hence $\gamma_{i_{q+1}} \in \overline{\mu}(w)$. By (1), we have $\delta_t = \delta_{i_q} = \gamma_{i_{q+1}}$. Thus $\delta_t \in \overline{\mu}(w)$. By induction hypothesis,

(25) for any $s \in L_1^* - I$, the colors γ_s and δ_s are T_s-SI under μ.

In particular, (25) holds for $s = t$, so $P_w(\gamma_t, \delta_t, \mu)$ is a path containing at most one vertex $v_n = v_t$ from T_t. Since none of the path involved in previous Kempe changes terminates at v_n, by (24) we have $\overline{\mu}(v_n) \cap (\cup_{i \in L_1^*} S_i) = \emptyset$. It follows that v_n is not an end of $P_w(\gamma_t, \delta_t, \mu)$.

Let $I' = I \cup \{t\}$ and $\mu' = \mu/P_w(\gamma_t, \delta_t, \mu)$. For each $s \in L_1^* - I'$, we have $s < t$. So $\epsilon(s) \leq t$. Since no color on any edge in $G[T_t]$ is changed under the Kempe change that transforms μ into μ', the colors γ_s and δ_s are still T_s-SI under μ' by (25). Hence (22) holds.

To justify Claim 4.4, recall from (22) that

(26) at each iteration $C(t)$, the path $P_w(\gamma_t, \delta_t, \mu)$ contains at most one vertex $v_n = v_t$ from T_t, but v_n is not an end of $P_w(\gamma_t, \delta_t, \mu)$.

Since $T_j \subseteq T_t$, we have $\mu(f) = \mu_0(f)$ (and hence $\varrho_3(f) = \mu_0(f)$) for each edge f incident to $T_j(v_n) - v_n$ by (26). It follows that ϱ_3 is a $(T_j(v_n) - v_n, D_{j-1}, \mu_0)$-stable coloring. By $(e1)$ in Claim 4.5, μ_0 is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. From Lemma 2.4 we see that $(d1)$ holds.

Let us first assume that $\beta \notin \overline{\varrho}_1(T_j - V(T_j(v_n)))$. Again, since $T_j \subseteq T_t$, from (26) we deduce that $\overline{\varrho}_3(v) = \overline{\mu}_0(v)$ for all $v \in V(T_j)$ and $\varrho_3(f) = \mu_0(f)$ for all $f \in E(T_j)$. By $(e6)$, we have $\overline{\mu}_0(T'_j - w) \cap S' = \overline{\mu}_0(v_n) \cap S' = \{\delta_n\}$ and $\mu_0\langle T'_j - T_j(v_n)\rangle \cap S' = \{\delta_n\}$. By (1) and (2), we obtain $\delta_n \notin \cup_{i \in L_1^*} S_i$. So at each iteration $C(t)$ the path $P_w(\gamma_t, \delta_t, \mu)$ neither contains any edge from $T'_j(w)$ nor terminate at a vertex in $T'_j(w) - w$. It follows that $\overline{\varrho}_3(v) = \overline{\mu}_0(v)$ for all $v \in V(T'_j(w) - w)$ and $\varrho_3(f) = \mu_0(f)$ for all edges f in $T'_j(w) - T_j(v_n)$. Hence $\overline{\varrho}_3(v) = \overline{\mu}_0(v)$ for all $v \in V(T_j \cup T'_j(w) - w)$ and $\varrho_3(f) = \mu_0(f)$ for all $f \in E(T_j \cup T'_j(w))$. Combining this with $(e2)$, we see that $(d2)$ holds.

Similarly, we can prove that if $\beta \in \overline{\varrho}_1(T_j - V(T_j(v_n)))$, then $(d3)$ is true.

By (22), we have $\delta_t \in \overline{\mu}(w)$ before each iteration $C(t)$. So γ_t becomes a missing color at w after performing iteration $C(t)$. It follows that $\gamma_{1_1} \in \overline{\varrho}_3(w)$. Hence $(d4)$ is established. This completes the proof of Claim 4.4 and hence of Lemma 4.4. ∎

Lemma 4.5. *(Assuming (4.1) and (4.3)) Theorem 3.10(v) holds for all ETTs with n rungs and satisfying MP; that is, for any (T_n, D_n, φ_n)-stable coloring σ_n and any defective color δ of T_n with respect to σ_n, if v is a vertex but not the smallest one (in the order $\prec$) in $I[\partial_{\sigma_n,\delta}(T_n)]$, then $v \preceq v_i$ for any supporting or extension vertex v_i with $i \geq m(v)$.*

Proof. By hypothesis, T is an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, and T satisfies MP under φ_n. Depending on the extension type Θ_n, we consider two cases.

Case 1. $\Theta_n = PE$. In this case, according to Step 4 in Algorithm 3.1, π'_{n-1} is a $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring, v_n is a $(T_n, \pi'_{n-1}, \{\gamma_n, \delta_n\})$-exit and $\varphi_n = \pi'_{n-1}/P_{v_n}(\gamma_n, \delta_n, \pi'_{n-1})$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, from Lemma 3.2(iv) we deduce that $\partial_{\sigma_n,\gamma_n}(T_n) = \{f_n\}$. So $\delta \neq \gamma_n$.

By Theorem 3.10(iv), $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$. Define $\sigma_{n-1} = \sigma_n/P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. Then

(1) σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable by Lemma 3.5 and hence it is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. Furthermore, $\partial_{\sigma_n,\delta}(T_n) \subseteq \partial_{\sigma_{n-1},\delta}(T_n)$ (as $\delta \neq \gamma_n$).

If $i < n$, then $v \in T_{n-1}$ because $m(v) \leq i < n$. Since v is not the smallest vertex in $I[\partial_{\sigma_n,\delta}(T_n)]$, from (1) it can be seen that δ is a defective color of T_{n-1} with respect to σ_{n-1}. Applying (4.3) and Theorem 3.10(v) to T_{n-1} and σ_{n-1} (see (1)), we obtain $v \preceq v_i$. So we assume that $i = n$. Since v_n the maximum defective vertex with respect to $(T_n, D_{n-1}, \varphi_{n-1})$ (see the definition above (3.1)), by (1) we also have $v \preceq v_n$.

Case 2. $\Theta_n = RE$ or SE. In this case, φ_n is $(T_n, D_{n-1}, \varphi_{n-1})$-stable (see Algorithm 3.1). Since σ_n is (T_n, D_n, φ_n)-stable and $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i), we deduce that σ_n is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. If $i < n$, then $m(v) < n$. Since $v \in T_{n-1}$, δ is a defective color of T_{n-1} with respect to σ_n. Thus $v \preceq v_i$ by

(4.3) and Theorem 3.10(v). So we assume that $i = n$. Since v_n the maximum defective vertex with respect to $(T_n, D_{n-1}, \varphi_{n-1})$, we also have $v \preceq v_n$. ∎

The proof of Theorem 3.10(vi) is based on the following technical lemma.

Lemma 4.6. *(Assuming (4.1) and (4.4)) Let $\mathcal{T}_1 = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$ (resp. $\mathcal{T}_2 = \{(T_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n\}$) be a Tashkinov series constructed from a k-triple (G, e, φ_0) (resp. (G, e, σ_0)) by using Algorithm 3.1. Suppose T_{n+1} satisfies MP under φ_n, and σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G-e)$ for $1 \le i \le n-1$. Furthermore, σ_{n-1} is a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring. If $\Theta_n = RE$, then there exists a Tashkinov series $\mathcal{T}_3 = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, such that $\sigma_n = \sigma_{n-1}$ and $T_i^* = T_i$ for $1 \le i \le n$.*

Proof. Since $\Theta_n = RE$, according to Step 1 in Algorithm 3.1, there exists a subscript $h \le n-1$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$, such that for all i with $h+1 \le i \le n-1$, if any, we have $\Theta_i = RE$ and $S_i = \{\delta_i, \gamma_i\} = S_h$, where $\delta_i = \delta_h$ and $\gamma_i = \gamma_h$, and such that some (γ_h, δ_h)-cycle C with respect to φ_{n-1} contains both an edge $f_n \in \partial_{\varphi_{n-1},\gamma_h}(T_n)$ and a segment L connecting $V(T_h)$ and v_n with $V(L) \subseteq V(T_n)$, where v_n is the end of f_n in T_n. According to Step 2 in this algorithm, $\varphi_n = \varphi_{n-1}$, T_{n+1} is a closure of $T_n + f_n$ under φ_n, $\delta_n = \delta_h$, $\gamma_n = \gamma_h$, $S_n = \{\delta_n, \gamma_n\}$, and $F_n = \{f_n\}$. Since $\Theta_i = RE$ for $h+1 \le i \le n-1$, from Algorithm 3.1 we further deduce that

(1) $\varphi_h = \varphi_{h+1} = \ldots = \varphi_n$ and $S_h = S_{h+1} = \ldots = S_n$.

Moreover,

(2) $\sigma_h = \sigma_{h+1} = \ldots = \sigma_{n-1}$.

Set $\sigma_n = \sigma_{n-1}$. As σ_i is a (T_i, D_i, φ_i)-stable coloring for $h \le i \le n-1$, by (2) we get

(3) σ_n is (T_h, D_h, φ_h)-stable.

Let f_n, L and C be as specified in the first paragraph of our proof. By the definition of D_{n-1}, we have $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} = \overline{\varphi}_{n-1}(T_n) \cup (\cup_{i \le n-1} S_i)$ (see (1) in the proof of Lemma 3.2). So $\{\delta_h, \gamma_h\} \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. Since σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable, $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for all $f \in E(L) \cup \{f_n\}$. Thus edges on L are also colored alternately by δ_h and γ_h in σ_n. Let C^* be the (δ_h, γ_h)-chain with respect to σ_n containing L. Then $f_n \in C^*$.

We claim that C^* is a cycle. Assume the contrary: C^* is a (δ_h, γ_h)-path with respect to σ_n. By Step 4 in Algorithm 3.1, we have $\delta_h \in \overline{\varphi}_h(v_h)$. Using (1), we obtain $\delta_h \in \overline{\varphi}_{n-1}(v_h)$, so v_h is outside C. It follows that L and hence C^* contains a vertex different from v_h in T_h. By (3) and Theorem 3.10(iv), $P_{v_h}(\delta_h, \gamma_h, \sigma_n)$ contains only vertex v_h from T_h. Thus C^* and $P_{v_h}(\delta_h, \gamma_h, \sigma_n)$ are two disjoint (δ_h, γ_h)-paths with respect to σ_n. Since $\sigma_h = \sigma_{h+1} = \ldots = \sigma_n$, we see that C^* and $P_{v_h}(\delta_h, \gamma_h, \sigma_n)$ are two disjoint (δ_h, γ_h)-paths with respect to σ_h intersecting T_{h+1}; this contradiction to Theorem 3.10(ii) justifies the claim.

The above claim and Algorithm 3.1 guarantee the existence of a Tashkinov series $\mathcal{T}_3 = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, such that $\sigma_n = \sigma_{n-1}$ and $T_i^* = T_i$ for $1 \le i \le n$. ∎

The lemma below actually states that ETTs along with the maximum property are also maintained under taking some stable colorings.

Lemma 4.7. *(Assuming (4.1) and (4.4)) Theorem 3.10(vi) holds for all ETTs with n rungs and satisfying MP; that is, every (T_n, D_n, φ_n)-stable coloring σ_n is a $\varphi_n \bmod T_n$ coloring. (So every ETT corresponding to (σ_n, T_n) satisfies MP under σ_n by Lemma 3.9.)*

Proof. By hypothesis, T is an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, and T satisfies MP under φ_n. We aim to show (recall Definition 3.7), by induction on $r(T)$, the existence of an extended Tashkinov tree T^* with corresponding Tashkinov series $\mathcal{T}^* = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$, satisfying $\sigma_0 \in \mathcal{C}^k(G-e)$ and the following conditions for all i with $1 \le i \le n$:

(1) $T_i^* = T_i$ and

(2) σ_i is a (T_i, D_i, φ_i)-stable coloring in $\mathcal{C}^k(G-e)$, where $D_i = \cup_{h \le i} S_h - \overline{\varphi}_i(T_i)$.

For this purpose, we shall define a $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable coloring σ_{n-1} based on σ_n, and apply induction hypothesis to σ_{n-1}.

Since T_n is an ETT constructed from the k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T}_n = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n\}$, with $r(T_n) = n-1$, and since T_n satisfies MP under φ_{n-1}, by (4.4) and Theorem 3.10(vi), σ_{n-1} is a $\varphi_{n-1} \bmod T_{n-1}$ coloring. So

(3) there exists a Tashkinov series $\mathcal{T}_n^* = \{(T_i^*, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n\}$, satisfying $\sigma_0 \in \mathcal{C}^k(G-e)$ and (1) and (2) for all i with $1 \le i \le n-1$.

Our objective is to find σ_{n-1}, such that

(4) T_n^* can be set to T_n, and an ETT T_{n+1}^* with respect to e and σ_n can be obtained from $T_n^* = T_n$ by using the same connecting edge, connecting color, and extension type Θ_n as T_{n+1} in $\mathcal{T}$.

Combining (3) and (4), we see that σ_n is indeed a $\varphi_n \bmod T_n$ coloring. To establish (4), we consider three cases, according to the extension type Θ_n.

Case 1. $\Theta_n = RE$. In this case, define $\sigma_{n-1} = \sigma_n$. By hypothesis, σ_n is a (T_n, D_n, φ_n)-stable coloring. So σ_{n-1} is also (T_n, D_n, φ_n)-stable. Since $\varphi_n = \varphi_{n-1}$ by Algorithm 3.1 and $\overline{\varphi}_{n-1}(T_n) \cup D_{n-1} \subseteq \overline{\varphi}_n(T_n) \cup D_n$ by Lemma 3.2(i), we deduce that σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. By Lemma 3.2(iii), we have $\sigma_n(f) = \varphi_n(f)$ for any edge f on T_n. It follows that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for any edge f on T_n. Thus we can set $T_n^* = T_n$. Therefore, by Lemma 4.6, an ETT T_{n+1}^* with respect to e and σ_n can be obtained from T_n by using the same connecting edge, connecting color, and extension type RE as T_{n+1} in $\mathcal{T}$.

Case 2. $\Theta_n = SE$. In this case, according to Step 3 of Algorithm 3.1, $\varphi_n = \pi_{n-1}$, T_{n+1} is a closure of $T_n + f_n$ under φ_n, $S_n = \{\delta_n\}$, and $F_n = \{f_n\}$, where π_{n-1} is a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring so that $v_{\pi_{n-1}} = v_n$, which is the maximum defective vertex with respect to $(T_n, D_{n-1}, \varphi_{n-1})$ (see the paragraph above (3.1)). By the definition of φ_n, we have

(5) φ_n is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. Moreover, $\partial_{\varphi_n, \delta_n}(T_n) = \partial_{\pi_{n-1}, \delta_n}(T_n)$.

Define $\sigma_{n-1} = \sigma_n$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, so is σ_{n-1}. In view of (5) and Lemma 3.2(i), we obtain

(6) σ_{n-1} is $(T_n, D_{n-1}, \varphi_{n-1})$-stable and hence is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$-stable. Moreover, $\partial_{\sigma_{n-1}, \delta_n}(T_n) = \partial_{\sigma_n, \delta_n}(T_n) = \partial_{\varphi_n, \delta_n}(T_n)$.

By Lemma 3.2(iii), we have $\varphi_{n-1}\langle T_n \rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. It follows from (6) that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for any edge f on T_n. Thus we can set $T_n^* = T_n$. Moreover, by (5), (6) and Lemma

2.4, v_n is also the maximum defective vertex with respect to $(T_n, D_{n-1}, \sigma_{n-1})$ (see the definition above (3.1)). We claim that

(7) for any $(T_n, D_{n-1} \cup \{\delta_n\}, \sigma_{n-1})$-stable coloring μ_{n-1}, there holds $\overline{\mu}_{n-1}(u_n) \cap \overline{\mu}_{n-1}(T_n) = \emptyset$, where u_n is the vertex of f_n outside T_n.

To justify this, note that $\sigma_{n-1} = \sigma_n$ is (T_n, D_n, φ_n)-stable. Since $\overline{\varphi}_n(T_n) \cup D_n = \overline{\varphi}_n(T_n) \cup D_{n-1} \cup \{\delta_n\}$, by the definition of stable colorings, σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \varphi_n)$-stable and hence $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable. Therefore μ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable by Lemma 2.4. From Step 1 in Algorithm 3.1 we see that $\overline{\mu}_{n-1}(u_n) \cap \overline{\mu}_{n-1}(T_n) = \emptyset$.

By (7), a tree sequence T^*_{n+1} with respect to e and σ_n can thus be obtained from T_n by using Step 3 in Algorithm 3.1 (with σ_{n-1} in place of both φ_{n-1} and π_{n-1}) and using the same connecting edge, connecting color, and extension type SE as T_{n+1} in $\mathcal{T}$.

Recall that RE has priority over both SE and PE in the construction of a Tashkinov series using Algorithm 3.1. To prove that T^*_{n+1} constructed above is an ETT, we still need to check that no ETT with respect to e and σ_n can be obtained from T_n by using RE. Assume the contrary: T^*_{n+1} (with a slight abuse of notation) is such an ETT. Since T satisfies MP, so does the ETT T^*_{n+1}. Let $\mathcal{T}_1$ be the Tashkinov series obtained from $\{(T_i, \sigma_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}$ by adding a tuple $(T^*_{n+1}, \sigma_n, S^*_n, F^*_n, \Theta^*_n)$ corresponding to T^*_{n+1}, where $\Theta^*_n = RE$, and let $\mathcal{T}_2 = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n\}$. Since σ_{n-1} is a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring by (6), it follows from Lemma 2.4 that φ_{n-1} is a $(T_n, D_{n-1}, \sigma_{n-1})$-stable coloring. Similarly, φ_i is a (T_i, D_i, σ_i)-stable coloring for $1 \leq i \leq n-1$, because σ_i is a (T_i, D_i, φ_i)-stable coloring by (2) and (3). Applying Lemma 4.6 to $\mathcal{T}_1$ and $\mathcal{T}_2$, we see that an ETT with respect to e and the coloring φ_{n-1} in $\mathcal{C}^k(G-e)$ can be obtained from T_n by using RE, contradicting the hypothesis of the present case.

Case 3. $\Theta_n = PE$. In this case, define $\sigma_{n-1} = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. Since σ_n is a (T_n, D_n, φ_n)-stable coloring, by (4.4) and Theorem 3.10(iv), we obtain $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$. Using Lemma 3.5, we have

(8) σ_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable and hence is $(T_n, D_{n-1}, \varphi_{n-1})$-stable.

By Lemma 3.2(iii), we have $\varphi_{n-1}\langle T_n \rangle \subseteq \overline{\varphi}_{n-1}(T_n) \cup D_{n-1}$. It follows from (8) that $\sigma_{n-1}(f) = \varphi_{n-1}(f)$ for any edge f on T_n. Thus we can set $T^*_n = T_n$. Moreover, by (8) and Lemma 2.4, v_n is also the maximum defective vertex with respect to $(T_n, D_{n-1}, \sigma_{n-1})$. We claim that

(9) for some $(T_n, D_{n-1} \cup \{\delta_n\}, \sigma_{n-1})$-stable coloring μ_{n-1}, there holds $\overline{\mu}_{n-1}(u_n) \cap \overline{\mu}_{n-1}(T_n) \neq \emptyset$, where u_n is the vertex of f_n outside T_n.

To justify this, let μ_{n-1} be a $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring for which $\overline{\mu}_{n-1}(u_n) \cap \overline{\mu}_{n-1}(T_n) \neq \emptyset$; such a coloring exists by Steps 1 and 4 in Algorithm 3.1. From (8) and Lemma 2.4 we deduce that μ_{n-1} is a $(T_n, D_{n-1} \cup \{\delta_n\}, \sigma_{n-1})$-stable coloring.

By (9), a tree sequence T^*_{n+1} with respect to e and σ_n can thus be obtained from T_n by using Step 4 in Algorithm 3.1 (with σ_{n-1} in place of both φ_{n-1} and π_{n-1}) and using the same connecting edge, connecting color, and extension type PE as T_{n+1} in $\mathcal{T}$.

Using the same argument as in Case 2, we conclude that no ETT with respect to e and σ_n can be obtained from T_n by using RE. So T^*_{n+1} constructed above is an ETT with respect to e and σ_n. ∎

5 Good Hierarchies

As is well known, Kempe changes play a fundamental role in edge-coloring theory. To ensure that ETTs are preserved under such operations, in this section we develop an effective control mechanism, the so-called good hierarchy of an ETT, which will serve as a powerful tool in the proof of Theorem 3.10(i). Throughout this section, we assume that

(5.1) Theorem 3.10(i) and (ii) holds for all ETTs with at most $n-1$ rungs and satisfying MP, and Theorem 3.10(iii)-(iv) hold for all ETTs with at most n rungs and satisfying MP.

Let J_n be a closure of $T_n(v_n)$ under a (T_n, D_n, φ_n)-stable coloring σ_n. We use $T_n \vee J_n$ to denote the tree sequence obtained from T_n by adding all vertices in $V(J_n) - V(T_n)$ to T_n one by one, following the linear order $\prec$ in J_n, and using edges in J_n.

Lemma 5.1. *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose $\Theta_n = PE$ and T enjoys MP under φ_n. If J_n is a closure of $T_n(v_n)$ under a (T_n, D_n, φ_n)-stable coloring σ_n, then $V(T_n \vee J_n)$ is elementary with respect to σ_n.*

Proof. Clearly, T_n is an ETT with corresponding Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n\}$, and satisfies MP under φ_{n-1}. Since $r(T_n) = n-1$, by (5.1) and Theorem 3.10(i), $V(T_n)$ is elementary with respect to φ_{n-1}. Let π_{n-1} and π'_{n-1} be as specified in Step 4 of Algorithm 3.1. Since π_{n-1} is a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring and π'_{n-1} is $(T_n, D_{n-1} \cup \{\delta_n\}, \pi_{n-1})$-stable coloring, by definition $V(T_n)$ is also elementary with respect to π'_{n-1}. As $\varphi_n = \pi'_{n-1}/P_{v_n}(\delta_n, \gamma_n, \pi'_{n-1})$ and $\delta_n \notin \pi'_{n-1}(T_n)$, we further obtain

(1) $V(T_n)$ is elementary with respect to φ_n and hence elementary with respect to σ_n.

As σ_n is a (T_n, D_n, φ_n)-stable coloring, it follows from (5.1) and Theorem 3.10(iii) that σ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable, where $j = m(v_n)$. So σ_n is a $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable coloring and hence is a φ_{j-1} mod T_{j-1} coloring by Theorem 3.10(vi). By Lemma 3.2(iii) and Lemma 4.2, we obtain $\sigma_n(f) = \varphi_n(f) = \varphi_{j-1}(f)$ for each edge f on T_j. By (5.1) and Theorem 3.10(vi), J_n is an ETT corresponding to σ_n, because it is obtained from T_{j-1} by using the same connecting edge, connecting color, and extension type as T_j. Clearly, J_n also satisfies the maximum property under σ_n. Since J_n has $j-1$ rungs, using (5.1), we obtain

(2) $V(J_n)$ is elementary with respect to σ_n.

Suppose on the contrary that $V(T_n \vee J_n)$ is not elementary with respect to σ_n. Then $T_n \vee J_n$ contains two distinct vertices u and v such that $\overline{\sigma}_n(u) \cap \overline{\sigma}_n(v) \neq \emptyset$. By (1) and (2), we may assume that $u \in V(T_n) - V(J_n)$ and $v \in V(J_n) - V(T_n)$. Let $\alpha \in \overline{\sigma}_n(u) \cap \overline{\sigma}_n(v)$. Then $\alpha \neq \delta_n$ by (2), because $\delta_n \in \overline{\varphi}_n(v_n) = \overline{\sigma}_n(v_n)$. Moreover, since $\gamma_n \in \overline{\varphi}_{n-1}(v_n)$ and $V(T_n)$ is elementary with respect to φ_{n-1}, from Step 4 of Algorithm 3.1 and the definition of stable colorings, we deduce that $\gamma_n \notin \overline{\varphi}_n(T_n)$ and hence $\gamma_n \notin \overline{\sigma}_n(T_n)$. So $\alpha \neq \gamma_n$. Consequently,

(3) $\alpha \notin S_n$.

Since $T_n(v_n)$ contains the uncolored edge e, it contains a vertex $w \neq v_n$. Note that w is contained in both T_n and J_n. Let $\beta \in \overline{\sigma}_n(w)$. Since $\delta_n \in \overline{\sigma}_n(v_n)$ and $\gamma_n \notin \overline{\sigma}_n(T_n)$, we obtain

(4) $\beta \notin S_n$ (see (2)).

As $V(J_n)$ is closed and elementary with respect to σ_n (see (2)), the other end of $P_v(\alpha, \beta, \sigma_n)$ is w. From (3), (4), and Step 4 of Algorithm 3.1, we see that $\partial(T_n)$ contains no edge colored by α or β under φ_n and hence under σ_n, because σ_n is (T_n, D_n, φ_n)-stable. Combining this with

(1), we conclude that the other end of $P_u(\alpha,\beta,\sigma_n)$ is also w. Thus $P_w(\alpha,\beta,\sigma_n)$ terminates at both u and v, a contradiction. ∎

Let T be an ETT as specified in Theorem 3.10; that is, T is constructed from a k-triple (G,e,φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. To prove that $V(T)$ is elementary with respect to φ_n, we shall turn to considering a restricted ETT T' with ladder $T_1 \subset T_2 \subset \ldots \subset T_n \subset T'$ and $V(T') = V(T_{n+1})$, and then show that $V(T')$ is elementary with respect to φ_n. For convenience, we may simply view T' as T.

In the remainder of this paper, we reserve the symbol R_n for a fixed closure of $T_n(v_n)$ under φ_n, if $\Theta_n = PE$. Let $T_n \vee R_n$ be the tree sequence as defined above Lemma 5.1. We assume hereafter that

(5.2) T_{n+1} is a closure of $T_n \vee R_n$ under φ_n, which is a special closure of T_n under φ_n (see Step 4 in Algorithm 3.1), when $\Theta_n = PE$.

By Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to φ_n, so we may further assume that

(5.3) $T \neq T_n \vee R_n$ if $\Theta_n = PE$, which together with (5.2) implies that $T_n \vee R_n$ is not closed with respect to φ_n.

(5.4) If $\Theta_n = PE$, then each color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$ is closed in $T_n \vee R_n$ with respect to φ_n.

To justify this, note that each color in $\overline{\varphi}_n(R_n)$ is closed in R_n under φ_n because R_n is a closure. By Lemma 3.2(iv), each color in $\overline{\varphi}_n(T_n) - \{\delta_n\}$ is closed in T_n under φ_n. Hence each color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \{\delta_n\}$ is closed in $T_n \vee R_n$ with respect to φ_n. Lemma 3.2(iv) also asserts that edges in $\partial_{\varphi_n,\delta_n}(T_n)$ are all incident to $V(T_n(v_n) - v_n)$. So δ_n is closed in $T_n \vee R_n$ as well, because it is closed in R_n. Hence (5.4) follows.

To prove Theorem 3.10(i), we shall appeal to a *hierarchy* of T of the form

(5.5) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$, such that $T_n \vee R_n \subset T_{n,1}$ if $\Theta_n = PE$ and $T_{n,i} = T(u_i)$ for $1 \le i \le q$, where $u_1 \prec u_2 \prec \ldots \prec u_q$ are some vertices in $T - V(T_n)$, called *dividers* of T. (So T has q dividers in total.)

As introduced in Algorithm 3.1, $D_n = \cup_{h \le n} S_h - \overline{\varphi}_n(T_n)$, where $S_h = \{\delta_h\}$ if $\Theta_h = SE$ and $S_h = \{\delta_h, \gamma_h\}$ otherwise. By Lemma 3.4, we have

(5.6) $|D_n| \le n$.

Write $D_n = \{\eta_1, \eta_2, \ldots, \eta_{n'}\}$. In Definition 5.2 given below and the remainder of this paper,

- $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise, and $T^*_{n,j} = T_{n,j}$ if $j \ge 1$;
- $D_{n,j} = \cup_{h \le n} S_h - \overline{\varphi}_n(T^*_{n,j})$ for $0 \le j \le q$;
- v_{η_h}, for $\eta_h \in D_n$, is defined to be the first vertex u of T in the order $\prec$ with $\eta_h \in \overline{\varphi}_n(u)$, if any, and defined to be the last vertex of T in the order $\prec$ otherwise;
- $\Lambda^0_h = \overline{\varphi}_n(T_n) - \varphi_n\langle T_{n,1}(v_{\eta_h}) - T^*_{n,0}\rangle$ for $\eta_h \in D_{n,0}$, where $T_{n,1}(v_{\eta_h}) = T_{n,1}$ if v_{η_h} is outside $T_{n,1}$;
- $\Lambda^j_h = \overline{\varphi}_n(T_{n,j}) - \varphi_n\langle T_{n,j+1}(v_{\eta_h}) - T_{n,j}\rangle$ for $1 \le j \le q$ and $\eta_h \in D_{n,j}$, where $T_{n,j+1}(v_{\eta_h}) = T_{n,j+1}$ if v_{η_h} is outside $T_{n,j+1}$; and
- $\Gamma^j = \cup_{\eta_h \in D_{n,j}} \Gamma^j_h$ for $0 \le j \le q$.

Let H be a subgraph of G and let C be a subset of $[k]$. We say that H is C-*closed* with respect to φ_n if $\partial_{\varphi_n,\alpha}(H) = \emptyset$ for any $\alpha \in C$, and say that H is C^--*closed* with respect to φ_n if it is $(\overline{\varphi}_n(H) - C)$-closed with respect to φ_n.

Definition 5.2. Hierarchy (5.5) of T is called *good* with respect to φ_n if for any j with $0 \le j \le q$ and any $\eta_h \in D_{n,j}$, there exists a 2-color subset $\Gamma^j_h = \{\gamma^j_{h_1}, \gamma^j_{h_2}\} \subseteq [k]$, such that

(i) $\Gamma^j_h \subseteq \Lambda^j_h$ (so $\Gamma^j_h \subseteq \overline{\varphi}_n(T_n)$ if $j = 0$ and $\Gamma^j_h \subseteq \overline{\varphi}_n(T_{n,j})$ if $j \ge 1$);

(ii) $\Gamma^j_g \cap \Gamma^j_h = \emptyset$ whenever η_g and η_h are two distinct colors in $D_{n,j}$;

(iii) for any j with $1 \le j \le q$, there exists precisely one color $\eta_g \in D_{n,j}$, such that $\Gamma^j_g \subseteq \overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$ (so $\Gamma^j_g \cap \Gamma^{j-1}_g = \emptyset$) and $\Gamma^j_h = \Gamma^{j-1}_h$ for all $\eta_h \in D_{n,j} - \{\eta_g\}$;

(iv) if $\Theta_n = PE$, then $T_n \vee R_n$ is not $(\Gamma^0)^-$-closed with respect to φ_n and, subject to this, $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0|$ is maximized (this maximum value is at least 4, as we shall see); and

(v) $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h)^-$-closed with respect to φ_n for all j with $1 \le j \le q$.

The sets Γ^j_h are referred to as Γ-*sets* of the hierarchy (or of T) under φ_n.

Some remarks may help to understand the concept of good hierarchies.

(5.7) From Condition (i) we see that neither the color $\gamma^j_{h_1}$ nor $\gamma^j_{h_2}$ can be used by edges on $T_{n,j+1}$ until after η_h becomes missing at the vertex v_{η_h} in $T_{n,j+1}$.

(5.8) Condition (iv) implies that $T_{n,1} \ne T_n \vee R_n$ if $\Theta_n = PE$.

(5.9) For $1 \le j \le q$, by definitions, $D_{n,j} \subseteq D_{n,j-1}$, so Γ^{j-1}_h is well defined for any $\eta_h \in D_{n,j}$ and $\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h \subseteq \Gamma^{j-1}$. In view of Condition (v), the first edge added to $T_{n,j+1} - T_{n,j}$ is colored by a color α in Γ^{j-1}_g for some g with $\eta_g \in D_{n,j}$. From Condition (i) and (5.7) we see that $\alpha \notin \Gamma^j_g$. So $\Gamma^j_g \ne \Gamma^{j-1}_g$. According to Condition (iii), now Γ^j_g consists of two colors in $\overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$. Thus $\Gamma^{j-1}_g \cap \Gamma^j_g = \emptyset$ and hence $\alpha \notin \Gamma^j$.

(5.10) If a color $\alpha \in \overline{\varphi}_n(T_{n,j} - V(T^*_{n,j-1}))$ for some j with $1 \le j \le q$, then $\alpha \notin \Gamma^{j-1}$ by Condition (i), and hence α is closed in $T_{n,j}$ with respect to φ_n by Condition (v). This simple observation will be used repeatedly in subsequent proofs.

(5.11) Note that not every ETT admits a good hierarchy. Suppose T does have such a hierarchy. To prove that $V(T)$ is elementary with respect to φ_n, as usual, we shall perform a sequence of Kempe changes. Since interchanging with colors in $D_{n,j}$ often results in a coloring which is not stable, in our proof we shall use colors in Γ^j_h as stepping stones to switch with the color η_h in $D_{n,j}$ while maintaining stable colorings in subsequent proofs. So we may think of Γ^j_h as a color set exclusively reserved for η_h and think of a good hierarchy as a control mechanism over Kempe changes.

We break the proof of Theorem 3.10(i) into the following two theorems. Although the first theorem appears to be weaker than Theorem 3.10(i), the second one implies that they are actually equivalent. We only present a proof of the second theorem in this section, and will give a proof of the first one in the next two sections.

Theorem 5.3. *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose T admits a good hierarchy and satisfies MP with respect to φ_n. Then $V(T)$ is elementary with respect to φ_n.*

Theorem 5.4. *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. If T satisfies MP under*

φ_n, then there exists a closed ETT T' constructed from T_n under φ_n by using the same connecting edge, connecting color, and extension type as T, with $r(T') = n$ and $V(T') = V(T_{n+1})$, such that T' admits a good hierarchy and satisfies MP with respect to φ_n.

Remark. As we shall see, our proof of Theorem 5.4 is based on Theorem 5.3, while the proof of Theorem 5.3 is completely independent of Theorem 5.4.

Proof of Theorem 5.4. By (5.1) and Theorem 3.10(i), $V(T_i)$ is elementary with respect to φ_{i-1} for $1 \le i \le n$. So each $|T_i|$ is an odd number. Thus $|T_i| - |T_{i-1}| \ge 2$ for each $1 \le i \le n$. By Theorem 2.9, if $|T_1| \le 10$, then G is an elementary multigraph, thereby proving Theorem 2.1 in this case. So we may assume that $|T_1| \ge 11$. Hence

(1) $|T_i| \ge 2i + 9$ for $1 \le i \le n$.

We shall actually construct an ETT T' from T_n by using the same connecting edge, connecting color, and extension type as T, which has a good hierarchy:

(2) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T'$, such that $T_n \vee R_n \subset T_{n,1}$ if $\Theta_n = PE$ and such that $V(T') = V(T_{n+1})$.

Since $V(T_n)$ is elementary with respect to φ_{n-1}, by (1) we have $|\overline{\varphi}_{n-1}(T_n)| \ge 2n + 11$ (as e is uncolored). From Algorithm 3.1 we see that $|\overline{\varphi}_{n-1}(T_n)| = |\overline{\varphi}_n(T_n)|$. So

(3) $|\overline{\varphi}_n(T_n)| \ge 2n + 11$. Moreover, $|D_{n,0}| \le |D_n| \le n$ by (5.6).

(4) If $\Theta_n = PE$, then we can find a 2-color set $\Gamma^0_h = \{\gamma^0_{h_1}, \gamma^0_{h_2}\} \subseteq \overline{\varphi}_n(T_n)$ for each $\eta_h \in D_{n,0} = \cup_{h \le n} S_h - \overline{\varphi}_n(T_n \vee R_n)$, such that $\Gamma^0_g \cap \Gamma^0_h = \emptyset$ whenever η_g and η_h are two distinct colors in $D_{n,0}$, and such that $T_n \vee R_n$ is not $(\Gamma^0)^-$-closed with respect to φ_n, where $\Gamma^0 = \cup_{\eta_h \in D_{n,0}} \Gamma^0_h$.

To justify this, let α be a color in $\overline{\varphi}_n(T_n \vee R_n)$ that is not closed in $T_n \vee R_n$ under φ_n; such a color exists by (5.3). In view of (3), $\overline{\varphi}_n(T_n) - \{\alpha\}$ contains at least $2n + 10$ colors. So (4) follows if we pick all colors in Γ^0 from $\overline{\varphi}_n(T_n) - \{\alpha\}$.

(5) If $\Theta_n = PE$, then there exists a 2-color set $\Gamma^0_h = \{\gamma^0_{h_1}, \gamma^0_{h_2}\} \subseteq \overline{\varphi}_n(T_n)$ for each $\eta_h \in D_{n,0}$ as described in (4), such that $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0|$ is maximized, which is at least 4.

To justify this, let α be as specified in the proof of (4). Then $\alpha \notin \overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$ by (5.4). If we pick all colors in Γ^0 from $\overline{\varphi}_n(T_n) - \{\alpha\}$, with priority given to those in $\overline{\varphi}_n(T_n) - \overline{\varphi}_n(R_n)$, then $|\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0| \ge 4$ by (3), because the ends of the uncolored edge e are contained in both T_n and R_n. So (5) is established.

Thus Definition 5.2(iv) is satisfied by these sets Γ^0_h. Using (3), we can similarly get the following statement.

(6) If $\Theta_n \ne PE$, then we can find a 2-color set $\Gamma^0_h = \{\gamma^0_{h_1}, \gamma^0_{h_2}\} \subseteq \overline{\varphi}_n(T_n)$ for each $\eta_h \in D_{n,0} = D_n$, such that $\Gamma^0_g \cap \Gamma^0_h = \emptyset$ whenever η_g and η_h are two distinct colors in $D_{n,0}$.

Note that the ETT T' to be constructed is not necessarily T, so $T_{n,j}$ may not be a segment of T for $1 \le j \le q$. Since T' is a tree sequence, we can obviously associate a linear order $\prec'$ with its vertices, so that $\prec'$ is identical with $\prec$ when restricted to $T^*_{n,0}$. Thus, in Algorithms 5.5 and 5.6, v_{η_h} is defined to be the first vertex of T' in the order $\prec'$ for which $\eta_h \in \overline{\varphi}_n(v_{\eta_h})$, if any, and defined to be the last vertex of T' in the order $\prec'$ otherwise; and $T_{n,j+1}(v_{\eta_h}) = T_{n,j+1}$ if v_{η_h} is not contained in $T_{n,j+1}$ for $0 \le j \le q$.

Given $\{\Gamma^0_h : \eta_h \in D_{n,0}\}$, let us construct $T_{n,1}$ using the following procedure.

Algorithm 5.5

Step 0. Set $T_{n,1} = T_n \vee R_n$ if $\Theta_n = PE$ and $T_{n,1} = T_n + f_n$ otherwise, where f_n is the connecting edge used in Step 2 or 3 of Algorithm 3.1, depending on Θ_n.

Step 1. While there exists $f \in \partial(T_{n,1})$ with $\varphi_n(f) \in \overline{\varphi}_n(T_{n,1})$, do: set $T_{n,1} = T_{n,1} + f$ if the resulting $T_{n,1}$ satisfies $\Gamma^0_h \cap \varphi_n\langle T_{n,1}(v_{\eta_h}) - T^*_{n,0}\rangle = \emptyset$ for all $\eta_h \in D_{n,0}$, where $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise.

Step 2. Return $T_{n,1}$.

Note that if $\Theta_n = PE$, then $T_n \vee R_n$ is not $(\Gamma^0)^-$-closed with respect to φ_n by (4) and (5). So Step 1 is applicable to $T_n \vee R_n$, and hence $T_{n,1} \neq T_n \vee R_n$. If $\Theta_n = RE$ or SE, then $T_{n,1} \neq T_n$ by the algorithm. For each $\eta_h \in D_{n,0}$, it follows from (5), (6), and Step 1 that $\Gamma^0_h \subseteq \overline{\varphi}_n(T_n) - \varphi_n\langle T_{n,1}(v_{\eta_h}) - T^*_{n,0}\rangle$. So $\Gamma^0_h \subseteq \Lambda^0_h$. Moreover, $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}}\Gamma^0_h)^-$-closed with respect to φ_n. To justify this, assume the contrary: there exists $f \in \partial(T_{n,1})$ with $\varphi_n(f) \in \overline{\varphi}_n(T_{n,1}) - (\cup_{\eta_h \in D_{n,1}}\Gamma^0_h)$. Then either $\varphi_n(f) \in \overline{\varphi}_n(T_{n,1}) - (\cup_{\eta_h \in D_{n,0}}\Gamma^0_h)$ or $\varphi_n(f) \in \Gamma^0_h$ for some $\eta_h \in D_{n,0}$ but $\eta_h \notin D_{n,1}$; in the latter case, η_h is a missing color at the vertex v_{η_h} in $T_{n,1}$. Thus we can further grow $T_{n,1}$ by using f and Step 1 in either case, a contradiction. Therefore, $T_{n,1}$ and $\{\Gamma^0_h : \eta_h \in D_{n,0}\}$ satisfy all the conditions stated in Definition 5.2.

Suppose we have constructed $T_{n,i}$ and $\{\Gamma^{i-1}_h : \eta_h \in D_{n,i-1}\}$ for all i with $1 \le i \le j$, which are as described in Definition 5.2. If $T_{n,j}$ is closed with respect to φ_n (equivalently $V(T_{n,j}) = V(T_{n+1})$), set $T' = T_{n,j}$. Otherwise, we proceed to the construction of $T_{n,j+1}$ and $\{\Gamma^j_h : \eta_h \in D_{n,j}\}$ using the following procedure.

Algorithm 5.6

Step 0. Set $\Gamma^j_h = \Gamma^{j-1}_h$ for each $\eta_h \in D_{n,j}$.

Step 1. Let f be an edge in $\partial(T_{n,j})$ with $\varphi_n(f) \in \Gamma^{j-1}_h$ for some $\eta_h \in D_{n,j}$, let $T_{n,j+1} = T_{n,j} + f$, and let $\{\gamma^j_{h_1}, \gamma^j_{h_2}\}$ be a 2-subset of $\overline{\varphi}_n(T_{n,j} - V(T_{n,j-1}))$. Replace Γ^j_h by $\{\gamma^j_{h_1}, \gamma^j_{h_2}\}$.

Step 2. While there exists $f \in \partial(T_{n,j+1})$ with $\varphi_n(f) \in \overline{\varphi}_n(T_{n,j+1})$, do: set $T_{n,j+1} = T_{n,j+1} + f$ if the resulting $T_{n,j+1}$ satisfies $\Gamma^j_h \cap \varphi_n\langle T_{n,j+1}(v_{\eta_h}) - T_{n,j}\rangle = \emptyset$ for all $\eta_h \in D_{n,j}$.

Step 3. Return $T_{n,j+1}$ and $\{\Gamma^j_h : \eta_h \in D_{n,j}\}$.

Let us make some observations about this algorithm and its output.

As $T_{n,j}$ is not closed with respect to φ_n, $V(T_{n,j})$ is a proper subset of $V(T_{n+1})$. By Definition 5.2(v), $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}}\Gamma^{j-1}_h)^-$-closed with respect to φ_n. So there exists a color $\beta \in \cup_{\eta_h \in D_{n,j}}\Gamma^{j-1}_h$, such that $\partial_{\varphi_n,\beta}(T_{n,j}) \neq \emptyset$. Hence the edge f specified in Step 1 is available.

For $1 \le i \le j$, we have $|\overline{\varphi}_n(T_{n,i})| \ge |\overline{\varphi}_n(T_n)| \ge 2n + 11$ and $|D_{n,i}| \le |D_{n,0}| \le |D_n| \le n$ by (3). So $\overline{\varphi}_n(T_{n,i}) - (\cup_{\eta_h \in D_{n,i}}\Gamma^{i-1}_h) \neq \emptyset$; let α be a color in this set. By Theorem 5.3 (see the remark right above the proof of this theorem), $V(T_{n,i})$ is elementary with respect to φ_n, which implies that $|T_{n,i}|$ is odd, because α is closed in $T_{n,j}$ under φ_n by Definition 5.2(v). It follows that $|T_{n,j}| - |T_{n,j-1}| \ge 2$. So $\overline{\varphi}_n(T_{n,j} - V(T_{n,j-1}))$ contains at least two distinct colors, and hence the 2-subset $\{\gamma^j_{h_1}, \gamma^j_{h_2}\}$ involved in Step 1 exists.

Note that each color in $\overline{\varphi}_n(T_{n,j+1}) - (\cup_{\eta_h \in D_{n,j+1}}\Gamma^j_h)$ is closed in $T_{n,j+1}$ with respect to φ_n, for otherwise, $T_{n,j+1}$ can be augmented further using Step 2 (see the paragraph succeeding Algorithm 5.5 for details). Thus $T_{n,j+1}$ is $(\cup_{\eta_h \in D_{n,j+1}} \Gamma_h^j)^-$-closed with respect to φ_n for $1 \le j \le q-1$. From the algorithm we see that $\Gamma_h^j \subseteq \overline{\varphi}(T_{n,j}) - \varphi_n\langle T_{n,j+1}(v_{\eta_h}) - T_{n,j}^*\rangle = \Lambda_h^j$ for all $\eta_h \in D_{n,j}$. So $T_{n,j+1}$ and $\{\Gamma_h^j : \eta_h \in D_{n,j}\}$ satisfy all the conditions in Definition 5.2 and hence are as desired.

Repeating the process, we can eventually get a closed ETT T', with $V(T') = V(T_{n+1})$, that admits a good hierarchy with respect to φ_n. Clearly, T' also satisfies MP under φ_n. ∎

Consider the case when $\Theta_n = PE$. By the definition of hierarchy (see (5.5)), $T_n \vee R_n$ is fully contained in $T_{n,1}$. To maintain the structure of $T_n \vee R_n$ under Kempe changes, we need the following concept in subsequent proofs. A coloring $\sigma \in \mathcal{C}^k(G-e)$ is called a $(T_n \oplus R_n, D_n, \varphi_n)$-*stable coloring* if it is both (T_n, D_n, φ_n)-stable and $(R_n, \emptyset, \varphi_n)$-stable; that is, the following conditions are satisfied:

- $\sigma(f) = \varphi_n(f)$ for any edge f incident to T_n with $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$;
- $\sigma(f) = \varphi_n(f)$ for any edge f incident to R_n with $\varphi_n(f) \in \overline{\varphi}_n(R_n)$; and
- $\overline{\sigma}(v) = \overline{\varphi}_n(v)$ for any $v \in V(T_n \cup R_n)$.

(5.12) If σ is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, then $\sigma(f) = \varphi_n(f)$ for any edge f on $T_n \cup R_n$. To justify this, note that, for any edge f on T_n, this equality holds by Lemma 3.2(iii). For any edge f in $R_n - T_n$, we have $\varphi_n(f) \in \overline{\varphi}_n(R_n)$ by the definition of R_n and TAA. It follows from the above definition that $\sigma(f) = \varphi_n(f)$.

From Lemma 2.4 it is clear that being $(T_n \oplus R_n, D_n, \cdot)$-stable is also an equivalence relation on $\mathcal{C}^k(G-e)$. Moreover, every $(T_n \vee R_n, D_n, \varphi_n)$-stable coloring is $(T_n \oplus R_n, D_n, \varphi_n)$-stable, but the converse need not hold.

Lemma 5.7. *Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose $\Theta_n = PE$ and T enjoys MP under φ_n. Let $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ be a hierarchy of T, and let σ_n be a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring. If T can be built from $T_n \vee R_n$ by using TAA under σ_n, then T is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of T under σ_n.*

Proof. Since σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, we have $\sigma(f) = \varphi_n(f)$ for any edge f on $T_n \vee R_n$ by (5.12). By definition, σ_n is a (T_n, D_n, φ_n)-stable coloring, so it is a φ_n mod T_n coloring by (5.1) and Theorem 3.10(vi). Thus T_n is an ETT corresponding to σ_n. As R_n is a closure of $T_n(v_n)$ under φ_n and σ_n is $(R_n, \emptyset, \varphi_n)$-stable, R_n is also a closure of $T_n(v_n)$ under σ_n. By hypothesis, T can be built from $T_n \vee R_n$ by using TAA under σ_n. So T is an ETT corresponding to the coloring σ_n and satisfies MP under σ_n by Theorem 3.10(vi). Obviously, $T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a hierarchy of T under σ_n. ∎

We define one more term before proceeding. Let T be a tree sequence with respect to G and e. A coloring $\pi \in \mathcal{C}^k(G-e)$ is called (T, φ_n)-*invariant* if $\pi(f) = \varphi_n(f)$ for any $f \in E(T-e)$ and $\overline{\pi}(v) = \overline{\varphi}_n(v)$ for any $v \in V(T)$. Clearly, being $(T, \cdot)$-invariant is also an equivalence relation on $\mathcal{C}^k(G-e)$. Note that for any subset C of $[k]$, a (T, C, φ_n)-stable coloring π is also (T, φ_n)-invariant, provided that $\pi\langle T\rangle \subseteq \overline{\varphi}_n(T) \cup C$.

Lemma 5.8. *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose T satisfies MP under φ_n. Let σ_n be obtained from φ_n by recoloring some (α, β)-chains fully contained in $G - V(T)$. Then the following statements hold:*

(i) σ_n is (T, D_n, φ_n)-stable. In particular, σ_n is (T, φ_n)-invariant. Furthermore, if $\Theta_n = PE$, then σ_n is $(T_n \oplus R_n, D_n, \varphi_n)$-stable.

(ii) T is an ETT satisfying MP with respect to σ_n.

(iii) If T admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ under φ_n, then this hierarchy of T remains good under σ_n, with the same Γ-sets (see Definition 5.2). Furthermore, if T is $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to φ_n, then T is also $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to σ_n.

Proof. Since the recolored (α, β)-chains are fully contained in $G - V(T)$, we have

(1) $\sigma_n(f) = \varphi_n(f)$ for each edge f incident to $V(T)$ and $\overline{\varphi}_n(v) = \overline{\sigma}_n(v)$ for each $v \in V(T)$.

Our proof relies heavily on this observation.

(i) By (1) and definitions, it is clear that σ_n is a (T, D_n, φ_n)-stable. In particular, σ_n is (T, φ_n)-invariant. Furthermore, if $\Theta_n = PE$, then σ_n is $(T_n \vee R_n, D_n, \varphi_n)$-stable, which implies that σ_n is $(T_n \oplus R_n, D_n, \varphi_n)$-stable.

(ii) In view of (1), we can construct T from T_n under σ_n in exactly the same way as under φ_n. From (1) we also deduce that σ_n is a (T_n, D_n, φ_n)-stable coloring. Hence, by Theorem 3.10(vi), T remains to be an ETT and satisfies MP under σ_n.

(iii) From (1), (5.5) and Lemma 5.7 (when $\Theta_n = PE$), we see that the given hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ is also a hierarchy of T under σ_n. By hypothesis, this hierarchy is good with respect to φ_n. Consider the Γ-sets specified in Definition 5.2 with respect to φ_n. Using (1) it is routine to check that these Γ-sets satisfy all the conditions in Definition 5.2 with respect to σ_n. So the given hierarchy of T remains good under σ_n, with the same Γ-sets. Furthermore, if T is $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to φ_n, then T is also $(\cup_{\eta_h \in D_{n,q+1}} \Gamma^q_h)^-$-closed with respect to σ_n. ∎

6 Basic Properties

As we have seen, Theorem 3.10(i) follows from Theorems 5.3 and 5.4. In the preceding section we have proved Theorem 5.4. The remainder of this paper is devoted to a proof of Theorem 5.3. In this section we make some technical preparations.

Let T is an ETT that admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ and satisfies MP with respect to the generating coloring φ_n. To prove Theorem 5.3 (that is, $V(T)$ is elementary with respect to φ_n), we apply induction on q, and the induction base is Theorem 3.10(i) for T_n. For convenience, we view $T_{n,0}$ as an ETT with -1 divider and n rungs in the following assumption. Throughout this section we assume that

(6.1) In addition to (5.1), Theorem 5.3 holds for every ETT that admits a good hierarchy and satisfies MP, with n rungs and at most $q-1$ dividers, where $q \ge 0$.

Let us first prove two technical lemmas that will be used in the proof of Theorem 5.3.

Lemma 6.1. *(Assuming (5.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose $\Theta_n = PE$ and T enjoys MP under φ_n. Let σ_n be a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring and let α and β be two colors in $[k]$. Then the following statements hold:*

(i) α and β are R_n-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(R_n)$;

(ii) α and β are T_n-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(T_n)$;

(iii) α and β are $T_n \vee R_n$-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(T_n \vee R_n)$ is closed in $T_n \vee R_n$ under σ_n; and

(iv) α and β are $T_n \vee R_n$-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(T_n)$ and $\beta \in \overline{\sigma}_n(R_n)$.

Proof. Since σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, it is (T_n, D_n, φ_n)-stable by definition. Let $j = m(v_n)$. It follows from (5.1) and Theorem 3.10(iii) that σ_n is a $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$-stable coloring. So σ_n is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$-stable and hence, by (5.1) and Theorem 3.10(vi), it is a φ_{j-1} mod T_{j-1} coloring. Furthermore, $\sigma(f) = \varphi_n(f)$ for any edge f in $T_n \cup R_n$ by (5.12) and $\overline{\sigma}_n(v) = \overline{\varphi}_n(v)$ for all $v \in V(T_n \cup R_n)$.

(i) Since R_n is a closure of $T_n(v_n)$ under φ_n and σ_n is $(R_n, \emptyset, \varphi_n)$-stable, R_n is also a closure of $T_n(v_n)$ under σ_n. Since R_n is obtained from T_{j-1} by using the same connecting edge, connecting color, and extension type as T_j, by (5.1) and Theorem 3.10(vi), R_n is an ETT corresponding to (σ_n, T_{j-1}) and satisfies MP under σ_n. Let α and β be as specified in the lemma. As $r(R_n) = j-1$, by (5.1) and Theorem 3.10(ii), there is at most one (α, β)-path with respect to σ_n intersecting R_n. Hence α and β are R_n-interchangeable under σ_n.

Let us make some observations before proving statements (ii) and (iii). By (5.4), each color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$ is closed in $T_n \vee R_n$ with respect to φ_n. Since σ_n is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, by definition we obtain

(1) each color in $\overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n)$ is closed in $T_n \vee R_n$ under σ_n.

(2) α and β are T_n-interchangeable under σ_n if $\alpha \in \overline{\sigma}_n(T_n)$, $\alpha \neq \delta_n$, and $\beta \neq \delta_n$.

To justify this, note that $\alpha \neq \gamma_n$, because $\gamma_n \notin \overline{\varphi}_n(T_n) = \overline{\sigma}_n(T_n)$. So $\alpha \notin S_n$. Nevertheless, there are two possibilities for β.

Let us first consider the case when $\beta \neq \gamma_n$. Since σ_n is (T_n, D_n, φ_n)-stable, $P_{v_n}(\gamma_n, \delta_n, \sigma_n) \cap T_n = \{v_n\}$ by (5.1) and Theorem 3.10(iv). Define $\sigma_n' = \sigma_n / P_{v_n}(\gamma_n, \delta_n, \sigma_n)$. By Lemma 3.5, σ_n' is $(T_n, D_{n-1}, \varphi_{n-1})$-stable. From (5.1) and Theorem 3.10(ii) we deduce that α and β are T_n-interchangeable under σ_n'. So they are T_n-interchangeable under σ_n because $\{\alpha, \beta\} \cap S_n = \emptyset$.

It remains to consider the case when $\beta = \gamma_n$. In this case, f_n is the only edge in $\partial_{\sigma_n, \gamma_n}(T_n) = \partial_{\varphi_n, \gamma_n}(T_n)$ by Lemma 3.2(iv). Since $V(T_n)$ is elementary with respect to φ_n, it is also elementary with respect to σ_n. As $\partial_{\sigma_n, \alpha}(T_n) = \emptyset$, there is at most one (α, γ_n)-path with respect to σ_n intersecting T_n. So α and β are T_n-interchangeable under σ_n. Thus (2) is established.

By (1), δ_n is closed in $T_n \vee R_n$ with respect to σ_n. So statement (ii) follows instantly from (2) and statement (iii).

(iii) Assume the contrary: there are at least two (α, β)-paths P_1 and P_2 with respect to σ_n intersecting $T_n \vee R_n$. We may assume that

(3) $\alpha \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n)$.

To justify this, let A be the set of four ends of P_1 and P_2. Then at least two vertices from A are outside $T_n \vee R_n$ because, by Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to σ_n.

Using (i), it is then routine to check that $P_1 \cup P_2$ contains two vertex-disjoint subpaths Q_1 and Q_2, which are T_n-exit paths with respect to σ_n. Let $u \in V(T_n) \cap V(R_n)$, let $\eta \in \overline{\sigma}_n(u)$, and let $\sigma_n' = \sigma_n/(G - T_n \vee R_n, \alpha, \eta)$. By (1), η is closed in $T_n \vee R_n$ with respect to σ_n; so is α by hypothesis. Hence σ_n' is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring, and Q_1 and Q_2 are two T_n-exit paths with respect to σ_n'. Since $P_u(\eta, \beta, \sigma_n')$ contains at most one of Q_1 and Q_2, replacing σ_n and α by σ_n' and η, respectively, we obtain (3).

Let v be a vertex in $V(T_n) \cap V(R_n)$ with $\alpha \in \overline{\sigma}_n(v)$. Clearly, we may assume that $P_1 = P_v(\alpha, \beta, \sigma_n)$. By (i), we may further assume that P_2 is disjoint from R_n. So P_2 intersects T_n. Therefore α and β are not T_n-interchangeable under σ_n. Since $\gamma_n \notin \overline{\varphi}_n(T_n) = \overline{\sigma}_n(T_n)$, we have $\alpha \neq \gamma_n$. By (2), we may assume that $\alpha = \delta_n$ or $\beta = \delta_n$.

Suppose $\beta = \delta_n$. By Lemma 3.2(iv) and the definition of stable colorings, edges in $\partial_{\sigma_n, \delta_n}(T_n)$ are all incident to $V(T_n) \cap V(R_n)$. Thus both P_1 and P_2 intersect $V(T_n) \cap V(R_n)$, contradicting statement (i).

Suppose $\alpha = \delta_n$. By (1), δ_n is closed in $T_n \vee R_n$ under σ_n. Since $V(T_n) \cap V(R_n)$ contains both ends of the uncolored edge e, there exists a color $\theta \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n) - \{\delta_n\}$. Let $\sigma_n'' = \sigma_n/(G - T_n \vee R_n, \delta_n, \theta)$. Then σ_n'' is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. From the existence of P_1 and P_2, we see that θ and β are not $T_n \vee R_n$-interchangeable under σ_n'', contradicting our observation (2) above the case when $\alpha \neq \delta_n$ and $\beta \neq \delta_n$.

(iv) Assume the contrary: there are at least two (α, β)-paths P_1 and P_2 with respect to σ_n intersecting $T_n \vee R_n$. Let u be a vertex in T_n with $\alpha \in \overline{\sigma}_n(u)$ and let v be a vertex in R_n with $\beta \in \overline{\sigma}_n(v)$. By (ii) (resp. (i)), $P_u(\alpha, \beta, \sigma_n)$ (resp. $P_v(\alpha, \beta, \sigma_n)$) is the only (α, β)-path with respect to σ_n intersecting T_n (resp. R_n). Hence $P_1 = P_u(\alpha, \beta, \sigma_n)$, $P_2 = P_v(\alpha, \beta, \sigma_n)$ (rename subscripts if necessary), and $P_u(\alpha, \beta, \sigma_n) \neq P_v(\alpha, \beta, \sigma_n)$. Moreover, neither $P_u(\alpha, \beta, \sigma_n)$ nor $P_v(\alpha, \beta, \sigma_n)$ has an end in $V(T_n) \cap V(R_n)$, which in turn implies that

(4) $u \in V(T_n) - V(R_n)$ and $v \in V(R_n) - V(T_n)$.

By (4) and statement (ii), $P_v(\alpha, \beta, \sigma_n)$ is disjoint from T_n. Let $\sigma_n' = \sigma_n/P_v(\alpha, \beta, \sigma_n)$. By Lemma 5.8, σ_n' is a (T_n, D_n, φ_n)-stable coloring. By Lemma 5.1, $V(T_n \vee R_n)$ is elementary with respect to σ_n. Since $\alpha \in \overline{\sigma}_n(u)$ and $\beta \in \overline{\sigma}_n(v)$, from TAA we see that no edge in $R_n(v) - T_n(v_n)$ is colored by α or β under both φ_n and σ_n. Thus edges in $R_n(v) - T_n(v_n)$ are colored exactly the same under σ_n' as under σ_n and $\overline{\sigma}_n(x) = \overline{\sigma}_n'(x)$ for any $x \in V(R_n(v) - v)) \cup V(T_n)$. Let R_n' be a closure of $T_n(v_n)$ under σ_n'. Then $v \in V(R_n')$. In view of Lemma 5.1, $V(T_n \vee R_n')$ is elementary with respect to σ_n'. However, $\alpha \in \overline{\sigma}_n'(u) \cap \overline{\sigma}_n'(v)$, a contradiction. ∎

As introduced in Section 5, $T_{n,0}^* = T_n \vee R_n$ if $\Theta_n = PE$ and $T_{n,0}^* = T_n$ otherwise. Throughout a coloring $\sigma_n \in \mathcal{C}^k(G - e)$ is called a $(T_{n,0}^*, D_n, \varphi_n)$-*strongly stable coloring* if it is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring when $\Theta_n = PE$ and is a (T_n, D_n, φ_n)-stable coloring when $\Theta_n \neq PE$. By Lemma 3.2(iii) and (5.12), every $(T_{n,0}^*, D_n, \varphi_n)$-strongly stable coloring is $(T_{n,0}^*, \varphi_n)$-invariant. It follows from Lemma 2.4 that being $(T_{n,0}^*, D_n, \cdot)$-strongly stable is an equivalence relation on $\mathcal{C}^k(G - e)$.

Lemma 6.2. *(Assuming (6.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$, and let σ_n be a $(T_{n,0}^*, D_n, \varphi_n)$-strongly stable coloring. Suppose T' is an ETT obtained from $T_{n,0}^*$ corresponding to (σ_n, T_n) (see Definition 3.7 and Theorem 3.10(vi)) that has a good hierarchy*

$T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p} = T'$, where $1 \le p \le q$ (see Definition 5.2 and (6.1)). Furthermore, T' is $(\cup_{\eta_h \in D_{n,p}} \Gamma_h^{p-1})^-$-closed with respect to σ_n. Let $\alpha \in \overline{\sigma}_n(T')$ and $\beta \in [k] - \{\alpha\}$. If α is closed in T' under σ_n, then α and β are T'-interchangeable under σ_n.

A very useful corollary of this lemma is given below.

Corollary 6.3. *(Assuming (6.1)) Let T be an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Suppose T has a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$. Let p be a subscript with $1 \le p \le q$, and let $\alpha \in \overline{\varphi}_n(T_{n,p})$ and $\beta \in [k] - \{\alpha\}$. If α is closed in $T_{n,p}$ under φ_n, then α and β are $T_{n,p}$-interchangeable under φ_n.*

Proof of Lemma 6.2. Assume the contrary: there are two (α, β)-paths Q_1 and Q_2 with respect to σ_n intersecting $T' = T_{n,p}$; subject to this, p is minimum. Let us make some simple observations about T' before proceeding. Since σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, by Theorem 3.10(vi) we have

(1) T' satisfies MP under σ_n, and hence $V(T')$ is elementary with respect to σ_n by (6.1) and Theorem 5.3.

By hypothesis, α is closed in T' with respect to σ_n, which together with (1) implies that

(2) $|T'|$ is odd.

In our proof we shall repeatedly use the following hypothesis:

(3) T' is $(\cup_{\eta_h \in D_{n,p}} \Gamma_h^{p-1})^-$-closed with respect to σ_n.

Depending on whether β is contained in $\overline{\sigma}_n(T')$, we consider two cases.

Case 1. $\beta \in \overline{\sigma}_n(T')$.

In this case, $|\partial_{\sigma_n,\beta}(T')|$ is even by (1) and (2). From the existence of Q_1 and Q_2, we see that G contains two vertex-disjoint $(T', \sigma_n, \{\alpha, \beta\})$-exit paths P_1 and P_2. For $i = 1, 2$, let a_i and b_i be the ends of P_i with $b_i \in V(T')$. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2$. We distinguish between two subcases according to the location of b_2.

Subcase 1.1. $b_2 \in V(T') - V(T^*_{n,p-1})$.

Since the edge on P_1 incident to b_1 is a boundary edge of T' and is colored by β, we have $\beta \in \Gamma_h^{p-1}$ for some h with $\eta_h \in D_{n,p}$ by (3), which together with Definition 5.2(i) implies that $v_\beta \in V(T_{n,p-1})$, where v_β is the vertex in T' (see (1)) for which $\beta \in \overline{\sigma}_n(v_\beta)$. Let $\gamma \in \overline{\sigma}_n(b_2)$. By the assumption of the present subcase and Definition 5.2(i), we have $\gamma \notin \Gamma^{p-1}$. Hence γ is closed with respect to σ_n in T' by (3). So

(4) both α and γ are closed in T' under σ_n.

Let $\mu_1 = \sigma_n/(G - T', \alpha, \gamma)$. Clearly, μ_1 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. By Lemma 5.8,

(5) the given hierarchy of T' remains good under μ_1, with the same Γ-sets as those under σ_n (see Definition 5.2). Furthermore, T' is $(\cup_{\eta_h \in D_{n,p}} \Gamma_h^{p-1})^-$-closed under μ_1.

Note that P_1 and P_2 are two $(T', \mu_1, \{\gamma, \beta\})$-exit paths. Let $\mu_2 = \mu_1/P_{b_2}(\gamma, \beta, \mu_1)$. Since $P_{b_2}(\gamma, \beta, \mu_1) \cap T' = \{b_2\}$, all edges incident to $V(T'(b_2) - b_2)$ are colored the same under μ_2 as under μ_1. By (5) and Lemma 5.8, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_2) - b_2$ is a good hierarchy of $T'(b_2) - b_2$ under μ_2, with the same Γ-sets as T' under σ_n. So

(6) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_2)$ is a good hierarchy of $T'(b_2)$ under μ_2, with the same Γ-sets as T' under σ_n.

Clearly, μ_2 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. By Theorem 3.10(vi), $T'(b_2)$ satisfies MP under μ_2. Thus from (6.1) we conclude that $V(T'(b_2))$ is elementary with respect to μ_2. However, $\beta \in \overline{\mu}_2(T_{n,p-1}) \cap \overline{\mu}_2(b_2)$, a contradiction.

Subcase 1.2. $b_2 \in V(T^*_{n,p-1})$.

We propose to show that

(7) there exists a color $\theta \in \overline{\sigma}_n(T_n)$ that is closed in both $T^*_{n,0}$ and $T_{n,1}$ under σ_n if $p = 1$, and a color $\theta \in \overline{\sigma}_n(T_{n,p-1})$ that is closed in both $T_{n,p-1}$ and $T_{n,p}$ under σ_n if $p \geq 2$.

Our proof is based on the following simple observation (see (3) in the proof of Theorem 5.4).

(8) $|\overline{\sigma}_n(T_n)| \geq 2n + 11$ and $|D_{n,i}| \leq |D_n| \leq n$ for $0 \leq i \leq q$.

Let us first assume that $p = 1$. When $\Theta_n \neq PE$, let θ be a color in $\overline{\sigma}_n(T_n) - (\cup_{\eta_h \in D_{n,1}} \Gamma^0_h)$; such a color exists by (8). From Algorithm 3.1 we see that T_n is closed under φ_n and hence under σ_n. By (3) and Definition 5.2(v), $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}} \Gamma^0_h)^-$-closed under σ_n. So θ is as desired. When $\Theta_n = PE$, we have $|\overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n) - \Gamma^0| \geq 4$ by Definition 5.2(iv). Let $\theta \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n) - \Gamma^0 - \{\delta_n\}$. By the hypothesis of the present lemma, σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. It follows from (5.4) that θ is closed in $T_n \vee R_n$ under σ_n. By Definition 5.2(v), $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}} \Gamma^0_h)^-$-closed with respect to σ_n. So θ is also as desired.

Next we assume that $p \geq 2$. By (8), we have $|\overline{\sigma}_n(T_{n,p-2})| \geq |\overline{\sigma}_n(T_n)| \geq 2n+11$ and $|D_{n,p-2}| \leq |D_n| \leq n$. So there exists a color θ in $\overline{\sigma}_n(T_{n,p-2}) - (\cup_{\eta_h \in D_{n,p-1}} \Gamma^{p-2}_h)$. Since $\overline{\sigma}_n(T_{n,p-2}) \subseteq \overline{\sigma}_n(T_{n,p-1})$, we have $\theta \in \overline{\sigma}_n(T_{n,p-1}) - (\cup_{\eta_h \in D_{n,p-1}} \Gamma^{p-2}_h)$. By Definition 5.2(v), θ is closed in $T_{n,p-1}$ under σ_n. From the definition of θ and Definition 5.2(iii), we also see that $\theta \notin \Gamma^{p-1}$. So $\theta \in \overline{\sigma}_n(T_{n,p}) - \Gamma^{p-1} \subseteq \overline{\sigma}_n(T_{n,p}) - (\cup_{\eta_h \in D_{n,p}} \Gamma^{p-1}_h)$. By (3), θ is closed in $T_{n,p}$ under σ_n. Hence (7) is established.

Let $\mu_3 = \sigma_n/(G - T', \alpha, \theta)$. Since both α and θ are closed in T' with respect to σ_n, by Lemma 5.8, μ_3 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. Furthermore, $T_{n,p}$ admits a good hierarchy and satisfies MP with respect to μ_3. Thus $T_{n,p-1}$ also admits a good hierarchy and satisfies MP with respect to μ_3 if $p \geq 2$. By (7), θ is closed in $T^*_{n,0}$ if $p = 1$ and closed in $T_{n,p-1}$ if $p \geq 2$ under μ_3. Note that both P_1 and P_2 are $(T^*_{n,p-1}, \mu_3, \{\theta, \beta\})$-exit paths. So θ and β are not $T^*_{n,0}$-interchangeable under μ_3 if $p = 1$ and not $T_{n,p-1}$-interchangeable under μ_3 if $p \geq 2$, which contradicts Lemma 6.1(iii) or the interchangeability property of T_n when $p = 1$ and the minimality assumption on p when $p \geq 2$.

Case 2. $\beta \notin \overline{\sigma}_n(T')$.

In this case, $|\partial_{\sigma_n, \beta}(T')|$ is odd and at least three by (1) and (2). From the existence of Q_1 and Q_2, we see that G contains at least three $(T, \sigma_n, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3. For $i = 1, 2, 3$, let a_i and b_i be the ends of P_i with $b_i \in V(T)$, and f_i be the edge of P_i incident to b_i. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2 \prec b_3$.

Subcase 2.1. $b_3 \in V(T') - V(T^*_{n,p-1})$.

For convenience, we call the tuple $(\sigma_n, T', \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* and use $\mathcal{K}$ to denote the set of all such counterexamples. With a slight abuse of notation, we still use $(\sigma_n, T', \alpha, \beta, P_1, P_2, P_3)$ to denote a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$. Let $\gamma \in \overline{\varphi}(b_3)$. By the hypothesis of the present subcase and Definition 5.2(i), we have $\gamma \notin \Gamma^{p-1}$. So γ is closed in T' under σ_n by (3). Note that γ might be some $\eta_h \in D_n$.

Let $\mu_4 = \sigma_n/(G - T', \alpha, \gamma)$. By Lemma 5.8, μ_4 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. Furthermore, T' admits a good hierarchy and satisfies MP under μ_4. Note that P_1, P_2, P_3 are three $(T', \mu_4, \{\gamma, \beta\})$-exit paths.

Consider $\mu_5 = \mu_4/P_{b_3}(\gamma, \beta, \mu_4)$. Clearly, $\beta \in \overline{\mu}_5(b_3)$ and $\beta \notin \Gamma^{p-1}$. Since $P_{b_3}(\gamma, \beta, \mu_4) \cap T' = \{b_3\}$, it is easy to see that μ_5 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring and all edges incident to $V(T'(b_3) - b_3)$ are colored the same under μ_5 as under μ_4. By (5.1) and Theorem 3.10(vi), $T'(b_3)$ is an ETT satisfying MP under μ_5. By Lemma 5.7 and Lemma 5.8, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_3) - b_3$ is a good hierarchy of $T'(b_3) - b_3$ under μ_5, with the same Γ-sets as T' under σ_n. So

(9) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T'(b_3)$ is a good hierarchy of $T'(b_3)$ under μ_5, with the same Γ-sets as T' under σ_n.

Let H be obtained from $T'(b_3)$ by adding f_1 and f_2. Since $\beta \notin \Gamma^{p-1}$, it can be seen from (9) that

(10) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset H$ is a good hierarchy of H under μ_5, with the same Γ-sets as T' under σ_n.

By (5.1) and Theorem 3.10(vi), H satisfies MP under μ_5. Set $T'' = H$. Let us grow T'' by using the following algorithm:

(11) While there exists $f \in \partial(T'')$ with $\mu_5(f) \in \overline{\mu}_5(T'')$, do: set $T'' = T'' + f$ if the resulting T'' satisfies $\Gamma^{p-1}_h \cap \mu_5\langle T''(v_{\eta_h}) - T_{n,p-1}\rangle = \emptyset$ for all $\eta_h \in D_{n,p-1}$.

Note that this algorithm is exactly the same as Step 2 in Algorithm 5.6. From (11) we see that

(12) T'' is $(\cup_{\eta_h \in D''_{n,p}} \Gamma^{p-1}_h)^-$-closed with respect to μ_5, where $D''_{n,p} = \cup_{h \le n} S_h - \overline{\mu}_5(T'')$ (so $D''_{n,p} \subseteq D_{n,p-1}$).

In view of (10) and (11), we conclude that

(13) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,p-1} \subset T''$ is a good hierarchy of H under μ_5, with the same Γ-sets as T' under σ_n.

Clearly, T'' satisfies MP under μ_5. By (13), (6.1), and Theorem 5.3, $V(T'')$ is elementary with respect to μ_5. Observe that none of a_1, a_2, a_3 is contained in T'', for otherwise, let $a_i \in V(T_2)$ for some i with $1 \le i \le 3$. Since $\{\beta, \gamma\} \cap \overline{\mu}_5(a_i) \neq \emptyset$ and $\beta \in \overline{\mu}_5(b_3)$, we obtain $\gamma \in \overline{\sigma}_2(a_i)$. Hence from TAA we see that P_1, P_2, P_3 are all entirely contained in $G[T'']$, which in turn implies $\gamma \in \overline{\sigma}_2(a_j)$ for $j = 1, 2, 3$. So $V(T'')$ is not elementary with respect to μ_5, a contradiction. Each P_i contains a subpath L_i, which is a T''-exit path with respect to μ_5. Since f_1 is not contained in L_1, we obtain $|L_1| + |L_2| + |L_3| < |P_1| + |P_2| + |P_3|$. Thus, in view of (12), the existence of the counterexample $(\mu_5, T'', \gamma, \beta, L_1, L_2, L_3)$ violates the minimality assumption on $(\sigma_n, T', \alpha, \beta, P_1, P_2, P_3)$.

Subcase 2.2. $b_3 \in V(T^*_{n,p-1})$.

The proof in this subcase is essentially the same as that in Subcase 1.2. Let θ be a color as described in (7). Consider $\mu_3 = \sigma_n/(G - T', \alpha, \theta)$. Then we can verify that θ and β are not $T^*_{n,0}$-interchangeable under μ_3 if $p = 1$ and not $T_{n,p-1}$-interchangeable under μ_3 if $p \ge 2$, which contradicts Lemma 6.1(iii) or the minimality assumption on p; for the omitted details, see the proof in Subcase 1.2. ∎

Let us make some further preparations before proving Theorem 5.3. Let $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ be a good hierarchy of T (see (5.5) and Definition 5.2). Recall that $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise, $T^*_{n,0} \subset T_{n,1}$ by (5.5), and $T^*_{n,i} = T_{n,i}$ if $i \ge 1$. Let T be constructed from $T^*_{n,q}$ using TAA by recursively adding edges $e_1, e_2, \ldots, e_p$ and vertices $y_1, y_2, \ldots, y_p$, where y_i is the end of e_i outside $T(y_{i-1})$ for $i \geq 1$, with $T(y_0) = T^*_{n,q}$. Write $T = T^*_{n,q} \cup \{e_1, y_1, e_2, ..., e_p, y_p\}$. The *path number* of T, denoted by $p(T)$, is defined to be the smallest subscript $i \in \{1, 2, ..., p\}$ such that the sequence $(y_i, e_{i+1}, ..., e_p, y_p)$ corresponds to a path in G.

A coloring $\sigma_n \in \mathcal{C}^k(G-e)$ is called a $(T^*_{n,i}, D_n, \varphi_n)$-*strongly stable coloring*, with $1 \leq i \leq q$, if it is both a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable and a $(T^*_{n,i}, \varphi_n)$-invariant coloring. Since every $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring is $(T^*_{n,0}, \varphi_n)$-invariant by Lemma 3.2(iii) and (5.12), this concept is a natural extension of $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable colorings. Let v be a vertex of G. By $T \prec v$ we mean that $u \prec v$ for any $u \in V(T)$. Given a color $\alpha \in [k]$, we use v_α to denote the first vertex u of T in the order $\prec$ for which $\alpha \in \overline{\varphi}_n(u)$, if any, and defined to be the last vertex of T in the order $\prec$ otherwise.

Recall that our proof of Theorem 5.3 proceeds by induction on q (see (6.1)). The induction step will be carried out by contradiction. Throughout the remainder of this section and Subsection 7.1, (T, φ_n) stands for a minimum counterexample to Theorem 5.3; that is,

(6.2) T is an ETT that admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1}$ $= T$ and satisfies MP with respect to the generating coloring φ_n;

(6.3) subject to (6.2), $V(T)$ is not elementary with respect to φ_n;

(6.4) subject to (6.2) and (6.3), $p(T)$ is minimum; and

(6.5) subject to (6.2)-(6.4), $|T| - |T_{n,q}|$ is minimum.

Our objective is to find another counterexample (T', σ_n) to Theorem 5.3, which violates the minimality assumption (6.4) or (6.5) on (T, φ_n).

The following fact will be used frequently in subsequent proof.

(6.6) $V(T(y_{p-1}))$ is elementary with respect to φ_n.

Let us exhibit some basic properties satisfied by the minimum counterexample (T, φ_n) as specified above.

Lemma 6.4. *For $0 \leq i \leq p$, the inequality*

$$|\overline{\varphi}_n(T(y_i))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_i) - T^*_{n,q}\rangle| \geq 2n + 11$$

holds, where $T(y_0) = T^*_{n,q}$. Furthermore, if

$$|\overline{\varphi}_n(T(y_i))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_i) - T^*_{n,q}\rangle| - |\Gamma^q \cup D_{n,q}| \leq 4,$$

then there exist 7 distinct colors $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_i))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_i) - T^*_{n,q}\rangle = \emptyset$, where Γ^q and Γ^q_h are introduced in Definition 5.2.

Proof. Since the number of vertices in $T(y_i) - V(T^*_{n,q})$ is i, and the number of edges in $T(y_i) - T^*_{n,q}$ is also i, we obtain $|\overline{\varphi}_n(T(y_i) - V(T^*_{n,q}))| \geq |\varphi_n\langle T(y_i) - T^*_{n,q}\rangle|$. Hence

$$
\begin{aligned}
& |\overline{\varphi}_n(T(y_i))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_i) - T^*_{n,q}\rangle| \\
\geq\ & |\overline{\varphi}_n(T(y_i))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\overline{\varphi}_n(T(y_i) - V(T^*_{n,q}))| \\
=\ & |\overline{\varphi}_n(T^*_{n,q})| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| \\
\geq\ & |\overline{\varphi}_n(T^*_{n,0})| - |\overline{\varphi}_n(T^*_{n,0}) - \overline{\varphi}_n(T_n)| \\
=\ & |\overline{\varphi}_n(T_n)| \\
\geq\ & 2n + 11,
\end{aligned}
$$

where the last inequality can be found in the proof of Theorem 5.4 (see (3) therein). So the first inequality is established.

Suppose the second inequality also holds. Then these two inequalities guarantee the existence of at least $2n+7$ colors in the intersection of $\overline{\varphi}_n(T(y_i)) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_i) - T^*_{n,q}\rangle$ and $\Gamma^q \cup D_{n,q}$. Let C denote this intersection. Then $|C| \geq 2n+7$. By (5.6), we have $|D_{n,q}| \leq |D_n| \leq n$ and $|\Gamma^q| \leq 2|D_{n,q}| \leq 2n$. So $|\Gamma^q \cup D_{n,q}| \leq 3n$. Since $|C| \leq |\Gamma^q \cup D_{n,q}|$, it follows that $2n+7 \leq 3n$, which implies $n \geq 7$. Note that $C = \cup_{\eta_h \in D_{n,q}} (\Gamma^q_h \cup \{\eta_h\}) \cap C$ and $|(\Gamma^q_h \cup \{\eta_h\}) \cap C| \leq 3$ for any η_h in $D_{n,q}$. Since $|C| \geq 2n+7$ and $n \geq 7$, by the Pigeonhole Principle, there exist at least 7 distinct colors η_h in $D_{n,q}$, such that $|(\Gamma^q_h \cup \{\eta_h\}) \cap C| = 3$, or equivalently, $\Gamma^q_h \cup \{\eta_h\} \subseteq C$. For each of these η_h, clearly $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_i))$ and $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_i) - T^*_{n,q}\rangle = \emptyset$. ∎

Lemma 6.5. *Suppose $q \geq 1$ and $\alpha \in \overline{\varphi}_n(T_{n,q})$. If there exists a subscript i with $0 \leq i \leq q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n, then $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$, where r is the largest such i. If there is no such subscript i, then $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h \subseteq \Gamma^{j-1}$ for $1 \leq j \leq q$, $\Theta_n = PE$, $v_\alpha \in V(T_n) - V(R_n)$, and $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$.*

Proof. Let us first assume the existence of a subscript i with $0 \leq i \leq q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. By definition, r is the largest such i. Suppose the contrary: $\alpha \in \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$. Then $r < q$ and there exists a subscript s with $r+1 \leq s \leq q$, such that $\alpha \in \varphi_n\langle T_{n,s} - T^*_{n,s-1}\rangle$. From the definition of r, we see that α is not closed in $T_{n,s}$ with respect to φ_n. It follows from Definition 5.2(v) that $\alpha \in \Gamma^{s-1}_h$ for some $\eta_h \in D_{n,s}$. By the definitions of $D_{n,s}$ and $D_{n,s-1}$, we have $D_{n,s} \subseteq D_{n,s-1}$. So $\eta_h \in D_{n,s-1}$. Since α in Γ^{s-1}_h is used by at least one edge in $T_{n,s} - T^*_{n,s-1}$, from Definition 5.2(i) (with $j = s-1$) we deduce that η_h is a color missing at some vertex in $T_{n,s}$ (see (5.7)). Thus $\eta_h \notin D_{n,s}$ by definition, a contradiction.

Next we assume that there exists no subscript i with $0 \leq i \leq q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. Since $\alpha \in \overline{\varphi}_n(T_{n,q})$, it follows from (5.10) that $\alpha \in \overline{\varphi}_n(T^*_{n,0})$. By Definition 5.2(v), we obtain

(1) $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h \subseteq \Gamma^{j-1}$ for $1 \leq j \leq q$.

Hence $\alpha \in \Gamma^j$ for all $0 \leq j \leq q-1$. From the definition of Γ^0, we see that $v_\alpha \in V(T_n)$. If $\Theta_n \neq PE$, then α would be closed in $T_n = T^*_{n,0}$ under φ_n, a contradiction. So $\Theta_n = PE$. Moreover, by the assumption on α, Algorithm 3.1 and (5.4), we have $v_\alpha \in V(T_n) - V(R_n)$. Since R_n is a closure of $T_n(v_n)$ under φ_n, using (6.6) and TAA we obtain

(2) $\alpha \notin \overline{\varphi}_n(R_n - V(T_n))$ and $\alpha \notin \varphi_n\langle R_n - T_n\rangle$.

(3) $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,0}\rangle$.

Assume the contrary: $\alpha \in \varphi_n\langle T_{n,q} - T^*_{n,0}\rangle$. Then there exists a subscript $1 \leq s \leq q$ such that $\alpha \in \varphi_n\langle T_{n,s} - T^*_{n,s-1}\rangle$. By (1), we have $\alpha \in \Gamma^{s-1}_h$ for some $\eta_h \in D_{n,s}$. As $D_{n,s} \subseteq D_{n,s-1}$, we obtain $\eta_h \in D_{n,s-1}$. Since α is used by at least one edge in $T_{n,s} - T^*_{n,s-1}$, from Definition 5.2(i) (with $j = s-1$) we deduce that η_h is a color missing at some vertex in $T_{n,s}$ (see (5.7)). Thus $\eta_h \notin D_{n,s}$ by definition, a contradiction.

Combining (2) and (3), we conclude that $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$. ∎

Our proof of Theorem 5.3 relies heavily on the following two technical lemmas.

Lemma 6.6. *Let α and β be two colors in $\overline{\varphi}_n(T(y_{p-1}))$. Suppose $v_\alpha \prec v_\beta$ and $\alpha \notin \varphi_n\langle T(v_\beta) - T^*_{n,q}\rangle$ if $\{\alpha,\beta\} - \overline{\varphi}_n(T^*_{n,q}) \neq \emptyset$. Then $P_{v_\alpha}(\alpha,\beta,\varphi_n) = P_{v_\beta}(\alpha,\beta,\varphi_n)$ if one of the following cases occurs:*

(i) $q \geq 1$, and $\alpha \in \overline{\varphi}_n(T_{n,q})$ or $\{\alpha,\beta\} \cap D_{n,q} = \emptyset$;

(ii) $q = 0$, and $\alpha \in \overline{\varphi}_n(T_n)$ or $\{\alpha,\beta\} \cap D_n = \emptyset$; and

(iii) $\alpha \in \overline{\varphi}_n(T^*_{n,q})$ and is closed in $T^*_{n,q}$ with respect to φ_n.

Furthermore, in Case (iii), $P_{v_\alpha}(\alpha,\beta,\varphi_n) = P_{v_\beta}(\alpha,\beta,\varphi_n)$ is the only (α,β)-path with respect to φ_n intersecting $T^*_{n,q}$.

Proof. Let $a = v_\alpha$ and $b = v_\beta$. We distinguish among three cases according to the locations of a and b.

Case 1. $\{a,b\} \subseteq V(T^*_{n,q})$.

By (6.6), $V(T^*_{n,q})$ is elementary with respect to φ_n. So a (resp. b) is the only vertex in $T^*_{n,q}$ missing α (resp. β). If both α and β are closed in $T^*_{n,q}$ with respect to φ_n, then no boundary edge of $T^*_{n,q}$ is colored by α or β. Hence $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$ is the only path intersecting $T^*_{n,q}$. So we may assume that α or β is not closed in $T^*_{n,q}$ with respect to φ_n. It follows that if $q = 0$, then $\Theta_n = PE$, for otherwise, Algorithm 3.1 would imply that both α and β are closed in $T_n = T^*_{n,0}$, a contradiction. Therefore

(1) $T^*_{n,0} = T_n \vee R_n$ if $q = 0$.

Let us first assume that precisely one of α and β is closed in $T^*_{n,q}$ with respect to φ_n. In this subcase, by Corollary 6.3 if $q \geq 1$ and by (1) and Lemma 6.1(iii) if $q = 0$, colors α and β are $T^*_{n,q}$-interchangeable under φ_n, so $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$ is the only path intersecting $T^*_{n,q}$.

Next we assume that neither α nor β is closed in $T^*_{n,q}$ with respect to φ_n. In this subcase, we only need to show that $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$. Symmetry allows us to assume that $a \prec b$. Let r be the subscript with $\beta \in \overline{\varphi}_n(T^*_{n,r} - V(T^*_{n,r-1}))$, where $0 \leq r \leq q$ and $T^*_{n,-1} = \emptyset$. Then $a, b \in V(T^*_{n,r})$. By (6.2), $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a good hierarchy of T. If $r \geq 1$, then β is closed in $T_{n,r}$ with respect to φ_n by Definition 5.2 (see (5.10)). From the above discussion about $T^*_{n,q}$ (with r in place of q), we similarly deduce that $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$. So we may assume that $r = 0$. If $\Theta_n \neq PE$, then both α and β are closed in T_n with respect to φ_n (see Algorithm 3.1), so $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$ by (6.6). If $\Theta_n = PE$, then it follows from Lemma 6.1(i), (ii) and (iv) that $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$.

Case 2. $\{a,b\} \cap V(T^*_{n,q}) = \emptyset$.

By the hypotheses of the present case and the present lemma, we have $\{\alpha,\beta\} \cap D_{n,q} = \emptyset$ if $q \geq 1$ and $\{\alpha,\beta\} \cap D_n = \emptyset$ if $q = 0$. So

(2) $\alpha,\beta \notin D_{n,q} \cup \overline{\varphi}_n(T^*_{n,q})$ if $q \geq 1$ and $\alpha,\beta \notin D_n \cup \overline{\varphi}_n(T^*_{n,0})$ if $q = 0$.

By the definitions of D_n and $D_{n,q}$, we have $D_n \cup \overline{\varphi}_n(T_n) \subseteq D_{n,q} \cup \overline{\varphi}_n(T^*_{n,q})$. Since $\alpha \notin \varphi_n\langle T(b) - T^*_{n,q}\rangle$ by hypothesis, from (2), Lemma 3.2(iii) and TAA we see that

(3) $\alpha, \beta \notin \varphi_n\langle T(b)\rangle$.
Suppose on the contrary that $P_a(\alpha,\beta,\varphi_n) \neq P_b(\alpha,\beta,\varphi_n)$. Consider $\sigma_n = \varphi_n/P_b(\alpha,\beta,\varphi_n)$. Using (2), (3) and (6.6), it is routine to check that σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring, and $T(b)$ is an ETT satisfying MP with respect to σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of $T(b)$ under σ_n, with the same Γ-sets as T under φ_n (see Definition 5.2). As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on (T, φ_n).

Case 3. $a \in V(T^*_{n,q})$ and $b \notin V(T^*_{n,q})$.

By the hypotheses of the present case and the present lemma, (6.6) and TAA, we obtain

(4) $\alpha \notin \varphi_n\langle T(b) - T^*_{n,q}\rangle$ and $\beta \notin \overline{\varphi}_n(T(b) - b)$. So β is not used by any edge in $T(b) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$).

Let us first assume that α is closed in $T^*_{n,q}$ with respect to φ_n. By Corollary 6.3 if $q \geq 1$ and by Lemma 6.1(iii) or Theorem 3.10(ii) (see (5.1)) if $q = 0$, colors α and β are $T^*_{n,q}$-interchangeable under φ_n. So $P_a(\alpha,\beta,\varphi_n)$ is the only (α,β)-path intersecting $T^*_{n,q}$. Suppose on the contrary that $P_a(\alpha,\beta,\varphi_n) \neq P_b(\alpha,\beta,\varphi_n)$. Then $P_b(\alpha,\beta,\varphi_n)$ is vertex-disjoint from $T^*_{n,q}$ and hence contains no edge incident to $T^*_{n,q}$.

Consider $\sigma_n = \varphi_n/P_b(\alpha,\beta,\varphi_n)$. It is routine to check that σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring, and $T(b)$ is an ETT satisfying MP with respect to σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of $T(b)$ under σ_n, with the same Γ-sets as T under φ_n, by (4). As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on (T, φ_n).

So we assume hereafter that

(5) α is not closed in $T^*_{n,q}$ with respect to φ_n.

Our objective is to show that $P_a(\alpha,\beta,\varphi_n) = P_b(\alpha,\beta,\varphi_n)$. Assume the contrary: $P_a(\alpha,\beta,\varphi_n) \neq P_b(\alpha,\beta,\varphi_n)$. We distinguish between two subcases according to the value of q.

Subcase 3.1. $q = 0$.

By the hypothesis of the present lemma, $\alpha \in \overline{\varphi}_n(T_n)$ or $\{\alpha,\beta\} \cap D_n = \emptyset$. So $\alpha \notin D_n$. From (5) and Algorithm 3.1 we deduce that $T^*_{n,0} \neq T_n$. Hence

(6) $\Theta_n = PE$, which together with (5) and (5.4) yields $a \notin V(T_n) \cap V(R_n)$.

Consider $\sigma_n = \varphi_n/P_b(\alpha,\beta,\varphi_n)$. We claim that

(7) σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring.

To justify this, note that if $a \in V(T_n) - V(R_n)$, then $\alpha, \beta \notin \overline{\varphi}_n(R_n)$ by (6.6) and the hypothesis of the present case. By definition, σ_n is $(R_n, \emptyset, \varphi_n)$-stable. In view of Lemma 6.1(ii), $P_b(\alpha,\beta,\varphi_n)$ is disjoint from T_n and hence contains no edge incident to T_n. So σ_n is (T_n, D_n, φ_n)-stable. Hence (7) holds. Suppose $a \in V(R_n) - V(T_n)$. By the hypothesis of the present lemma, $\{\alpha,\beta\} \cap D_n = \emptyset$. By (6.6), we also have $\alpha, \beta \notin \overline{\varphi}_n(T_n)$. Thus $\alpha, \beta \notin \overline{\varphi}_n(T_n) \cup D_n$. By definition, σ_n is (T_n, D_n, φ_n)-stable. Using Lemma 6.1(i), $P_b(\alpha,\beta,\varphi_n)$ is disjoint from R_n and hence contains no edge incident to R_n. By definition, σ_n is $(R_n, \emptyset, \varphi_n)$-stable. Therefore (7) is true.

From (4), (7) and (6.6) we see that $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ (see the remark above lemma 6.2) and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$. Furthermore, $T(b)$ is an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T(b)$ is a good hierarchy of $T(b)$ under σ_n, with the same Γ-sets as T under φ_n. As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the pair $(T(b), \sigma_n)$ is a counterexample to Theorem 5.3, which contradicts the minimality assumption (6.5) on (T, φ_n).

Subcase 3.2. $q \geq 1$.

Let us first assume that α is closed in $T^*_{n,i}$ with respect to φ_n for some i with $0 \leq i \leq q$. Let r be the largest subscript i with this property. Then $r \leq q-1$ by (5). By Lemma 6.5, we have $\alpha \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$, which together with (4) yields

(8) $\alpha \notin \varphi_n\langle T(b) - T^*_{n,r}\rangle$.

By Corollary 6.3 if $r \geq 1$ and by Theorem 3.10(ii) or Lemma 6.1(iii) if $r = 0$, colors α and β are $T^*_{n,r}$-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting $T^*_{n,r}$. Hence $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T^*_{n,r}$ and therefore contains no edge incident to $T^*_{n,r}$. Consider $\sigma_n = \varphi_n / P_b(\alpha, \beta, \varphi_n)$. By Lemma 5.8, σ_n is a $(T^*_{n,r}, D_n, \varphi_n)$-strongly stable coloring, and $T^*_{n,r}$ is an ETT having a good hierarchy and satisfying MP with respect to σ_n. By (4) and TAA, β is not used by any edge in $T(b) - T^*_{n,r}$, except possibly e_1 when $r = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$). Since σ_n is (T_n, D_n, φ_n)-stable, it follows from (8) and (6.6) that $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$. So $T(b)$ is an ETT satisfying MP with respect to σ_n. Moreover,

(9) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of $T(b)$ under σ_n, with the same Γ-sets as T under φ_n.

To justify this, it suffices to verify that Definition 5.2(v) is satisfied with respect to σ_n; that is, $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h)^-$-closed with respect to σ_n for $1 \leq j \leq q$. As the statement holds trivially if $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from $T_{n,j}$, we may assume that $P_b(\alpha, \beta, \varphi_n)$ intersects $T_{n,j}$. Thus $r+1 \leq j \leq q$. Observe that $\alpha \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h$, for otherwise, α is closed in $T_{n,j}$ with respect to φ_n by Definition 5.2(v), contradicting the definition of r. By (6.6), we also obtain $\beta \notin \overline{\varphi}_n(T_j)$. Consequently, $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h)^-$-closed with respect to σ_n. (Note that α may become closed in $T_{n,j}$ with respect to σ_n. Yet, even in this situation the desired statement is true.) This proves (9).

As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the existence of $(T(b), \sigma_n)$ contradicts the minimality assumption (6.5) on (T, φ_n).

Next we assume that α is not closed in $T^*_{n,i}$ with respect to φ_n for any i with $0 \leq i \leq q$. In view of Lemma 6.5, we obtain

(10) $\alpha \in (\cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h) \subseteq \Gamma^{j-1}$ for $1 \leq j \leq q$, $\Theta_n = PE$, $a \in V(T_n) - V(R_n)$, and $\alpha \notin \varphi_n\langle T_{n,q} - T_n\rangle$.

It follows from (4), (10) and TAA that

(11) $\alpha \notin \varphi_n\langle T(b) - T_n\rangle$ and $\beta \notin \varphi_n\langle T(b) - T^*_{n,0}\rangle$.

Since R_n is a closure of $T_n(v_n)$ under φ_n, using (10), (6.6) and TAA we obtain

(12) $\alpha, \beta \notin \overline{\varphi}_n(R_n)$ and $\beta \notin \varphi_n\langle R_n - T_n\rangle$.

By Lemma 6.1(ii), colors α and β are T_n-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting T_n. Hence $P_b(\alpha, \beta, \varphi_n)$ is vertex-disjoint from T_n and therefore contains no edge incident to T_n. Consider $\sigma_n = \varphi_n / P_b(\alpha, \beta, \varphi_n)$. By Lemma 5.8, σ_n is a (T_n, D_n, φ_n)-stable coloring, and T_n is an ETT satisfying MP with respect to σ_n. From (11) and (12) we further deduce that σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(b))$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(b) - b)$. So $T(b)$ is an ETT satisfying MP with respect to σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(b)$ is a good hierarchy of $T(b)$ under σ_n, with the same Γ-sets as T under φ_n (see (10) and the proof of (9) for omitted details). As $\alpha \in \overline{\sigma}_n(a) \cap \overline{\sigma}_n(b)$, the existence of $(T(b), \sigma_n)$ contradicts the minimality assumption (6.5) on (T, φ_n). ∎

Lemma 6.7. *Let α and β be two colors in $\overline{\varphi}_n(T(y_{p-1}))$, let Q be an (α,β)-chain with respect to φ_n, and let $\sigma_n = \varphi_n/Q$. Suppose one of the following cases occurs:*

1) $q \ge 1$, $\alpha \in \overline{\varphi}_n(T_{n,q})$, and Q is an (α,β)-path disjoint from $P_{v_\alpha}(\alpha,\beta,\varphi_n)$;

2) $q = 0$, $\alpha \in \overline{\varphi}_n(T_n)$, or $\alpha \in \overline{\varphi}_n(T^*_{n,0})$ with $\alpha,\beta \notin D_n$, and Q is an (α,β)-path disjoint from $P_{v_\alpha}(\alpha,\beta,\varphi_n)$; and

3) $T^*_{n,q} \prec v_\alpha \prec v_\beta$, $\alpha,\beta \notin D_{n,q}$, $\alpha \notin \varphi_n\langle T(v_\beta) - T(v_\alpha)\rangle$, and Q is an arbitrary (α,β)-chain.

Then the following statements hold:

(i) σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring;

(ii) $T^*_{n,q}$ is an ETT satisfying MP with respect to σ_n; and

(iii) if $q \ge 1$, then $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ is a good hierarchy of $T_{n,q}$ under σ_n, with the same Γ-sets (see Definition 5.2) as T under φ_n, and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma^{q-1}_h)^-$-closed with respect to σ_n.

Furthermore, in Case 3, T is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets (see Definition 5.2) as T under φ_n.

Remark. To prove Theorem 5.3, we shall perform a series of Kempe changes as described in Lemma 6.7 starting from φ_n and T. Let σ' be a resulting coloring and let T' be a resulting ETT. By the above statement (iii), to show that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under σ', with the same Γ-sets as T under φ_n, it suffices to verify that Definition 5.2(i) is satisfied, which is fairly straightforward in our proof, as we shall see.

Proof of Lemma 6.7. Write $a = v_\alpha$ and $b = v_\beta$. Let us consider the three cases described in the lemma separately.

Case 1. $q \ge 1$, $\alpha \in \overline{\varphi}_n(T_{n,q})$, and Q is an (α,β)-path disjoint from $P_{v_\alpha}(\alpha,\beta,\varphi_n)$.

We distinguish between two subcases according to the location of b.

Subcase 1.1. $b \in V(T_{n,q})$.

Let us first assume that there exists a subscript i with $0 \le i \le q$, such that α or β is closed in $T^*_{n,i}$ with respect to φ_n. Let r be the largest such i. By (5.10) and Lemma 6.5, we have

(1) $\{a,b\} \subseteq V(T^*_{n,r})$ and $\alpha,\beta \notin \varphi_n\langle T_{n,q} - T^*_{n,r}\rangle$.

(2) α and β are $T^*_{n,r}$-interchangeable under φ_n. So $P_\alpha(\alpha,\beta,\varphi_n) = P_\beta(\alpha,\beta,\varphi_n)$.

To justify this, note that if $r \ge 1$, then (2) holds by Corollary 6.3. So we assume that $r = 0$. Then α or β is closed in $T^*_{n,0}$ with respect to φ_n. Hence, by Lemma 6.1(iii) if $\Theta_n = PE$ and by (5.1) and Theorem 3.10(ii) otherwise, α and β are $T^*_{n,0}$-interchangeable under φ_n. This proves (2).

It follows from (2) that Q is vertex-disjoint from $T^*_{n,r}$ and hence contains no edge incident to $T^*_{n,r}$. By Lemma 5.8, $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,r}, D_n, \varphi_n)$-strongly stable coloring, and $T^*_{n,r}$ is an ETT satisfying MP with respect to σ_n. By (1) and (6.6), we obtain $\sigma_n(f) = \varphi_n(f)$ for each edge f of $T_{n,q}$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each vertex u of $T_{n,q}$. Therefore σ_n is a $(T_{n,q}, D_n, \varphi_n)$-strongly stable coloring. By the definition of r, for any $r+1 \le j \le q$ and $\theta \in \{\alpha,\beta\}$, we have $\partial_{\varphi_n,\theta}(T_{n,j}) \ne \emptyset$, so $\theta \in \cup_{\eta_h \in D_{n,j}} \Gamma^{j-1}_h$ by Definition 5.2(v). It is then routine to check that

$T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ is a good hierarchy of $T_{n,q}$ under σ_n, with the same Γ-sets as T under ${\varphi_n}$[1], and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$-closed with respect to σ_n.

Next we assume that there exists no subscript i with $0 \le i \le q$, such that α or β is closed in $T^*_{n,i}$ with respect to φ_n. By Lemma 6.5, we have

(3) $\alpha, \beta \in (\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1}) \subseteq \Gamma^{j-1}$ for $1 \le j \le q$, $\Theta_n = PE$, $v_\alpha, v_\beta \in V(T_n) - V(R_n)$, and $\alpha, \beta \notin \varphi_n \langle T_{n,q} - T_n \rangle$.

Since R_n is a closure of $T_n(v_n)$ under φ_n, using (6.6) and TAA we obtain

(4) $\alpha, \beta \notin \overline{\varphi}_n(R_n)$.

By Lemma 6.1(ii), colors α and β are T_n-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting T_n. Hence Q is vertex-disjoint from T_n and therefore contains no edge incident to T_n. By Lemma 5.8, $\sigma_n = \varphi_n / Q$ is a (T_n, D_n, φ_n)-stable coloring, and T_n is an ETT satisfying MP with respect to σ_n. By (3), (4) and (6.6), we further deduce that σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-stable coloring, $\sigma_n(f) = \varphi_n(f)$ for each edge f of $T_{n,q}$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each vertex u of $T_{n,q}$. It is then routine to check that the desired statements hold.

Subcase 1.2. $b \notin V(T_{n,q})$.

Let us first assume that there exists a subscript i with $0 \le i \le q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. Let r be the largest such i. By (5.10), Lemma 6.5 and TAA, we have

(5) $a \subseteq V(T^*_{n,r})$ and $\alpha \notin \varphi_n \langle T_{n,q} - T^*_{n,r} \rangle$. Furthermore, no edge in $T_{n,q} - T^*_{n,r}$ is colored by β, except possibly e_1 when $r = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\beta \in D_n$).

Using the same argument as that of (2), we obtain

(6) α and β are $T^*_{n,r}$-interchangeable under φ_n.

It follows from (6) that Q is vertex-disjoint from $T^*_{n,r}$ and hence contains no edge incident to $T^*_{n,r}$. By Lemma 5.8, $\sigma_n = \varphi_n / Q$ is a $(T^*_{n,r}, D_n, \varphi_n)$-strongly stable coloring, and $T^*_{n,r}$ is an ETT satisfying MP with respect to σ_n. Using (5), we obtain $\sigma_n(f) = \varphi_n(f)$ for each edge f of $T_{n,q}$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each vertex u of $T_{n,q}$. Therefore σ_n is a $(T_{n,q}, D_n, \varphi_n)$-strongly stable coloring, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ is a good hierarchy of $T_{n,q}$ under σ_n, with the same Γ-sets as T under φ_n, and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$-closed with respect to σ_n (see the justification of (9) in the proof of Lemma 6.6 for omitted details).

Next we assume that there exists no subscript i with $0 \le i \le q$, such that α is closed in $T^*_{n,i}$ with respect to φ_n. By Lemma 6.5, we have

(7) $\alpha \in (\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1}) \subseteq \Gamma^{j-1}$ for $1 \le j \le q$, $\Theta_n = PE$, $v_\alpha \in V(T_n) - V(R_n)$, and $\alpha \notin \varphi_n \langle T_{n,q} - T_n \rangle$.

It follows that (4) also holds. By Lemma 6.1(ii), colors α and β are T_n-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path with respect to φ_n intersecting T_n. Hence Q is vertex-disjoint from T_n and therefore contains no edge incident to T_n. By Lemma 5.8, $\sigma_n = \varphi_n / Q$ is a (T_n, D_n, φ_n)-stable coloring, and T_n is an ETT satisfying MP with respect to σ_n. Since $b \notin V(T_{n,q})$, no edge in $T_{n,q} - T^*_{n,0}$ is colored by β by TAA, because $T^*_{n,0} = T_n \vee R_n$ by (7). Using (4) and (7), it is routine to check that the desired statements hold.

[1]See the justification of (9) in the proof of Lemma 6.6 for omitted details. Note that α or β may become closed in $T_{n,j}$ with respect to σ_n for some j with $r + 1 \le j \le q$. Yet, even in this situation Definition 5.2(v) remains valid with respect to σ_n.

Case 2. $q = 0$, $\alpha \in \overline{\varphi}_n(T_n)$, or $\alpha \in \overline{\varphi}_n(T^*_{n,0})$ with $\alpha, \beta \notin D_n$, and Q is an (α, β)-path disjoint from $P_{v_\alpha}(\alpha, \beta, \varphi_n)$.

Let us first assume that α or β is closed in $T^*_{n,0}$ with respect to φ_n. By Lemma 6.1(iii) or Theorem 3.10(ii) (see (5.1)), colors α and β are $T^*_{n,0}$-interchangeable under φ_n. So $P_a(\alpha, \beta, \varphi_n)$ is the only (α, β)-path intersecting $T^*_{n,0}$, and hence Q is vertex-disjoint from $T^*_{n,0}$. It is then routine to check that $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, and $T^*_{n,0}$ is an ETT satisfying MP with respect to σ_n by Theorem 3.10(vi). So we assume hereafter that

(8) neither α nor β is closed in $T^*_{n,0}$ with respect to φ_n.

By the hypothesis of the present case, $\alpha \in \overline{\varphi}_n(T_n)$ or $\{\alpha, \beta\} \cap D_n = \emptyset$. So $\alpha \notin D_n$. From (8) and Algorithm 3.1 we deduce that $T^*_{n,0} \neq T_n$. Hence

(9) $\Theta_n = PE$, which together with (5.4) yields $a, b \notin V(T_n) \cap V(R_n)$.

Let us show that

(10) $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring.

To justify this, note that if one of a and b is contained in $V(T_n) - V(R_n)$ and the other is contained in $V(R_n) - V(T_n)$, then α and β are $T^*_{n,0}$-interchangeable under φ_n by Lemma 6.1(iv). So Q is vertex-disjoint from $T^*_{n,0}$ and hence (10) holds. In view of (9), we may assume that

(11) if $a, b \in V(T^*_{n,0})$, then either $a, b \in V(T_n) - V(R_n)$ or $a, b \in V(R_n) - V(T_n)$.

Let us first assume that $a \in V(T_n) - V(R_n)$. Then $\alpha \notin \overline{\varphi}_n(R_n)$ by (6.6) and $b \in V(T_n) - V(R_n)$ if $b \in V(T^*_{n,0})$ by (11). So α and β are T_n-interchangeable under φ_n by Lemma 6.1(ii) and $\beta \notin \overline{\varphi}_n(R_n)$ by (6.6). It follows that Q is vertex-disjoint from T_n and that $\sigma_n(f) = \varphi_n(f)$ for any edge f incident to R_n with $\varphi_n(f) \in \overline{\varphi}_n(R_n)$. Hence (10) holds.

Next we assume that $a \in V(R_n) - V(T_n)$. Then $\alpha \notin \overline{\varphi}_n(T_n)$ by (6.6) and $b \in V(R_n) - V(T_n)$ if $b \in V(T^*_{n,0})$ by (11). So α and β are R_n-interchangeable under φ_n by Lemma 6.1(i) and $\beta \notin \overline{\varphi}_n(T_n)$ by (6.6). It follows that Q is vertex-disjoint from R_n. By the hypothesis of the present case, $\{\alpha, \beta\} \cap D_n = \emptyset$. So $\alpha, \beta \notin \overline{\varphi}_n(T_n) \cup D_n$ and hence (10) holds.

From (10) we deduce that $T^*_{n,0}$ is an ETT satisfying MP with respect to σ_n.

Case 3. $T^*_{n,q} \prec v_\alpha \prec v_\beta$, $\alpha, \beta \notin D_{n,q}$, $\alpha \notin \varphi_n\langle T(v_\beta) - T(v_\alpha)\rangle$, and Q is an arbitrary (α, β)-chain.

By (6.6), $V(T(y_{p-1}))$ is elementary with respect to φ_n. So $\alpha, \beta \notin \overline{\varphi}_n(T^*_{n,q})$. By hypothesis, $\alpha, \beta \notin D_{n,q}$. Hence

(12) $\alpha, \beta \notin \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$.

By the definitions of D_n and $D_{n,q}$, we have $D_n \cup \overline{\varphi}_n(T_n) \subseteq D_{n,q} \cup \overline{\varphi}_n(T^*_{n,q})$. So $\alpha, \beta \notin \overline{\varphi}_n(T_n) \cup D_n$. From Lemma 3.2(iii), TAA and the hypothesis of the present case, we further deduce that

(13) $\alpha, \beta \notin \varphi_n\langle T(b)\rangle$.

In view of Lemma 6.6, we obtain

(14) $P_a(\alpha, \beta, \varphi) = P_b(\alpha, \beta, \varphi)$. (Possibly Q is this path.)

Since $T^*_{n,q} \prec a \prec b$, using (12)-(14), it is straightforward to verify that $\sigma_n = \varphi_n/Q$ is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring.

From (12) and (13) we also see that $T(b)$ can be obtained from $T^*_{n,q}$ by using TAA, no matter whether $Q = P_a(\alpha, \beta, \varphi)$. Thus T is an ETT corresponding to (σ_n, T_n). It is clear that T also satisfies MP under σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as T under φ_n. ∎

7 Elementariness and Interchangeability

In Section 5 we have developed a control mechanism over Kempe changes; that is, a good hierarchies of an ETT. In Section 6 we have derived some properties satisfied by such hierarchies. Now we are ready to present a proof of Theorem 5.3 by using a novel recoloring technique based on these hierarchies.

7.1 Proof of Theorem 5.3

By hypothesis, T is an ETT constructed from a k-triple (G, e, φ) by using the Tashkinov series $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \le i \le n+1\}$. Furthermore, T admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ and satisfies MP with respect to φ_n. Our objective is to show that $V(T)$ is elementary with respect to φ_n.

As introduced in the preceding section, $T = T^*_{n,q} \cup \{e_1, y_1, e_2, ..., e_p, y_p\}$, where y_i is the end of e_i outside $T(y_{i-1})$ for $i \ge 1$, with $T(y_0) = T^*_{n,q}$. Suppose on the contrary that $V(T)$ is not elementary with respect to φ_n. Then

(7.1) $\overline{\varphi}_n(T(y_{p-1})) \cap \overline{\varphi}_n(y_p) \neq \emptyset$ by (6.6).

For ease of reference, recall that (see (3) in the proof of Theorem 5.4)

(7.2) $|\overline{\varphi}_n(T_n)| \ge 2n + 11$ and $|D_{n,j}| \le |D_n| \le n$ for $0 \le j \le q$.

In our proof we shall frequently make use of a coloring $\sigma_n \in \mathcal{C}^k(G-e)$ with properties (i)-(iii) as described in Lemma 6.7; that is,

(7.3) σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring, and $T^*_{n,q}$ is an ETT satisfying MP with respect to σ_n. Furthermore, if $q \ge 1$, then $T_{n,q}$ admits a good hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q}$ under σ_n, with the same Γ-sets (see Definition 5.2) as T under φ_n, and $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma^{q-1}_h)^-$-closed with respect to σ_n (see the remark succeeding Lemma 6.7).

Claim 7.1. $p \ge 2$.

Assume the contrary: $p = 1$; that is, $T = T^*_{n,q} \cup \{e_1, y_1\}$. Then

(1) there exists a color α in $\overline{\varphi}_n(T^*_{n,q}) \cap \overline{\varphi}_n(y_1)$ by (7.1).

We consider two cases according to the value of q.

Case 1. $q = 0$. In this case, from (1) and Algorithm 3.1 we see that $\Theta_n \neq SE$. Let us first assume that $\Theta_n = RE$. Let δ_n, γ_n be as specified in Step 2 of Algorithm 3.1. Since $\alpha, \delta_n \in \overline{\varphi}_n(T_n)$, both of them are closed in T_n with respect to φ_n. Hence $P_{y_1}(\alpha, \delta_n, \varphi_n)$ is vertex-disjoint from T_n. Let $\sigma_n = \varphi_n / P_{y_1}(\alpha, \delta_n, \varphi_n)$. Then $\delta_n \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(y_1)$. By Lemma 5.8, σ_n is a (T_n, D_n, φ_n)-stable coloring. It follows from Theorem 3.10(vi) that σ_n is a $\varphi_n \bmod T_n$ coloring. From Definition 3.7 and Step 1 of Algorithm 3.1, we see that $f_n = e_1$ is still an RE connecting edge under σ_n and is contained in a (δ_n, γ_n)-cycle under σ_n, which is impossible because $\delta_n \in \overline{\sigma}_n(y_1)$.

So we may assume that $\Theta_n = PE$. Let $\beta = \varphi_n(e_1)$. From TAA we see that $\beta \in \overline{\varphi}_n(T^*_{n,0})$. Let $\theta \in \overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n)$. Then θ is closed in $T^*_{n,0}$ under φ_n by (5.4). In view of Lemma 6.1(iii), $P_{v_\theta}(\alpha, \theta, \varphi_n)$ is the only (α, θ)-path intersecting $T^*_{n,0}$. Thus $P_{y_1}(\alpha, \theta, \varphi_n) \cap T^*_{n,0} = \emptyset$. Let $\sigma_n = \varphi_n / P_{y_1}(\alpha, \theta, \varphi_n)$. By Lemma 6.7 (the second case), σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, so θ is also closed in $T^*_{n,0}$ with respect to σ_n. In view of Lemma 6.1(iii), β and θ are $T^*_{n,0}$-interchangeable under σ_n. As $P_{y_1}(\theta, \beta, \sigma_n) \cap T^*_{n,0} \neq \emptyset$, there are at least two (θ, β)-paths with respect to σ_n intersecting $T^*_{n,0}$, a contradiction.

Case 2. $q \geq 1$. In this case, by Definition 5.2(v), we have

(2) $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$-closed with respect to φ_n

So e_1 is colored by some color γ_1 in $\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1}$. By Definition 5.2(i) and (5.9), we have $\gamma_1 \notin \Gamma^q$. Let $\theta \in \overline{\varphi}_n(T_{n,q}) - \overline{\varphi}_n(T^*_{n,q-1})$. Then $\theta \notin \Gamma^{q-1}$ (so $\theta \neq \gamma_1$) by Definition 5.2(i). Furthermore, θ is closed in $T_{n,q}$ under φ_n by (2). In view of Corollary 6.3, α and θ are $T_{n,q}$-interchangeable under φ_n. So $P_{v_\theta}(\alpha, \theta, \varphi_n) = P_{v_\alpha}(\alpha, \theta, \varphi_n)$ is the unique (α, θ)-path intersecting $T_{n,q}$. Hence $P_{y_1}(\alpha, \theta, \varphi_n) \cap T_{n,q} = \emptyset$. Let $\sigma_n = \varphi_n / P_{y_1}(\alpha, \theta, \varphi_n)$. Then σ_n satisfies all the properties described in (7.3) by Lemma 6.7. Since e_1 is still colored by $\gamma_1 \in \Gamma^{q-1}$ under σ_n and $\gamma_1 \notin \Gamma^q$, we can obtain T from $T_{n,q}$ by TAA under σ_n, so T is an ETT satisfying MP under σ_n. Moreover, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). As $P_{y_1}(\theta, \gamma_1, \sigma_n) \cap T_{n,q} \neq \emptyset$, there are at least two (θ, γ_1)-paths with respect to σ_n intersecting $T_{n,q}$, contradicting Lemma 6.6(iii) (with σ_n in place of φ_n), because $\theta, \gamma_1 \in \overline{\sigma}_n(T_{n,q})$ and θ is also closed in $T_{n,q}$ under σ_n by (2). Hence Claim 7.1 is justified.

Recall that the path number $p(T)$ of T is the smallest subscript $i \in \{1, 2, ..., p\}$, such that the sequence $(y_i, e_{i+1}, ..., e_p, y_p)$ corresponds to a path in G, where $p \geq 2$ by Claim 7.1. Depending on the value of $p(T)$, we distinguish among three situations, labeled as Situation 7.1, Situation 7.2, and Situation 7.3.

Situation 7.1. $p(T) = 1$. Now $T - V(T^*_{n,q})$ is a path obtained by using TAA under φ_n.

Claim 7.2. *We may assume that $\overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p) \neq \emptyset$ for some i with $1 \leq i \leq p-1$.*

To justify this, let $\alpha \in \overline{\varphi}_n(T(y_{p-1})) \cap \overline{\varphi}_n(y_p)$ (see (7.1)). If $\alpha \in \overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p)$ for some i with $1 \leq i \leq p-1$, we are done. So we assume that

(1) $\alpha \in \overline{\varphi}_n(T^*_{n,q}) \cap \overline{\varphi}_n(y_p)$ and $\alpha \notin \overline{\varphi}_n(y_i)$ for all $1 \leq i \leq p-1$.

(2) If $\Theta_n = PE$ and $q = 0$, then we may further assume that $\alpha \in \overline{\varphi}_n(T_n)$.

By (1), we have $\alpha \in \overline{\varphi}_n(T^*_{n,0})$. Suppose $\alpha \in \overline{\varphi}_n(R_n - V(T_n))$. Then $\alpha \notin \Gamma^0$ by Definition 5.2(i). In view of (7.2), we have $|\overline{\varphi}_n(T_n)| \geq 11 + 2n$ and $|\Gamma^0| \leq 2|D_{n,0}| \leq 2n$. So there exists $\beta \in \overline{\varphi}_n(T_n) - \Gamma^0$. By Lemma 6.1(iv), α and β are $T^*_{n,0}$-interchangeable under φ_n. Thus $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$ and $P_{y_p}(\alpha, \beta, \varphi_n)$ is disjoint from $T^*_{n,0}$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.7 (the second case), σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, and $T^*_{n,0}$ is an ETT satisfying MP with respect to σ_n. Note that T can also be obtained from $T^*_{n,0}$ by TAA under σ_n, because $\alpha, \beta \in \overline{\sigma}_n(T^*_{n,0})$. Hence T satisfies MP under σ_n as well. Since $\alpha, \beta \notin \Gamma^0$ and $\alpha, \beta \notin \overline{\varphi}_n(T(y_{p-1}) - V(T^*_{n,0}))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). As $\beta \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(y_p)$, replacing φ_n by σ_n and α by β if necessary, we see that (2) holds.

Depending on whether α is used by edges in $T - T^*_{n,q}$, we consider two cases.

Case 1. $\alpha \notin \varphi_n \langle T - T^*_{n,q} \rangle$. In this case, let $\beta \in \overline{\varphi}_n(y_{p-1})$. Then β is not used by any edge in $T - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_n$). By (1) and (2), we have $\alpha \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \overline{\varphi}_n(T_n)$ if $q = 0$. It follows from Lemma 6.6 that $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{y_{p-1}}(\alpha, \beta, \varphi_n)$. So $P_{y_p}(\alpha, \beta, \varphi_n)$ is disjoint from $P_{v_\alpha}(\alpha,\beta,\varphi_n)$. Let $\sigma_n = \varphi_n/P_{y_p}(\alpha,\beta,\varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \beta \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha,\beta,\varphi_n)$. So $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. Thus T can be obtained from $T^*_{n,q}+e_1$ by TAA and satisfies MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T,σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). As $\beta \in \overline{\sigma}_n(y_{p-1}) \cap \overline{\sigma}_n(y_p)$, replacing φ_n by σ_n if necessary, we see that Claim 7.2 is true.

Case 2. $\alpha \in \varphi_n\langle T - T^*_{n,q}\rangle$. In this case, let e_j be the edge with the smallest subscript in $T - T^*_{n,q}$ such that $\varphi(e_j) = \alpha$. We distinguish between two subcases according to the value of j.

Subcase 2.1. $j \ge 2$. In this subcase, let $\beta \in \overline{\varphi}_n(y_{j-1})$. Then β is not used by any edge in $T(y_j) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \beta \in D_n$). By (1) and (2), we have $\alpha \in \overline{\varphi}_n(T_{n,q})$ if $q \ge 1$ and $\alpha \in \overline{\varphi}_n(T_n)$ if $q = 0$. It follows from Lemma 6.6 that $P_{v_\alpha}(\alpha,\beta,\varphi_n) = P_{y_{j-1}}(\alpha,\beta,\varphi_n)$. So $P_{y_p}(\alpha,\beta,\varphi)$ is disjoint from $P_{v_\alpha}(\alpha,\beta,\varphi_n)$. Let $\sigma_n = \varphi_n/P_{y_p}(\alpha,\beta,\varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \beta \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha,\beta,\varphi_n)$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ_n and hence satisfies MP under σ_n.

Note that $\beta \notin \Gamma^q$ by Definition 5.2(i) and that $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$ by (6.6). If $\alpha \notin \Gamma^q$, then clearly $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. If $\alpha \in \Gamma^q$, say $\alpha \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$, then Definition 5.2(i) implies that $\eta_h \in \overline{\varphi}_n(w)$ for some $w \preceq y_{j-1}$. Since only edges outside $T(w)$ may change colors between α and β as we transform φ_n into σ_n, it follows that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T,σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\beta \in \overline{\sigma}_n(y_{j-1}) \cap \overline{\sigma}_n(y_p)$, replacing φ_n by σ_n if necessary, we see that Claim 7.2 holds.

Subcase 2.2. $j = 1$. In this subcase, $\alpha = \varphi(e_1)$. Note that $\alpha \notin \Gamma^q$ by Definition 5.2(i) and (5.9). We propose to show that

(3) there exists a color γ in $\overline{\varphi}_n(T_{n,q}) - \Gamma^q$ if $q \ge 1$ and in $\overline{\varphi}_n(T_n) - \Gamma^0$ if $q = 0$, such that γ is closed in $T^*_{n,q}$ with respect to φ_n.

Let us first assume that $q \ge 1$. By (7.2), we obtain $|\overline{\varphi}_n(T_{n,q})| \ge |\overline{\varphi}_n(T_n)| \ge 2n + 11$ and $|\Gamma^{q-1}| \le 2|D_{n,q-1}| \le 2n$. So $|\overline{\varphi}_n(T_{n,q}) - \Gamma^{q-1}| \ge 11$. By Definition 5.2(iii), we have $|\Gamma^q - \Gamma^{q-1}| = 2$. So $|\overline{\varphi}_n(T_{n,q}) - (\Gamma^{q-1} \cup \Gamma^q)| \ge 9$. Let γ be a color in $\overline{\varphi}_n(T_{n,q}) - (\Gamma^{q-1} \cup \Gamma^q)$. By Definition 5.2(v), γ is closed in $T_{n,q}$ with respect to φ_n.

Next we assume that $q = 0$. Again, by (7.2), we have $|\overline{\varphi}_n(T_n)| \ge 2n+11$ and $|\Gamma^0| \le 2|D_{n,0}| \le 2|D_n| \le 2n$. Let γ be a color in $\overline{\varphi}_n(T_n) - \Gamma^0$ if $\Theta_n \ne PE$ and a color in $\overline{\varphi}_n(T_n) \cap \overline{\varphi}_n(R_n) - \Gamma^0$ if $\Theta_n = PE$ (see Definition 5.2(iv)). By Algorithm 3.1 and (5.4), γ is closed in $T^*_{n,0}$ with respect to φ_n. So (3) holds.

By (3) and Lemma 6.6, $P_{v_\alpha}(\alpha,\gamma,\varphi_n) = P_{v_\gamma}(\alpha,\gamma,\varphi_n)$ is the only (α,γ)-path intersecting $T^*_{n,q}$. So $P_{y_p}(\alpha,\gamma,\varphi_n)$ is disjoint from $T^*_{n,q}$ and hence it does not contain e_1. Let $\sigma_n = \varphi_n/P_{y_p}(\alpha,\gamma,\varphi_n)$. Then σ_n satisfies all the properties described in (7.3) by Lemma 6.7. Moreover, $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for all $u \in V(T(y_{p-1}))$. Since $\alpha,\gamma \in \overline{\varphi}_n(T^*_{n,q})$, we have $\alpha,\gamma \in \overline{\sigma}_n(T^*_{n,q})$. Hence we can obtain T from $T^*_{n,q}+e_1$ by using TAA under σ_n, so T satisfies MP under σ_n. Since $\alpha,\gamma \notin \Gamma^q$, the hierarchy

$T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since e_1 is outside $P_{y_p}(\alpha, \gamma, \varphi_n)$, we have $\sigma_n(e_1) = \alpha$. As $\gamma \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(v)$ for some $v \in V(T_{n,q})$ and $\alpha \neq \gamma$, the present subcase reduces to Case 1 if $\gamma \notin \sigma_n\langle T - T^*_{n,q}\rangle$ or to Subcase 2.1 if $\gamma \in \sigma_n\langle T - T^*_{n,q}\rangle$. This proves Claim 7.2.

Claim 7.3. *We may assume that* $\overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p) \neq \emptyset$.

To justify this, let $\mathcal{K}$ be the set of all minimum counterexamples (T, φ_n) to Theorem 5.3 (see (6.2)-(6.5)), and let i be the largest subscript with $1 \leq i \leq p-1$, such that there exists a member (T, μ_n) of $\mathcal{K}$ with $\overline{\mu}_n(y_i) \cap \overline{\mu}_n(y_p) \neq \emptyset$; this i exists by Claim 7.2. We aim to show that $i = p-1$. Thus Claim 7.3 follows by replacing φ_n with μ_n, if necessary.

With a slight abuse of notation, we assume that $\overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p) \neq \emptyset$ and assume, on the contrary, that $i \leq p-2$. Let $\alpha \in \overline{\varphi}_n(y_i) \cap \overline{\varphi}_n(y_p)$. Using (6.6) and TAA, we obtain

(1) $\alpha \notin \overline{\varphi}_n(T(y_{i-1}))$, where $T(y_0) = T^*_{n,q}$. So α is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \alpha \in D_n$).

Recall that Definition 5.2 involves $\Gamma^q_h = \{\gamma^q_{h_1}, \gamma^q_{h_2}\}$ for each $\eta_h \in D_{n,q}$. Nevertheless, in our proof we only consider a fixed $\eta_h \in D_{n,q}$. For simplicity, we abbreviate its corresponding $\gamma^q_{h_j}$ to γ_j for $j = 1, 2$. By Definition 5.2(i) and (5.9), we have

(2) $\gamma_j \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\gamma_j \in \overline{\varphi}_n(T_n)$ if $q = 0$. Moreover, if $\eta_h \in \overline{\varphi}_n(y_t)$ for some $t \geq 1$, then $\gamma_j \notin \varphi_n\langle T(y_t) - T^*_{n,q}\rangle$ for $j = 1, 2$.

Depending on whether $\alpha \in D_{n,q}$, we consider two cases.

Case 1. $\alpha \notin D_{n,q}$. In this case, let $\theta \in \overline{\varphi}_n(y_{i+1})$. From TAA and (6.6) it follows that

(3) $\theta \notin \overline{\varphi}_n(T(y_i))$, so θ is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \theta \in D_n$).

If $\theta \notin D_{n,q}$, then $\{\alpha, \theta\} \cap D_{n,q} = \emptyset$. By the definitions of D_n and $D_{n,q}$, we have $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, which together with (1) and (3) implies $\{\alpha, \theta\} \cap D_n = \emptyset$. Hence $P_{y_i}(\alpha, \theta, \varphi_n) = P_{y_{i+1}}(\alpha, \theta, \varphi_n)$ by Lemma 6.6. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \theta, \varphi_n)$. Since both y_i and y_{i+1} are contained in $T - V(T^*_{n,q})$ and (1) holds, by Lemma 6.7 (the third case), σ_n satisfies all the properties described in (7.3). Furthermore, T is also an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_{i+1})$, we reach a contradiction to the maximality assumption on i.

So we may assume that $\theta \in D_{n,q}$. Let $\theta = \eta_h \in D_{n,q}$. In view of (2) and Lemma 6.6, we obtain $P_{v_{\gamma_1}}(\alpha, \gamma_1, \varphi_n) = P_{y_i}(\alpha, \gamma_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \gamma_1, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \gamma_1, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \alpha \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha, \gamma_1, \varphi_n)$. By (6.6), (1) and (2), we have $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma_n(f) = \varphi_n(f)$ for each edge f in $T(y_{i+1})$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ_n, and hence satisfies MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Hence (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), with $\gamma_1 \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T_{n,q})$.

Using (2) and Lemma 6.6, we obtain $P_{v_{\gamma_1}}(\eta_h, \gamma_1, \sigma_n) = P_{y_{i+1}}(\eta_h, \gamma_1, \sigma_n)$, which is disjoint from $P_{y_p}(\eta_h, \gamma_1, \sigma_n)$. Let $\sigma'_n = \sigma_n / P_{y_p}(\eta_h, \gamma_1, \sigma_n)$. By Lemma 6.7, σ'_n satisfies all the properties described in (7.3) (with σ'_n in place of σ_n). In particular, if $e_1 = f_n$ and $\sigma_n(e_1) = \eta_h \in D_n$, then $\sigma'_n(e_1) = \sigma_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_h, \gamma_1, \sigma_n)$. By (6.6), (2) and (3), we have $\overline{\sigma}'_n(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma'_n(f) = \sigma_n(f)$ for each edge f in $T(y_{i+1})$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ'_n, and hence satisfies MP under σ'_n. Furthermore, since $\eta_h \in \overline{\sigma}'_n(y_{i+1})$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ'_n, with the same Γ-sets as those under φ_n. Therefore (T, σ'_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\eta_h \in \overline{\sigma}'_n(y_p) \cap \overline{\sigma}'_n(y_{i+1})$, we reach a contradiction to the maximality assumption on i.

Case 2. $\alpha \in D_{n,q}$. In this case, let $\alpha = \eta_h \in D_{n,q}$. Then $\Gamma^q_h = \{\gamma_1, \gamma_2\}$ (see the paragraph above (2)). Renaming subscript if necessary, we may assume that $\varphi_n(e_{i+1}) \neq \gamma_1$. By (1) and (2), we have

(4) $\gamma_1 \notin \varphi_n\langle T(y_{i+1}) - T^*_{n,q}\rangle$ and η_h is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (4) and Lemma 6.6, we obtain $P_{v_{\gamma_1}}(\eta_h, \gamma_1, \varphi_n) = P_{y_i}(\eta_h, \gamma_1, \varphi_n)$, which is disjoint from the path $P_{y_p}(\eta_h, \gamma_1, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\eta_k, \gamma_1, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_h \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_k, \gamma_1, \varphi_n)$. By (6.6) and (4), we have $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$ and $\sigma_n(f) = \varphi_n(f)$ for each edge f in $T(y_{i+1})$. So T can be obtained from $T^*_{n,q} + e_1$ by TAA under σ_n, and hence satisfies MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), with $\gamma_1 \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T_{n,q})$. Let $\theta \in \overline{\sigma}_n(y_{i+1})$. From TAA we see that

(5) θ is not used by any edge in $T(y_{i+1}) - T^*_{n,q}$ under σ_n, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\sigma_n(e_1) = \theta \in D_n$).

By (6.6), we have $\theta \neq \gamma_1$. Using (4) and Lemma 6.6, we get $P_{v_{\gamma_1}}(\theta, \gamma_1, \sigma_n) = P_{y_{i+1}}(\theta, \gamma_1, \sigma_n)$. Let $\sigma'_n = \sigma_n / P_{y_p}(\theta, \gamma_1, \sigma_n)$. By Lemma 6.7, σ'_n satisfies all the properties described in (7.3) (with σ'_n in place of σ_n). In particular, if $e_1 = f_n$ and $\sigma_n(e_1) = \theta \in D_n$, then $\sigma'_n(e_1) = \sigma_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\theta, \gamma_1, \sigma_n)$. From (6.6) and (4) we deduce that $\overline{\sigma}'_n(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$, and $\sigma'_n(f) = \sigma_n(f)$ for each edge f in $T(y_{i+1})$. So T can also be obtained from $T^*_{n,q} + e_1$ by TAA under σ'_n, and hence satisfies MP under σ'_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of those under σ'_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ'_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \overline{\sigma}'_n(y_p) \cap \overline{\sigma}'_n(y_{i+1})$, we reach a contradiction to the maximality assumption on i. Hence Claim 7.3 is established.

By Claim 7.1, $p \geq 2$. By Claim 7.3, $\overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p) \neq \emptyset$. Let $\alpha \in \overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p)$ and $\beta = \varphi_n(e_p)$. Let σ_n be obtained from φ_n by recoloring e_p with α and let $T' = T(y_{p-1})$. Then $\beta \in \overline{\sigma}_n(y_{p-1}) \cap \overline{\sigma}_n(T'(y_{p-2}))$ and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under σ_n. So (T', σ_n) is a counterexample to Theorem 5.3 (see (6.2)-(6.4)), which violates the minimality assumption (6.5) on (T, φ_n). This completes our discussion about Situation 7.1.

Situation 7.2. $p(T) = p$. Now e_p is not incident to y_{p-1}.

By (7.1), there exists a color $\alpha \in \overline{\varphi}_n(T(y_{p-1})) \cap \overline{\varphi}_n(y_p)$. We divide this situation into 3 cases and further into 6 subcases (see figure below), depending on whether $v_\alpha = y_{p-1}$ and $\alpha \in D_{n,q}$. Our proof of Subcase 1.1 is self-contained. Yet, in our discussion Subcase 1.2 may be redirected to Subcase 1.1 and Subcase 2.1, and Subcase 2.1 may be redirected to Subcase 1.1, etc. Figure 1 illustrates such redirections (note that no cycling occurs).

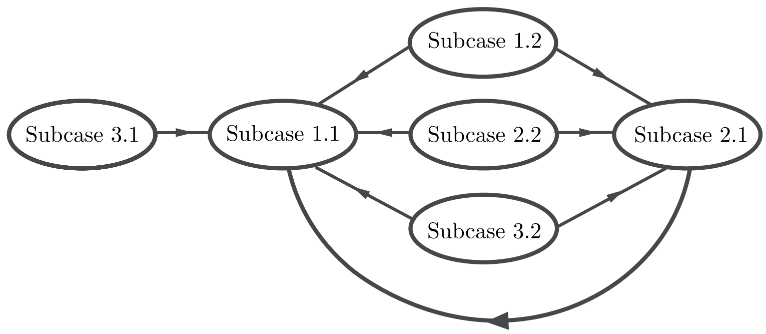

Figure 1. Redirections

Throughout this situation we reserve the symbol θ for $\varphi_n(e_p)$. Clearly, $\theta \neq \alpha$.

Case 1. $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_{p-1})$ and $\alpha \in D_{n,q}$.

Let $\alpha = \eta_m \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma^q_{m_1}$ and $\gamma^q_{m_2}$ in Γ^q_m (see Definition 5.2) to γ_1 and γ_2, respectively. Since $\eta_m \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_{p-1})$, from TAA and Definition 5.2(i) we see that

(1) $\gamma_1, \gamma_2 \notin \varphi_n\langle T(y_{p-1}) - T^*_{n,q}\rangle$ and η_m is not used by any edge in $T - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_m \in D_{n,q} \subseteq D_n$).

By (1) and Lemma 6.6 (with respect to (T, φ_n)), we have

(2) $P_{v_{\gamma_j}}(\eta_m, \gamma_j, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_j, \varphi_n)$ for $j = 1, 2$.

Let us consider two subcases according to whether $\theta \in \overline{\varphi}_n(y_{p-1})$.

Subcase 1.1. $\theta \notin \overline{\varphi}_n(y_{p-1})$.

In our discussion about this subcase, we shall appeal to the following two tree sequences:

- $T^- = (T^*_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$ and
- $T^* = (T^*_{n,q}, e_1, y_1, e_2, \ldots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$.

Note that T^- is obtained from T by deleting y_{p-1} and T^* arises from T by interchanging the order of (e_{p-1}, y_{p-1}) and (e_p, y_p). We propose to show that both T^- and T^* are ETTs corresponding to φ_n. Indeed, if $T(y_{p-2}) \neq T_n$, then both T^- and T^* can be obtained from $T(y_{p-2})$ by using TAA under φ_n. So we assume that $T(y_{p-2}) = T_n$. By the hypothesis of the present subcase, $\varphi_n(e_p) = \theta \notin \overline{\varphi}_n(y_{p-1})$. From Algorithm 3.1 we deduce that now $\Theta_n = PE$. Hence both T^- and T^* can be obtained from $T(y_{p-2})$ by using TAA under φ_n as well. Therefore both T^- and T^* are ETTs corresponding to φ_n. In view of the maximum property enjoyed by T, we further conclude that both T^- and T^* are ETTs satisfying MP with respect to φ_n.

Let us first assume that $\theta \notin \Gamma^q$. Now it is easy to see that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-$ is a good hierarchy of T^- under φ_n, with the same Γ-sets (see Definition 5.2) as T. (If $\theta \in \Gamma^q$, say $\theta \in \Gamma^q_h$, and $\eta_h \in \overline{\varphi}_n(y_{p-1})$, then T^- no longer satisfies Definition 5.2(i).) Observe that

$\gamma_1 \notin \overline{\varphi}_n(y_p)$, for otherwise, γ_1 is missing at two vertices in T^-. Thus (T^-, φ_n) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, φ_n). Let us turn to considering T^*. Since $\theta \notin \overline{\varphi}_n(y_{p-1})$ and $\theta \notin \Gamma^q$, it is clear that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^*$ is a good hierarchy of T^* under φ_n, with the same Γ-sets as T. Moreover, by (1), we have $\gamma_1 \notin \varphi_n\langle T^*(y_p) - T^*_{n,q}\rangle$. It follows from Lemma 6.6 (with respect to (T^*, φ_n)) that $P_{v_{\gamma_1}}(\eta_m, \gamma_1, \varphi_n) = P_{y_p}(\eta_m, \gamma_1, \varphi_n)$, contradicting (2).

Next we assume that $\theta \in \Gamma^q$. Then $\theta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$. If $\eta_h \notin \overline{\varphi}_n(y_{p-1})$, then $\eta_h \in \overline{\varphi}_n(T(y_{p-2}))$ by Definition 5.2(i). So we can still ensure that both T^- and T^* have good hierarchies under φ_n. Thus, using the same argument as employed in the preceding paragraph, we can reach a contradiction. Hence we may assume that $\eta_h \in \overline{\varphi}_n(y_{p-1})$.

Clearly, $\theta \neq \gamma_1$ or γ_2. Renaming subscripts if necessary, we may assume that

(3) $\theta \neq \gamma_2$.

Since $P_{v_{\gamma_2}}(\eta_m, \gamma_2, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_2, \varphi_n)$ by (2), this path is disjoint from $P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_m, \gamma_2, \varphi_n)$. By (1) and (3), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1; thereby T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which γ_2 is missing at two vertices.

By Lemma 6.4, we have $|\overline{\mu}_1(T(y_{p-2}))| - |\overline{\mu}_1(T^*_{n,0} - V(T_n))| - |\mu_1\langle T(y_{p-2}) - T^*_{n,q}\rangle| \geq 2n + 11$, where $T(y_0) = T^*_{n,q}$. It follows that $|\overline{\mu}_1(T(y_{p-2}))| - |\overline{\mu}_1(T^*_{n,0} - V(T_n))| - |\mu_1\langle T - T^*_{n,q}\rangle| \geq 2n + 9$. As $|\Gamma^q| \leq 2|D_{n,q}| \leq 2|D_n| \leq 2n$ by Lemma 3.4, using (6.6) we obtain

(4) there exists a color β in $\overline{\mu}_1(T(y_{p-2})) - \overline{\mu}_1(T^*_{n,0} - V(T_n)) - \mu_1\langle T - T^*_{n,q}\rangle - \Gamma^q$.

By Lemma 6.6 (with γ_2 in place of α), $P_{v_{\gamma_2}}(\beta, \gamma_2, \mu_1) = P_{v_\beta}(\beta, \gamma_2, \mu_1)$, so it is disjoint from $P_{y_p}(\beta, \gamma_2, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\beta, \gamma_2, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). By (1), (3) and (4), we have $\beta, \gamma_2 \notin \mu_1\langle T(y_p) - T^*_{n,q}\rangle$. So $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$ and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. Hence we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2; thereby T satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which β is missing at two vertices. Since $\theta \in \Gamma^q_h$ and $\eta_h \in \overline{\varphi}_n(y_{p-1}) = \overline{\mu}_1(y_{p-1}) = \overline{\mu}_2(y_{p-1})$, we obtain

(5) $\theta \notin \mu_2\langle T(y_{p-1}) - T^*_{n,q}\rangle$.

By (4), we also have

(6) $\beta \notin \mu_2\langle T - T^*_{n,q}\rangle$.

It follows from (5) and Lemma 6.6 (with θ in place of α) that $P_{v_\theta}(\beta, \theta, \mu_2) = P_{v_\beta}(\beta, \theta, \mu_2)$, so it is disjoint from $P_{y_p}(\beta, \theta, \mu_2)$. Finally, set $\mu_3 = \mu_2 / P_{y_p}(\beta, \theta, \mu_2)$. By Lemma 6.7, μ_3 satisfies all the properties described in (7.3) (with μ_3 in place of σ_n). From (5) and (6) we see that T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_3. Hence T satisfies MP under μ_3. Note that $\mu_3(f) = \mu_2(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_3(e_p) = \beta$, and $\overline{\mu}_3(u) = \overline{\mu}_2(u)$ for each $u \in V(T(y_{p-1}))$. Moreover, $\beta \notin \Gamma^q$ by (4). It is a routine matter to check that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_3, with the same Γ-sets as those under μ_2. Since $\mu_3(e_p) = \beta \notin \Gamma^q$ and $v_\beta \prec y_{p-1}$, we see that T^- has a good hierarchy and satisfies MP with respect to μ_3. As θ is missing at two vertices in T^-, we conclude that (T^-, μ_3) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which contradicts the minimality assumption (6.4) or (6.5) on (T, φ_n).

Subcase 1.2. $\theta \in \overline{\varphi}_n(y_{p-1})$.

In this subcase, from (6.6) and TAA we see that

(7) $\theta \notin \overline{\varphi}_n(T(y_{p-2}))$, so $\theta \notin \Gamma^q$ and hence $\theta \neq \gamma_1, \gamma_2$. Furthermore, θ is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \theta \in D_n$).

Since $P_{v_{\gamma_1}}(\eta_m, \gamma_1, \varphi_n) = P_{y_{p-1}}(\eta_m, \gamma_1, \varphi_n)$ by (2), this path is disjoint from $P_{y_p}(\eta_m, \gamma_1, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\eta_m, \gamma_1, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_m, \gamma_1, \varphi_n)$. By (1) and (6.6), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which γ_1 is missing at two vertices.

From (1) and the definition of μ_1, we see that

(8) $\gamma_1 \notin \mu_1\langle T - T^*_{n,q}\rangle$.

From (8) and Lemma 6.6 (with γ_1 in place of α), we deduce that $P_{v_{\gamma_1}}(\theta, \gamma_1, \mu_1) = P_{y_{p-1}}(\theta, \gamma_1, \mu_1)$, which is disjoint from $P_{y_p}(\theta, \gamma_1, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\theta, \gamma_1, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \theta \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\theta, \gamma_1, \mu_1)$. In view of (7), (8) and (6.6), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_2(e_p) = \gamma_1$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. Moreover, $\theta \notin \Gamma^q$. So T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_2, and hence satisfies MP under μ_2. It is a routine matter to check that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$ and $\mu_2(e_p) = \gamma_1 \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 1.1 if $\theta \in D_{n,q}$ and reduces to Subcase 2.1 (to be discussed below) if $\theta \notin D_{n,q}$.

Case 2. $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_{p-1})$ and $\alpha \notin D_{n,q}$.

By the definitions of D_n and $D_{n,q}$, we have $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$. Using (6.6) and this set inclusion, we obtain

(9) $\alpha \notin \overline{\varphi}_n(T(y_{p-2}))$ and $\alpha \notin D_n$. So $\alpha \notin \varphi_n\langle T - T^*_{n,q}\rangle$ by TAA (see, for instance, (1)).

Recall that $T(y_0) = T^*_{n,q}$ and $\theta = \varphi_n(e_p)$. We consider two subcases according to whether $\theta \in \overline{\varphi}_n(y_{p-1})$.

Subcase 2.1. $\theta \notin \overline{\varphi}_n(y_{p-1})$.

In our discussion about this subcase, we shall also appeal to the following two tree sequences:

- $T^- = (T^*_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$ and
- $T^* = (T^*_{n,q}, e_1, y_1, e_2, \ldots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$.

As stated in Subcase 1.1, T^- is obtained from T by deleting y_{p-1} and T^* arises from T by interchanging the order of (e_{p-1}, y_{p-1}) and (e_p, y_p). Furthermore, both T^- and T^* are ETTs satisfying MP with respect to φ_n. Observe that

(10) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T^*$ is a good hierarchy of T^* under φ_n, unless $\theta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$ such that $\eta_h \in \overline{\varphi}_n(y_{p-1})$.

Let us first assume that the exceptional case in (10) does not occur; that is, there exists no $\eta_h \in D_{n,q}$ such that $\eta_h \in \overline{\varphi}_n(y_{p-1})$ and $\theta \in \Gamma^q_h$. It is easy to see that now $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-$ is a good hierarchy of T^- under φ_n.

By Lemma 6.4, we have $|\overline{\varphi}_1(T(y_{p-2}))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_1\langle T(y_{p-2}) - T^*_{n,q}\rangle| \geq 2n + 11$ holds, where $T(y_0) = T^*_{n,q}$. Since $|\Gamma^q| \leq 2|D_{n,q}| \leq 2|D_n| \leq 2n$ by Lemma 3.4, using (6.6) we obtain

(11) there exists a color β in $\overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - \Gamma^q$.

Note that $\beta \notin \overline{\varphi}_n(y_p)$, for otherwise, (T^-, φ_n) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, σ_n). Since $\alpha, \beta \notin \varphi_n\langle T - T^*_{n,q}\rangle$ by (9) and (11), applying Lemma 6.6 to (T, φ_n) and (T^*, φ_n), respectively, we obtain $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{p-1}}(\alpha, \beta, \varphi_n)$ and $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_p}(\alpha, \beta, \varphi_n)$, a contradiction.

So we assume that the exceptional case in (10) occurs; that is, there exists $\eta_h \in D_{n,q}$ such that $\eta_h \in \overline{\varphi}_n(y_{p-1})$ and $\theta \in \Gamma^q_h$. For simplicity, we abbreviate the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2) to γ_1 and γ_2, respectively. Renaming subscripts if necessary, we may assume that $\theta = \gamma_1$. By Definition 5.2(i) and TAA, we have

(12) $\gamma_2 \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and η_h is not used by any edge in $T - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (12) and Lemma 6.6 (with α in place of β), we obtain $P_{v_{\gamma_2}}(\alpha, \gamma_2, \varphi_n) = P_{y_{p-1}}(\alpha, \gamma_2, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \gamma_2, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \gamma_2, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Since $\alpha, \gamma_2 \notin \varphi_n\langle T(y_p) - T^*_{n,q}\rangle$ by (9) and (12), we have $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which γ_2 is missing at two vertices.

If $\eta_h \in \overline{\mu}_1(y_p)$, then $\eta_h \in \overline{\mu}_1(y_p) \cap \overline{\mu}_1(y_{p-1})$, $\eta_h \in D_{n,q}$, and $\mu_1(e_p) = \gamma_1 \notin \overline{\varphi}_n(y_{p-1})$. Thus the present subcase reduces to Subcase 1.1. So we may assume that $\eta_h \notin \overline{\mu}_1(y_p)$. By (12) and the definition of μ_1, we have

(13) $\gamma_2 \notin \mu_1\langle T - T^*_{n,q}\rangle$ and η_h is not used by any edge in $T - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_h \in D_n$).

By (13) and Lemma 6.6 (with γ_2 in place of α), we obtain $P_{v_{\gamma_2}}(\eta_h, \gamma_2, \mu_1) = P_{y_{p-1}}(\eta_h, \gamma_2, \mu_1)$, which is disjoint from $P_{y_p}(\eta_h, \gamma_2, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\eta_h, \gamma_2, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_h \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_h, \gamma_2, \mu_1)$. By (13), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$ and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2, and hence T satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\eta_h \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_h \in D_{n,q}$, and $\mu_2(e_p) = \gamma_1 \notin \overline{\mu}_2(y_{p-1})$. Thus the present subcase reduces to Subcase 1.1.

Subcase 2.2. $\theta \in \overline{\varphi}_n(y_{p-1})$.

Let us first assume that $\theta \in D_{n,q}$; that is, $\theta = \eta_m$ for some $\eta_m \in D_{n,q}$. For simplicity, we use ε_1 and ε_2 to denote the two colors $\gamma^q_{m_1}$ and $\gamma^q_{m_2}$ in Γ^q_m (see Definition 5.2), respectively. By Definition 5.2(i) and TAA, we have

(14) $\varepsilon_1, \varepsilon_2 \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and η_m is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_m \in D_n$).

By (14) and Lemma 6.6, we obtain $P_{v_{\varepsilon_1}}(\alpha, \varepsilon_1, \varphi_n) = P_{y_{p-1}}(\alpha, \varepsilon_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. Let $\mu_1 = \varphi_n/P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). By (9) and (14), we have

(15) $\alpha, \varepsilon_1 \notin \mu_1\langle T - T^*_{n,q}\rangle$ and η_m is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_m \in D_n$).

So $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. Thus T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which ε_1 is missing at two vertices.

By (15) and Lemma 6.6 (with ε_1 in place of α), we obtain $P_{v_{\varepsilon_1}}(\eta_m, \varepsilon_1, \mu_1) = P_{y_{p-1}}(\eta_m, \varepsilon_1, \mu_1)$, which is disjoint from $P_{y_p}(\eta_m, \varepsilon_1, \mu_1)$. Let $\mu_2 = \mu_1/P_{y_p}(\eta_m, \varepsilon_1, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_m \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\eta_m, \varepsilon_1, \mu_1)$. In view of (15), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_2(e_p) = \varepsilon_1$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. So T can be obtained from $T^*_{n,q} + e_1$ by using TAA under μ_2, and hence satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\eta_m \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_m \in D_{n,q}$, and $\mu_2(e_p) = \varepsilon_1 \notin \overline{\mu}_2(y_{p-1})$. Thus the present subcase reduces to Subcase 1.1.

Next we assume that $\theta \notin D_{n,q}$. Set $T(y_0) = T^*_{n,q}$. We propose to show that

(16) there exists a color $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - D_{n,q}$, such that either $\beta \notin \Gamma^q$ or $\beta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{p-2}))$.

To justify this, note that if $|\overline{\varphi}_n(T(y_{p-2}))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_{p-2}) - T^*_{n,q}\rangle| - |\Gamma^q \cup D_{n,q}| \geq 5$, then $|\overline{\varphi}_n(T(y_{p-2}))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T - T^*_{n,q}\rangle| - |\Gamma^q \cup D_{n,q}| \geq 3$, because $T - T(y_{p-2})$ contains precisely two edges. Thus there exists a color $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - D_{n,q}$, such that $\beta \notin \Gamma^q$.

So we assume that $|\overline{\varphi}_n(T(y_{p-2}))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_{p-2}) - T^*_{n,q}\rangle| - |\Gamma^q \cup D_{n,q}| \leq 4$. By Lemma 6.4, there exist 7 distinct colors $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{p-2}))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_{p-2}) - T^*_{n,q}\rangle = \emptyset$. Let β be an arbitrary color in such a Γ^q_h. From Definition 5.2, we see that $\Gamma^q_h \subseteq \overline{\varphi}_n(T^*_{n,q}) \subseteq \overline{\varphi}_n(T(y_{p-2}))$, $\Gamma^q_h \cap \overline{\varphi}_n(T^*_{n,0} - V(T_n)) = \emptyset$, and $\Gamma^q_h \cap D_{n,q} = \emptyset$. So $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{p-2}) - T^*_{n,q}\rangle - D_{n,q}$. Since $T - T(y_{p-2})$ contains precisely two edges, there exists $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - D_{n,q}$, such that $\beta \in \Gamma^q_h$ for some $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{p-2}))$. Hence (16) is established.

By the definitions of D_n and $D_{n,q}$, we have $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$. By (16), $\beta \notin \overline{\varphi}_n(T^*_{n,0} - V(T_n)) \cup D_{n,q}$. It follows from these two observations that

(17) if $q \geq 1$, then $\beta \in \overline{\varphi}_n(T^*_{n,q})$ or $\beta \notin D_n$; if $q = 0$, then $\beta \in \overline{\varphi}_n(T_n)$ or $\beta \notin D_n$.

By (9), (17) and Lemma 6.6, we obtain $P_{v_\beta}(\alpha,\beta,\varphi_n)=P_{y_{p-1}}(\alpha,\beta,\varphi_n)$, which is disjoint from $P_{y_p}(\alpha,\beta,\varphi_n)$. Let $\mu_3=\varphi_n/P_{y_p}(\alpha,\beta,\varphi_n)$. By Lemma 6.7, μ_3 satisfies all the properties described in (7.3) (with μ_3 in place of σ_n). By (9) and (16), we have $\alpha,\beta\notin\varphi_n\langle T-T^*_{n,q}\rangle$. So

(18) $\alpha,\beta\notin\mu_3\langle T-T^*_{n,q}\rangle$,

$\mu_3(f)=\varphi_n(f)$ for each $f\in E(T)$, and $\overline{\mu}_3(u)=\overline{\varphi}_n(u)$ for each $u\in V(T(y_{p-1}))$. Thus we can obtain T from $T^*_{n,q}+e_1$ by using TAA under μ_3, and hence T satisfies MP under μ_3. Furthermore, $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be a good hierarchy of T under μ_3, with the same Γ-sets as those under φ_n. Therefore, (T,μ_3) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which β is missing at two vertices.

Since $\theta\in\overline{\varphi}_n(y_{p-1})$, it follows from (6.6) that $\theta\notin\overline{\varphi}_n(T^*_{n,q})$. By assumption, $\theta\notin D_{n,q}$. As $\overline{\varphi}_n(T_n)\cup D_n\subseteq\overline{\varphi}_n(T^*_{n,q})\cup D_{n,q}$, we obtain

(19) $\theta\notin D_n$ and hence $\theta\notin\mu_3\langle T(y_{p-1})-T^*_{n,q}\rangle$ by TAA.

By (17)-(19) and Lemma 6.6, we obtain $P_{v_\beta}(\theta,\beta,\mu_3)=P_{y_{p-1}}(\theta,\beta,\mu_3)$, which is disjoint from $P_{y_p}(\theta,\beta,\mu_3)$. Let $\mu_4=\mu_3/P_{y_p}(\theta,\beta,\mu_3)$. By Lemma 6.7, μ_4 satisfies all the properties described in (7.3) (with μ_4 in place of σ_n). By (18) and (19), we have $\mu_4(f)=\mu_3(f)$ for each $f\in E(T(y_{p-1}))$ and $\overline{\mu}_4(u)=\overline{\mu}_3(u)$ for each $u\in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q}+e_1$ by using TAA under μ_4, and hence T satisfies MP under μ_4. Since either $\beta\notin\Gamma^q$ or $\beta\in\Gamma^q_h$ for some $\eta_h\in D_{n,q}\cap\overline{\mu}_3(T(y_{p-2}))$ by (16), it follows that $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be a good hierarchy of T under μ_4, with the same Γ-sets as those under μ_3. Therefore, (T,μ_4) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta\in\overline{\mu}_4(y_p)\cap\overline{\mu}_4(y_{p-1})$, $\theta\notin D_{n,q}$, and $\mu_4(e_p)=\beta\notin\overline{\mu}_4(y_{p-1})$. Thus the present subcase reduces to Subcase 2.1.

Case 3. $\alpha\in\overline{\varphi}_n(y_p)\cap\overline{\varphi}_n(v)$ for some vertex $v\prec y_{p-1}$.

Set $T(y_0)=T^*_{n,q}$. Let us first impose some restrictions on α.

(20) We may assume that $\alpha\in\overline{\varphi}_n(T(y_{p-2}))-\varphi_n\langle T-T^*_{n,q}\rangle$, such that either $\alpha\notin D_{n,q}\cup\Gamma^q$ if $q\ge1$ and $\alpha\notin D_n\cup\Gamma^0$ if $q=0$, or α is some $\eta_h\in D_{n,q}$ satisfying $\Gamma^q_h\cap\varphi_n\langle T-T^*_{n,q}\rangle=\emptyset$.

To justify this, note that if $|\overline{\varphi}_n(T(y_{p-2}))|-|\overline{\varphi}_n(T^*_{n,0}-V(T_n))|-|\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle|-|\Gamma^q\cup D_{n,q}|\ge5$, then $|\overline{\varphi}_n(T(y_{p-2}))|-|\overline{\varphi}_n(T^*_{n,0}-V(T_n))|-|\varphi_n\langle T-T^*_{n,q}\rangle|-|\Gamma^q\cup D_{n,q}|\ge3$, because $T-T(y_{p-2})$ contains precisely two edges. Thus there exists a color $\beta\in\overline{\varphi}_n(T(y_{p-2}))-\overline{\varphi}_n(T^*_{n,0}-V(T_n))-\varphi_n\langle T-T^*_{n,q}\rangle-(\Gamma^q\cup D_{n,q})$. Clearly, $\beta\in\overline{\varphi}_n(T(y_{p-2}))-\varphi_n\langle T-T^*_{n,q}\rangle$ and $\beta\notin D_{n,q}\cup\Gamma^q$ if $q\ge1$ and $\beta\notin D_n\cup\Gamma^0$ if $q=0$ (see the definitions of D_n and $D_{n,0}$).

If $|\overline{\varphi}_n(T(y_{p-2}))|-|\overline{\varphi}_n(T^*_{n,0}-V(T_n))|-|\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle|-|\Gamma^q\cup D_{n,q}|\le4$, then, by Lemma 6.4, there exist 7 distinct colors $\eta_h\in D_{n,q}\cap\overline{\varphi}_n(T(y_{p-2}))$ such that $(\Gamma^q_h\cup\{\eta_h\})\cap\varphi_n\langle T(y_{p-2})-T^*_{n,q}\rangle=\emptyset$. Since $T-T(y_{p-2})$ contains precisely two edges, there exists one of these η_h, denoted by β, such that $(\Gamma^q_h\cup\{\eta_h\})\cap\varphi_n\langle T-T^*_{n,q}\rangle=\emptyset$.

Combining the above observations, we conclude that

(21) there exists $\beta\in\overline{\varphi}_n(T(y_{p-2}))-\varphi_n\langle T-T^*_{n,q}\rangle$, such that either $\beta\notin D_{n,q}\cup\Gamma^q$ if $q\ge1$ and $\beta\notin D_n\cup\Gamma^0$ if $q=0$, or β is some $\eta_h\in D_{n,q}$ satisfying $\Gamma^q_h\cap\varphi_n\langle T-T^*_{n,q}\rangle=\emptyset$.

If $\beta\in\overline{\varphi}_n(y_p)$, then (20) holds by replacing α with β. So we assume hereafter that $\beta\notin\overline{\varphi}_n(y_p)$. Let $Q=P_{y_p}(\alpha,\beta,\varphi_n)$ and let $\sigma_n=\varphi_n/Q$. We propose to show that one of the following statements (a) and (b) holds:

(a) σ_n is a $(T^*_{n,q},D_n,\varphi_n)$-strongly stable coloring, T is also an ETT satisfying MP with respect to σ_n, and $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be a hierarchy of T

under σ_n, with the same Γ-sets (see Definition 5.2) as those under φ_n. Moreover, (20) holds with respect to (T, σ_n).

(b) There exists an ETT T' satisfying MP with respect to φ_n, such that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. Moreover, $V(T')$ is not elementary with respect to φ_n and $p(T') < p(T)$.

Note that if (b) holds, then (T', φ_n) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) on (T, φ_n).

Let us first assume that Q is vertex-disjoint from $T(y_{p-1})$. By Lemma 5.8, σ_n is both $(T(y_{p-1}), D_n, \varphi_n)$-stable and $(T(y_{p-1}), \varphi_n)$-invariant. If $\Theta_n = PE$, then σ_n is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. Furthermore, $T(y_{p-1})$ is an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(y_{p-1})$ is a good hierarchy of $T(y_{p-1})$, with the same Γ-sets as T under σ_n. By definition, σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring. By the hypothesis of Case 3 and assumption on β, we have $\varphi_n(e_p) \neq \alpha, \beta$. Thus it is clear that (a) is true, and (20) follows if we replace φ_n by σ_n and α by β.

Next we assume that Q and $T(y_{p-1})$ have vertices in common. Let u be the first vertex of Q contained in $T(y_{p-1})$ as we traverse Q from y_p. Define $T' = T(y_{p-1}) \cup Q[u, y_p]$ if $u = y_{p-1}$ and $T' = T(y_{p-2}) \cup Q[u, y_p]$ otherwise. By the hypothesis of Case 3 and (21), we have $\alpha, \beta \in \overline{\varphi}_n(T(y_{p-2}))$. So T' can be obtained from $T(y_{p-2})$ by using TAA under φ_n, with $p(T') < p(T)$. It follows that T' is an ETT satisfying MP with respect to φ_n.

By Definition 5.2, we have $D_{n,q} \cap \Gamma^q = \emptyset$. Thus

(22) $\beta \notin \Gamma^q$ by (21).

Let us proceed by considering three possibilities for α.

• $\alpha \notin \Gamma^q$. Since both α and β are outside Γ^q (see (22)), it is easy to see that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. Hence (b) holds.

• $\alpha \in \Gamma^q \cap \varphi_n\langle T - T^*_{n,q}\rangle$. Let $\alpha \in \Gamma^q_h$ for some $\eta_h \in D_{n,q}$. Since $\varphi(e_p) \neq \alpha$, we have $\alpha \in \varphi_n\langle T(y_{p-1}) - T^*_{n,q}\rangle$. Hence $\eta_h \in \overline{\varphi}_n(T(y_{p-2}))$ by Definition 5.2(i). Furthermore, $\beta \in \overline{\varphi}_n(T(y_{p-2}))$ and $\beta \notin \Gamma^q$ by (21) and (22). Therefore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. Hence (b) holds.

• $\alpha \in \Gamma^q - \varphi_n\langle T - T^*_{n,q}\rangle$. By the definition of Γ^q, we have $\alpha \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\alpha \in \overline{\varphi}_n(T_n)$ if $q = 0$. It follows from Lemma 6.6 that $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$, which is disjoint from Q. By Lemma 6.7, $\sigma_n = \varphi_n/Q$ satisfies all the properties described in (7.3). Since $\alpha, \beta \notin \varphi_n\langle T - T^*_{n,q}\rangle$ by the assumption on α and (21), we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n, and hence T satisfies MP under σ_n. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which β is missing at two vertices. So (a) holds and therefore (20) is established by replacing φ_n with σ_n and β with α.

Let α be a color as specified in (20). Recall that $\theta = \varphi_n(e_p)$. We consider two subcases according to whether $\theta \in \overline{\varphi}_n(y_{p-1})$.

Subcase 3.1. $\theta \notin \overline{\varphi}_n(y_{p-1})$.

Consider the tree sequence $T^- = (T^*_{n,q}, e_1, y_1, e_2, \ldots, e_{p-2}, y_{p-2}, e_p, y_p)$. As stated in Subcase 1.1, T^- arises from T by deleting y_{p-1}, and T^- is an ETT satisfying MP with respect to φ_n.

Observe that

(23) $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T^-$ is a good hierarchy of T^- under φ_n, unless $\theta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ such that $\eta_m \in \overline{\varphi}_n(y_{p-1})$.

It follows that the exceptional case stated in (23) must occur, for otherwise, (T^-, φ_n) would be a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, φ_n). So $\theta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ such that $\eta_m \in \overline{\varphi}_n(y_{p-1})$.

Since $\alpha \in \overline{\varphi}_n(T(y_{p-2}))$, we have $\alpha \neq \eta_m$ by (6.6). From Definition 5.2(i), we see that

(24) $\theta \notin \varphi_n\langle T(y_{p-1}) - T^*_{n,q}\rangle$.

By the definition of Γ^q, we have $\theta \in \overline{\varphi}_n(T_{n,q})$ if $q \geq 1$ and $\theta \in \overline{\varphi}_n(T_n)$ if $q = 0$. Thus, by (20), (24) and Lemma 6.6, we obtain $P_{v_\alpha}(\alpha, \theta, \varphi_n) = P_{v_\theta}(\alpha, \theta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \theta, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \theta, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Using (20) and (24), we get

(25) $\alpha, \theta \notin \mu_1\langle T(y_{p-1}) - T^*_{n,q}\rangle$,

$\mu_1(f) = \varphi_n(f)$ for each $f \in E(T(y_{p-1}))$, $\mu_1(e_p) = \alpha \notin \Gamma^q$ (see (20)), and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which θ is missing at two vertices.

By (25) and Lemma 6.6, we obtain $P_{v_\theta}(\eta_m, \theta, \mu_1) = P_{y_{p-1}}(\eta_m, \theta, \mu_1)$, which is disjoint from $P_{y_p}(\eta_m, \theta, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\eta_m, \theta, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). Note that η_m is not used by any edge in $T - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_m \in D_n$). So e_1 is outside $P_{y_p}(\eta_m, \theta, \mu_1)$. Hence $\mu_2(f) = \mu_1(f)$ for each $f \in E(T)$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. It follows that T can be obtained from $T^*_{n,q} + e_1$ by using TAA and hence satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\eta_m \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_m \in D_{n,q}$, and $\mu_2(e_p) = \alpha \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 1.1.

Subcase 3.2. $\theta \in \overline{\varphi}_n(y_{p-1})$.

We first assume that $\theta \in D_{n,q}$. Let $\theta = \eta_m \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma^q_{m_1}$ and $\gamma^q_{m_2}$ in Γ^q_m (see Definition 5.2) to γ_1 and γ_2, respectively. By (20) and Definition 5.2(i), we have

(26) $\{\alpha, \gamma_1, \gamma_2\} \cap \varphi_n\langle T - T^*_{n,q}\rangle = \emptyset$.

By (26) and Lemma 6.6, we obtain $P_{v_\alpha}(\alpha, \gamma_1, \varphi_n) = P_{v_{\gamma_1}}(\alpha, \gamma_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \gamma_1, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\alpha, \gamma_1, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Since $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$, and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under μ_1. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which γ_1 is missing at two vertices. In view of (26) and Definition 5.2(i), we get

(27) $\{\alpha, \gamma_1, \gamma_2\} \cap \mu_1\langle T - T^*_{n,q}\rangle = \emptyset$, and η_m is not used by any edge in $T - T^*_{n,q}$ under μ_1,

except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_m \in D_{n,q} \subseteq D_n$).

By (27) and Lemma 6.6, we obtain $P_{v_{\gamma_1}}(\gamma_1, \eta_m, \mu_1) = P_{y_{p-1}}(\gamma_1, \eta_m, \mu_1)$, which is disjoint from $P_{y_p}(\gamma_1, \eta_m, \mu_1)$. Let $\mu_2 = \mu_1/P_{y_p}(\gamma_1, \eta_m, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In particular, if $e_1 = f_n$ and $\mu_1(e_1) = \eta_m \in D_n$, then $\mu_2(e_1) = \mu_1(e_1)$, which implies that e_1 is outside $P_{y_p}(\gamma_1, \eta_m, \mu_1)$. Since $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$ by (27), and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2 and hence T satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\eta_m \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\eta_m \in D_{n,q}$, and $\mu_2(e_p) = \gamma_1 \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 1.1.

Next we assume that $\theta \notin D_{n,q}$. By (6.6) and the hypothesis of the present subcase, we have $\theta \notin \overline{\varphi}_n(T^*_{n,q})$. So $\theta \notin \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, which implies $\theta \notin \overline{\varphi}_n(T_n) \cup D_n$. In particular,

(28) $\theta \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\theta \notin D_n \cup \Gamma^0$ if $q = 0$. Furthermore, θ is not used by any edge in $T(y_{p-1}) - T^*_{n,q}$ by TAA (see, for instance, (1)).

We proceed by considering two possibilities for α.

• $\alpha \notin D_{n,q}$. Now it follows from (20) that

(29) $\alpha \notin D_{n,q} \cup \Gamma^q$ if $q \geq 1$ and $\alpha \notin D_n \cup \Gamma^0$ if $q = 0$.

By (20) and Lemma 6.6, we obtain $P_{v_\alpha}(\alpha, \theta, \varphi_n) = P_{y_{p-1}}(\alpha, \theta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \theta, \varphi_n)$. Let $\sigma_n = \varphi_n/P_{y_p}(\alpha, \theta, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). Since $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{p-1}))$ by (20) and (28), and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q}+e_1$ by using TAA under σ_n and hence T satisfies MP under σ_n. In view of (28) and (29), $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_{p-1})$, $\theta \notin D_{n,q}$, and $\sigma_n(e_p) = \alpha \notin \overline{\sigma}_n(y_{p-1})$, the present subcase reduces to Subcase 2.1.

• $\alpha \in D_{n,q}$. Let $\alpha = \eta_h \in D_{n,q}$. For simplicity, we use ε_1 and ε_2 to denote the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2), respectively. By (20), we have

(30) $\{\alpha, \varepsilon_1, \varepsilon_2\} \cap \varphi_n\langle T - T^*_{n,q}\rangle = \emptyset$.

By (30) and Lemma 6.6, we obtain $P_{v_\alpha}(\alpha, \varepsilon_1, \varphi_n) = P_{v_{\varepsilon_1}}(\alpha, \varepsilon_1, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. Let $\mu_1 = \varphi_n/P_{y_p}(\alpha, \varepsilon_1, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). Since $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T)$ by (30), and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1 and hence T satisfies MP under μ_1. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_1, with the same Γ-sets as those under φ_n. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which ε_1 is missing at two vertices. From (30) and Definition 5.2(i) we see that

(31) $\varepsilon_1 \notin \mu_1\langle T - T^*_{n,q}\rangle$.

By (31) and Lemma 6.6, we obtain $P_{v_{\varepsilon_1}}(\theta, \varepsilon_1, \mu_1) = P_{y_{p-1}}(\theta, \varepsilon_1, \mu_1)$, which is disjoint from $P_{y_p}(\theta, \varepsilon_1, \mu_1)$. Let $\mu_2 = \mu_1/P_{y_p}(\theta, \varepsilon_1, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). In view of (28) and (31), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{p-1}))$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So T can be obtained from $T^*_{n,q} + e_1$ by using TAA and hence satisfies MP under μ_2. Furthermore, $T_n = T_{n,0} \subset T_{n,1} \subset$

$\ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)). Since $\theta \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(y_{p-1})$, $\theta \notin D_{n,q}$, and $\mu_2(e_p) = \varepsilon_1 \notin \overline{\mu}_2(y_{p-1})$, the present subcase reduces to Subcase 2.1. This completes our discussion about Situation 7.2.

Situation 7.3. $2 \le p(T) \le p - 1$.

Recall that $T = T^*_{n,q} \cup \{e_1, y_1, e_2, ..., e_p, y_p\}$, and the path number $p(T)$ of T is the smallest subscript $t \in \{1, 2, ..., p\}$ such that the sequence $(y_t, e_{t+1}, ..., e_p, y_p)$ corresponds to a path in G. Set $I_{\varphi_n} = \{1 \le t \le p-1 : \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_t) \ne \emptyset\}$. We use $\max(I_{\varphi_n})$ to denote the maximum element of I_{φ_n} if $I_{\varphi_n} \ne \emptyset$. For convenience, set $\max(I_{\varphi_n}) = -1$ if $I_{\varphi_n} = \emptyset$.

If $\max(I_{\varphi_n}) \ge p(T)$, then we may assume that $\max(I_{\varphi_n}) = p - 1$ (the proof is exactly the same as that of Claim 7.2). Let $\alpha \in \overline{\varphi}_n(y_{p-1}) \cap \overline{\varphi}_n(y_p)$ and $\beta = \varphi_n(e_p)$. Let σ_n be obtained from φ_n by recoloring e_p with α and let $T' = T(y_{p-1})$. Then $\beta \in \overline{\sigma}_n(y_{p-1}) \cap \overline{\sigma}_n(T')$ and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under σ_n. So (T', σ_n) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) or (6.5) on (T, φ_n).

So we may assume hereafter that $\max(I_{\varphi_n}) < p(T)$. Let $i = \max(I_{\varphi_n})$ if $I_{\varphi_n} \ne \emptyset$, and let $j = p(T)$. Then e_j is not incident to y_{j-1}. In our proof we reserve y_0 for the maximum vertex (in the order $\prec$) in $T^*_{n,q}$.

Claim 7.4. *We may assume that there exists $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(T(y_{j-2}))$, such that either $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$ or $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$.*

To establish this statement, we consider two cases, depending on whether I_φ is nonempty.

Case 1. $I_\varphi \ne \emptyset$.

By assumption, $\max(I_{\varphi_n}) < p(T)$. So $i \le j-1$. Let $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(y_i)$. By (6.6), we obtain

(1) $\alpha \notin \overline{\varphi}_n(T^*_{n,q})$. So $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$.

If $i \le j-2$, then $\alpha \in \overline{\varphi}_n(T(y_{j-2}))$, as desired. Thus we may assume that $i = j-1$.

(2) There exists a color $\beta \in \overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$ or a color $\beta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ and $(\Gamma^q_m \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle = \emptyset$.

To justify this, note that if $|\overline{\varphi}_n(T(y_{j-2}))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_{j-2}) - T^*_{n,q}\rangle| - |\Gamma^q \cup D_{n,q}| \ge 5$, then there exists a color β in $\overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$, because $T(y_{j-1}) - T(y_{j-2})$ contains only one edge.

If $|\overline{\varphi}_n(T(y_{j-2}))| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\varphi_n\langle T(y_{j-2}) - T^*_{n,q}\rangle| - |\Gamma^q \cup D_{n,q}| \le 4$, then, by Lemma 6.4, there exist 7 distinct colors $\eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{j-2}))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_{j-2}) - T^*_{n,q}\rangle = \emptyset$. Since $T(y_{j-1}) - T(y_{j-2})$ contains only one edge, there exists at least one of these η_h, say η_m, such that $(\Gamma^q_m \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle = \emptyset$. So (2) is true.

Depending on whether α is contained in $D_{n,q}$, we distinguish between two subcases.

Subcase 1.1. $\alpha \in D_{n,q}$. In this subcase, let $\alpha = \eta_h \in D_{n,q}$. For simplicity, we abbreviate the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2) to γ_1 and γ_2, respectively. Since $\eta_h \in \overline{\varphi}_n(y_{j-1})$, by Definition 5.2(i) and TAA, we have

(3) $\gamma_1, \gamma_2 \notin \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle$, and η_h is not used by any edge in $T(y_{j-1}) - T^*_{n,q}$, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

By (3) and Lemma 6.6, we obtain $P_{v_{\gamma_1}}(\gamma_1, \eta_h, \varphi_n) = P_{y_{j-1}}(\gamma_1, \eta_h, \varphi_n)$, which is disjoint from $P_{y_p}(\gamma_1, \eta_h, \varphi_n)$. Let $\mu_1 = \varphi_n / P_{y_p}(\gamma_1, \eta_h, \varphi_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3) (with μ_1 in place of σ_n). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_h \in D_n$, then $\mu_1(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\gamma_1, \eta_h, \varphi_n)$. Using (3) and (6.6), we get $\mu_1(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$ and $\overline{\mu}_1(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_1, and hence T satisfies MP under μ_1. Furthermore, since $\eta_h \in \overline{\mu}_1(y_{p-1})$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_1, with the same Γ-sets as those under μ_1. Therefore, (T, μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which γ_1 is missing at two vertices.

From (3) we see that

(4) $\gamma_1, \gamma_2 \notin \mu_1\langle T(y_{j-1}) - T^*_{n,q}\rangle$, and η_h is not used by any edge in $T(y_{j-1}) - T^*_{n,q}$ under μ_1, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_1(e_1) = \eta_h \in D_{n,q} \subseteq D_n$).

Let β be a color as specified in (2). Note that

(5) $\beta \notin \mu_1\langle T(y_{j-1}) - T^*_{n,q}\rangle$, $\beta \notin D_{n,q}$, and $\beta \neq \eta_h = \alpha$.

Since $\gamma_1 \in \overline{\mu}_1(T_{n,q})$ if $q \geq 1$ and $\gamma_1 \in \overline{\mu}_1(T_n)$ if $q = 0$, from (4) and Lemma 6.6 we deduce that $P_{v_{\gamma_1}}(\gamma_1, \beta, \mu_1) = P_{v_\beta}(\gamma_1, \beta, \mu_1)$, which is disjoint from $P_{y_p}(\gamma_1, \beta, \mu_1)$. Let $\mu_2 = \mu_1 / P_{y_p}(\gamma_1, \beta, \mu_1)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3) (with μ_2 in place of σ_n). By (4), (5) and (6.6), we have $\mu_2(f) = \mu_1(f)$ for each $f \in E(T(y_{j-1}))$, and $\overline{\mu}_2(u) = \overline{\mu}_1(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_2 and hence T satisfies MP under μ_2. If $\beta \notin \Gamma^q$, then clearly $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. So we assume that $\beta \in \Gamma^q$. By (2), we have $\beta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ and $(\Gamma^q_m \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle = \emptyset$. It follows that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ is still a good hierarchy of T under μ_2, with the same Γ-sets as those under μ_1. Therefore, (T, μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta \in \overline{\mu}_2(y_p) \cap \overline{\mu}_2(T(y_{j-2}))$. From (2) and the definitions of μ_1 and μ_2, we see that either $\beta \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$ or $\beta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$. Thus Claim 7.4 holds by replacing φ_n with μ_2 and α with β.

Subcase 1.2. $\alpha \notin D_{n,q}$. In this subcase, using (1) and the set inclusion $\overline{\varphi}_n(T_n) \cup D_n \subseteq \overline{\varphi}_n(T^*_{n,q}) \cup D_{n,q}$, we get

(6) $\alpha \notin D_n$. So α is not used by any edge in $T(y_{j-1}) - T^*_{n,q}$ by TAA.

Let β be a color as specified in (2). Then there are two possibilities for β.

- $\beta \in \overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$. Now it follows from Lemma 6.6 that $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{j-1}}(\alpha, \beta, \varphi_n)$, so this path is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). By (6), the assumption on β and (6.6), we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T satisfies MP under σ_n. Since $\alpha, \beta \notin \Gamma^q$ (see (1)), the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T(y_{j-2}))$. Thus Claim 7.4 holds by replacing φ_n with σ_n and α with β.

• $\beta \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ and $(\Gamma^q_m \cup \{\eta_m\}) \cap \varphi_n\langle T(y_{j-1}) - T^*_{n,q}\rangle = \emptyset$. Note that $\eta_m \in \overline{\varphi}_n(T(y_{j-2}))$ and hence $\alpha \neq \eta_m$ by (6.6). In view of Lemma 6.6, we obtain $P_{v_\beta}(\alpha, \beta, \varphi_n) = P_{y_{j-1}}(\alpha, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). By (6), the assumption on β and (6.6), we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_{j-1}))$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T satisfies MP under σ_n. Since $\alpha \notin \Gamma^q$ (see (1)) and $\eta_m \in \overline{\varphi}_n(T(y_{j-2}))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T(y_{j-2}))$. Thus Claim 7.4 holds by replacing φ_n with σ_n and α with β.

Case 2. $I_\varphi = \emptyset$.

Let $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(T(y_{p-1}))$. By the hypothesis of the present case, we have $\alpha \in \overline{\varphi}_n(T^*_{n,q})$. If $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$, we are done. So we assume that $\alpha \in \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$.

Subcase 2.1. $\alpha \in \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q$. Let us first show that

(7) there exists a color $\beta \in \overline{\varphi}_n(T^*_{n,q}) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \Gamma^q$.

Indeed, since $V(T^*_{n,q})$ is elementary with respect to φ_n, we have $|\overline{\varphi}_n(T^*_{n,q})| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\Gamma^q| \geq |\overline{\varphi}_n(T^*_{n,0})| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\Gamma^q| = |\overline{\varphi}_n(T_n)| - |\Gamma^q|$. In view of (7.2), we obtain $|\overline{\varphi}_n(T_n)| \geq 2n+11$ and $|\Gamma^q| \leq 2|D_{n,q}| \leq 2n$. So $|\overline{\varphi}_n(T^*_{n,q})| - |\overline{\varphi}_n(T^*_{n,0} - V(T_n))| - |\Gamma^q| \geq 11$, which implies (7).

By (7) and Lemma 6.6, we obtain $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). Since $\alpha, \beta \in \overline{\varphi}_n(T^*_{n,q})$, we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T^*_{n,q})$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T satisfies MP under σ_n. As $\alpha, \beta \notin \Gamma^q$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(T(y_{j-2}))$. Thus Claim 7.4 holds by replacing φ_n with σ_n and α with β.

Subcase 2.2. $\alpha \in \Gamma^q$. Let $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$. Depending on whether η_m is contained in $\overline{\varphi}_n(T(y_{p-1}))$, we consider two possibilities.

• $\eta_m \notin \overline{\varphi}_n(T(y_{p-1}))$. By Definition 5.2(i), we have $\alpha \notin \varphi_n\langle T - T^*_{n,q}\rangle$. Since $T - T(y_{p-2})$ contains precisely two edges, Lemma 6.4 guarantees the existence of a color β in $\overline{\varphi}_n(T(y_{p-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$ or a color $\beta = \eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{p-2}))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T - T^*_{n,q}\rangle = \emptyset$. Note that $\beta \in \overline{\varphi}_n(T(y_{p-2})) - \varphi_n\langle T - T^*_{n,q}\rangle$. By Lemma 6.6, we obtain $P_{v_\alpha}(\alpha, \beta, \varphi_n) = P_{v_\beta}(\alpha, \beta, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \beta, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \beta, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). Since $\alpha, \beta \notin \varphi_n\langle T - T^*_{n,q}\rangle$ and $\alpha, \beta \in \overline{\varphi}_n(T(y_{p-2}))$, we have $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T)$, and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n and hence T satisfies MP under σ_n. Furthermore, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\beta \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(v_\beta)$. Thus Claim 7.4 holds if $v_\beta \preceq y_{j-2}$, the present subcase reduces to the case when $\max(I_{\sigma_n}) \geq p(T)$ if $y_j \preceq v_\beta$ (see the paragraphs above Claim 7.4), and the present subcase reduces to Case 1 (where $I_{\sigma_n} \neq \emptyset$) if $y_{j-1} = v_\beta$.

• $\eta_m \in \overline{\varphi}_n(T(y_{p-1}))$. Note that $\eta_m \notin \overline{\varphi}_n(T^*_{n,q})$ because $\eta_m \in D_{n,q}$. So $\eta_m \in \overline{\varphi}_n(y_t)$ for some $1 \le t \le p-1$. If $t \le j-2$, then Claim 7.4 holds. Thus we may assume that $t \ge j-1$. Since $\eta_m \in \overline{\varphi}_n(y_t)$, it is not used by any edge in $T(y_t) - T^*_{n,q}$, except possibly e_1 when $q=0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\varphi_n(e_1) = \eta_m \in D_{n,q} \subseteq D_n$). Since $\alpha \in \Gamma^q_m$, by Definition 5.2(i), α is not used by any edge in $T(y_t) - T^*_{n,q}$. It follows from Lemma 6.6 that $P_{v_\alpha}(\alpha, \eta_m, \varphi_n) = P_{y_t}(\alpha, \eta_m, \varphi_n)$, which is disjoint from $P_{y_p}(\alpha, \eta_m, \varphi_n)$. Let $\sigma_n = \varphi_n / P_{y_p}(\alpha, \eta_m, \varphi_n)$. By Lemma 6.7, σ_n satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\varphi_n(e_1) = \eta_m \in D_n$, then $\sigma_n(e_1) = \varphi_n(e_1)$, which implies that e_1 is outside $P_{y_p}(\alpha, \eta_m, \varphi_n)$. Since $\sigma_n(f) = \varphi_n(f)$ for each $f \in E(T(y_t))$ and $\overline{\sigma}_n(u) = \overline{\varphi}_n(u)$ for each $u \in V(T(y_{p-1}))$, we can obtain T from $T^*_{n,q} + e_1$ by using TAA under σ_n, so T satisfies MP under σ_n. Furthermore, As $\alpha, \eta_m \in \overline{\sigma}_n(T(y_t))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under σ_n, with the same Γ-sets as those under φ_n. Therefore, (T, σ_n) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\eta_m \in \overline{\sigma}_n(y_p) \cap \overline{\sigma}_n(y_t)$. Thus the present subcase reduces to the case when $\max(I_{\sigma_n}) \ge p(T)$ if $j \preceq t$ (see the paragraphs above Claim 7.4), and reduces to Case 1 (where $I_{\sigma_n} \neq \emptyset$) if $t = j-1$. This proves Claim 7.4.

Let α be a color as specified in Claim 7.4; that is, $\alpha \in \overline{\varphi}_n(y_p) \cap \overline{\varphi}_n(T(y_{j-2}))$, such that either $\alpha \notin \Gamma^q \cup \overline{\varphi}_n(T^*_{n,0} - V(T_n))$ or $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$. Since $T(y_j) - T(y_{j-2})$ contains precisely two edges, Lemma 6.4 guarantees the existence of a color β in $\overline{\varphi}_n(T(y_{j-2})) - \overline{\varphi}_n(T^*_{n,0} - V(T_n)) - \varphi_n\langle T(y_j) - T^*_{n,q}\rangle - (\Gamma^q \cup D_{n,q})$ or a color $\beta = \eta_h \in D_{n,q} \cap \overline{\varphi}_n(T(y_{j-2}))$ such that $(\Gamma^q_h \cup \{\eta_h\}) \cap \varphi_n\langle T(y_j) - T^*_{n,q}\rangle = \emptyset$. Note that

(8) $\beta \notin \varphi_n\langle T(y_j) - T^*_{n,q}\rangle \cup \Gamma^q$.

Let $Q = P_{y_p}(\alpha, \beta, \varphi_n)$. We consider two cases, depending on whether Q intersects $T(y_{j-1})$.

Case 1. Q and $T(y_{j-1})$ have vertices in common. Let u be the first vertex of Q contained in $T(y_{j-1})$ as we traverse Q from y_p. Define $T' = T(y_{j-1}) \cup Q[u, y_p]$ if $u = y_{j-1}$ and $T' = T(y_{j-2}) \cup Q[u, y_p]$ otherwise. By the choices of α and β, we have $\alpha, \beta \in \overline{\varphi}_n(T(y_{j-2}))$. So T' can be obtained from $T(y_{j-2})$ by using TAA under φ_n. It follows that T' is an ETT satisfying MP with respect to φ_n, with $p(T') < p(T)$. If $\alpha \notin \Gamma^q$, then both α and β are outside Γ^q (see (8)), so $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. If $\alpha \in \Gamma^q$, then $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ by Claim 7.4. Since $\alpha, \eta_m \in \overline{\varphi}_n(T(y_{j-2}))$ and $\beta \notin \Gamma^q$, it is clear that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T'$ is also a good hierarchy of T' under φ_n, with the same Γ-sets as T under φ_n. So (T', φ_n) is a counterexample to Theorem 5.3 (see (6.2) and (6.3)), which violates the minimality assumption (6.4) on (T, φ_n).

Case 2. Q is vertex-disjoint from $T(y_{j-1})$. Let $\sigma_n = \varphi_n / Q$. By Lemma 5.8, σ_n is $(T(y_{j-1}), D_n, \varphi_n)$-stable. In particular, σ_n is $(T(y_{j-1}), \varphi_n)$-invariant. If $\Theta_n = PE$, then σ_n is also $(T_n \oplus R_n, D_n, \varphi_n)$-stable. Furthermore, $T(y_{j-1})$ is an ETT satisfying MP with respect to σ_n, and $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T(y_{j-1})$ is a good hierarchy of $T(y_{j-1})$ under σ_n, with the same Γ-sets as T under φ_n. By definition, σ_n is a $(T^*_{n,q}, D_n, \varphi_n)$-strongly stable coloring. If $\alpha \notin \Gamma^q$, then both α and β are outside Γ^q (see (8)), so $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T$ is a good hierarchy of T under σ_n, with the same Γ-sets as T under φ_n. If $\alpha \in \Gamma^q$, then $\alpha \in \Gamma^q_m$ for some $\eta_m \in D_{n,q}$ with $v_{\eta_m} \preceq y_{j-2}$ by Claim 7.4. Since $\alpha, \eta_m \in \overline{\varphi}_n(T(y_{j-2}))$ and $\beta \notin \Gamma^q$, it is clear that $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T$ is also a good hierarchy of T under σ_n, with the same Γ-sets as T under φ_n. So (T, σ_n) is a counterexample to Theorem 5.3, in which β is missing at two vertices.

From the choice of β above (8) and the definition of σ_n, we see that

(9) either $\beta \notin \overline{\sigma}_n(T^*_{n,0}-V(T_n))\cup\sigma_n\langle T(y_j)-T^*_{n,q}\rangle\cup(\Gamma^q\cup D_{n,q})$ or $\beta=\eta_h\in D_{n,q}\cap\overline{\sigma}_n(T(y_{j-2}))$, such that $(\Gamma^q_h\cup\{\eta_h\})\cap\sigma_n\langle T(y_j)-T^*_{n,q}\rangle=\emptyset$.

Let $\theta\in\overline{\sigma}_n(y_j)$. Then $\theta\notin\Gamma^q$. We proceed by considering two subcases.

Subcase 2.1. $\theta\notin D_{n,q}$. In this subcase, using (6.6) and the set inclusion $\overline{\varphi}_n(T_n)\cup D_n\subseteq\overline{\varphi}_n(T^*_{n,q})\cup D_{n,q}$, we obtain

(10) $\theta\notin\overline{\sigma}_n(T(y_{j-1}))$ and $\theta\notin D_n$. So θ is not assigned to any edge in $T(y_j)-T^*_{n,q}$ by TAA.

As described in (9), there are two possibilities for β.

• $\beta\notin\overline{\sigma}_n(T^*_{n,0}-V(T_n))\cup\sigma_n\langle T(y_j)-T^*_{n,q}\rangle\cup(\Gamma^q\cup D_{n,q})$. Observe that $\beta\notin D_n$ if $q=0$. By Lemma 6.6, we obtain $P_{v_\beta}(\beta,\theta,\sigma_n)=P_{y_j}(\beta,\theta,\sigma_n)$, which is disjoint from $P_{y_p}(\beta,\theta,\sigma_n)$. Let $\mu_1=\sigma_n/P_{y_p}(\beta,\theta,\sigma_n)$. By Lemma 6.7, μ_1 satisfies all the properties described in (7.3). By (10), the assumption on β and (6.6), we have $\mu_1(f)=\sigma_n(f)$ for each $f\in E(T(y_j))$ and $\overline{\mu}_1(u)=\overline{\sigma}_n(u)$ for each $u\in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q}+e_1$ by using TAA under μ_1 and hence T satisfies MP under μ_1. As $\beta,\theta\notin\Gamma^q$, the hierarchy $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be good under μ_1, with the same Γ-sets as those under σ_n. Therefore, (T,μ_1) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta\in\overline{\mu}_1(y_p)\cap\overline{\mu}_1(y_j)$. Thus the present subcase reduces to the case when $\max(I_{\mu_1})\ge p(T)$ (see the paragraphs above Claim 7.4).

• $\beta=\eta_h\in D_{n,q}\cap\overline{\sigma}_n(T(y_{j-2}))$, such that $(\Gamma^q_h\cup\{\eta_h\})\cap\sigma_n\langle T(y_j)-T^*_{n,q}\rangle=\emptyset$. For simplicity, we abbreviate the two colors $\gamma^q_{h_1}$ and $\gamma^q_{h_2}$ in Γ^q_h (see Definition 5.2) to γ_1 and γ_2, respectively. By Lemma 6.6, we obtain $P_{v_\beta}(\beta,\gamma_1,\sigma_n)=P_{v_{\gamma_1}}(\beta,\gamma_1,\sigma_n)$, which is disjoint from $P_{y_p}(\beta,\gamma_1,\sigma_n)$. Let $\mu_2=\sigma_n/P_{y_p}(\beta,\gamma_1,\sigma_n)$. By Lemma 6.7, μ_2 satisfies all the properties described in (7.3). By the assumption on β, neither β nor γ_1 is used by any edge in $T(y_j)-T^*_{n,q}$. So $\mu_2(f)=\sigma_n(f)$ for each $f\in E(T(y_j))$. By (6.6), we get $\overline{\mu}_2(u)=\overline{\sigma}_n(u)$ for each $u\in V(T(y_{p-1}))$. It follows that T can be obtained from $T^*_{n,q}+e_1$ by using TAA under μ_2 and hence T satisfies MP under μ_2. Furthermore, the hierarchy $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be good under μ_2, with the same Γ-sets as those under σ_n. Therefore, (T,μ_2) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which γ_1 is missing at both y_p and v_{γ_1}.

From the assumption on β and the definition of μ_2, we deduce that

(11) $\beta=\eta_h\in D_{n,q}\cap\overline{\mu}_2(T(y_{j-2}))$, such that $(\Gamma^q_h\cup\{\eta_h\})\cap\mu_2\langle T(y_j)-T^*_{n,q}\rangle=\emptyset$.

By (11) and Lemma 6.6, we obtain $P_{v_{\gamma_1}}(\theta,\gamma_1,\mu_2)=P_{y_j}(\theta,\gamma_1,\mu_2)$, which is disjoint from $P_{y_p}(\theta,\gamma_1,\mu_2)$. Let $\mu_3=\mu_2/P_{y_p}(\theta,\gamma_1,\mu_2)$. By Lemma 6.7, μ_3 satisfies all the properties described in (7.3). By (10), (11) and (6.6), we have $\mu_3(f)=\mu_2(f)$ for each $f\in E(T(y_j))$ and $\overline{\mu}_3(u)=\overline{\mu}_2(u)$ for each $u\in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q}+e_1$ by using TAA under μ_3 and hence T satisfies MP under μ_3. Furthermore, the hierarchy $T_n=T_{n,0}\subset T_{n,1}\subset\ldots\subset T_{n,q}\subset T_{n,q+1}=T$ remains to be good under μ_3, with the same Γ-sets as those under μ_2. Therefore, (T,μ_3) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which θ is missing at both y_p and y_j. Thus the present subcase reduces to the case when $\max(I_{\mu_3})\ge p(T)$ (see the paragraphs above Claim 7.4).

Subcase 2.2. $\theta\in D_{n,q}$. Let $\theta=\eta_t\in D_{n,q}$. For simplicity, we use ε_1 and ε_2 to denote the two colors $\gamma^q_{t_1}$ and $\gamma^q_{t_2}$ in Γ^q_t (see Definition 5.2), respectively. Then

(12) $\varepsilon_1,\varepsilon_2\notin\sigma_n\langle T(y_j)-T^*_{n,q}\rangle$ and η_t is not used by any edge in $T(y_j)-T^*_{n,q}$ under σ_n, except possibly e_1 when $q=0$ and $T^*_{n,0}=T_n$ (now $e_1=f_n$ in Algorithm 3.1 and $\sigma_n(e_1)=\eta_t\in D_{n,q}\subseteq$

D_n).

By (12) and Lemma 6.6 (with ε_1 in place of α), we obtain $P_{v_{\varepsilon_1}}(\varepsilon_1, \beta, \sigma_n) = P_{v_\beta}(\varepsilon_1, \beta, \sigma_n)$, which is disjoint from $P_{y_p}(\varepsilon_1, \beta, \sigma_n)$. Let $\mu_4 = \sigma_n / P_{y_p}(\varepsilon_1, \beta, \sigma_n)$. By Lemma 6.7, μ_4 satisfies all the properties described in (7.3). By (9), we have $\beta \notin \sigma_n \langle T(y_j) - T^*_{n,q} \rangle$, which together with (12) and (6.6) implies $\mu_4(f) = \sigma_n(f)$ for each $f \in E(T(y_j))$ and $\overline{\mu}_4(u) = \overline{\sigma}_n(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_4 and hence T satisfies MP under μ_4. Since $\beta \notin \Gamma^q$ by (9) and $\eta_t \in \overline{\mu}_4(y_j)$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_4, with the same Γ-sets as those under σ_n. Therefore, (T, μ_4) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which ε_1 is missing at both y_p and v_{ε_1}.

From (12) and (6.6) it can be seen that

(13) $\varepsilon_1, \varepsilon_2 \notin \mu_4 \langle T(y_j) - T^*_{n,q} \rangle$ and $\eta_t \notin \overline{\mu}_4(T(y_{j-1}))$. So η_t is not used by any edge in $T(y_j) - T^*_{n,q}$ under μ_4, except possibly e_1 when $q = 0$ and $T^*_{n,0} = T_n$ (now $e_1 = f_n$ in Algorithm 3.1 and $\mu_4(e_1) = \eta_t \in D_{n,q} \subseteq D_n$).

By (13) and Lemma 6.6, we obtain $P_{v_{\varepsilon_1}}(\varepsilon_1, \eta_t, \mu_4) = P_{y_j}(\varepsilon_1, \eta_t, \mu_4)$, which is disjoint from $P_{y_p}(\varepsilon_1, \eta_t, \mu_4)$. Let $\mu_5 = \mu_4 / P_{y_p}(\varepsilon_1, \eta_t, \mu_4)$. By Lemma 6.7, μ_5 satisfies all the properties described in (7.3). In particular, if $e_1 = f_n$ and $\mu_4(e_1) = \eta_t \in D_n$, then $\mu_5(e_1) = \mu_4(e_1)$, which implies that e_1 is outside $P_{y_p}(\varepsilon_1, \eta_t, \mu_4)$. By (13) and (6.6), we have $\mu_5(f) = \mu_4(f)$ for each $f \in E(T(y_j))$ and $\overline{\mu}_5(u) = \overline{\mu}_4(u)$ for each $u \in V(T(y_{p-1}))$. So we can obtain T from $T^*_{n,q} + e_1$ by using TAA under μ_5 and hence T satisfies MP under μ_5. Since $\eta_t, \varepsilon_1 \in \overline{\mu}_5(T(y_j))$, the hierarchy $T_n = T_{n,0} \subset T_{n,1} \subset \ldots \subset T_{n,q} \subset T_{n,q+1} = T$ remains to be good under μ_5, with the same Γ-sets as those under μ_4. Therefore, (T, μ_5) is also a minimum counterexample to Theorem 5.3 (see (6.2)-(6.5)), in which $\theta = \eta_t$ is missing at both y_p and y_j. Thus the present subcase reduces to the case when $\max(I_{\mu_5}) \geq p(T)$ (see the paragraphs above Claim 7.4).

This completes our discussion about Situation 7.3 and hence our proof of Theorem 5.3. ∎

7.2 Proof of Theorem 3.10(ii)

In the preceding subsection we have proved Theorem 5.3 and hence Theorem 3.10(i). To complete the proof of Theorem 3.10, we still need to establish the interchangeability property as described in Theorem 3.10(ii).

Lemma 7.1. *Suppose Theorem 3.10(i), (iii)-(vi) hold for all ETTs with n rungs and satisfying MP, and suppose Theorem 3.10(ii) holds for all ETTs with $n-1$ rungs and satisfying MP. Then Theorem 3.10(ii) holds for all ETTs with n rungs and satisfying MP; that is, T_{n+1} has the interchangeability property with respect to φ_n.*

Proof. Let $T = T_{n+1}$, let σ_n be a (T, D_n, φ_n)-stable coloring, and let α and β be two colors in $[k]$ with $\alpha \in \overline{\sigma}_n(T)$ (equivalently $\alpha \in \overline{\varphi}_n(T)$). We aim to prove that α and β are T-interchangeable under σ_n. Recalling (5.2), we may assume that T_{n+1} is a closure of $T_n \vee R_n$ under φ_n, which is a special closure of T_n under φ_n, if $\Theta_n = PE$. As introduced in Section 5, $T^*_{n,0} = T_n \vee R_n$ if $\Theta_n = PE$ and $T^*_{n,0} = T_n$ otherwise. From definitions we see that σ_n is also a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring (see the remark right above Lemma 6.2).

Assume the contrary: there are at least two (α, β)-paths Q_1 and Q_2 with respect to σ_n intersecting T. By Theorem 3.10(i), $V(T)$ is elementary with respect to φ_n, so it is also elementary with respect to σ_n. Since $T = T_{n+1}$ is closed with respect to φ_n, is it also closed with respect to σ_n. As $\alpha \in \overline{\sigma}_n(T)$, it follows that $|V(T)|$ is odd and β is outside $\overline{\sigma}_n(T)$. From the existence of Q_1 and Q_2, we see that $|\partial_{\sigma_n,\beta}(T)|$ is odd and at least three. Thus G contains at least three $(T, \sigma_n, \{\alpha, \beta\})$-exit paths P_1, P_2, P_3.

We call the tuple $(\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)$ a *counterexample* if σ_n is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring, and T is a closed ETT corresponding to (σ_n, T_n) (see Theorem 3.10(vi) and Definition 3.7) with n rungs, with $T^*_{n,0} \subset T$, and satisfying MP under σ_n. Moreover, P_1, P_2, P_3 are three $(T, \sigma_n, \{\alpha, \beta\})$-exit paths. We use $\mathcal{K}$ to denote the set of all such counterexamples. With a slight abuse of notation, let $(\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)$ be a counterexample in $\mathcal{K}$ with the minimum $|P_1| + |P_2| + |P_3|$. For $i = 1, 2, 3$, let a_i and b_i be the ends of P_i with $b_i \in V(T)$, and f_i be the edge of P_i incident to b_i. Renaming subscripts if necessary, we may assume that $b_1 \prec b_2 \prec b_3$. We propose to show that

(1) $b_3 \notin V(T_n)$ if $\Theta_n = SE$ or RE.

Otherwise, $b_3 \in V(T_n)$. Let $\gamma \in \overline{\sigma}_n(T_n)$. Since $T = T_{n+1}$ is closed with respect to σ_n, both α and γ are closed in T with respect to σ_n. Let $\mu_1 = \sigma_n/(G - T, \alpha, \gamma)$. By Lemma 5.8, μ_1 is a (T, D_n, φ_n)-stable coloring. By definition, μ_1 is a (T_n, D_n, φ_n)-stable coloring. Since $\Theta_n = SE$ or RE, by Algorithm 3.1, μ_1 is also a $(T_n, D_{n-1}, \varphi_{n-1})$-stable coloring. By Theorem 3.10(vi), T_n is an ETT corresponding to μ_1 (see see Theorem 3.10(vi) and Definition 3.7) and satisfies MP under μ_1, with $n-1$ rungs. Since P_1, P_2, P_3 are three $(T_n, \mu_1, \{\gamma, \beta\})$-exit paths, there are at least two (γ, β)-paths with respect to μ_1 intersecting T_n. Hence γ and β are not T_n-interchangeable under μ_1, contradicting Theorem 3.10(ii) because T_n has $n - 1$ rungs. So (1) is established.

(2) $b_3 \notin V(T_n \vee R_n)$ if $\Theta_n = PE$.

The proof is similar to that of (1). Assume the contrary: $b_3 \in V(T_n \vee R_n)$. Let $\gamma \in \overline{\sigma}_n(T_n) \cap \overline{\sigma}_n(R_n)$ and let $\mu_1 = \sigma_n/(G - T, \alpha, \gamma)$. By Lemma 5.8, μ_1 is a $(T_n \oplus R_n, D_n, \varphi_n)$-stable coloring. Since P_1, P_2, P_3 are three $(T_n \vee R_n, \mu_1, \{\gamma, \beta\})$-exit paths, there are at least two (γ, β)-paths with respect to μ_1 intersecting $T_n \vee R_n$, contradicting Lemma 6.1(iii). So (2) holds.

Let $\gamma \in \overline{\sigma}_n(b_3)$ and let $\mu_2 = \sigma_n/(G - T, \alpha, \gamma)$. By Lemma 5.8, μ_2 is a (T, D_n, φ_n)-stable coloring. So μ_2 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. By Theorem 3.10(vi), T is an ETT corresponding to μ_2 (see Definition 3.7) and satisfies MP under μ_2. Note that f_i is colored by β under both μ_2 and σ_n for $i = 1, 2, 3$.

Consider $\mu_3 = \mu_2/P_{b_3}(\beta, \gamma, \mu_2)$. Clearly, $\beta \in \overline{\mu}_3(b_3)$. By (1), (2) and Lemma 5.8, μ_3 is a $(T^*_{n,0}, D_n, \mu_2)$-strongly stable coloring. It follows from Lemma 2.4 that μ_3 is a $(T^*_{n,0}, D_n, \varphi_n)$-strongly stable coloring. By Theorem 3.10(vi), $T(b_3)$ is an ETT corresponding to μ_3 (see Definition 3.7) and satisfies MP under μ_3. Let T' be obtained from $T(b_3)$ by adding f_1 and f_2 and let T'' be a closure of T' under μ_3. Obviously, both T' and T'' are ETTs corresponding to μ_3 and satisfies MP under μ_3. By Theorem 3.10(i), $V(T'')$ is elementary with respect to μ_3, because T'' has n rungs.

Observe that none of a_1, a_2, a_3 is contained in T'', for otherwise, let $a_i \in V(T'')$ for some i with $1 \le i \le 3$. Since $\{\beta, \gamma\} \cap \overline{\mu}_3(a_i) \neq \emptyset$ and $\beta \in \overline{\mu}_3(b_3)$, we obtain $\gamma \in \overline{\mu}_3(a_i)$. Hence from TAA we see that P_1, P_2, P_3 are all entirely contained in $G[T'']$, which in turn implies $\gamma \in \overline{\mu}_3(a_j)$ for $j = 1, 2, 3$. So $V(T'')$ is not elementary with respect to μ_3, a contradiction. Each P_i contains a subpath Q_i, which is a T''-exit path with respect to μ_3. Since f_1 is not contained in Q_1, we obtain $|Q_1| + |Q_2| + |Q_3| < |P_1| + |P_2| + |P_3|$. In view of (1) and (2), we have $T_n \subseteq T''$ if $\Theta_n \neq PE$ and $T_n \vee R_n \subseteq T''$ if $\Theta_n = PE$. Thus the existence of the counterexample $(\mu_3, T'', \gamma, \beta, Q_1, Q_2, Q_3)$

violates the minimality assumption on $(\sigma_n, T, \alpha, \beta, P_1, P_2, P_3)$.

This completes our proof of Lemma 7.1 and hence of Theorem 3.10. ∎

References

[1] L. Andersen, On edge-colorings of graphs, *Math. Scand.* **40** (1977), 161-175.

[2] J. Asplund and J. McDonald, On a limit of the method of Tashkinov trees for edge-coloring, *Discrete Math.* **339** (2016), 2231-2238.

[3] G. Chen, Y. Gao, R. Kim, L. Postle, and S. Shan, Chromatic index determined by fractional chromatic index, *J. Combin. Theory Ser. B* **131** (2018), 85-108.

[4] G. Chen and G. Jing, Structural properties of edge-chromatic critical multigraphs, revised for *J. Combin. Theory Ser. B.*

[5] G. Chen, X. Yu, and W. Zang, Approximating the chromatic index of multigraphs, *J. Comb. Optim.* **21** (2009), 219-246.

[6] X. Chen, W. Zang, and Q. Zhao, Densities, matchings, and fractional edge-colorings, *SIAM J. Optim.* **29** (2019), 240-261.

[7] J. Edmonds, Maximum matching and a polyhedron with $0, 1$-vertices, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 125-130.

[8] L. Favrholdt, M. Stiebitz, and B. Toft, *Graph Edge Colouring: Vizing's Theorem and Goldberg's Conjecture*, Preprint, 2006, No. 20, IMADA, University of Southern Denmark, 91 pages.

[9] M. Goldberg, On multigraphs of almost maximal chromatic class (in Russian), *Diskret. Analiz.* **23** (1973), 3-7.

[10] M. Goldberg, Edge-coloring of multigraphs: recoloring technique, *J. Graph Theory* **8** (1984), 123-137.

[11] R. Gupta, *Studies in the Theory of Graphs*, Ph.D. Thesis, Tata Institute of Fundamental Research, Bombay, 1967.

[12] P. Haxell, M. Krivelevich, and G. Kronenberg, Goldberg's conjecture is true for random multigraphs, arXiv:1803.00908v1 [math.CO] 2 Mar 2018.

[13] P. Haxell and J. McDonald, On characterizing Vizing's edge-coloring bound, *J. Graph Theory* **69** (2012), 160-168.

[14] D. Hochbaum, T. Nishizeki, and D. Shmoys, A better than "best possible" algorithm to edge color multigraphs, *J. Algorithms* **7** (1986), 79-104

[15] I. Holyer, The NP-completeness of edge-colorings, *SIAM J. Comput.* **10** (1980), 718-720.

[16] I. Jakobsen, On critical graphs with chromatic index 4, *Discrete Math.* **9** (1974), 265-279.

[17] I. Jakobsen, On critical graphs with respect to edge-coloring, in: *Infinite and Finite Sets* (A. Hajnal, R. Rado, and V. Sós, eds.), Vol. II, pp. 927-934, North-Holland, Amsterdam, 1975.

[18] T. Jensen and B. Toft, Unsolved graph edge coloring problems, in: *Topics in Chromatic Graph Theory* (L. Beineke and R. Wilson, eds.), pp. 327-357, Cambridge University Press, 2015.

[19] J. Kahn, Asymptotics of the chromatic index for multigraphs, *J. Combin. Theory Ser. B* **68** (1996), 233-254.

[20] H. Kierstead, On the chromatic index of multigraphs without large triangles, *J. Combin. Theory Ser. B* **36** (1984), 156-160.

[21] A. Letchford, G. Reinelt, and D. Theis, Odd matching cut sets and b-matchings revisited, *SIAM J. Discrete Math.* **22** (2008), 1480-1487.

[22] O. Marcotte, On the chromatic index of multigraphs and a conjecture of Seymour, II, in: *Polyhedral Combinatorics* (W. Cook and P. Seymour, eds.), DIMACS Series in Discrete Mathematics and Theoretical Computer Science **1** (1990), 245-279.

[23] J. McDonald, Edge-colorings, in: *Topics in Chromatic Graph Theory* (L. Beineke and R. Wilson, eds.), pp. 94-113, Cambridge University Press, 2015.

[24] G. Nemhauser and S. Park, A polyhedral approach to edge coloring, *Oper. Res. Lett.* **10** (1991), 315-322.

[25] T. Nishizeki and K. Kashiwagi, On the 1.1 edge-coloring of multigraphs, *SIAM J. Discrete Math.* **3** (1990), 391-410.

[26] M. Padberg and R. Rao, Odd minimum cutsets and b-matchings, *Math. Oper. Res.* **7** (1982), 67-80.

[27] M. Padberg and L. Wolsey, Fractional covers for forests and matchings, *Math. Programming* **29** (1984), 1-14.

[28] M. Plantholt, A sublinear bound on the chromatic index of multigraphs, *Discrete Math.* **202** (1999), 201-213.

[29] M. Plantholt, A combined logarithmic bound on the chromatic index of multigraphs, *J. Graph Theory* **73** (2013), 239-259.

[30] D. Scheide, *Kantenfärbungen von Multigraphen*, Diploma Thesis, TU Ilmenau, Ilmenau, 2007.

[31] D. Scheide, Graph edge coloring: Tashkinov trees and Goldberg's conjecture, *J. Combin. Theory Ser. B* **100** (2010), 68-96.

[32] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, New York, 1986.

[33] A. Schrijver, *Combinatorial Optimization – Polyhedra and Efficiency*, Springer-Verlag, Berlin, 2003.

[34] P. Seymour, On multi-colorings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. London Math. Soc.* **38** (1979), 423-460.

[35] C. Shannon, A theorem on coloring lines of a network, *J. Math. Phys.* **28** (1949), 148-151.

[36] M. Stiebitz, D. Scheide, B. Toft, and L. Favrholdt, *Graph Edge Colouring: Vizing's Theorem and Goldberg's Conjecture*, John Wiley & Sons, 2012.

[37] V. Tashkinov, On an algorithm for the edge coloring of multigraphs (in Russian), *Diskretn. Anal. Issled. Oper. Ser. 1* **7** (2000), 72-85.

[38] V. Vizing, On an estimate of the chromatic class of a p-graph (in Russian), *Diskret. Analiz.* **3** (1964), 25-30.

Subject Index