Local Well-Posedness for the Free-Boundary MHD Equations with Surface Tension

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Inviscid free-boundary fluid model (Euler equations)

Let v be the velocity of the fluid that occupies $\mathcal{D}_t \subset \mathbb{R}^3$. Its motion is described by

$$\partial_t v + \nabla_v v + \nabla p = 0, \quad \operatorname{div} v = 0, \qquad \text{in } \mathcal{D}.$$

Here, p is the pressure, $\mathcal{D} = \cup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$, $\mathcal{D}_t \approx B(\mathbf{0}, 1)$.
The unknowns are $v = v(t, x)$, $p = p(t, x)$ and $\mathcal{D}_t$.

Initial and Boundary conditions:

$$\begin{cases} \{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \\ v = v_0 \quad \text{in } \{0\} \times \mathcal{D}_0, \end{cases} \quad \begin{cases} D_t := \partial_t + \nabla_v|_{\partial \mathcal{D}} \in \mathcal{T}(\partial \mathcal{D}) \\ p = \sigma \mathcal{H} \quad \text{on } \partial \mathcal{D}, \end{cases}$$

where $\sigma > 0$ and $\mathcal{H} =$ mean curvature of $\partial \mathcal{D}_t$.
LWP is well-known, e.g., Coutand-Shkoller (JAMS, 07), Shatah-Zeng (ARMA, 11), etc.

Real life examples

In many circumstances, the fluid domain moves with the velocity of the fluid, e.g., the ocean surface and a metallic liquid droplet.

Motion of a conducting fluid

We consider the motion of a conducting fluid (i.e., a plasma, a liquid metal) described by

$$\begin{aligned} \partial_t v + \nabla_v v + \nabla p = \mathcal{J} \times B, &\quad \text{in } \mathcal{D}, \\ \partial_t B = -\operatorname{curl} E, &\quad \text{in } \mathcal{D}, \\ \operatorname{div} v = 0, \quad \operatorname{div} B = 0 &\quad \text{in } \mathcal{D}. \end{aligned}$$

Here,

1. v = velocity, and $\mathcal{D} = \cup_{0 \le t \le T} \{t\} \times \mathcal{D}_t$.
2. B = magnetic field of the fluid.
3. E = electric field generated by B.
4. $\mathcal{J}$ = current density, and thus $\mathcal{J} \times B$ is the Lorentz force.

Free-boundary MHD equations

We can simplify the system by removing E and $\mathcal{J}$. Indeed, by invoking

1. Ohm's law of ideal plasma $E = -v \times B$, and
2. Ampère's law $\mathcal{J} = \operatorname{curl} B$,

the above system becomes (recall $D_t := \partial_t + \nabla_v$)

$$\begin{cases} D_t v + \nabla(p + \frac{1}{2}|B|^2) = (B \cdot \nabla)B, & \text{in } \mathcal{D}, \\ D_t B = (B \cdot \nabla)v, & \text{in } \mathcal{D}, \\ \operatorname{div} v = 0, \quad \operatorname{div} B = 0 & \text{in } \mathcal{D}. \end{cases}$$

Here, we define $P := p + \frac{1}{2}|B|^2$ to be the total pressure.

IBVP

We consider

$$\begin{cases} D_t v + \nabla P = (B \cdot \nabla) B, & \text{in } \mathcal{D}, \\ D_t B = (B \cdot \nabla) v, & \text{in } \mathcal{D}, \\ \operatorname{div} v = 0, \quad \operatorname{div} B = 0 & \text{in } \mathcal{D}, \end{cases}$$

equipped with

$$\begin{cases} \{x : (0, x) \in \mathcal{D}\} = \mathcal{D}_0, \\ v = v_0, \quad B = b_0 \quad \text{in } \{0\} \times \mathcal{D}_0, \end{cases} \qquad \begin{cases} D_t|_{\partial \mathcal{D}} \in \mathcal{T}(\partial \mathcal{D}), \\ B \cdot \mathcal{N} = 0, \quad \text{on } \partial \mathcal{D}, \\ P = \sigma \mathcal{H} \quad \text{on } \partial \mathcal{D}, \end{cases}$$

where $\mathcal{N}$ is the outward unit normal of $\partial \mathcal{D}_t$. **This system reduces to Euler equations in the absence of B.**

Several remarks (Goedbloed-Poedts (2004))

i). The IBVP presented above can be viewed as a special case of the general plasma-vacuum interface problem. In the general getting, $\mathcal{D}_t$ is confined in a vacuum region Ω_{vac} with another magnetic field $\hat{B}$ verifies

$$\text{curl}\, \hat{B} = \mathbf{0}, \quad \text{div}\, \hat{B} = 0, \quad \text{in } \Omega_{vac}.$$

ii). We assume that our fluid is a perfect conductor, then $B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N} = 0$. This leads to the boundary condition (the pressure balance law)

$$p + \frac{1}{2}|B|^2 - \frac{1}{2}|\hat{B}|^2 = \sigma\mathcal{H}, \quad \text{on } \partial\mathcal{D}_t.$$

iii). If we neglect the magnetic field in Ω_{vac}, i.e., $\hat{B} = \mathbf{0}$, we get the model we are considering.

The surface tension

Surface tension results from the attraction of fluid molecules to each other. The surface tension is what gives cohesion to the fluid surface, e.g., what keeps the fluid body together.

In particular, the surface tension "stabilizes" the motion of the fluid. In fact, the free-boundary MHD Equations are **ill-posed** when $\sigma = 0$ [Hao-Luo, CMP 20], unless the physical sign condition

$$-\nabla_{\mathcal{N}} P \geq c_0 > 0, \quad \text{on } \partial\mathcal{D}_t.$$

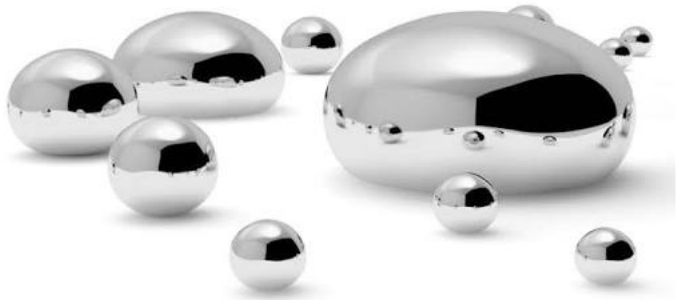
is assumed.

Importance of the surface tension

- All fluids have surface tension. The case without surface tension is merely a simplified model.
- In the case of a conducting fluid, the effect of surface tension is crucially important for modeling liquid metals (See,e.g., Molokov-Reed (2000)).

However, the surface tension brings severe difficulty as it yields top order terms on the boundary.

Mercury droplets



Small droplets take spherical shapes owing to the surface tension, while large droplets are rather flat due to the gravity.

Our main result (rough form)

Theorem (Gu-L.-Zhang, 21')

Let $v_0 \in H^{4.5}$ and $b_0 \in H^{4.5}$ be divergence-free vector fields with $(b_0 \cdot \mathcal{N})|_{\partial \mathcal{D}_0} = 0$. Then the free-boundary MHD equations with initial data (v_0, b_0) admit a unique local strong solution.

The solution is in fact expressed in the Lagrangian coordinates in terms of (η, v, q), where η is the flow map of the velocity v.

Some previous results

Here comes a *non-exclusive* list of results concerning the free-boundary MHD equations.

No surface tension

- A priori estimate: Hao-Luo (ARMA, 14), Hao (ARMA, 17)
- LWP: Gu-Wang (JMPA, 19)
- Compressible MHD (LWP in anisotropic Sobolev space): Trakhinin-Wang (ARMA, 21), a priori estimate by Lindblad-Zhang (21').

With surface tension

- A priori estimate: L.-Zhang (SIMA, 21).
- Compressible MHD (LWP in anisotropic Sobolev space): Trakhinin-Wang (Math. Ann, 21).

The Lagrangian Coordinates

It is convenient to study the free boundary MHD equations in the Lagrangian coordinates (t, y), in which the free boundary becomes fixed.

$$\eta(t, y) = \text{flow map}, \quad \frac{d\eta(t, y)}{dt} = v(t, \eta(t, y)),$$
$$\eta(t, \cdot) : \Omega(= B(\mathbf{0}, 1)) \to \mathcal{D}_t, \quad \eta(0, y) = y.$$

The boundary becomes fixed under (t, y) coordinates.

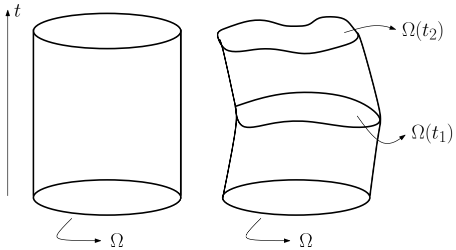


Figure: $\Omega(t) = \mathcal{D}_t$

The Lagrangian variables and derivatives

Let $a := [\partial_y \eta]^{-1}$ cofactor matrix. We introduce

- $D_t = \partial_t$.
- Lagrangian spatial derivative $\partial_\mu = \frac{\partial}{\partial y_\mu}$.
- Eulerian spatial derivative $\nabla_\alpha = (\nabla_a)_\alpha = \frac{\partial}{\partial x_\alpha} = a^\mu{}_\alpha \partial_\mu$.

Set $v = v(t, x(t, y))$ (after a slight abuse of notation), $b = B(t, x(t, y))$, and $q = P(t, x(t, y))$ to be the Lagrangian velocity, magnetic field, and pressure, respectively. Then

- $B \cdot \nabla = b_\beta a^{\mu\beta} \partial_\mu$.
- $\text{div}_a v = a^{\mu\alpha} \partial_\mu v_\alpha$, $\text{div}_a b = a^{\mu\alpha} \partial_\mu b_\alpha$.

The MHD equations in the Lagrangian coordinates

$$\begin{cases} D_t v - (B\cdot\nabla)B + \nabla P = 0, \quad \text{in } \mathcal{D}, \\ D_t B = (B\cdot\nabla)v, \quad \text{in } \mathcal{D}, \\ \operatorname{div} v = 0, \quad \operatorname{div} B = 0 \quad \text{in } \mathcal{D}, \end{cases}$$

becomes

$$\begin{cases} \partial_t v_\alpha - b_\beta a^{\mu\beta}\partial_\mu b_\alpha + a^\mu{}_\alpha \partial_\mu q = 0 & \text{in } [0,T]\times\Omega; \\ \partial_t b_\alpha - b_\beta a^{\mu\beta}\partial_\mu v_\alpha = 0 & \text{in } [0,T]\times\Omega; \\ a^{\mu\alpha}\partial_\mu v_\alpha = 0, \quad a^{\mu\alpha}\partial_\mu b_\alpha = 0 & \text{in } [0,T]\times\Omega. \end{cases}$$

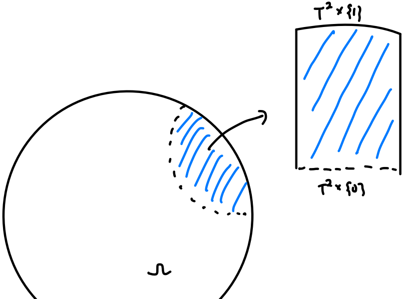
The first simplification: reference domain

For the sake of simplicity and clean notation, we take

$$\Omega = \mathbb{T}^2 \times (0,1),$$

where $\mathbb{T}^2$ is the 2-torus, $\partial\Omega = \Gamma_0 \cup \Gamma$ and $\Gamma = \mathbb{T}^2 \times \{1\}$ is the top boundary (corresponds to the moving boundary), $\Gamma_0 = \mathbb{T}^2 \times \{0\}$ is the fixed bottom. **The choice of Ω allows us to work in one coordinate patch**.

Under this setting, (y_1, y_2)-direction is tangential to Γ, y_3-direction is normal to Γ, and the outward unit normal to Γ is $N = (0,0,1)$.



The Lagrangian boundary condition

Under this setting, $P = \sigma \mathcal{H}$ becomes

$$\underbrace{a^{\mu\alpha} N_\mu}_{a^{3\alpha}} q + \sigma(\sqrt{g}\Delta_g \eta^\alpha) = 0, \quad \text{on } \Gamma,$$

where $g_{ij} = \overline{\partial}_i \eta^\mu \overline{\partial}_j \eta_\mu$, $\Delta_g(\cdot) = \frac{1}{\sqrt{g}} \overline{\partial}_i(\sqrt{g} g^{ij} \overline{\partial}_j(\cdot))$, with $g = det(g_{ij})$.

Set $\hat{n}^\alpha = \mathcal{N}^\alpha \circ \eta = \frac{a^{3\alpha}}{\sqrt{g}}$ =unit outer normal to $\eta(\Gamma)$. Then the BC on Γ can be re-formulated as

$$a^{3\alpha} q = -\sigma(\sqrt{g}\Delta_g \eta \cdot \hat{n})\hat{n}^\alpha, \quad \text{on } \Gamma.$$

The second simplification: a remarkable identity

Let's recall the transport equations verified by b, namely

$$\partial_t b_\alpha - b_\beta a^{\mu\beta} \partial_\mu v_\alpha = 0.$$

By contracting this equation with $a^{\nu\alpha}$, we have

$$a^{\nu\alpha}\partial_t b_\alpha - b_\beta a^{\mu\beta} a^{\nu\alpha} \partial_\mu \partial_t \eta_\alpha = 0.$$

Since $a^{\nu\alpha}\partial_\mu \eta_\alpha = \delta^\nu_\mu$, the equation above becomes

$$a^{\nu\alpha}\partial_t b_\alpha + b_\beta a^{\mu\beta} (\partial_t a^{\nu\alpha}) \partial_\mu \eta_\alpha = 0,$$

which yields $\partial_t(a^{\nu\alpha} b_\alpha) = 0$. Thus

$$a^{\nu\alpha} b_\alpha = a^{\nu\alpha} b_\alpha|_{t=0} = b_0^\nu. \tag{0.1}$$

By plugging (0.1) back to $\partial_t b_\alpha - b_\beta a^{\mu\beta} \partial_\mu v_\alpha = 0$, we obtain

$$\partial_t b_\alpha - b_0^\mu \partial_\mu \partial_t \eta_\alpha = 0,$$

which is

$$\partial_t (b_\alpha - b_0^\mu \partial_\mu \eta_\alpha) = 0.$$

This implies

$$b_\alpha = b_0^\mu \partial_\mu \eta_\alpha = (b_0 \cdot \partial) \eta_\alpha. \tag{0.2}$$

Notice that this identity allows us to *represent b in terms of its initial data and* η.

The reformulated MHD equations

$$
\begin{cases}
\partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_a q = 0 & \text{in } [0,T] \times \Omega; \\
\operatorname{div}_a v = 0, & \text{in } [0,T] \times \Omega; \\
\operatorname{div} b_0 = 0 & \text{in } \{t=0\} \times \Omega; \\
v^3 = b_0^3 = 0 & \text{on } \Gamma_0; \\
a^{3\alpha} q + \sigma(\sqrt{g} \Delta_g \eta^\alpha) = 0 & \text{on } \Gamma; \\
b_0^3 = 0 & \text{on } \Gamma, \\
(\eta, v) = (\text{id}, v_0) & \text{on } \{t=0\} \times \overline{\Omega}.
\end{cases} \tag{0.3}
$$

We only need to solve η (which characterizes the moving domain) and v.

The main result

Theorem (Gu-L.-Zhang, 21')

Let $\eta_0 \in H^{4.5}(\Omega)$. Let $v_0 \in H^{4.5}(\Omega)$ and $b_0 \in H^{4.5}(\Omega)$ be divergence-free vector fields with $(b_0 \cdot N)|_{\Gamma \cup \Gamma_0} = 0$. Then there exists some $T > 0$, only depending on σ, v_0, b_0, such that the system (0.3) with initial data (η_0, v_0, b_0) has a unique strong solution (η, v, q) with the energy estimates

$$\sup_{0 \le t \le T} \left(\|\eta(t)\|_{4.5}^2 + \|v(t)\|_{4.5}^2 + \|(b_0 \cdot \partial)\eta(t)\|_{4.5}^2 \right) \le \mathcal{C},$$

where $\mathcal{C}$ is a constant depends on $\|v_0\|_{4.5}, \|b_0\|_{4.5}$.

MHD vs Euler

1. No extra regularity can be assumed on the vorticity.
   For Euler equations, one can impose extra regularity on the initial vorticity $\operatorname{curl} v_0$ and this leads to extra regularity on the flow-map η. As a consequence, η may be more regularity than v.
   However, η cannot be more regular than v in MHD equations.
2. There is a strong coupling between the velocity and magnetic fields.
   In particular, $\operatorname{curl} v$ and $\operatorname{curl} (b_0 \cdot \partial)\eta$ have to be studied together. As a consequence, **the full (interior) Sobolev norm of v (and its time derivatives) must be controlled together with the corresponding Sobolev norm of $(b_0 \cdot \partial)\eta$.**

We will revisit these two points when we construct the energy functional.

The property of $(b_0 \cdot \partial)$ derivative

There is a simple yet remarkable observation on the $(b_0 \cdot \partial)$ derivative. The energy estimate

$$\sup_{0 \le t \le T} \left(\|\eta(t)\|_{4.5}^2 + \|v(t)\|_{4.5}^2 + \|(b_0 \cdot \partial)\eta(t)\|_{4.5}^2 \right) \le \mathcal{C}(\|v_0\|_{4.5}, \|b_0\|_{4.5})$$

yields that $\partial_t \eta = v$ and $(b_0 \cdot \partial)\eta$ are **both** controlled in $H^{4.5}$; in other words, $(b_0 \cdot \partial)$ behaves exactly like the time derivative ∂_t when acting on η!

This observation plays a crucial role in the energy estimate.

The general strategy on constructing solutions

By replacing the nonlinear coefficients with given functions, the linearized equations take the form:

$$\begin{cases} \partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_{\mathring{a}} q = 0, \quad \text{div}_{\mathring{a}} v = 0, \quad \text{in } \Omega, \\ \mathring{a}^{3\alpha} q + \sigma(\sqrt{\mathring{g}} \Delta_{\mathring{g}} \eta^\alpha) = 0, \quad \text{on } \Gamma, \end{cases}$$

where $\mathring{a} = [\partial \mathring{\eta}]^{-1}$ and $\mathring{g}_{ij} = \overline{\partial}_i \mathring{\eta}^\mu \overline{\partial}_j \mathring{\eta}_\mu$. Unfortunately, the a priori energy estimate for free-boundary MHD equations cannot be carried over to the linearized problem since it destroys the symmetry.

To overcome this, we will consider a suitable approximate problem which is asymptotically consistent with the a priori estimates. **We will construct the approximate problem by adapting Coutand and Shkoller's idea, i.e., the artificial viscosity and smoothed κ-problem.**

The artificial viscosity

To overcome the above problem, we introduce an artificial viscosity term in the Laplace-Young BC

$$a^{3\alpha} q = -\sigma(\sqrt{g}\Delta_g \eta \cdot \hat{n})\hat{n}^\alpha + \kappa(1 - \overline{\Delta})(v \cdot \hat{n})\hat{n}^\alpha.$$

Here, $\overline{\Delta} = \overline{\partial}_1^2 + \overline{\partial}_2^2$. With the addition of the artificial viscosity term, the

BC is converted into a parabolic-type BC, and thus finding a solution (κ fixed) for the following linearized equations

$$\begin{cases} \partial_t \eta = v & \text{in } \Omega; \\ \partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_{\mathring{a}} q = 0 & \text{in } \Omega; \\ \mathring{a}^{3\alpha} q = -\sigma(\sqrt{\mathring{g}}\Delta_{\mathring{g}} \mathring{\eta} \cdot \mathring{n})\mathring{n}^\alpha + \kappa(1 - \overline{\Delta})(v \cdot \mathring{n})\mathring{n}^\alpha & \text{on } \Gamma, \end{cases}$$

becomes easy (e.g., by Galerkin's method).

Loss of regularity on Γ

The next step is to use the solution to the linearized equations to approximate the solution to the nonlinear equations. This requires to study the energy estimate.

The standard Hodge elliptic estimate yields that we need to control $|v^3|_4$ (boundary Sobolev norms). However, the BC

$$\mathring{a}^{3\alpha} q = -\sigma(\sqrt{\mathring{g}} \Delta_{\mathring{g}} \mathring{\eta} \cdot \mathring{n}) \mathring{n}^\alpha + \kappa(1 - \overline{\Delta})(v \cdot \mathring{n}) \mathring{n}^\alpha$$

yields merely the control of $v \cdot \mathring{n}$. Therefore, we need to control

$$|v \cdot (N - \mathring{n})|_4.$$

But this yields a loss of regularity since $N - \mathring{n} = -\int_0^t \partial_t \mathring{n} \sim -\int_0^t \partial \mathring{v}$, which cannnot be controlled in $H^4(\Gamma)$.

The tangential smoothing

To overcome this issue, we replace $\mathring{n}$ by its "tangentially smoothed" version. Specifically, let $\zeta = \zeta(y_1, y_2) \in C_c^\infty(\mathbb{R}^2)$ be a standard cut-off function supported in the unit ball. For each $\kappa > 0$, Let

$$\zeta_\kappa(y_1, y_2) = \frac{1}{\kappa^2}\zeta\left(\frac{y_1}{\kappa}, \frac{y_2}{\kappa}\right).$$

Define

$$\Lambda_\kappa f(y_1, y_2, y_3) := \int_{\mathbb{R}^2} \zeta_\kappa(y_1 - z_1, y_2 - z_2) f(z_1, z_2)\, dz_1\, dz_2,$$

and let

$$\begin{cases} -\Delta\widetilde{\eta} = -\Delta\eta, & \text{in } \Omega, \\ \widetilde{\eta} = \Lambda_\kappa^2 \eta & \text{on } \partial\Omega. \end{cases}$$

Smoothed equations

We are now able to define the smoothed cofactor matrix $\widetilde{a} := [\partial\widetilde{\eta}]^{-1}$, as well as the smoothed $\tilde{g}_{ij}$ and $\tilde{n}$. Given these, we have the smoothed MHD equations (κ-problem)

$$\begin{cases} \partial_t \eta = v & \text{in } \Omega; \\ \partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_{\widetilde{a}} q = 0 & \text{in } \Omega; \\ \widetilde{a}^{3\alpha} q = -\sigma(\sqrt{g}\Delta_g \eta \cdot \tilde{n})\tilde{n}^{\alpha} + \kappa(1-\overline{\Delta})(v \cdot \tilde{n})\tilde{n}^{\alpha} & \text{on } \Gamma, \end{cases}$$

as well as the linearized equations

$$\begin{cases} \partial_t \eta = v & \text{in } \Omega; \\ \partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_{\mathring{\tilde{a}}} q = 0 & \text{in } \Omega; \\ \mathring{\tilde{a}}^{3\alpha} q = -\sigma(\sqrt{\mathring{\tilde{g}}}\Delta_{\mathring{\tilde{g}}} \mathring{\eta} \cdot \mathring{\tilde{n}})\mathring{\tilde{n}}^{\alpha} + \kappa(1-\overline{\Delta})(v \cdot \mathring{\tilde{n}})\mathring{\tilde{n}}^{\alpha} & \text{on } \Gamma. \end{cases}$$

Notice that $|N - \mathring{\tilde{n}}|_4 \lesssim \int_0^t |\partial \mathring{\tilde{v}}|_4 \lesssim_{\kappa^{-1}} \int_0^t |\mathring{v}|_4$.

Uniform-in-κ energy estimate for the nonlinear κ-problem

Now, each κ-problem (with fixed κ) admits a solution in $[0, T_\kappa]$.

Our final step is to prove an uniform-in-κ energy estimate in some fixed time interval $[0, T]$, which then allows us to obtain a solution to the original problem by letting $\kappa \to 0$.

The crucial step is to come up with a correct energy functional. We recall
i). η has to be as regular as v.
ii). In Ω, v and $(b_0 \cdot \partial)\eta$ have to be controlled in the same Sobolev spaces due to coupling. Indeed, the evolution equation verified by $\mathrm{curl}_{\tilde{a}} v$ reads

$$\partial_t(\mathrm{curl}_{\tilde{a}} v) - (b_0 \cdot \partial)\mathrm{curl}_{\tilde{a}}((b_0 \cdot \partial)\eta) = \text{commutator}.$$

More observations

iii). Mixed space-time derivatives: Our energy consists terms with mixed space-time derivatives. The time derivatives are needed since we will study the elliptic estimates of q using Neumann BC

$$\tilde{n} \cdot \nabla_{\tilde{a}} q = \partial_t v \cdot \tilde{n} + (b_0 \cdot \partial)^2 \eta \cdot \tilde{n},$$

obtained by taking the $\tilde{n}$-normal component of $\partial_t v - (b_0 \cdot \partial)^2 \eta + \nabla_{\tilde{a}} q = 0$.

iv). The full Sobolev norms of v and $(b_0 \cdot \partial)\eta$: The div-curl estimate

$$\|X\|_s \lesssim \|\text{div}\, X\|_{s-1} + \|\text{curl}\, X\|_{s-1} + |\overline{\partial} X^3|_{s-\frac{3}{2}}.$$

will be used to study the estimate of v and $(b_0 \cdot \partial)\eta$ and their time derivatives. Here, $|\overline{\partial} X^3|_{s-\frac{3}{2}}$ is comparable with $|\overline{\partial}(\Pi X)^3|_{s-\frac{3}{2}}$, $\Pi^\alpha_\lambda = \hat{n}^\alpha \hat{n}_\lambda$. This term appears in the tangential energy estimates. In particular, we will study the tangential energy estimates with

$$\partial_t^4, \overline{\partial}\partial_t^3, \overline{\partial}^2\partial_t^2, \overline{\partial}^3\partial_t, \overline{\partial}^3(b_0 \cdot \partial)$$

which yield

$$|\overline{\partial}(\Pi\partial_t^3 v)^3|_0, |\overline{\partial}(\Pi\overline{\partial}\partial_t^2 v)^3|_0, |\overline{\partial}(\Pi\overline{\partial}^2\partial_t v)^3|_0,$$
$$|\overline{\partial}(\Pi\overline{\partial}^3 v)^3|_0, |\overline{\partial}(\Pi\overline{\partial}^3(b_0 \cdot \partial)\eta)^3|_0,$$

respectively.

v). The weighted energy functional: Let's consider the tangential energy estimate with ∂_t^4. The term corresponds to the artificial viscosity is

$$-\kappa \int_0^T \int_\Gamma \partial_t^4 v_\alpha \partial_t^4 \left((1-\overline{\Delta})(v\cdot\tilde{n})\tilde{n}^\alpha\right).$$

Up to the top order, this yields the positive energy term $\kappa \int_0^T \int_\Gamma |\partial_t^4 v \cdot \tilde{n}|_1^2$ with the error term

$$\kappa \int_0^T \int_\Gamma (\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot \partial_t^4\overline{\partial}\tilde{n}) \sim \kappa \int_0^T \int_\Gamma (\overline{\partial}\partial_t^4 v\cdot\tilde{n})(v\cdot \overline{\partial}^2\partial_t^3\tilde{v}).$$

We have to control $\overline{\partial}^2\partial_t^3 v$ directly by the trace lemma, and this suggests that our energy should contain $\int_0^T \|\sqrt{\kappa}\partial_t^3 v\|_{2.5}$, and in view of ii). we also need $\int_0^T \|\sqrt{\kappa}\partial_t^3(b_0\cdot\partial)\eta\|_{2.5}$.

The full energy

Let $E = E^{(1)} + \kappa E^{(2)} + \kappa E^{(3)}$, where

$$E^{(1)} := \|\eta\|_{4.5}^2 + \|v\|_{4.5}^2 + \|\partial_t v\|_{3.5}^2 + \left\|\partial_t^2 v\right\|_{2.5}^2 + \left\|\partial_t^3 v\right\|_{1.5}^2 + \left\|\partial_t^4 v\right\|_0^2$$
$$+ \|(b_0 \cdot \partial)\eta\|_{4.5}^2 + \|\partial_t(b_0 \cdot \partial)\eta\|_{3.5}^2 + \left\|\partial_t^2(b_0 \cdot \partial)\eta\right\|_{2.5}^2 + \left\|\partial_t^3(b_0 \cdot \partial)\eta\right\|_{1.5}^2$$
$$+ \left\|\partial_t^4(b_0 \cdot \partial)\eta\right\|_0^2 + \left|\overline{\partial}\left(\Pi\partial_t^3 v\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}\partial_t^2 v\right)\right|_0^2 + \left|\overline{\partial}\left(\Pi\overline{\partial}^2\partial_t v\right)\right|_0^2$$
$$+ \left|\overline{\partial}(\Pi\overline{\partial}^3 v)\right|_0^2 + \left|\overline{\partial}(\Pi\overline{\partial}^3(b_0 \cdot \partial)\eta)\right|_0^2,$$

$$E^{(2)} := \sigma \int_0^T \left(\left|\partial_t^4 v \cdot \tilde{n}\right|_1^2 + \left|\partial_t^3 v \cdot \tilde{n}\right|_2^2 + \left|\partial_t^2 v \cdot \tilde{n}\right|_3^2 \right.$$
$$\left. + |\partial_t v \cdot \tilde{n}|_4^2 + |(b_0 \cdot \partial)v \cdot \tilde{n}|_4^2 \right) dt,$$

$$E^{(3)} := \int_0^T \left(\left\|\partial_t^4 v\right\|_{1.5}^2 + \left\|\partial_t^4(b_0 \cdot \partial)\eta\right\|_{1.5}^2 + \left\|\partial_t^3 v\right\|_{2.5}^2 + \left\|\partial_t^3(b_0 \cdot \partial)\eta\right\|_{2.5}^2\right.$$
$$\left. + \left\|\partial_t^2 v\right\|_{3.5}^2 + \left\|\partial_t^2(b_0 \cdot \partial)\eta\right\|_{3.5}^2 + \left\|\partial_t v\right\|_{4.5}^2 + \left\|\partial_t(b_0 \cdot \partial)\eta\right\|_{4.5}^2\right) dt.$$

Thank you!