Periodic Maxwell-Chern-Simons vortices with concentrating property

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Superconductivity

- Superconductivity (1911, Heike Kamerlingh Onnes):

  "Electrical resistance$= 0$" & "Magnetic flux fields are expelled".

- The classical Abelian Maxwell-Higgs (Abelian Higgs, AH) model describes

the superconductivity phenomena at low temperature.

Abelian-Higgs (AH) model

- Minkowski space $(\mathbb{R}^{1+d}, g)$ with metric tensor $g = \mathrm{diag}(1, -1, \cdots, -1)$

The Lagrangean $\mathcal{L}^{AH}$ for (AH) model

$$\mathcal{L}^{AH}(A, \phi) = -\frac{1}{4q^2} F_{\alpha\beta} F^{\alpha\beta} + D_\alpha \phi \overline{(D^\alpha \phi)} - \frac{q^2}{2} \left(|\phi|^2 - 1\right)^2.$$

- The Higgs field $\phi : \mathbb{R}^{1+d} \to \mathbb{C}$

$|\phi|$ measures density of superconducting electron pairs (Cooper pairs)

The gauge potential field $A = -iA_\alpha dx^\alpha$, $A_\alpha : \mathbb{R}^{1+d} \to \mathbb{R}$

The Maxwell gauge field $F_A = -\frac{i}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$.

$D_A \phi = D_\alpha \phi dx^\alpha$, $D_\alpha \phi = \partial_\alpha \phi - iA_\alpha \phi$.

Euler Lagrange equations

- The invariance of $\mathcal{L}^{AH}$ under the following gauge transformations:

$$\begin{cases} \phi \to e^{i\omega}\phi, \\ A \to A - id\omega, \end{cases}$$

for any smooth real function ω over $\mathbb{R}^{1+d}$.

The gauge group is given by the abelian group of rotations in $\mathbb{R}^2$, $U(1)$.

- Euler-Lagrange equations

$$\begin{cases} D_\mu D^\mu \phi = -2\frac{\partial V}{\partial \bar{\phi}}, \\ \partial_\nu F^{\mu\nu} = \frac{i}{2}\left(\bar{\phi} D^\mu \phi - \overline{D^\mu \phi}\phi\right). \end{cases}$$

If $\phi \equiv 0$, then $\partial_\nu F^{\mu\nu} = 0$ is Maxwell's equations in a vacuum.

- Vortices: bi-dimensional soliton solutions of Euler-Lagrange equations.

Chern-Simons (CS) model

- The first high-temperature superconductor: Bednorz and Müller (1986).

- [Hong-Kim-Pac, Jackiw-Weinberg (1990)] independently proposed the

Chern-Simons (CS) model for the high critical temperature superconductivity.

- Lagrangean $\mathcal{L}^{CS}$ for (CS) model

$$\mathcal{L}^{CS}(A,\phi) = -\frac{\mu}{4q^2}\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{(D^\alpha\phi)} - \frac{q^4}{\mu^2}|\phi|^2\left(|\phi|^2-1\right)^2.$$

- $\varepsilon^{\alpha\beta\gamma}$: the totally skew-symmetric tensor fixed so that $\varepsilon^{012}=1$

q: the electric charge

μ: Chern-Simons mass scale

Maxwell-Chern-Simons (MCS) model

- [Lee, Lee, and Min (1990)]

- introduced Maxwell-Chern-Simons (MCS) model as a unified self-dual system of Abelian-Higgs (AH) model and Chern-Simons (CS) model.

- showed formally that the self-dual equation of (MCS) owns both (AH) model and (CS) model as limiting problems depending on the electric charge q and the Chern-Simons mass scale μ.

Limit to Abelian-Higgs (AH) model

- The Lagrangian $\mathcal{L}^{MCS}$ for (MCS) model:

$$\mathcal{L}^{MCS}(A,\phi,\frac{N}{2}) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} - \frac{\mu}{4q^2}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{(D^\alpha\phi)} + \frac{1}{8q^2}\partial_\alpha N\partial^\alpha N$$
$$-|\phi|^2\left(\frac{N}{2}-\frac{q^2}{\mu}\right)^2 - \frac{q^2}{2}\left(|\phi|^2 - \frac{\mu}{q^2}\frac{N}{2}\right)^2.$$

- Fix $q^2 = \frac{\lambda\mu}{2}$, and assume the identity $\frac{N}{2} = \frac{q^2}{\mu}$.

As $\mu \to 0$, a "limiting" model would be (AH) model, whose Lagrangean $\mathcal{L}^{AH}$ is

$$\mathcal{L}^{AH}(A,\phi) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} + D_\alpha\phi\overline{(D^\alpha\phi)} - \frac{q^2}{2}\left(|\phi|^2-1\right)^2.$$

Limit to Chern-Simons (CS) model

- The Lagrangian $\mathcal{L}^{MCS}$ for (MCS) model:

$$\mathcal{L}^{MCS}(A,\phi,\frac{N}{2}) = -\frac{1}{4q^2}F_{\alpha\beta}F^{\alpha\beta} - \frac{\mu}{4q^2}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{(D^\alpha\phi)} + \frac{1}{8q^2}\partial_\alpha N\partial^\alpha N$$

$$-|\phi|^2\left(\frac{N}{2}-\frac{q^2}{\mu}\right)^2 - \frac{q^2}{2}\left(|\phi|^2-\frac{\mu}{q^2}\frac{N}{2}\right)^2.$$

- Fix $\lambda = \frac{2q^2}{\mu}$, and insert the identity $\frac{N}{2} = \frac{q^2}{\mu}|\phi|^2$ into the potential of $\mathcal{L}^{MCS}$.

As $\mu \to \infty$, a "limiting" model would be the (CS) model, whose Lagrangean $\mathcal{L}^{CS}$ is

$$\mathcal{L}^{CS}(A,\phi) = -\frac{\mu}{4q^2}\varepsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{(D^\alpha\phi)} - \frac{q^4}{\mu^2}|\phi|^2\left(|\phi|^2-1\right)^2.$$

Notations

$$\mathcal{L}^{MCS}(A, \phi, \frac{N}{2}) = -\frac{1}{4q^2} F_{\alpha\beta}F^{\alpha\beta} - \frac{\mu}{4q^2}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} + D_\alpha\phi\overline{(D^\alpha\phi)} + \frac{1}{8q^2}\partial_\alpha N \partial^\alpha N$$
$$- |\phi|^2 \left(\frac{N}{2} - \frac{q^2}{\mu}\right)^2 - \frac{q^2}{2}\left(|\phi|^2 - \frac{\mu}{q^2}\frac{N}{2}\right)^2.$$

- $\alpha, \beta, \gamma \in \{0, 1, 2\}$.
- $\epsilon^{\alpha\beta\gamma}$ is the totally skew-symmetric tensor fixed so that $\epsilon^{012} = 1$
- $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex valued Higgs field.
- $N : \mathbb{R}^{1+2} \to \mathbb{R}$ is the neutral scalar field.
- $A_\alpha : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field.
- $D_\alpha = \partial_\alpha - iA_\alpha$ is the covariant derivative with $i = \sqrt{-1}$.
- $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Maxwell gauge field strength.
- The contant $q > 0$ denotes the eletric charge.
- The contant $\mu > 0$ is the Chern-Simons mass scale.

The elliptic PDE for (MCS) model

- By the Jaffe-Taubes argument (1980),

$$\begin{cases} \Delta u = \lambda\mu e^u - \mu N + 4\pi \sum_{j=1}^{s} n_j \delta_{p_j}, \\ \\ \Delta N = \mu\left(\mu + \lambda e^u\right) N - \lambda\mu(\mu + \lambda)e^u. \end{cases} \qquad (MCS)$$

- e^u : the density of superconducting electron pairs (the Cooper pairs).

- δ_{p_j} : Dirac measure at p_j.

- p_j : vortex point (the absence of electron pairs, i.e. $e^{u(p_j)} = 0$).

- N : the neutral scalar field.

- $\mu > 0$: the Chern-Simons mass scale.

- $\lambda = \frac{2q^2}{\mu}$, where $q > 0$ denotes the eletric charge.

The class of solutions for (MCS) model

- The elliptic PDE for (MCS) model

$$\begin{cases} \Delta u = \lambda\mu e^u - \mu N + 4\pi \sum_{j=1}^{s} n_j \delta_{p_j}, \\ \\ \Delta N = \mu\left(\mu + \lambda e^u\right) N - \lambda\mu(\mu+\lambda)e^u. \end{cases} \qquad (MCS)$$

- In $\mathbb{R}^2$,

- topological solution: $u(\infty) = 0$ and $N(\infty) = \lambda$.

- nontopological solution: $u(\infty) = -\infty$ and $N(\infty) = 0$.

- In a flat two torus Ω,

(i) (CS) limit ($\mu \to \infty$)

- topological solution: $u \to 0$ and $\frac{N}{\lambda} \to 1$ a.e. as $\lambda \to \infty$.

- nontopological solution: $u \to -\infty$ and $\frac{N}{\lambda} \to 0$ a.e. $\lambda \to \infty$.

(ii) (AH) limit ($\mu \to 0$)

- unique periodic solution.

Mathematically rigorous proofs

(i) Chae and Kim (1997)
- the existence and the convergence of topological solutions to the (CS) model and (AH) model in a full space $\mathbb{R}^2$, and on a flat two torus Ω.

(ii) Ricciardi and Tarantello (2000)
- the existence and the convergence of topological solution and mountain pass solution to the (CS) model and (AH) model on Ω.
(here, the convergence of mountain pass solution to (CS) model was only proved when the total number of vortex points is one).

(iii) Ricciardi (2002)
- (CS) convergence in C^n regularity, $\forall n \geq 0$, for an arbitrary sequence of solutions on Ω while $\lambda = 1$.

(iv) Chae and Imanuvilov (2002)
- the existence of non-topological solutions in $\mathbb{R}^2$ by the perturbation theory.

(v) Han and Kim (2005)
- the convergence to the (CS) model and (AH) model for the nonself-dual case.

Our main goals

• to improve and complete the (CS) limit result of (MCS) model without any restriction on either a particular class of solutions, the number of vortex points, or the Chern-Simons parameter λ.

• to derive the relation between the density of superconducting electron pairs e^u and the neutral scalar field N.

• to establish the existence of periodic solutions of (MCS) satisfying the concentrating property so that we could answer the open problem raised by [Tarantello (2004)].

Main result I (Asymptotic behavior of solutions)

Theorem

We assume that $\{(u_{\lambda,\mu}, N_{\lambda,\mu})\}$ is a sequence of solutions of (MCS). Then

$$\lim_{\lambda,\mu\to\infty,\ \frac{\lambda}{\mu}\to 0} \left\| e^{u_{\lambda,\ \mu}} - \frac{N_{\lambda,\mu}}{\lambda} \right\|_{L^\infty(\Omega)} = 0.$$

- The idea of the proof:

Step 1. By applying the Green's representation formula,

$$\left\| \nabla \left(u_{\lambda,\mu} - u_0 + \frac{N_{\lambda,\mu}}{\mu} \right) \right\|_{L^\infty(\Omega)} = O(\lambda).$$

Step 2. By using a suitable scaling, and the nondegeneracy of $-\Delta + 1$ in $\mathbb{R}^2$,

$$\lim_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0} \left\| e^{u_{\lambda,\mu}} - \frac{N_{\lambda,\mu}}{\lambda} \right\|_{L^\infty(\Omega)} = 0.$$

The relation between (MCS) and (CS)

- (MCS) is equivalent to

$$\begin{cases} \Delta(u + \frac{N}{\mu}) = -\lambda^2 e^u \left(1 - \frac{N}{\lambda}\right) + 4\pi \sum_{j=1}^{s} n_j \delta_{p_j}, \\ \Delta N = \mu \left(\mu + \lambda e^u\right) N - \lambda\mu(\mu+\lambda) e^u. \end{cases} \qquad (MCS)$$

- By $\lim_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0} \left\| e^{u_{\lambda,\mu}} - \frac{N_{\lambda,\mu}}{\lambda} \right\|_{L^\infty(\Omega)} = 0$,

$\lambda^2 e^u \left(1 - \frac{N}{\lambda}\right)$ would be a perturbation of $\lambda^2 e^u \left(1 - e^u\right)$ in the following elliptic PDE obtained from (CS) model.

$$\Delta u = -\lambda^2 e^u \left(1 - e^u\right) + 4\pi \sum_{j=1}^{s} n_j \delta_{p_j}. \qquad (CS)$$

Main result II (Asymptotic behavior of solutions)

Theorem

We assume that $\{(u_{\lambda,\mu}, N_{\lambda,\mu})\}$ *is a sequence of solutions of (MCS). As* $\lambda, \mu \to \infty$, $\frac{\lambda}{\mu} \to 0$, *up to subsequences, one of the following holds:*

(i) $u_{\lambda,\mu} \to 0$ *uniformly on any compact subset of* $\Omega \setminus \cup_i \{p_i\}$;

(ii) $u_{\lambda,\mu} + 2\ln\lambda - u_0 \to \hat{w}$ *in* $C^1_{loc}(\Omega)$, *where* $\hat{w}$ *satisfies* $\Delta\hat{w} + e^{\hat{w}+u_0} = 4\pi\mathfrak{M}$;

(iii) there exists a nonempty finite set $B = \{\hat{q}_1, \cdots, \hat{q}_k\} \subset \Omega$ *such that*

$$\lambda^2 e^{u_{\lambda,\mu}} \left(1 - \frac{N_{\lambda,\mu}}{\lambda}\right) \to \sum_j \alpha_j \delta_{\hat{q}_j}, \quad \alpha_j \geq 8\pi,$$

in the sense of measure.

- The idea of the proof:

Blow up analysis developed in Brezis-Merle (1991), Li-Shafrir (1994), Bartolucci-Tarantello (2002), Choe-Kim (2008).

Blow up solutions

$B := \{\hat{q}_j\}_{j=1}^{k}$ and $\{(u_{\lambda,\mu}, N_{\lambda,\mu})\}$ is a family of solutions of (MCS) satisfying

(i) $\lim_{\lambda,\mu\to\infty,\ \frac{\lambda}{\mu}\to 0} (u_{\lambda,\mu} + 2\ln\lambda)(q^j_{\lambda,\mu}) = +\infty$, and

(ii) $\lim_{\lambda,\mu\to\infty,\ \frac{\lambda}{\mu}\to 0} q^j_{\lambda,\mu} = \hat{q}_j$, $j = 1, \cdots, k$.

then B is called a blow-up set and $\{(u_{\lambda,\mu}, N_{\lambda,\mu})\}$ is called a family of blow up solutions (or bubbling solutions) of (MCS) at B.

Main result III (Blow up solutions with lower bound)

• Let $\mathfrak{M} = \sum_{i=1}^{n} m_i$, and $u_0(x) = -4\pi \sum_{i=1}^{n} m_i G(x, p_i)$,
where $G(x, y)$ is the Green's function satisfying

$$-\Delta_x G(x,y) = \delta_y - \frac{1}{|\Omega|}, \quad \int_\Omega G(x,y)dy = 0.$$

Theorem

Assume $\mathfrak{M} > 2$, and $1 \ll (\ln \lambda)\lambda^2 \ll \mu$.

Let $\hat{q}$ be a non-degenerate critical point of u_0.

Then (MCS) has a solution $(u_{\lambda,\mu}, N_{\lambda,\mu})$ satisfying

(i) $\lambda^2 e^{u_{\lambda,\mu}}\left(1 - \frac{N_{\lambda,\mu}}{\lambda}\right) \to 4\pi\mathfrak{M}\delta_{\hat{q}}$ *in the sense of measure as $\lambda, \mu \to \infty$,*

(ii) $\max_{y\in\Omega} u_{\lambda,\mu}(y) \geq c$ *for some constant $c \in \mathbb{R}$,*

(iii) $\frac{N_{\lambda,\mu}}{\lambda} \to 0$ *uniformly on any compact subset of $\Omega \setminus \{\hat{q}\}$ as $\lambda, \mu \to \infty$.*

Idea of the proof for Main result III

- Motivation: (MCS) is equivalent to

$$\begin{cases} \Delta(u + \frac{N}{\mu}) = -\lambda^2 e^u \left(1 - \frac{N}{\lambda}\right) + 4\pi \sum_{j=1}^{s} n_j \delta_{p_j}, \\ \Delta N = \mu \left(\mu + \lambda e^u\right) N - \lambda\mu(\mu + \lambda) e^u. \end{cases} \qquad (MCS)$$

By $\lim_{\lambda,\mu\to\infty, \frac{\lambda}{\mu}\to 0} \left\| e^{u_{\lambda,\mu}} - \frac{N_{\lambda,\mu}}{\lambda} \right\|_{L^\infty(\Omega)} = 0$,

$\lambda^2 e^u \left(1 - \frac{N}{\lambda}\right)$ would be a perturbation of $\lambda^2 e^u \left(1 - e^u\right)$ in the elliptic PDE obtained from (CS) model.

- Approximation solution

$$U_q(y) = w(\lambda|y-q|) - u_0(q) + 4\pi\mathfrak{M}(\gamma(y,q) - \gamma(q,q))(1-\theta) + o(1),$$

where w is the radially symmetric solution of

$$\begin{cases} \Delta w + e^w(1 - e^w) = 0, \text{ in } \mathbb{R}^2, \\ w'(t) \to -\frac{2\mathfrak{M}}{t} + \frac{a_1(2\mathfrak{M}-2)}{t^{2\mathfrak{M}-1}} + O(\frac{1}{t^{2\mathfrak{M}+1}}), \ t \gg 1, \\ w(t) = -2\mathfrak{M} \ln t + I_1 - \frac{a_1}{t^{2\mathfrak{M}-2}} + O(\frac{1}{t^{2\mathfrak{M}}}), \ t \gg 1. \end{cases} \qquad (CS)$$

- Apply the contraction mapping theorem, and observe ∇u_0 as the main error term in the Lyapunov-Schmidt reduction method.

Main result IV (Blow up solutions without lower bound)

• The main error term related to the translation invariance of limiting equation: $G^*(\mathbf{q}) = \sum_{i=1}^{k} u_0(q_i) + 8\pi \sum_{j\neq i} G(q_j, q_i)$, for $\mathbf{q} = (q_1, ..., q_k)$, $q_i \in \Omega$.

• The main error term related to the scaling invariance of limiting equation: $D(\mathbf{q}) = \lim_{r\to 0} \left(\sum_{i=1}^{k} \rho_i \left(\int_{\Omega_i \setminus B_r(q_i)} \frac{e^{f_{\mathbf{q},i}}-1}{|y-q_i|^4} - \int_{\mathbb{R}^2\setminus\Omega_i} \frac{1}{|y-q_i|^4} \right) \right)$, where

(i) $f_{\mathbf{q},i} = 8\pi(\gamma(y, q_i) - \gamma(q_i, q_i) + \sum_{j\neq i}(G(y, q_j) - G(q_i, q_j))) + u_0(y) - u_0(q_i)$.

(ii) $\rho_i = \rho_i(\mathbf{q}) = e^{8\pi(\gamma(q_i,q_i)+\sum_{j\neq i} G(q_i,q_j))+u_0(q_i)}$.

• Let $\mathfrak{M} = 2k \in 2\mathbb{N}$, and $\hat{\mathbf{q}} = (\hat{q}_1, ..., \hat{q}_k)$ be a non-degenerate critical point of $G^*(\mathbf{q})$.

Theorem

Assume that $1 \ll \lambda \ll \mu$ *and* $D(\hat{\mathbf{q}}) < 0$.
Then (MCS) has a solution $(u_{\lambda,\mu}, N_{\lambda,\mu})$ *satisfying*
(i) $\lambda^2 e^{u_{\lambda,\mu}} \left(1 - \frac{N_{\lambda,\mu}}{\lambda}\right) \to 8\pi \sum_{j=1}^{k} \delta_{\hat{q}_j}$, *and* $\frac{e^{u_{\lambda,\mu}}}{\int_\Omega e^{u_{\lambda,\mu}} dx} \to \frac{1}{k} \sum_{j=1}^{k} \delta_{\hat{q}_j}$ *in the sense of measure as* $\lambda, \mu \to \infty$,,
(ii) $\lim_{\lambda,\mu\to\infty} (\max_\Omega u_{\lambda,\mu}) = -\infty$,
(iii) $\lim_{\lambda,\mu\to\infty} \frac{\|N_{\lambda,\mu}\|_{L^\infty(\Omega)}}{\lambda} = 0$.

Idea of the proof for Main result IV

- By $\lim_{\lambda,\mu\to\infty,\frac{\lambda}{\mu}\to 0}\left\|e^{u_{\lambda,\mu}}-\frac{N_{\lambda,\mu}}{\lambda}\right\|_{L^\infty(\Omega)}=0$,

if $\lim_{\lambda,\mu\to\infty}\left(\max_\Omega u_{\lambda,\mu}\right)=-\infty$,

then the blow up profile for $\lambda^2 e^{u_{\lambda,\mu}}\left(1-\frac{N_{\lambda,\mu}}{\lambda}\right)$ is obtained from

$$V_{x_i,\eta_i}(y)=\ln\frac{8\eta_i^2}{(1+\eta_i^2|y-x_i|^2)^2},\ x_i\in\mathbb{R}^2,\ \eta_i>0,$$

which is a solution of Liouville equation:

$$\begin{cases}\Delta V_{x_i,\eta_i}+e^{V_{x_i,\eta_i}}=0 \text{ in } \mathbb{R}^2\\ \int_{\mathbb{R}^2}e^{V_{x_i,\eta_i}}dy=8\pi.\end{cases}$$

Future works

(i) the stability of solutions for the (MCS) model.

(ii) uniqueness or multiplicity of stable solutions, blow up solutions, etc.

(iii) simple / nonsimple blow up phenomena near the singularities.

Thank you for your attention!