Some results on the blow-up solutions of the fast diffusion equation

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Abstract

In this talk I will discuss some recent results on the blow-up solutions of the fast diffusion equation. I will also discuss my recent result on the existence and large time behaviour of the one point and finite points blow-up solutions of the fast diffusion equation. This is joint work with Sunghoon Kim, Soojung Kim and Jinwan Park.

The equation

$$u_t = \Delta u^m \tag{FDE}$$

arises in many physical models and has many geometric applications.

- For $m > 1$, (FDE) is called the porous medium equation which arises in the flow of gases through porous media or oil passing through sand etc.
- Recently Huang, Pan and Wang (Arch. Rat. Mech. Anal. 2011), T. Luo and H. Zeng (Comm. Pure Applied Math. 2015) have shown that (FDE) is also the large time asymptotic limit solution of the compressible Euler equation with damping.
- F. Golse and F. Salvarani (Nonlinearity, 2007), B. Choi and K. Lee (arxiv: 1510.08997v1) have shown that (FDE) also appears as the nonlinear diffusion limit for the generalized Carleman models.
- For $m = 1$, (FDE) is the heat equation.

- For $0 < m < 1$, (FDE) is called the fast diffusion equation. When $m = \frac{n-2}{n+2}$ and $n \geq 3$, the equation (FDE) arises in the study of the Yamabe flow

$$\frac{\partial g}{\partial t} = -Rg \tag{YF}$$

on $\mathbb{R}^n$ where R is the scalar curvature of the metric $g(x, t)$ at time t. In fact if $g = u^{\frac{4}{n+2}} dx^2$ is a metric on $\mathbb{R}^n$, $n \geq 3$, which evolves by the Yamabe flow, then the scalar curvature is given by

$$R = -\frac{4(n-1)}{(n-2)} u^{-1} \Delta u^{\frac{n-2}{n+2}} \tag{1}$$

By (YF) and (1), u satisfies

$$u_t = \frac{n-1}{m} \Delta u^m$$

with $m = \frac{n-2}{n+2}$ which is equivalent to (FDE) after a rescaling.

- Since

$$\Delta u^m = m \,\mathrm{div}\,(u^{m-1}\nabla u)$$

and

$$u^{m-1} \to \begin{cases} \infty & \text{as } u \to \infty \\ 0 & \text{as } u \to 0 \end{cases} \qquad \text{when } m > 1$$

$$u^{m-1} \to \begin{cases} 0 & \text{as } u \to \infty \\ \infty & \text{as } u \to 0 \end{cases} \qquad \text{when } 0 < m < 1,$$

the equation

$$u_t = \Delta u^m \tag{FDE}$$

is not uniformly parabolic for $m > 0$ and $m \neq 1$. Hence standard theory of parabolic partial differential equation cannot be applied directly to study this equation. This makes the study of (FDE) a hard problem.

- Recently there are a lot of interest on the study of singular solutions of the Yamabe flow and the fast diffusion equation

$$u_t = \Delta u^m \tag{FDE}$$

for the case $0 < m < \frac{n-2}{n}$.

- J.L. Vazquez and M. Winkler (SIAM J. Math. Anal. 2011) studied the one point singularity solutions of the fast diffusion equation (FDE) in bounded domains in $\mathbb{R}^n$.

  Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^3$, $\psi \in C^1(\overline{\Omega})$ be a given function that is positive on $\partial\Omega$. Let u_0 be a function on Ω that satisfies

$$c_1|x|^{-\gamma_1} \leq u_0(x) \leq c_2|x|^{-\gamma_2} \quad \forall x \in \Omega \setminus \{0\}$$

  for some constants $c_1 > 0$, $c_2 > 0$, $\gamma_2 \geq \gamma_1 > 0$. J.L. Vazquez and M. Winkler proved that the following boundary value problem,

$$\begin{cases} u_t = \Delta u^m & \text{in } (\Omega \setminus \{0\}) \times (0, \infty) \\ u(x,t) = \psi(x) & \text{on } \partial\Omega \times (0, \infty) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{BP}$$

  has a solution which has the following properties:

- If $\gamma_2 < \frac{2}{1-m}$, then immediate regularization occur and there exists a solution u of (BP) which can be extended to a smooth function on $\Omega \times (0, \infty)$.

- If $\frac{2}{1-m} < \gamma_1 \leq \gamma_2 < \frac{n-2}{m}$, then u exhibits an infinite time blow-down. That is for all $T > 0$ there exists a constant $C(T) > 0$

such that

$$c(T)|x|^{-\gamma_1} \le u(x,t) \le c_2|x|^{-\gamma_2} \quad \forall x \in \Omega, 0 < t < T.$$

and there exists a constant $c_3 > 0$ such that for any $r > 0$,

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega\setminus B_r(0))} \le c_3.$$

- If $\frac{n-2}{m} < \gamma_1 \le \gamma_2$, u blows up in infinite time in the sense that

$$u(x,t) \to \infty \quad \text{locally uniformly in } \Omega \setminus \{0\} \quad \text{as } t \to \infty.$$

- A natural question to ask is whether similar results still hold for solutions of the fast diffusion equation (FDE) with a finite number of singularities. We answer this question in the affirmative.

- Let $n \geq 3$, $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $a_1, a_2, \ldots, a_{i_0} \in \Omega$,

$$\widehat{\Omega} = \Omega \setminus \{a_1, a_2, \ldots, a_{i_0}\} \quad \text{and} \quad \widehat{\mathbb{R}^n} = \mathbb{R}^n \setminus \{a_1, a_2, \ldots, a_{i_0}\}.$$

Let

$$\delta_0 = \delta_0(\Omega) = \frac{1}{3} \min_{1 \leq i, j \leq i_0} \left(\mathbf{dist}(a_i, \Omega), \left| a_i - a_j \right| \right)$$

or

$$\delta_0 = \delta_0(\mathbb{R}^n) = \frac{1}{3} \min_{1 \leq i, j \leq i_0} |a_i - a_j|.$$

- We found the blow-up rate of such solutions near the blow-up points

$$a_1, a_2, \ldots, a_{i_0},$$

When $0 < m \leq \frac{n-2}{n}$, existence of singular solutions of (FDE) in $\widehat{\Omega} \times (0, T)$ which blows up at $\{a_1, a_2, \ldots, a_{i_0}\} \times (0, T)$ was proved by K.M. Hui and S. Kim (Discrete and Contin. Dynamical Systems-Series A, 2015) when the initial value u_0 satisfies

$$u_0(x) \approx |x - a_i|^{-\gamma_i} \quad \text{for } x \approx a_i \quad \forall i = 1, 2, \ldots, i_0$$

for some constants $\gamma_i > \max\left(\frac{n}{2m}, \frac{n-2}{m}\right)$ for any $i = 1, 2, \ldots, i_0$.

- Later K.M. Hui and Sunghoon Kim (Calc. Var. PDE 2018, 57: 112) proved the existence of solution u of (FDE) in $\widehat{\Omega} \times (0, \infty)$ and $\widehat{\mathbb{R}^n} \times (0, \infty)$ which satisfies

$$u(x,t) \to \infty \quad \text{as } x \to a_i \quad \forall t > 0, i = 1, \cdots, i_0,$$

when $0 < m < \frac{n-2}{n}$, $n \geq 3$, and the initial value satisfies

$$0 \leq u_0 \in L^p_{loc}(\widehat{\Omega}) \quad (u_0 \in L^p_{loc}(\widehat{\mathbb{R}^n}) \text{ respectively})$$

for some constant $p > \frac{n(1-m)}{2}$ such that

$$u_0(x) \geq \frac{\lambda_i}{|x - a_i|^{\gamma_i}} \qquad \forall 0 < |x - a_i| < \delta_1, \; i = 1, \cdots, i_0$$

holds for some constants $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$ and

$$\gamma_i > \frac{2}{1-m} \quad \forall i = 1, \cdots, i_0.$$

We also obtain the asymptotic large time behaviour of such singular solutions.

- K.M. Hui (Proc. the Royal Society of Edin. Section A: Math., 2020) also proved the uniqueness of such singular solutions.

- K.M. Hui and Sunghoon Kim find that the asymptotic large time behaviour of such solutions depends on the blow-up rate of the initial value u_0 at the singular points $a_1, a_2, \ldots, a_{i_0}$, and the lower bound of u_0.

- If the initial value satisfies

$$u_0 \geq \mu_0 \quad \text{on } \widehat{\Omega} \quad (\widehat{\mathbb{R}^n} \text{ respectively})$$

for some constant $\mu_0 > 0$,

$$u_0(x) \geq \frac{\lambda_i}{|x - a_i|^{\gamma_i}} \qquad \forall 0 < |x - a_i| < \delta_1, \ \ i = 1, \cdots, i_0$$

for some constants $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$, and

$$\gamma_1 > \frac{n-2}{m}, \quad \gamma_i > \frac{2}{1-m} \quad \forall i = 2, \ldots, i_0,$$

then the singular solution converges locally uniformly on every compact subset of $\widehat{\Omega}$ (or $\widehat{\mathbb{R}^n}$ respectively) to infinity as $t \to \infty$.

- When $u_0 \geq \mu_0$ on $\widehat{\Omega}$ ($\widehat{\mathbb{R}^n}$, respectively) for some constant $\mu_0 > 0$ satisfies satisfy

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_1, \ i = 1, \cdots, i_0,$$

  for some constants $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$, and

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < \frac{n-2}{m} \quad \forall i = 1, 2, \ldots, i_0,$$

  we prove that u converges converges in $C^2(K)$ for any compact subset K of $\overline{\Omega} \setminus \{a_1, a_2, \ldots, a_{i_0}\}$ (or $\widehat{\mathbb{R}^n}$ respectively) to a harmonic function as $t \to \infty$.

- More precisely we prove the following existence, uniqueness and convergence results.

Theorem 1. *Let* $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \delta_0$, $0 \leq f \in L^\infty(\partial\Omega \times [0,\infty))$ *and* $0 \leq u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ *for some constant* $p > \frac{n(1-m)}{2}$ *be such that*

$$u_0(x) \geq \frac{\lambda_i}{|x-a_i|^{\gamma_i}} \qquad \forall 0 < |x-a_i| < \delta_1, \; i = 1, \cdots, i_0.$$

holds for some constants $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$ *and* $\gamma_1, \cdots, \gamma_{i_0} \in \left(\frac{2}{1-m}, \infty\right)$. *Then there exists a solution* u *of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u = f & \text{on } \partial\Omega \times (0,\infty) \\ u(a_i, t) = \infty & \forall t > 0, i = 1, \cdots, i_0 \\ u(x,0) = u_0(x) & \text{in } \widehat{\Omega} \end{cases} \tag{DP}$$

such that for any $T > 0$ *and* $\delta_2 \in (0, \delta_1)$ *there exists a constant* $C_1 > 0$ *such that*

$$u(x,t) \geq \frac{C_1}{|x-a_i|^{\gamma_i}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T.$$

Moreover if there exists a constant $T_0 \geq 0$ *such that*

$$f(x,t) \text{ is monotone decreasing in } t \text{ on } \partial\Omega \times (T_0, \infty),$$

then u *satisfies*

$$u_t \leq \frac{u}{(1-m)(t-T_0)} \qquad \text{in } \widehat{\Omega} \times (T_0, \infty). \tag{AB}$$

Theorem 2. *Let* $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \delta_0$ *and* $0 \leq u_0 \in L^p_{loc}\left(\widehat{\mathbb{R}^n}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *such that*

$$u_0(x) \geq \frac{\lambda_i}{|x - a_i|^{\gamma_i}} \qquad \forall 0 < |x - a_i| < \delta_1, \ i = 1, \cdots, i_0.$$

holds for some constants $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$ *and* $\gamma_1, \cdots, \gamma_{i_0} \in \left(\frac{2}{1-m}, \infty\right)$. *Then there exists a solution* u *of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\mathbb{R}^n} \times (0, \infty) \\ u(a_i, t) = \infty & \forall i = 1, \cdots, i_0, \ t > 0 \\ u(x, 0) = u_0(x) & \text{in } \widehat{\mathbb{R}^n} \end{cases} \tag{CP}$$

such that for any $T > 0$ *and* $\delta_2 \in (0, \delta_1)$ *there exists a constant* $C_1 > 0$ *such that* (1) *and*

$$u_t \leq \frac{u}{(1 - m)t} \quad \text{in } \widehat{\mathbb{R}^n} \times (0, \infty)$$

hold.

Theorem 3. *Let* $n \geq 3$, $0 < m < \frac{n-2}{n}$, $p > \frac{n(1-m)}{2}$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $f_1, f_2 \in C^3(\partial\Omega \times (0,\infty)) \cap L^\infty(\partial\Omega \times (0,\infty))$ *be such that*

$$f_2 \geq f_1 \geq \mu_0 \quad \text{on } \partial\Omega \times (0,\infty)$$

and

$$\mu_0 \leq u_{0,1} \leq u_{0,2} \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$$

be such that

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_{0,1}(x) \leq u_{0,2} \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}}$$

holds for any $0 < |x-a_i| < \delta_1, i = 1, \cdots, i_0$ *and some constants* $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ *and*

$$\gamma_i' \geq \gamma_i > \frac{2}{1-m} \quad \forall i = 1, 2, \ldots, i_0.$$

Suppose u_1, u_2, *are the solutions of* (DP) *with* $u_0 = u_{0,1}, u_{0,2}$, $f = f_1, f_2$, *respectively which satisfy*

$$u_j(x,t) \geq \mu_0 \quad \forall x \in \widehat{\Omega}, t > 0, j = 1, 2 \tag{2}$$

such that for any constants $T > 0$ *and* $\delta_2 \in (0, \delta_1)$ *there exist constants* $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, *such that*

$$\frac{C_1}{|x-a_i|^{\gamma_i}} \leq u_j(x,t) \leq \frac{C_2}{|x-a_i|^{\gamma_i'}} \tag{3}$$

holds for any $0 < |x-a_i| < \delta_2, 0 < t < T, i = 1, 2, \dots, i_0, j = 1, 2.$ *Suppose* u_1, u_2, *also satisfy*

$$\|u_i(\cdot, t) - u_{0,i}\|_{L^1(\Omega_\delta)} \to 0 \quad \text{as } t \to 0 \quad \forall 0 < \delta < \delta_0, i = 1, 2. \tag{4}$$

Then

$$u_1(x, t) \le u_2(x, t) \quad \forall x \in \widehat{\Omega}, t > 0. \tag{5}$$

Theorem 4. *Let* $n \ge 3$, $0 < m < \frac{n-2}{n}$, $p > \frac{n(1-m)}{2}$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$,

$$\mu_0 \le f_1 \le f_2 \in L^\infty(\partial\Omega \times (0, \infty)),$$

$$\mu_0 \le u_{0,1} \le u_{0,2} \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}),$$

and

$$\frac{\lambda_i}{|x - a_i|^{\gamma_i}} \le u_{0,1}(x) \le u_{0,2} \le \frac{\lambda_i'}{|x - a_i|^{\gamma_i'}}$$

holds for any $0 < |x - a_i| < \delta_1, i = 1, \cdots, i_0$ *and some constants* $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ *and*

$$\frac{2}{1-m} < \gamma_i \le \gamma_i' < \frac{n-2}{m} \quad \forall i = 1, 2, \dots, i_0. \tag{6}$$

Suppose u_1, u_2, *are the solutions of* (DP) *with* $u_0 = u_{0,1}, u_{0,2}$, $f = f_1, f_2$, *respectively which satisfy* (2), (3) *and* (4). *Then* $u_1(x, t) \le u_2(x, t)$ *for any* $x \in \widehat{\Omega}, t > 0$.

Theorem 5. *Let* $n \geq 3$, $0 < m < \frac{n-2}{n}$, $p > \frac{n(1-m)}{2}$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $R_1 > R_0$ *and* $\mu_0 \leq u_{0,1} \leq u_{0,2} \in L^p_{loc}(\widehat{\mathbb{R}^n})$ *such that*

$$\int_{\mathbb{R}^n \setminus B_{R_1}} |u_{0,j} - \mu_0| \, dx < \infty \quad \forall j = 1, 2. \tag{7}$$

Let

$$\frac{\lambda_i}{|x - a_i|^{\gamma_i}} \leq u_{0,1}(x) \leq u_{0,2} \leq \frac{\lambda_i'}{|x - a_i|^{\gamma_i'}}$$

hold for some constants $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ *and*

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < n \quad \forall i = 1, 2, \ldots, i_0.$$

Suppose u_1, u_2, *are the solutions of* (CP) *with* $u_0 = u_{0,1}, u_{0,2}$ *respectively which satisfy*

$$u_j(x,t) \geq \mu_0 \quad \forall x \in \widehat{\mathbb{R}^n}, t > 0, j = 1, 2$$

and

$$\int_{\widehat{\mathbb{R}^n}} |u_j(x,t) - \mu_0| \, dx \leq \int_{\widehat{\mathbb{R}^n}} |u_{0,j} - \mu_0| \, dx \quad \forall t > 0, j = 1, 2 \tag{8}$$

such that

$$\frac{C_1}{|x - a_i|^{\gamma_i}} \leq u_j(x,t) \leq \frac{C_2}{|x - a_i|^{\gamma_i'}}$$

holds for any $0 < |x - a_i| < \delta_2, 0 < t < T, i = 1, 2, \ldots, i_0$, $j = 1, 2$. *Then*

$$u_1(x,t) \leq u_2(x,t) \quad \forall x \in \widehat{\mathbb{R}^n}, t > 0. \tag{9}$$

Theorem 6. *Suppose that $n \geq 3$, $0 < m < \frac{n-2}{n}$ and $\mu_0 > 0$. Let $\mu_0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}\right)$ for some constant $p > \frac{n(1-m)}{2}$ satisfy*

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_1, \ i = 1, \cdots, i_0,$$

for some constants $0 < \delta_1 < \min(\delta_0, 1)$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ and

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < n \quad \forall i = 1, 2, \ldots, i_0.$$

and let $f \in C^\infty(\partial\Omega \times (0, \infty))$ satisfy

$$f \geq \mu_0 \quad \text{on } \partial\Omega \times (0, \infty)$$

and

$$f(x,t) \to \mu_0 \qquad \text{uniformly on } \partial\Omega \quad \text{as } t \to \infty.$$

Let u be the solution of (DP) *given by Theorem 1. Then*

$$u(x,t) \to \mu_0 \quad \text{in } C^2(K) \quad \text{as } t \to \infty$$

for any compact subset K of $\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}$.

Theorem 7. *Suppose that* $n \geq 3$, $0 < m < \frac{n-2}{n+2}$ *and* $\mu_1 \geq \mu_0 > 0$. *Let* $\mu_0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *satisfy*

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_1, \ i = 1, \cdots, i_0,$$

for some constants $0 < \delta_1 < \min(1, \delta_0)$, $\lambda_1, \cdots, \lambda_{i_0}$ *and*

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < \frac{n}{m+1} \quad \forall i = 1, 2, \ldots, i_0.$$

and $f \in C^\infty(\partial\Omega \times (0, \infty))$, $f_t \in L^1(\partial\Omega \times (0, \infty))$, *satisfies*

$$f \geq \mu_0 \quad \text{on } \partial\Omega \times (0, \infty)$$

and

$$f(x.t) \to \mu_1 \qquad \text{as } t \to \infty.$$

Let u *be the solution of* (DP) *given by Theorem 1. Then*

$$u(x,t) \to \mu_1 \qquad \text{in } C^2(K) \qquad \text{as } t \to \infty.$$

for any compact subset K *of* $\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}$.

Theorem 8. *Suppose that* $n \geq 3$, $0 < m < \frac{n-2}{n}$ *and* $\mu_0 > 0$. *Let* $\mu_0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *satisfy*

$$\frac{\lambda_i}{|x - a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x - a_i|^{\gamma_i'}} \quad \forall 0 < |x - a_i| < \delta_1, \; i = 1, \cdots, i_0,$$

for some constants satisfying

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < \frac{n-2}{m} \quad \forall i = 1, 2, \ldots, i_0,$$

and $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ *and* $f \in C^3\left(\partial\Omega \times (0, \infty)\right)$ *satisfies*

$$f \geq \mu_0 \quad \text{on } \partial\Omega \times (0, \infty)$$

and

$$f(x.t) \to g(x) \qquad \text{uniformly in } C^3\left(\partial\Omega\right) \text{ as } t \to \infty.$$

for some function $g \in C^3\left(\partial\Omega\right)$, $g \geq \mu_0$ *on* $\partial\Omega$. *Let* u *be the solution of* (DP) *given by Theorem 1. Let* ϕ *be the solution of*

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \phi = g^m & \text{on } \partial\Omega. \end{cases} \tag{HE}$$

Then u converges in $C^2(K)$ to $\phi^{\frac{1}{m}}$ as $t \to \infty$ for any compact subset K of $\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}$.

Theorem 9. *Suppose that $n \geq 3$, $0 < m < \frac{n-2}{n}$ and $\mu_0 > 0$. Let $\mu_0 \leq u_0 \in L^p_{loc}\left(\widehat{\mathbb{R}^n}\right)$ for some constant $p > \frac{n(1-m)}{2}$ satisfy*

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_1,\ i = 1, \cdots, i_0,$$

for some constants satisfying

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < n \quad \forall i = 1, 2, \ldots, i_0.$$

$0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$. Suppose there exist constants $R_1 > R_0$ and $C_1 > 0$ such that

$$u_0(x) \leq C_1 \qquad \forall |x| \geq R_1.$$

and

$$\int_{\mathbb{R}^n \setminus B_{R_1}} |u_0 - \mu_0|\, dx < \infty$$

hold. Let u be the solution of (CP) *given by Theorem 2. Then*

$$u(x,t) \to \mu_0 \quad \text{in } C^2(K) \quad \text{as } t \to \infty$$

holds for any compact subset K of $\widehat{\mathbb{R}^n}$.

Theorem 10. *Suppose that* $n \geq 3$, $0 < m < \frac{n-2}{n}$ *and* $\mu_0 > 0$. *Let* $\mu_0 \leq u_0 \in L^p_{loc}\left(\widehat{\mathbb{R}^n}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *satisfy*

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_1, \ i = 1, \cdots, i_0,$$

for some constants satisfying

$$\frac{2}{1-m} < \gamma_i \leq \gamma_i' < \frac{n-2}{m} \quad \forall i = 1, 2, \ldots, i_0,$$

and $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$. *Suppose* u_0 *satisfies*

$$u_0(r\sigma) \to \mu_0 \qquad \text{uniformly in } S^{n-1} \text{ as } |x| = r \to \infty.$$

Let u *be the solution of* (CP) *given by Theorem 2. Then*

$$u(x,t) \to \mu_0 \quad \text{in } C^2(K) \quad \text{as } t \to \infty$$

holds for any compact subset K *of* $\widehat{\mathbb{R}^n}$.

Theorem 11. *Suppose that* $n \geq 3$, $0 < m < \frac{n-2}{n}$ *and* $\mu_0 > 0$. *Let* $\mu_0 \leq u_0 \in L^p_{loc}\left(\widehat{\Omega}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *satisfy*

$$u_0(x) \geq \frac{\lambda_i}{|x - a_i|^{\gamma_i}} \qquad \forall 0 < |x - a_i| < \delta_1,\ i = 1, \cdots, i_0.$$

for some constants satisfying

$$\gamma_1 > \frac{n-2}{m}, \quad \gamma_i > \frac{2}{1-m} \quad \forall i = 2, \ldots, i_0,$$

and $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$ *and let* $f \in C^\infty(\partial\Omega \times (0, \infty))$ *satisfy*

$$f \geq \mu_0 \quad \text{on } \partial\Omega \times (0, \infty).$$

Let u *be the solution of* (DP) *given by Theorem 1. Then*

$$u(x,t) \to \infty \quad \text{on } K \quad \text{as } t \to \infty$$

for any compact subset K *of* $\widehat{\Omega}$.

Theorem 12. *Suppose that* $n \geq 3$, $0 < m < \frac{n-2}{n}$ *and* $\mu_0 > 0$. *Let* $\mu_0 \leq u_0 \in L^p_{loc}\left(\widehat{\mathbb{R}^n}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *satisfy*

$$u_0(x) \geq \frac{\lambda_i}{|x - a_i|^{\gamma_i}} \qquad \forall 0 < |x - a_i| < \delta_1, \ i = 1, \cdots, i_0.$$

for some constants satisfying

$$\gamma_1 > \frac{n-2}{m}, \quad \gamma_i > \frac{2}{1-m} \quad \forall i = 2, \ldots, i_0,$$

and $0 < \delta_1 < \delta_0$, $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$. *Let* u *be the solution of* (CP) *given by Theorem 2. Then*

$$u(x,t) \to \infty \quad \text{on } K \quad \text{as } t \to \infty$$

holds for any compact subset K *of* $\widehat{\mathbb{R}^n}$.

Theorem 13. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$ and $\mu_0 > 0$. There there exists $u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$, $u_0 \geq \mu_0$ in $\widehat{\Omega}$, such that*

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \qquad \forall 0 < |x-a_i| < \delta_1, i = 1, \cdots, i_0 \tag{10}$$

for some constants satisfying (3) *and $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ such that*

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0, \infty) \\ u = \mu_0 & \text{on } \partial\Omega \times (0, \infty) \\ u(a_i, t) = \infty & \forall t > 0, i = 1, \cdots, i_0 \\ u(x, 0) = u_0(x) & \text{in } \widehat{\Omega} \end{cases} \tag{11}$$

has a unique solution u with the property that u oscillates between μ_0 and infinity as $t \to \infty$.

- Another natural question to ask is do we have similar results in $\mathbb{R}^n$ if we remove the requirement on the initial value u_0 that it satisfies

$$u_0 \geq \mu_0$$

  for some constant $\mu_0 > 0$.

- For the case $\gamma = \frac{n-2}{m}$, $A|x|^{-\frac{n-2}{m}}$ is a particular solution of (FDE) in $(\mathbb{R}^n \setminus \{0\}) \times (0, \infty)$ for any $A > 0$.

- The singular Barenblatt solution,

$$B_0(x,t) = \left(\frac{C^*(T-t)}{|x|^2} \right)^{\frac{1}{1-m}}$$

  where $C_* = 2m(n-2-nm)/(1-m)$ remains singular at the origin for all time $t < T$ with

$$B_0(x,0) = (C^* T)^{\frac{1}{1-m}} |x|^{-\frac{2}{1-m}}$$

  and $B_0(x,T) \equiv 0$.

- For $n \geq 3$, $0 < m < \frac{n-2}{n}$ and $\gamma = 2/(1-m)$, extinction profile of the general one point blow up solution of the (FDE) with initial value

$$u_0(x) \approx C|x|^{-2/(1-m)} \quad \text{as } x \to 0$$

which blow up at the origin were studied by T. Jin and J. Xiong (arxiv:2008.02059).

- For $n \geq 3$, $0 < m < \frac{n-2}{n}$ and

$$\frac{2}{1-m} < \gamma < \frac{n-2}{m},$$

existence and asymptotic large time behaviour of singular solutions of the fast diffusion equation

$$\begin{cases} u_t = \Delta u^m, u > 0, & \text{in } (\mathbb{R}^n \setminus \{0\}) \times (0, \infty) \\ u(x,0) = u_0 & \text{in } \mathbb{R}^n \setminus \{0\} \end{cases} \tag{IVP}$$

which blows up at the origin $x = 0$ for all time with initial value u_0 satisfying the growth condition

$$A_1|x|^{-\gamma} \leq u_0(x) \leq A_2|x|^{-\gamma} \quad \text{in } \mathbb{R}^n \setminus \{0\}$$

for some constants $A_2 > A_1 > 0$ was proved by me and Soojung Kim (Discrete Contin. Dyn. Syst. Series A, 2017). These solutions has the behaviour

$$u(x,t) \approx C|x|^{-\gamma} \quad \text{as } |x| \to 0 \quad \forall t > 0.$$

- For $0 < m < \frac{n-2}{n}$ and $n \geq 3$, existence and uniqueness of radially symmetric eternal self-similar solutions of (FDE) in $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}$ of the form

$$U(x,t) := e^{-\alpha t} f(e^{-\beta t} x) \quad \forall x \in \mathbb{R}^n \setminus \{0\}, t \in \mathbb{R}$$

that blow up at $\{0\} \times \mathbb{R}$, where f is a radially symmetric function satisfying

$$\frac{n-1}{m} \Delta f^m + \alpha f + \beta x \cdot \nabla f = 0, \quad f > 0, \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

or equivalently,

$$\frac{n-1}{m} \left((f^m)_{rr} + \frac{n-1}{r} (f^m)_r \right) + \alpha f + \beta r f_r = 0, \quad f > 0, \quad \forall r > 0$$

with

$$\beta < 0 \quad \text{and} \quad \alpha = \frac{2\beta}{1-m}$$

and f blows up at the origin was proved by me and Jinwan Park (Discrete Contin. Dyn. Syst. Series A, 2021). Furthermore we prove that such function f satisfies

$$\lim_{r \to 0} \frac{r^2 f(r)^{1-m}}{\log r^{-1}} = \frac{2(n-1)(n-2-nm)}{(1-m)|\beta|}$$

and

$$\lim_{r \to \infty} r^{\frac{n-2}{m}} f(r) = A$$

for some constant $A > 0$.

- M. Fila, P. Macková, J. Takahashi and E. Yanagida (arxiv:2111.08068) have proved that for any $n \geq 2$, $0 < m < 1$, γ, μ, ν, and any function $0 < f \in C^2(S^{n-1})$ that satisfy

$$\gamma > \frac{2}{1-m}, \quad (1-m)\gamma - 2 - m(\gamma - \nu) > 0, \quad \mu > \gamma > \nu > 0$$

and

$$m\mu(m\mu + 2 - n) \min_{S^{n-1}} f^m - \max_{S^{n-1}} |\Delta_{S^{n-1}}(f^m)| > 0$$

where $0 < u_0 \in C(\mathbb{R}^n \setminus \{0\})$ satisfies

$$u_0^m(x) = f^m(\omega)|x|^{-m\gamma} + O(|x|^{-m\nu}) \quad \text{ as } x \to 0 \quad \forall \omega \in S^{n-1}$$

and

$$C^{-1} \leq u_0 \leq C \quad \forall |x| \geq 1$$

for some constant $C > 0$, there exists a solution $u \in C^{2,1}((\mathbb{R}^n \setminus \{0\}) \times (0, \infty)) \cap C((\mathbb{R}^n \setminus \{0\}) \times [0, \infty))$ of (FDE) in $(\mathbb{R}^n \setminus \{0\}) \times (0, \infty)$ with initial value u_0 that satisfies

$$u^m(x,t) = f^m(\omega)|x|^{-m\gamma} + O(|x|^{-m\nu}) \quad \text{ as } x \to 0$$

for any $\omega \in S^{n-1}, t \geq 0$.

- For $n \geq 3$ and $m > \frac{n-2}{n}$, solution of the (FDE) with moving singularities in bounded domains have been constructed by M. Fila, J. Takahashi and E. Yanagida (Nonlinear Analysis, 2019).

- **Sketch of proof of Theorem 1**:
- **Step 1**: For any non-negative functions $u_0 \in L^1_{loc}(\widehat{\Omega})$ or $L^1_{loc}(\widehat{\mathbb{R}^n})$, $0 \le f \in L^\infty(\partial\Omega \times (0, \infty))$, and constants $M > 0$, $0 < \varepsilon < 1$, let

$$\begin{cases} u_{0,M}(x) = \min\left(u_0(x), M\right) \\ u_{0,\varepsilon,M}(x) = \min\left(u_0(x), M\right) + \varepsilon \end{cases}$$

and

$$f_\varepsilon(x,t) = f(x,t) + \varepsilon \qquad \forall (x,t) \in \partial\Omega \times (0,\infty).$$

Then for any $\varepsilon > 0$ and $M > 0$ there exists a unique solution $u_{\varepsilon,M}$ of

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega \times (0,\infty) \\ u(x,t) = f_\varepsilon & \text{on } \partial\Omega \times (0,\infty) \\ u(x,0) = u_{0,\varepsilon,M} & \text{in } \Omega. \end{cases}$$

which satisfies

$$\varepsilon \le u_{\varepsilon,M} \le \max\left(M, \|f\|_{L^\infty(\partial\Omega\times[0,\infty))}\right) + \varepsilon.$$

Moreover if there exists a constant $T_0 > 0$ such that $f(x,t)$ is monotone decreasing in t on $\partial\Omega\times[T_0,\infty)$, then $u_{\varepsilon,M}$ also satisfies the Aronson-Benilan inequality,

$$u_t \le \frac{u}{(1-m)(t-T_0)}$$

in $\Omega \times (T_0, \infty)$.

- We claim that
$$u_M(x,t) = \lim_{\varepsilon\to 0} u_{\varepsilon,M}(x,t)$$
and
$$u(x,t) = \lim_{\varepsilon\to 0} u_M(x,t)$$
exists and u is the solution we want.

- **Step 2**: Since u_0 satisfies
$$u_0(x) \le \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \qquad \forall 0 < |x-a_i| < \delta_3, i = 1,\cdots,i_1$$
for some integer $1 \le i_1 \le i_0$ and constants $\lambda_1',\cdots,\lambda_{i_1}' \in \mathbb{R}^+$, $\gamma_i',\cdots,\gamma_{i_1}' \in \left[\frac{2}{1-m},\infty\right)$. Then one can prove that there exists a constant $A_0 > 0$ such that
$$u_{\varepsilon,M}(x,t) \le \phi_{i,A_0}(x-a_i,t)$$
for any $0 < |x-a_i| < \delta_3$, $t \ge 0$, $0 < \varepsilon < 1$, $M > 0$ and $i = 1,\cdots,i_1$ where
$$\phi_{i,A_0}(x,t) = \frac{A_0(1+t)^{\frac{1}{1-m}}}{|x|^{\gamma_i'}(\delta_3-|x|)^{\frac{2}{1-m}}} \qquad \forall i = 1,\cdots,i_1.$$

- **Step 3**: solution of the (FDE) has the $L^p - L^\infty$ regularizing property.

- **Step 4**: Positivity estimate: Let $\delta_0 > \delta_1 > \delta_2 > 0$ and $0 \le \eta \in C_c^\infty(B_{\delta_0})$ be such that

$$\begin{cases} \eta(x) > 0 & \forall |x| < \delta_1 \\ \eta(x) = 0 & \forall \delta_1 \le |x| \le \delta_0 \\ \eta(x) = \delta_2^{-\beta_1/b_1} & \forall |x| = \delta_2 \end{cases}$$

for some constants $b_1 > \frac{2}{1-m}$ and $\beta_1 \in \left(0, n - \frac{2}{1-m}\right)$. Let $\psi \in C^\infty(\widehat{B_{\delta_0}})$ be such that

$$\begin{cases} \psi(x) = |x|^{-\beta_1} & \forall 0 < |x| \le \delta_2 \\ \psi(x) = \eta^{b_1}(x) & \forall \delta_2 \le |x| \le \delta_0. \end{cases}$$

Then there exists a constant $C_\psi > 0$ such that

$$\int_{\widehat{B}_{\delta_1}} \psi^{-\frac{m}{1-m}} |\Delta \psi|^{\frac{1}{1-m}} \, dx \le C_\psi < \infty.$$

Let $\psi_{x_0}(x) = \psi(x - x_0)$ for any $x_0 \in \Omega$. Then there exists a constant $C_1 > 0$ such that for any $\varepsilon \in (0,1)$ and $M > 0$

$$\begin{aligned} &\int_{\widehat{B}_{\delta_1}(a_i)} u_{\varepsilon,M}(x,t)\psi_{a_i}(x)\, dx \\ \ge & e^{-t} \int_{\widehat{B}_{\delta_1}(a_i)} u_{0,\varepsilon,M}(x)\psi_{a_i}(x)\, dx - C_1\left(1 - e^{-t}\right) \end{aligned}$$

for any $t > 0$ and $i = 1, 2, \cdots, i_0$. Then for any $T > 0$ and $C_2 > 0$ there exists a constant $M_0 = M_0(T, C_2) > 0$ such that for any $M \geq M_0$,

$$\int_{\widehat{B}_{\delta_1}(a_i)} u_{\varepsilon,M}(x,t)\psi_{a_i}(x)\,dx > C_2$$

for any $0 < t < T$, $0 < \varepsilon < 1$, $i = 1, 2, \cdots, i_0$. Hence

$$\int_{\widehat{B}_{\delta_1}(a_i)} u(x,t)\psi_{a_i}(x)\,dx = \infty \qquad \forall t > 0,\ i = 1, 2, \cdots, i_0.$$

From these one can deduce that the limits u_M and u exists and are positive solutions of (FDE).

- **Step 5**: By using the the fact that if $0 \leq u_0 \in L^\infty_{loc}(\Omega)$ and u is a bounded solution, then

$$\lim_{\substack{|x-x_0|\leq\alpha\sqrt{t} \\ t\to 0}} u(x,t) = u_0(x_0)$$

for any point $x_0 \in \Omega$ of continuity of u_0 and comparison with the solution constructed by Vazquez and Winkler one can show that for any $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exists a constant $0 < C_1 \leq \min_{1\leq i\leq i_0} \lambda_i$ such that

$$u(x,t) \geq \frac{C_1}{|x-a_i|^{\gamma_i}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T$$

holds and the theorem follows.

- **Sketch of proof of the convergence theorems**: Let $\delta_2 = \delta_1/2$, $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}^+$ be a sequence such that $t_k \to \infty$ as $k \to \infty$, and $u_k(x,t) = u(x, t + t_k)$.

  We first observe that for any $C_0 > 0$ and $\eta \in (0,1)$, if

$$v_\eta(x) = \left[C_0^m + \sum_{i=1}^{i_0} \left(v_{i,\eta}(x) \right)^m \right]^{\frac{1}{m}} \qquad \forall x \in \mathbb{R}^n$$

  where

$$v_{i,\eta}(x) = A_i \left(|x - a_i|^2 + \eta \right)^{-\frac{\gamma_i'}{2}} \qquad \forall i = 1, \cdots, i_0.$$

  Then

$$\Delta v_\eta^m \le 0 \qquad \text{in } \mathbb{R}^n.$$

  Then by the maximum principle one can show that there exists constants $C_0 > 0$, $A_i > 0$, such that for any $0 < \eta < \eta_1(M)$, $0 < \varepsilon < 1$, $M > M_2$,

$$u_{\varepsilon,M}(x,t) \le v_\eta(x) \quad \text{in } \Omega \times [t_0, \infty)$$

$$\Rightarrow \quad u_{\varepsilon,M}(x,t) \le \left[C_0^m + \sum_{i=1}^{i_0} \left(A_i \, |x - a_i|^{-\gamma_i'} \right)^m \right]^{\frac{1}{m}} \text{ in } \Omega \times [t_0, \infty)$$

$$\Rightarrow \quad u_M(x,t) \le \left[C_0^m + \sum_{i=1}^{i_0} \left(A_i \, |x - a_i|^{-\gamma_i'} \right)^m \right]^{\frac{1}{m}} \text{ in } \Omega \times [t_0, T_M)$$

as $\varepsilon \to 0$. Hence

$$u(x,t) \le \left[C_0^m + \sum_{i=1}^{i_0} \left(A_i \, |x - a_i|^{-\gamma_i'} \right)^m \right]^{\frac{1}{m}} \quad \text{in } \Omega \times [t_0, \infty) \text{ as } M \to \infty.$$

Thus there exist constants $C_2 > 0$, $C_3 > 0$ such that

$$u_k \le C_2 \quad \text{in } \overline{\Omega_{\delta_2}} \times \left(\frac{1}{2} - t_k, \infty \right) \tag{12}$$

and

$$u_k(x,t) \le C_3 \, |x - a_i|^{-\gamma_i'} \ \forall 0 < |x - a_i| \le \delta_2, t \ge \frac{1}{2} - t_k, i = 1, \cdots, i_0. \tag{13}$$

Hence the equation (FDE) for the sequence $\{u_k\}_{k=N_1}^{\infty}$ is uniformly parabolic on $\overline{\Omega_\delta} \times (1 - N_1, \infty)$ for any $0 < \delta < \delta_0$. By the Schauder estimates, Ascoli theorem and a diagonalization argument the sequence $\{u_k\}_{k=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^{2,1}(K)$ on every compact subset K of $(\overline{\Omega} \backslash \{a_1, \cdots, a_{i_0}\}) \times (-\infty, \infty)$ to a solution u_∞ of (FDE) in $\widehat{\Omega} \times (-\infty, \infty)$ as $k \to \infty$ which satisfies

$$u_\infty \ge \mu_0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}. \tag{14}$$

- **Sketch of proof of Theorem 6**: Note that

$$u_\infty = \mu_0 \qquad \text{on } \partial\Omega \times \mathbb{R}.$$

We now divide the proof into two cases.

Case 1: There exists a constant $t_0 > 0$ such that $f \equiv \mu_0$ on $\partial\Omega \times (t_0, \infty)$.

Then

$$u_{\varepsilon,M} = \mu_0 + \varepsilon \quad \text{on } \partial\Omega \times (t_0, \infty).$$

Let $\gamma_0 = \max_{1\le i\le i_0} \gamma_i'$. Then

$$\frac{2}{1-m} < \gamma_0 < n.$$

Let $v_0(x) = u_\infty(x, 0)$ and $1 < p_1 < n/\gamma_0$. Then

$$\begin{aligned}\int_{\widehat{\Omega}} u^{p_1}(x, t_0)\, dx \le & C_3^p \omega_n \sum_{i=1}^{i_0} \int_0^{\delta_1} r^{n-p_1\gamma_0-1}\, dr + \|u(\cdot, t_0)\|_{L^\infty(\Omega_{\delta_1})}^{p_1} |\Omega| \\ \le & C_1(1 + \delta_1^{n-p_1\gamma_0}) < \infty\end{aligned}$$

for some constant $C_1 > 0$ where ω_n is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$. Hence $u(\cdot, t_0) \in L^{p_1}(\widehat{\Omega})$. Now

$$\frac{\partial u_{\varepsilon,M}}{\partial \nu} \le 0 \qquad \text{on } \partial\Omega \times (t_0, \infty)$$

where $\frac{\partial}{\partial \nu}$ is the derivative with respect to the unit outward normal on $\partial\Omega \times (t_0, \infty)$. Then

$$
\begin{aligned}
&\frac{1}{p_1}\left(\int_{\Omega_\delta} u_{\varepsilon,M}^{p_1}(x,t)\,dx - \int_{\Omega_\delta} u_{\varepsilon,M}^{p_1}(x,t_1)\,dx\right)\\
=&\frac{1}{p_1}\int_{t_0}^{t}\frac{d}{d\tau}\left[\int_{\Omega_\delta} u_{\varepsilon,M}^{p_1}(x,\tau)\,dx\right]d\tau\\
=&\int_{t_0}^{t}\int_{\Omega_\delta} u_{\varepsilon,M}^{p_1-1}(u_{\varepsilon,M})_\tau\,dx\,d\tau = \int_{t_0}^{t}\int_{\Omega_\delta} u_{\varepsilon,M}^{p_1-1}\Delta u_{\varepsilon,M}^{m}\,dx\,d\tau\\
\le&-\int_{t_0}^{t}\int_{\Omega_\delta}\nabla u_{\varepsilon,M}^{m}\cdot\nabla u_{\varepsilon,M}^{p_1-1}\,dx\,d\tau\\
=&-m(p_1-1)\int_{t_0}^{t}\int_{\Omega_\delta} u_{\varepsilon,M}^{m+p_1-3}|\nabla u_{\varepsilon,M}|^2\,dx\,d\tau \quad \forall t > t_0, 0 < \delta \le \delta_1.
\end{aligned}
$$

Hence as $\varepsilon \to 0$, $M \to \infty$,

$$
\begin{aligned}
&\frac{1}{p_1}\int_{\Omega_\delta} u^{p_1}(x,t)\,dx + m(p_1-1)\int_{t_0}^{t}\int_{\Omega_\delta} u^{m+p_1-3}|\nabla u|^2\,dx\,d\tau\\
\le&\frac{1}{p_1}\int_{\Omega_\delta} u^{p_1}(x,t_0)\,dx \quad \forall t > t_0, 0 < \delta \le \delta_1
\end{aligned}
$$

$$
\Rightarrow \quad \int_{t_0}^{\infty}\int_{\Omega_\delta}\frac{1}{u^{3-m-p_1}}|\nabla u|^2\,dx\,dt < C\int_{\widehat{\Omega}} u^{p_1}(x,t_0)\,dx < \infty
$$

as $t \to \infty$. Let $0 < \delta < \delta_1$. Hence

$$
\int_{t_0}^{\infty}\int_{\Omega_\delta}|\nabla u|^2\,dx\,dt \le C'_\delta
$$

for some constant $C'_\delta > 0$ depending on $\delta > 0$. Then

$$\int_0^\infty \int_{\Omega_\delta} |\nabla u_k(x,t)|^2\, dx\, dt = \int_{t_k}^\infty \int_{\Omega_\delta} |\nabla u(x,s)|^2\, dx\, ds \to 0$$

as $k \to \infty$. Thus

$$\int_0^\infty \int_{\Omega_\delta} |\nabla u_\infty|^2\, dx\, dt = 0 \quad \forall 0 < \delta < \delta_1$$
$$\Rightarrow \quad \nabla u_\infty = 0 \qquad \text{in } \widehat{\Omega} \times [0,\infty).$$

Hence

$$u_\infty = \mu_0 \quad \text{on } (\overline{\Omega}\backslash\{a_1, \cdots, a_{i_0}\}) \times [0,\infty)$$
$$\Rightarrow \qquad v_0 = \mu_0 \quad \text{on } \overline{\Omega}\backslash\{a_1, \cdots, a_{i_0}\}.$$

Since the sequence $\{t_k\}_{k=1}^\infty$ is arbitrary,

$$u(x,t) \to \mu_0 \quad \text{in } C^2(K) \quad \text{as } t \to \infty$$

for any compact subset K of $\overline{\Omega}\backslash \left\{a_1, \cdots, a_{i_0}\right\}$.

Case 2: f satisfies $f \geq \mu_0$ on $\partial\Omega \times (0,\infty)$ and $f \to \mu_0$ uniformly on $\partial\Omega$ as $t \to \infty$.

Note that for any $i \geq 2$ there exists a constant $T_i > 0$ such that

$$\mu_0 \leq f \leq \mu_0\left(1 + \frac{1}{i}\right) \qquad \text{on } \partial\Omega \times (T_i, \infty).$$

For any $i \geq 2$, let $\underline{f}_i$, $\overline{f}_i$, be given by

$$\begin{cases} \underline{f}_i(x,t) = \mu_0 & \text{on } \partial\Omega \times (0,\infty) \\ \overline{f}_i(x,t) = \max\left(f(x,t), \mu_0\left(1 + \dfrac{1}{i}\right)\right) & \text{on } \partial\Omega \times (0,\infty) \end{cases}$$

and $\underline{u}_{0,i}$, $\overline{u}_{0,i}$ be given by

$$\underline{u}_{0,i}(x,0) = u_0 - \frac{\mu_0}{i} \quad \text{and} \quad \overline{u}_{0,i}(x,0) = u_0 + \frac{\mu_0}{i}.$$

Let $\underline{u}_i$, $\overline{u}_i$, be solutions of (DP) with $f = \underline{f}_i, \overline{f}_i$, and $u_0 = \underline{u}_{0,i}, \overline{u}_{0,i}$, respectively given by Theorem 1. Since

$$\underline{f}_i(x,t) = \mu_0 \quad \text{and} \quad \overline{f}_i(x,t) = \mu_0\left(1 + \frac{1}{i}\right) \quad \text{on } \partial\Omega \times (T_i, \infty),$$

by case 1,

$$\begin{cases} \underline{u}_i \to \mu_0 & \text{in } C^2(K) \text{ as } t \to \infty \\ \overline{u}_i \to \mu_0\left(1 + \dfrac{1}{i}\right) & \text{in } C^2(K) \text{ as } t \to \infty \end{cases}$$

for any compact subset K of $\overline{\Omega} \setminus \{a_1, \cdots, a_n\}$. Hence

$$
\begin{aligned}
& \underline{u}_i \le u \le \overline{u}_i \quad \text{in } \Omega \times (0,\infty) \quad \forall i \ge 2 \\
\Rightarrow \quad & \underline{u}_i(x, t + t_k) \le u_k(x,t) \le \overline{u}_i(x, t + t_k) \\
\Rightarrow \quad & \mu_0 \le u_\infty(x,t) \le \mu_0 \left(1 + \frac{1}{i}\right) \qquad \forall (x,t) \in \widehat{\Omega} \times \mathbb{R}, i \ge 2 \\
\Rightarrow \quad & u_\infty = \mu_0 \quad \text{in } \widehat{\Omega} \times \mathbb{R} \quad \text{as } i \to \infty
\end{aligned}
$$

and the theorem follows.

- **Sketch of proof of Theorem 8**: Let $t_0 = \frac{1}{2}$. Note that

$$u_\infty = g \qquad \text{on } \partial\Omega \times \mathbb{R}. \tag{15}$$

Letting $k \to \infty$ in (12) and (13),

$$u_\infty \le C_2 \qquad \text{in } \overline{\Omega_{\delta_2}} \times (-\infty, \infty) \tag{16}$$

and

$$u_\infty(x,t) \le C_3 \, |x - a_i|^{-\gamma_i'} \quad \forall 0 < |x - a_i| \le \delta_2, t \in \mathbb{R}, i = 1, \cdots, i_0. \tag{17}$$

We now divide the proof into two cases.

Case 1: There exists a constant $T_0 \ge 0$ such that (1) holds. By (1) and Theorem 1,

$$u_{k,t} \le \frac{u_k}{(1-m)\,(t + t_k - T_0)} \qquad \text{in } \widehat{\Omega} \times (T_0 - t_k, \infty). \tag{18}$$

Letting $k \to \infty$ in (18), by (12) and (13),

$$u_{\infty,t} \le 0 \qquad \text{in } \widehat{\Omega} \times \mathbb{R}. \tag{19}$$

Thus by (14), (16) and (17), the equation (FDE) for u_∞ is uniformly parabolic in $\overline{\Omega_\delta} \times (-\infty, \infty)$ for any $0 < \delta < \delta_2$. By the Schauder estimates the family $\{u_\infty(\cdot, t)\}_{t \in \mathbb{R}}$ is equi-Holder continuous in $C^2(K)$ for any compact subset K of $\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}$.

Hence by (14), (15), (16), (17) and (19) $u_\infty(\cdot, t)$ decreases (increases respectively) and converges uniformly in $C^2(K)$ on every compact subset K of $\overline{\Omega}\backslash\{a_1, \cdots, a_{i_0}\}$ to some function

$$w_1 \in C^2(\overline{\Omega}\backslash\{a_1, \cdots, a_{i_0}\}) \quad \text{as } t \to \infty$$

($w_2 \in C^2(\overline{\Omega}\backslash\{a_1, \cdots, a_{i_0}\})$, as $t \to -\infty$ respectively) which satisfies

$$\begin{cases} w_j \ge \mu_0 & \text{in } \widehat{\Omega} \quad \forall j = 1, 2 \\ w_j = g & \text{on } \partial\Omega \quad \forall j = 1, 2 \\ w_j \le C_2 & \text{on } \overline{\Omega_{\delta_2}} \quad \forall j = 1, 2 \\ w_j(x) \le C_3|x - a_i|^{-\gamma_i'} \ \forall 0 < |x - a_i| \le \delta_2, i = 1, \cdots, i_0, j = 1, 2. \end{cases} \tag{20}$$

By (19) and (20),

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{\Omega_\delta} \left|\Delta u_\infty^m\right| dx\,dt &= \int_{-\infty}^{\infty}\int_{\Omega_\delta} |u_{\infty,t}|\,dx\,dt \\ &= -\int_{-\infty}^{\infty}\int_{\Omega_\delta} u_{\infty,t}\,dx\,dt \\ &\le \int_{\Omega_\delta} w_2\,dx \le C_\delta\,|\Omega| < \infty \quad \forall 0 < \delta < \delta_2 \end{aligned}$$

where $C_\delta = \max(C_2, C_3 \max_{1\le i\le i_0} \delta^{-\gamma_i'})$. Hence there exist sequences $s_i \to \infty$ and $s_i' \to -\infty$ as $i \to \infty$ such that

$$\int_{\Omega_\delta} \left|\Delta u_\infty^m(x, s_i)\right| dx \to 0 \quad \text{and} \quad \int_{\Omega_\delta} \left|\Delta u_\infty^m(x, s_i')\right| dx \to 0$$

as $i \to \infty$. Hence

$$\int_{\Omega_\delta} \left|\Delta w_j^m\right| dx = 0 \qquad \forall 0 < \delta < \delta_0, \ \ j = 1, 2$$
$$\Rightarrow \quad \Delta w_j^m = 0 \qquad \text{in } \widehat{\Omega} \quad \forall j = 1, 2. \tag{21}$$

Then

$$0 \le w_j^m(x) \le C_3^m |x - a_i|^{-m\gamma_i'} = o\left(|x - a_i|^{2-n}\right)$$

for any $x \in \widehat{B}_{\delta_2}(a_i), i = 1, \cdots, i_0, \ \ j = 1, 2$. Hence a_i is a removable singularity of w_j^m for all $i = 1, \cdots, i_0$, $j = 1, 2$. Thus w_j can be extended to a function on $\overline{\Omega}$ for $j = 1, 2$, such that

$$\Delta w_j^m = 0 \quad \text{in } \Omega \quad \forall j = 1, 2. \tag{22}$$

By (20), (22), and the maximum principle for harmonic functions,

$$w_j^m = \phi \quad \text{in } \Omega \quad \forall j = 1, 2. \tag{23}$$

By (19) and (23), $\forall x \in \overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}, t \in \mathbb{R}$,

$$
\begin{aligned}
& \phi^{\frac{1}{m}}(x) = w_1(x) \le u_\infty(x,t) \le w_2(x) = \phi^{\frac{1}{m}}(x) \\
\Rightarrow \quad & u_\infty(x,t) = \phi^{\frac{1}{m}}(x) \qquad (24)\\
\Rightarrow \quad & \lim_{k\to\infty} u(x,t_k) = \phi^{\frac{1}{m}}(x). \qquad (25)
\end{aligned}
$$

Since the sequence $\{t_k\}$ is arbitrary, u converges in $C^2(K)$ to $\phi^{\frac{1}{m}}$ as $t \to \infty$ for any compact subset K of $\overline{\Omega} \setminus \left\{a_1, \cdots, a_{i_0}\right\}$.

Case 2: $f \ge \mu_0$ and (8).

By (8) for any $i \ge 2$ there exists a constant $T_i > 0$ such that

$$
g - \frac{\mu_0}{i} \le f \le g + \frac{\mu_0}{i} \qquad \text{on } \partial\Omega \times (T_i, \infty).
$$

For any $i \ge 2$, let $\underline{f}_i$, $\underline{g}_i$, $\overline{f}_i$ and $\overline{g}_i$ be given by

$$
\begin{cases}
\underline{f}_i(x,t) = \min\left(f(x,t), g(x) - \dfrac{\mu_0}{i}\right) \\
\overline{f}_i(x,t) = \max\left(f(x,t),\ g(x) + \dfrac{\mu_0}{i}\right) \\
\quad \underline{g}_i(x) = g(x) - \dfrac{\mu_0}{i}, \quad \overline{g}_i(x) = g(x) + \dfrac{\mu_0}{i} \quad \text{on } \partial\Omega,
\end{cases} \qquad (26)
$$

and

$$
\underline{u}_{0,i}(x,0) = u_0 - \frac{\mu_0}{i} \quad \text{and} \quad \overline{u}_{0,i}(x,0) = u_0 + \frac{\mu_0}{i}.
$$

Let $\underline{u}_i$, $\overline{u}_i$ be solutions of (DP) with $f = \underline{f}_i, \overline{f}_i$ and $u_0 = \underline{u}_i, \overline{u}_i$, respectively given by Theorem 1. Let $\underline{\phi}_i$, $\overline{\phi}_i$, be the solutions of (HE) with $= \underline{g}_i, \overline{g}_i$, respectively. Since

$$f_i(x,t) = g(x) - \frac{\mu_0}{i} \text{ and } \overline{f}_i(x,t) = g(x) + \frac{\mu}{i} \quad \text{on } \partial\Omega \times (T_i, \infty),$$

by case 1,

$$\begin{cases} \underline{u}_i \to \underline{\phi}_i^{\frac{1}{m}} & \text{in } C^2(K) \text{ as } t \to \infty \\ \overline{u}_i \to \overline{\phi}_i^{\frac{1}{m}} & \text{in } C^2(K) \text{ as } t \to \infty \end{cases} \tag{27}$$

for any compact subset K of $\overline{\Omega}\setminus\{a_1, \cdots, a_n\}$. Since by (26),

$$\begin{cases} \underline{f}_i \le f \le \overline{f}_i & \text{on } \partial\Omega \times (0,\infty) \quad \forall i \ge 2 \\ \underline{g}_i \le g \le \overline{g}_i & \text{on } \partial\Omega \quad \forall i \ge 2, \end{cases}$$

by (27) for any $k \in \mathbb{Z}^+$, $x \in \overline{\Omega}\setminus\left\{a_1, \cdots, a_{i_0}\right\}$, $t \in \mathbb{R}$, $i \ge i_2$,

$$\begin{aligned} & \underline{u}_i(x, t+t_k) \le u_k(x,t) \le \overline{u}_i(x, t+t_k) \\ \Rightarrow \quad & \underline{\phi}_i^{\frac{1}{m}}(x) \le u_\infty(x,t) \le \overline{\phi}_i^{\frac{1}{m}}(x) \quad \text{as } k \to \infty. \end{aligned} \tag{28}$$

Since both $\underline{g}_i$ and $\overline{g}_i$ converges uniformly on $\partial\Omega$ to g as $i \to \infty$, both $\underline{\phi}_i$ and $\overline{\phi}_i$ converges uniformly on $\overline{\Omega}$ to ϕ as $i \to \infty$. Hence letting $i \to \infty$ in (28), we get (24) and (25). Since the sequence $\{t_k\}_{k=1}^\infty$ is arbitrary, u converges in $C^2(K)$ to $\phi^{\frac{1}{m}}$ as $t \to \infty$ for any compact subset K of $\overline{\Omega}\setminus\left\{a_1, \cdots, a_{i_0}\right\}$ and the theorem follows.

Thank you!