Detached shocks past a blunt body

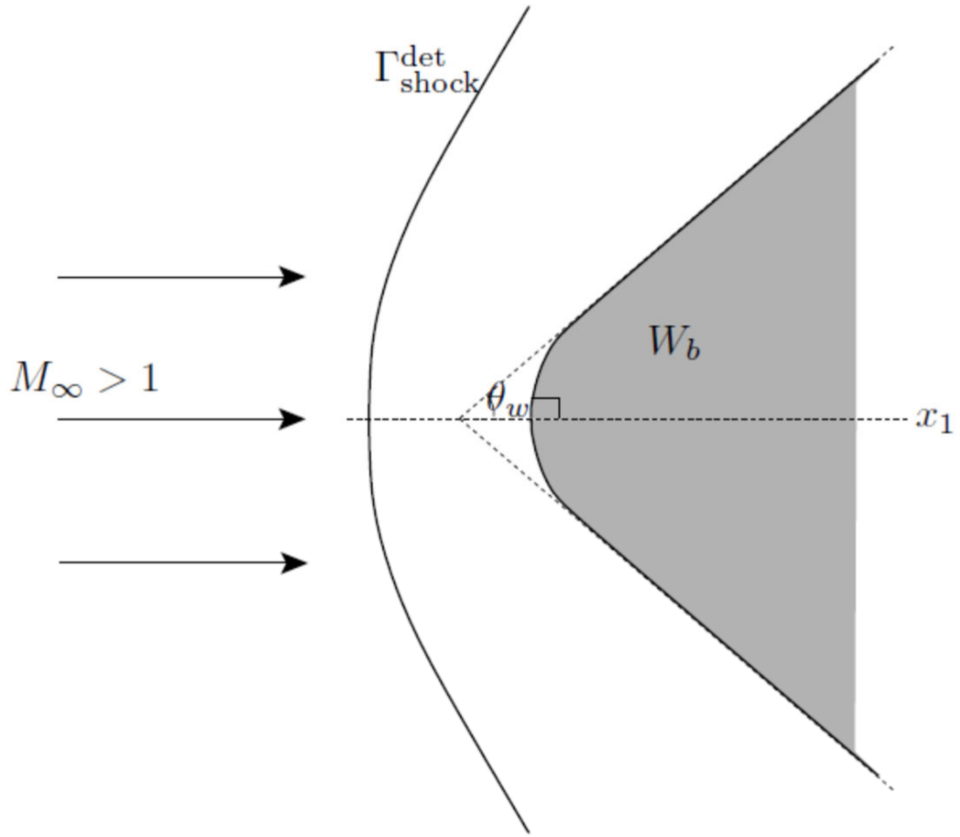
-Joint work with Wei Xiang(CUHK)-

Myoungjean Bae (KAIST)

March 16th , 2022

Main problem

Given incoming supersonic flow with uniform state, find a weak solution to the steady compressible Euler system with a (detached) shock $\Gamma^{\text{det}}_{\text{shock}}$ in $\mathbb{R}^2 \setminus W_b$.



Basics

ρ : density, $\mathbf{u}$: velocity, p : pressure

Steady Euler system for inviscid compressible flow of ideal polytropic gas

$$\mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0$$

$$\mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbb{I}) = \mathbf{0}$$

$$\mathrm{div}_{\mathbf{x}}\left(\rho\mathbf{u}(\frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma-1)\rho})\right) = 0 \text{ for an adiabatic exponent } \gamma > 1$$

Some important quantities to remember

Sound speed $c(\rho, p) = \sqrt{\frac{\gamma p}{\rho}}$

Mach number $M = \frac{\text{flow speed}}{\text{sound speed}} = \frac{|\mathbf{u}|}{c}$

Physical (Mathematical) classifications of flow types

M<1: Subsonic (Elliptic-Hyperbolic)

M=1: Sonic (Degenerate-Hyperbolic)

M>1: Supersonic (Hyperbolic)

Let Ω be a domain in $\mathbb{R}^2$.
Suppose that a non self-intersecting C^1 −curve Γ divides Ω into two open and connected subsets $\Omega^\pm$ s.t.

$$\Omega^- \cap \Omega^+ = \emptyset, \quad \text{and} \quad \Omega^- \cup \Gamma \cup \Omega^+ = \Omega.$$

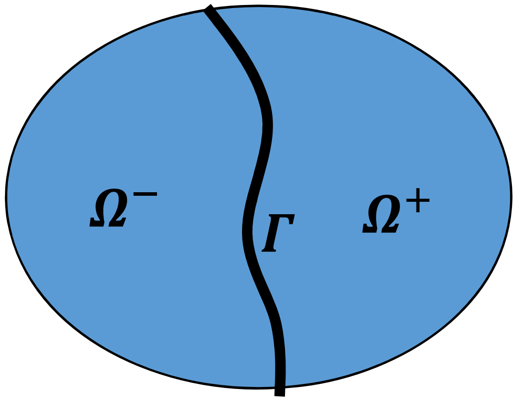
$(\rho, \boldsymbol{u}, p)$ with $\boldsymbol{u} = (u_1, u_2)$ is an **entropy solution in Ω with a shock Γ** if

- $(\rho, \boldsymbol{u}, p) \in C^1(\Omega^\pm) \cap C^0\left(\overline{\Omega^\pm}\right)$;
- $(\rho, \boldsymbol{u}, p)$ is a weak solution to steady Euler system in Ω;
- $\rho^\pm > 0$ in $\overline{\Omega^\pm}$ and $0 < \mathbf{u}^+ \cdot \boldsymbol{n} < \boldsymbol{u}^- \cdot \boldsymbol{n}$ on Γ for $\boldsymbol{n} = \frac{\boldsymbol{u}^- - \boldsymbol{u}^+}{|\boldsymbol{u}^- - \boldsymbol{u}^+|}$.

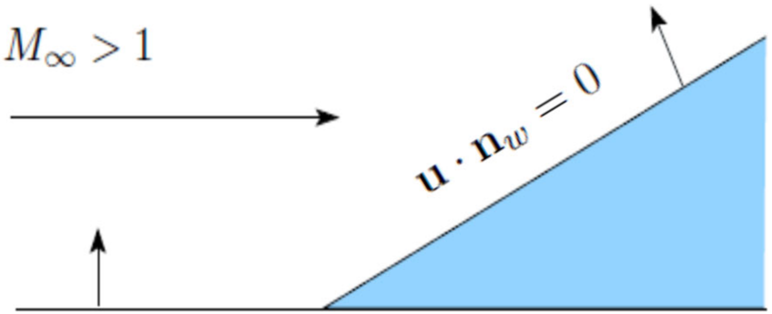
Rankine-Hugoniot conditions

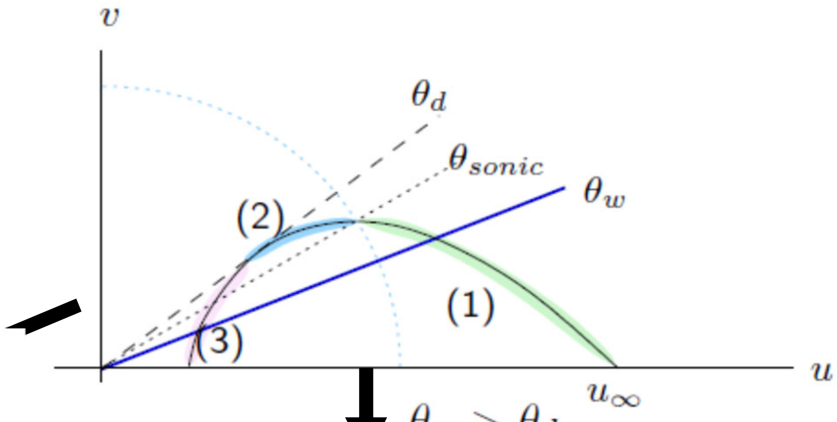
$$[\rho \mathbf{u} \cdot \mathbf{n}]_\Gamma = [\mathbf{u} \cdot \boldsymbol{\tau}]_\Gamma = [\rho(\mathbf{u} \cdot \mathbf{n})^2 + p]_\Gamma = [B]_\Gamma = 0$$

for $B = \frac{1}{2}|\mathbf{u}|^2 + \frac{\gamma p}{(\gamma - 1)\rho}$.



Motivation



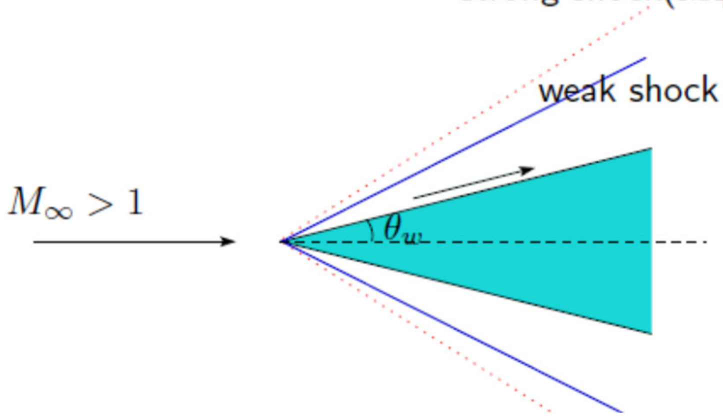


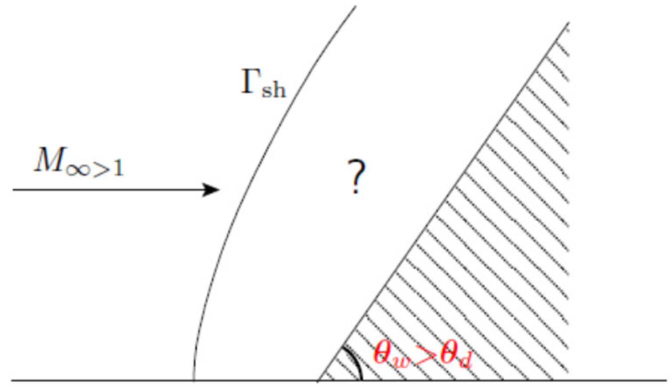
$\theta_w \leq \theta_d$

$\theta_w > \theta_d$

Attached oblique shocks

Detached shocks



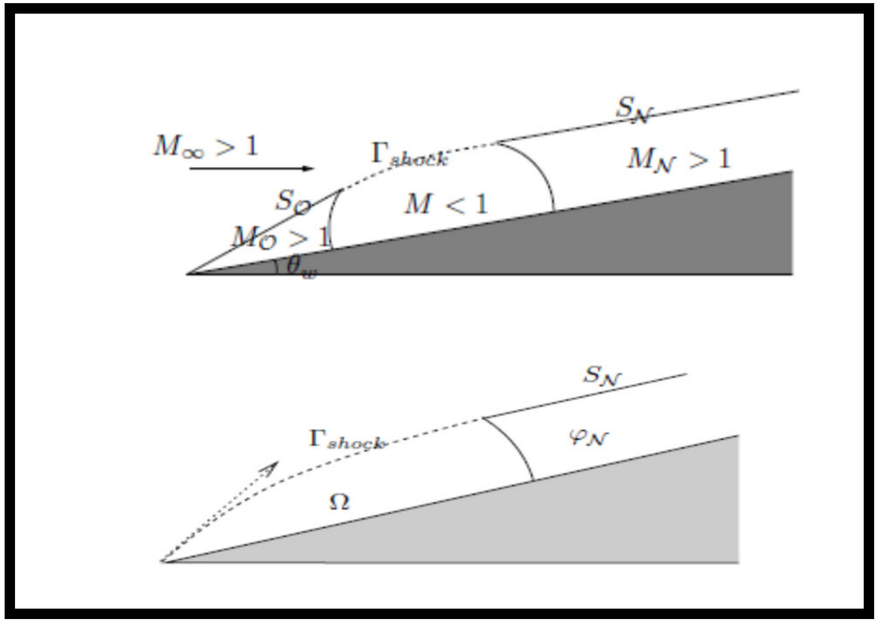


Questions.

1. ($\theta_w \le \theta_d$) **Prove Prandtl's conjecture.**

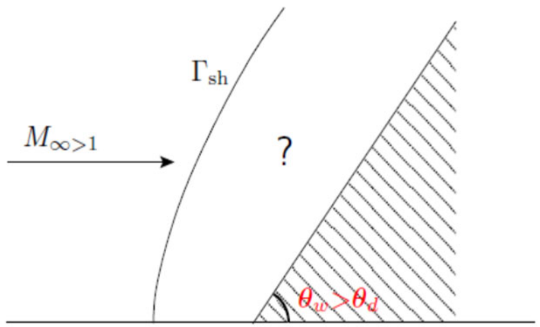
➡ Find **a global-in-time weak solution** of unsteady Euler system, and show that the solution **converges to the weak shock solution** as time tends to infinity.

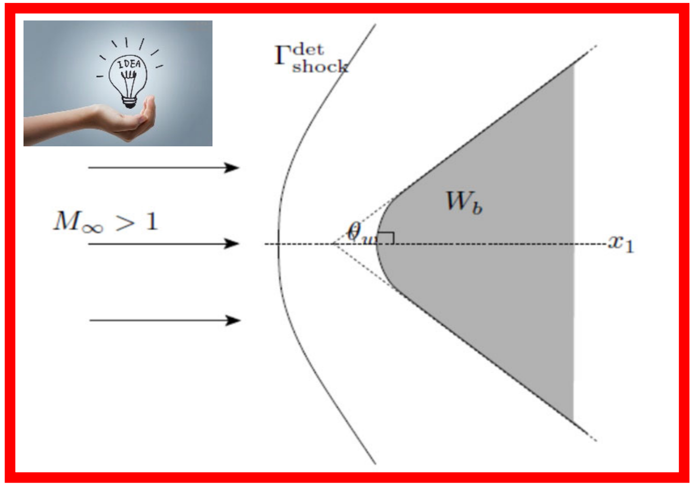
- Elling-Liu 2008(CPAM)
- B.-Chen-Feldman 2013(QAM) & 2020(*To appear in Mem. of AMS)*



2. ($\theta_w > \theta_d$) Construct a detached shock solution of steady Euler system.

Detached shocks





Description of a blunt body W_b

For fixed $\theta_w \in (0, \frac{\pi}{2})$ and $h_0 > 0$(not necessarily small), $b : \mathbb{R} \to \mathbb{R}^+$ satisfies

(b_1) $b(x_2) = b(-x_2)$ for all $x_2 \in \mathbb{R}$;

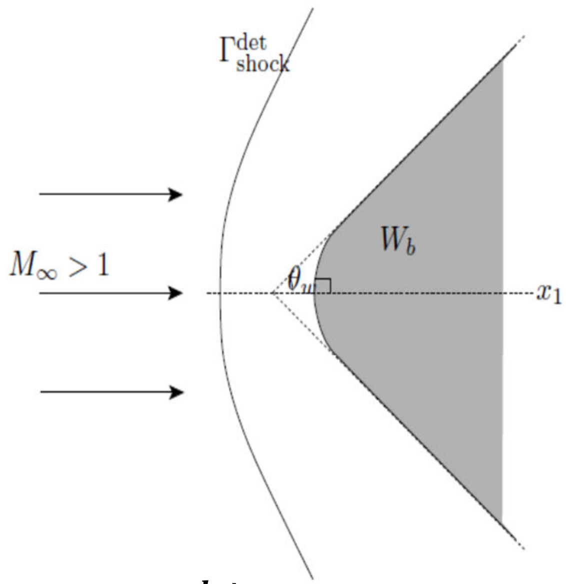
(b_2) $b \in C^3(\mathbb{R})$;

This condition turns out to be important !!!

(b_3) $b'(x_2) > 0$ and $b''(x_2) \geq 0$ for all $x_2 > 0$;

(b_4) $b(x_2) = x_2 \cot \theta_w$ for $x_2 \geq h_0$.

$$W_b := \{\mathrm{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq b(x_2)\}.$$



Γ^{det}_{shock}: $x_1 = b(x_2)$

(Note) The half-wedge angle θ_w is arbitrary in $(0, \frac{\pi}{2})$.

Model equations

Assumption

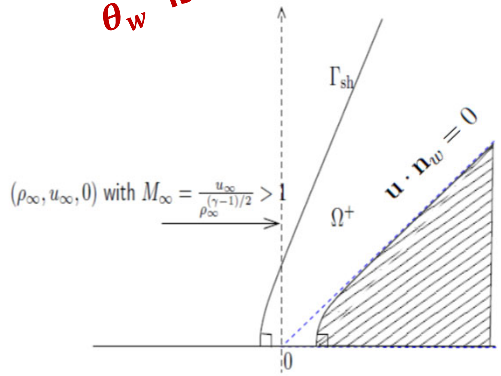
Irrotational flow $\nabla \times \boldsymbol{u} = \mathbf{0}$($\Rightarrow p = S_0\rho^{\gamma}$ for some constant $S_0 > 0$)

Steady Euler system for irrotational flow

$$
\begin{aligned}
\partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) &= 0 \quad \text{(Conservation of mass)} \\
\partial_{x_1} u_2 - \partial_{x_2} u_1 &= 0 \quad \text{(Irrotationality)} \\
\frac{1}{2}|\mathbf{u}|^2 + \frac{\rho^{\gamma-1}}{\gamma - 1} &= B_0 \quad \text{(Bernoulli law)}
\end{aligned}
$$

Detached shock problem past a blunt body W_b

Note. θ_w is arbitrary in $(0, \frac{\pi}{2})$.



For a fixed $d_0 > 0$, find an entropy solution $(\rho, \boldsymbol{u})$ with a shock $\Gamma_{sh} = \{x_1 = f_{sh}(x_2), x_2 \geq 0\}$ s.t. the following properties hold:

(i) $b(0) - f_{\rm sh}(0) = d_0;$ (detached distance)

(ii) $(\rho, u_1, u_2) = (\rho_\infty, u_\infty, 0)$ in $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 < f_{\rm sh}(x_2)\};$

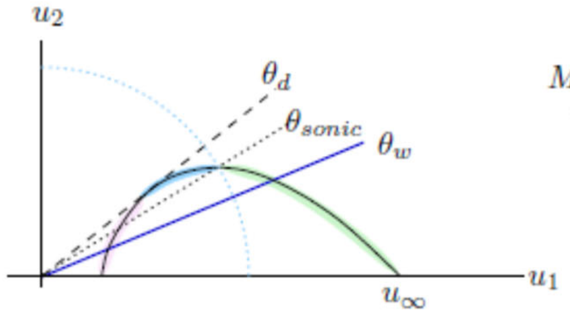
(iii) $u_2 = 0$ on $\Gamma_{\rm sym} := \{(x_1, 0) : x_1 \leq b(0)\};$

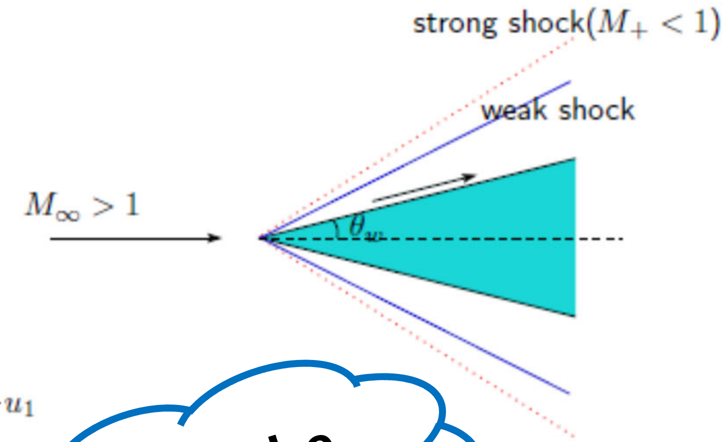
(iv) $(u_1, u_2) \cdot \mathbf{n}_w = 0$ on $\Gamma_w := \{(b(x_2), x_2) : x_2 \geq 0\}$ for a unit normal $\mathbf{n}_w$ on $\Gamma_w;$

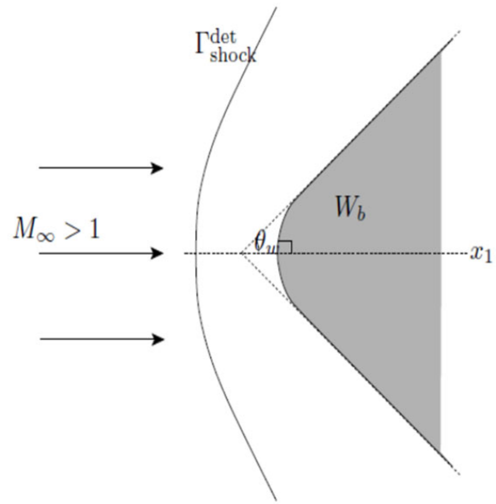
(v) (ρ, u_1, u_2) uniformly converges to a piecewise constant state as $|\mathrm{x}| \to \infty$

Asymptotic state at $|(x_1, x_2)| = \infty$

If the asymptotic state of (ρ, u_1, u_2) at $|x_1, x_2| = \infty$ is given as a piecewise constant state, then it must be determined by the shock polar curve associated with the incoming state (ρ_∞, u_∞).







Note. θ_w is arbitrary in $(0, \frac{\pi}{2})$.

How do we make the problem well-posed and solvable?

Lemma

For any given $\theta_w \in (0, \frac{\pi}{2})$, *there exists a small constant* $\varepsilon_0 \in (0,1)$ *depending on* (γ, B_0, θ_w) *s.t. if*

$$M_\infty \geq \frac{1}{\varepsilon_0}(> 1),$$

then,

$$\theta_w < \theta_d^{M_\infty}.$$

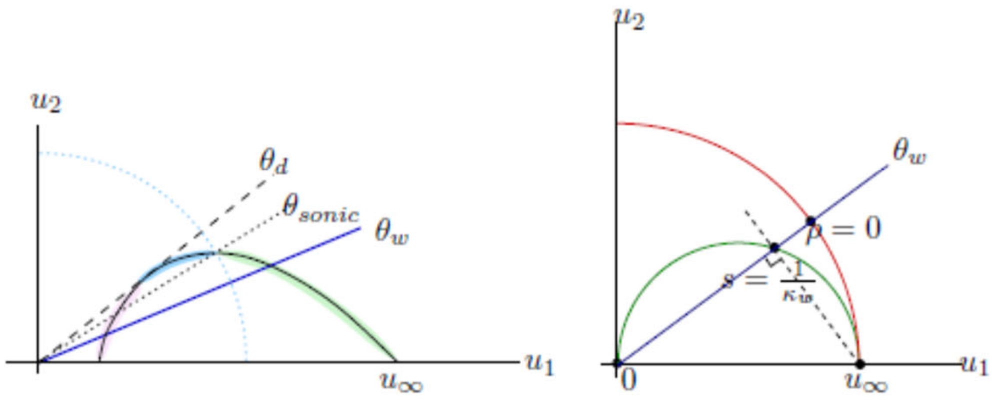
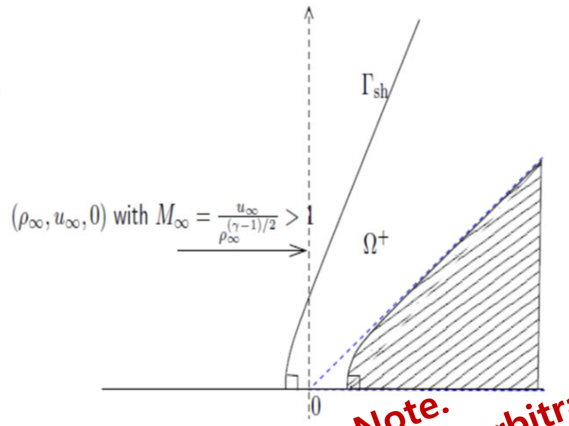


Figure: Shock polar for $M_\infty = \infty \Rightarrow \theta_{\text{det}}^{M_\infty} = \frac{\pi}{2}$

Detached shock problem past a blunt body W_b

(Revisit of the main problem)

For a fixed $d_0 > 0$, find an entropy solution $(\rho, \boldsymbol{u}$
with a shock $\Gamma_{sh} = \{x_1 = f_{sh}(x_2), x_2 \geq 0\}$ s.t. the following properties hold:



Note. θ_w is arbitrary in $(0, \frac{\pi}{2})$.

(i) $b(0) - f_{\text{sh}}(0) = d_0;$

(ii) $(\rho, u_1, u_2) = (\rho_\infty, u_\infty, 0)$ in $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 < f_{\text{sh}}(x_2)\};$

(iii) $u_2 = 0$ on $\Gamma_{\text{sym}} := \{(x_1, 0) : x_1 \leq b(0)\};$

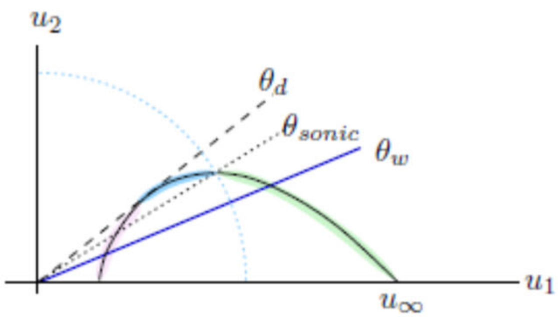
(iv) $(u_1, u_2) \cdot \mathbf{n}_w = 0$ on $\Gamma_w := \{(b(x_2), x_2) : x_2 \geq 0\}$ for a unit normal $\mathbf{n}_w$ on $\Gamma_w;$

(v) (ρ, u_1, u_2) uniformly converges to a piecewise constant state as $|\mathrm{x}| \to \infty$

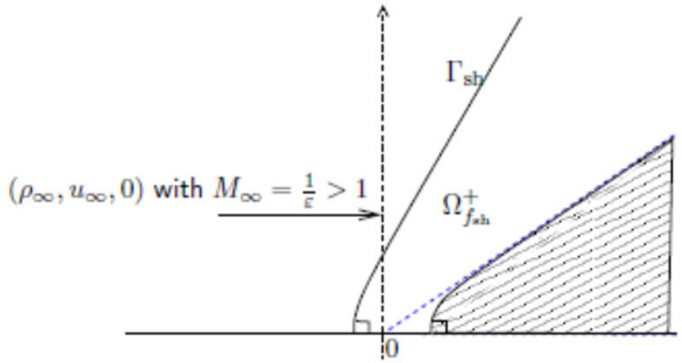
Formulation step 1

1. Given $\theta_w \in (0, \frac{\pi}{2})$, fix $(\rho_\infty, u_\infty, 0)$ with $M_\infty > 1$ sufficiently large s.t. $\theta_w < \theta_d^{M_\infty}$.

2. Choose the far-field asymptotic state as the strong shock point $(\mathbf{u}_{\rm st}, \rho_{\rm st})$ corresponding to θ_w on the shock polar.



Formulation step 2



3. Find a solution (ρ, u_1, u_2, f_{sh}) to the Free Boundary Problem:

$$\begin{cases} \partial_{x_1}(\rho u_1) + \partial_{x_2}(\rho u_2) = 0 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 = 0 \\ \frac{1}{2}|\mathbf{u}|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = B_0 \end{cases} \quad \text{in } \Omega^+_{f_{sh}} := \{\mathrm{x} \in R^2_+ \setminus \overline{W_b} : x_1 > f_{sh}(x_2)\}$$

$$\rho(u_1, u_2) \cdot \boldsymbol{\nu} = \rho_\infty(u_\infty, 0) \cdot \boldsymbol{\nu} \text{ on } \quad \Gamma_{sh} := \{x_1 = f_{sh}(x_2) : x_2 > 0\}$$

$$u_2 = 0 \text{ on } \Gamma_{sym}$$

$$(u_1, u_2) \cdot \mathbf{n}_w = 0 \text{ on } \Gamma_w := \{(b(x_2, x_2)) : x_2 \geq 0\}$$

$$\lim_{R\to\infty} \|(\rho, \mathbf{u}) - (\rho^\varepsilon_{st}, \mathbf{u}^\varepsilon_{st})\|_{C^0\left(\overline{\Omega^+_{f_{sh}} \setminus B_R(\mathbf{0})}\right)} = 0$$

Formulation step 3

with the free boundary condition

$$f'_{\mathrm{sh}}(x_2) = \frac{u_2(f_{\mathrm{sh}}(x_2), x_2)}{(u_\infty - u_1)(f_{\mathrm{sh}}(x_2), x_2)} \quad \text{for } x_2 > 0,$$

$$f_{\mathrm{sh}}(0) = b_0 - d_0 \quad \text{for } b_0 := b(0).$$

The free boundary condition is derived from

$$(\mathbf{u}_\infty - \mathbf{u}) \cdot (f'_{\mathrm{sh}}(x_2), 1) = 0 \quad \text{for } \mathbf{u}_\infty = (u_\infty, 0).$$

$$\left(\Leftrightarrow \mathbf{u} \cdot \boldsymbol{\tau} = \mathbf{u}_\infty \cdot \boldsymbol{\tau}, \quad \boldsymbol{\tau} = \frac{(f'_{\mathrm{sh}}(x_2), 1)}{\sqrt{(f'_{\mathrm{sh}}(x_2))^2 + 1}} \right)$$

Result part 1 (Existence, B.-Xiang, submitted)

Fix $\gamma > 1$, $B_0 > 0$ and $\beta \in (0,1)$. For any fixed constant $d_0 > 0$, $\exists$ a small constant $\bar{\varepsilon} > 0$ depending on (γ, B_0, d_0) s.t. if $M_\infty = \frac{1}{\varepsilon}$ for $\varepsilon \in (0, \bar{\varepsilon}]$, then the FBP has a solution $(\rho, \mathbf{u}, f_{\text{sh}})$ that satisfies the following properties:

(i) $f_{\text{sh}}(0) = b(0) - d_0$;

(ii) (Detached shock) $\exists\ \delta > 0$ depending only on (γ, B_0, d_0) such that

$$b(x_2) - f_{\text{sh}}(x_2) \geq \delta \quad \text{for all } x_2 \geq 0;$$

(iii)

$$\lim_{\substack{|\mathbf{x}|\to\infty \\ \mathbf{x}\in\Omega^+_{f_{\text{sh}}}}} |(\rho,\mathbf{u})(\mathbf{x}) - (\rho^\varepsilon_{\text{st}}, \mathbf{u}^\varepsilon_{\text{st}})| = 0, \quad \text{and} \quad \lim_{x_2\to\infty} |f'_{\text{sh}}(x_2) - s^\varepsilon_{\text{st}}| = 0;$$

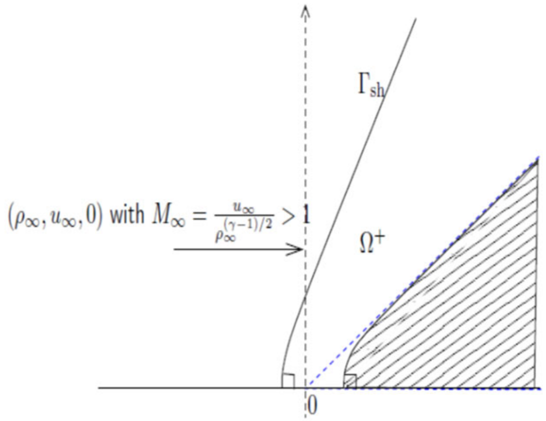
(iv) $\exists\ C > 0$ and $\alpha \in (0,1)$ s.t.

$$\|f_{\text{sh}} - f_0\|^{(-\beta)}_{2,\alpha,\mathbb{R}+} + \|\mathbf{u} - \mathbf{u}^\varepsilon_{\text{st}}\|^{(1-\beta)}_{1,\alpha,\Omega^+_{f_{\text{sh}}}} \leq C|(\rho^\varepsilon_{\text{st}}, u^\varepsilon_{\text{st}}, s^\varepsilon_{\text{st}}) - (\rho^0_{\text{st}}, 0, 0)|$$

for $f_0(x_2) = s^\varepsilon_{\text{st}} x_2 + b_0 - d_0$.

(transonic shock solution)

If $d_0 \geq d_*$ for some $d_* > 0$, then $\bar{\varepsilon}$ can be fixed depending only on d_*.



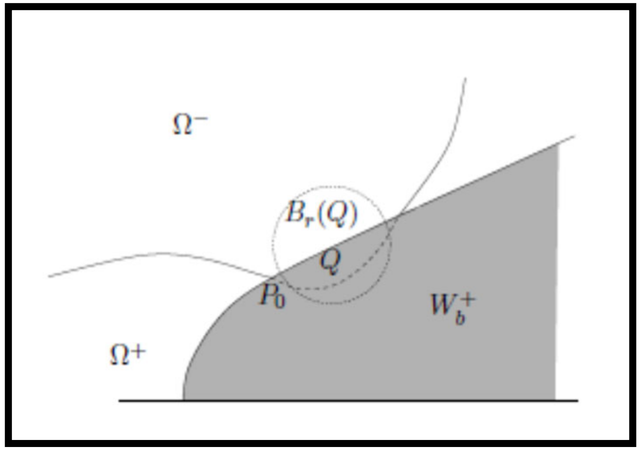
$|(f_{sh} - f_0)(x_2)| \approx (1 + x_2)^\beta$

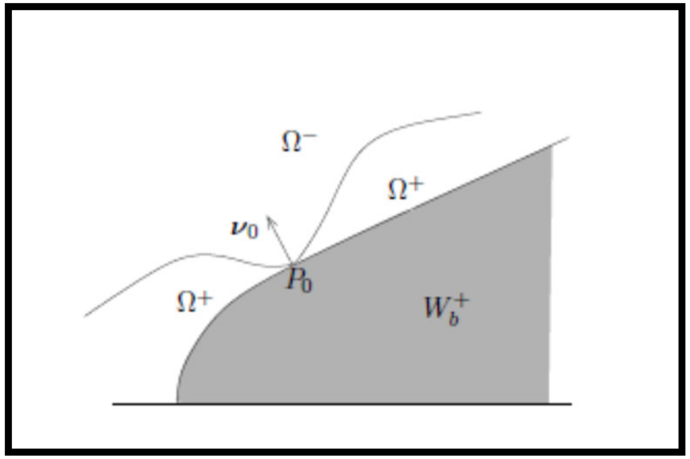
$|(f_{sh} - f_0)'(x_2)| \approx (1 + x_2)^{-1+\beta}$

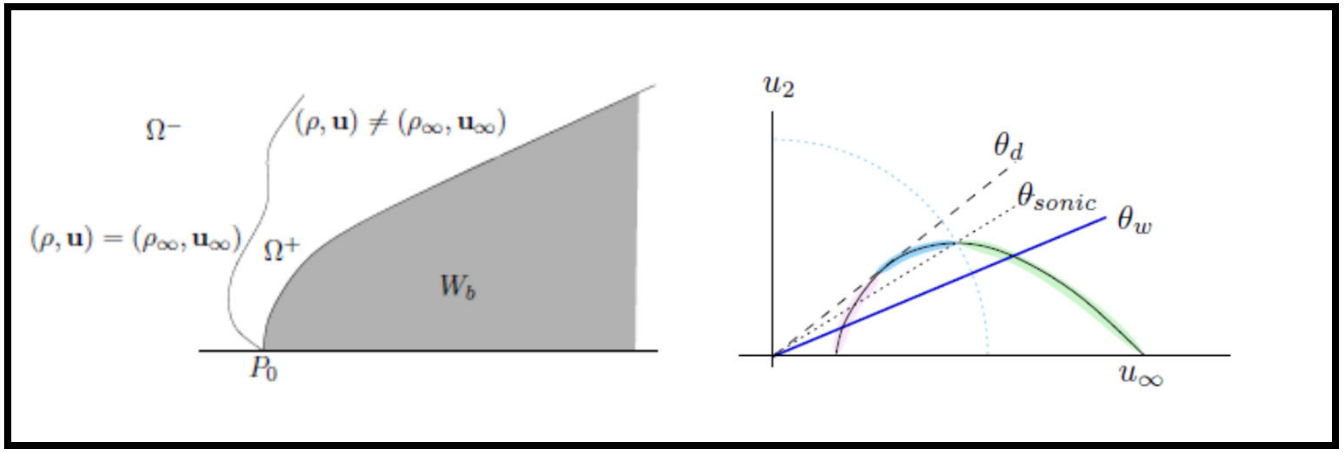
$|(\boldsymbol{u} - \boldsymbol{u}_{st})(x_2)| \approx (1 + x_2)^{-1+\beta}$

…

Why a shock must be DETACHED globally from the blunt body?







Idea of a proof:

1. Stream function for 2D flow

$\partial_{x_1}(\rho u_1)+\partial_{x_2}(\rho u_2) = 0 \Rightarrow \nabla^{\perp}\psi = (\rho u_1, \rho u_2)$ for $\nabla^{\perp}\psi := (\psi_{x_2}, -\psi_{x_1})$.

$$\partial_{x_2} u_1 - \partial_{x_1} u_2 = 0 \Rightarrow \mathrm{Div}\left(\frac{\nabla\psi}{\rho}\right) = 0,$$
$$\frac{1}{2}\frac{|\nabla\psi|^2}{\rho^2} + \frac{\rho^{\gamma-1}}{\gamma-1} = B_0 \Rightarrow \rho = \rho(|\nabla\psi|^2)$$

if either $M > 1$ or $M < 1$.

Steady Euler system for irrotational flow $\Rightarrow \mathrm{Div}\left(\frac{\nabla\psi}{\rho(|\nabla\psi|^2)}\right) = 0.$

2. Reformulation of FBP

(i)

$$|\nabla\psi|^2 < -2H(\rho_{\text{sonic}}) \quad \text{in} \quad \overline{\Omega^+_{f_{\text{sh}}}} (\Leftrightarrow M < 1 \Leftrightarrow \text{elliptc equation for } \psi)$$

(ii) (Equation for ψ)

$$(c^2 - u_1^2)\psi_{x_1x_1} - 2u_1u_2\psi_{x_1x_2} + (c^2 - u_2^2)\psi_{x_2x_2} = 0 \quad \text{in} \quad \Omega^+_{f_{\text{sh}}}$$

(iii) (Boundary conditions for ψ)

$$\begin{aligned} \psi = \psi_\infty (:= \rho_\infty u_\infty x_2) \quad &\text{on } \Gamma_{\text{sh}}, \\ \psi = 0 \quad &\text{on } \Gamma_{\text{sym}} \cup \Gamma_w. \end{aligned} \tag{1}$$

(Asymptotic boundary condition)

$$\lim_{\substack{|\mathrm{x}|\to\infty \\ \mathrm{x}\in\Omega_{f_{shock}}}} |\nabla^\perp \psi(\mathrm{x}) - \rho^\varepsilon_{\text{st}} \mathbf{u}^\varepsilon_{\text{st}}| = 0. \tag{2}$$

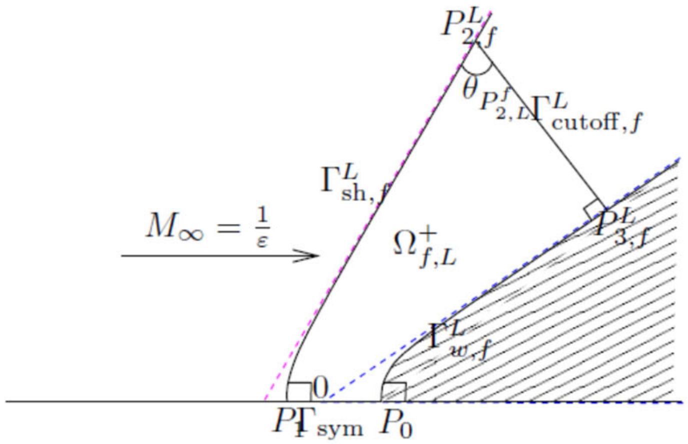
(iv) (Free boundary condition)

$$\begin{aligned} f'_{\text{sh}}(x_2) &= \frac{(\psi_{x_1}/\rho(|\nabla\psi|^2))(f_{\text{sh}}(x_2), x_2)}{(\psi_{x_2}/\rho(|\nabla\psi|^2))(f_{\text{sh}}(x_2), x_2) - u_\infty} \quad \text{for all } x_2 > 0, \\ f_{\text{sh}} &= b_0 - d_0. \end{aligned} \tag{3}$$

Free boundary problem in cut-off domains

How to execute actually?
(Iteration frame work)

1. **Fix L sufficiently large.**
2. **Fix an approximated shock f of "the height" L. Then, $\Omega^+_{f,L}$ is fixed.**
3. **Solve a NLBVP in $\Omega^+_{f,L}$.**
4. **Update the shock.**



Boundary condition on the cut-off boundary $\Gamma^L_{\text{cutoff},f}$

$$\nabla\psi \cdot \mathbf{n}_c = 0 \quad \text{on } \Gamma^f_{\text{cutoff,L}} \text{ for } \mathbf{n}_c = \langle \cos\theta_w, \sin\theta_w \rangle$$

This corresponds to

$$\mathbf{u} \cdot \boldsymbol{\tau}_c = 0 \quad \text{on } \Gamma^f_{\text{cutoff,L}}.$$

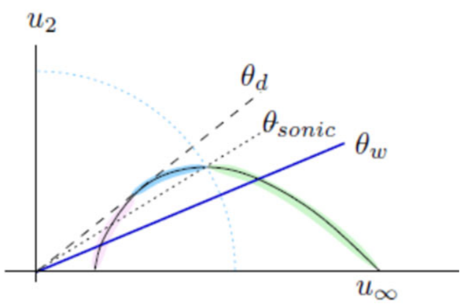
Result part 2(Convexity of the detached shock)

For a fixed $d_0 > 0$, one can find $\hat{\varepsilon} \in (0, \bar{\varepsilon}]$ s.t. if

$$M_\infty \geq \frac{1}{\hat{\varepsilon}},$$

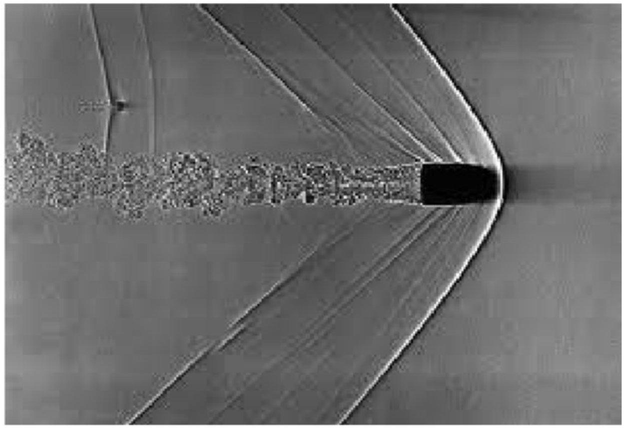
then the detached shock $x_1 = f_{sh}(x_2)$ satisfies that

$$f''_{sh}(x_2) \geq 0 \text{ for all } x_2 \geq 0.$$



The key assumption to achieve the convexity of the shock is

$$b''(x_2) \geq 0 \quad \text{for } x_2 > 0.$$



Q. Can the condition $b'' \geq 0$ be removed?

Summary & Further questions to study…

Summary

1. For a fixed (convex) blunt body W_b, we have proven the existence of a global detached shock solution.

2. The existence of a global detached shock solution can be proven for any given shock detached distance from the tip of W_b provided that M_∞ is sufficiently large.

3. The detached shock solutions that we have constructed converge to the strong shock solution at $|\mathrm{x}| = \infty$.

Questions

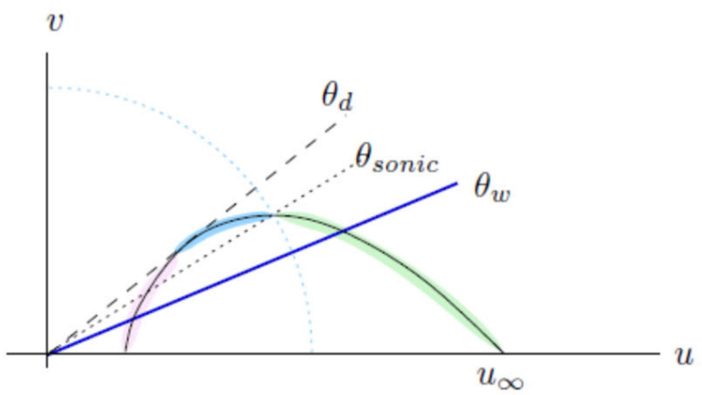
Q1. The uniqueness for a fixed d_0?

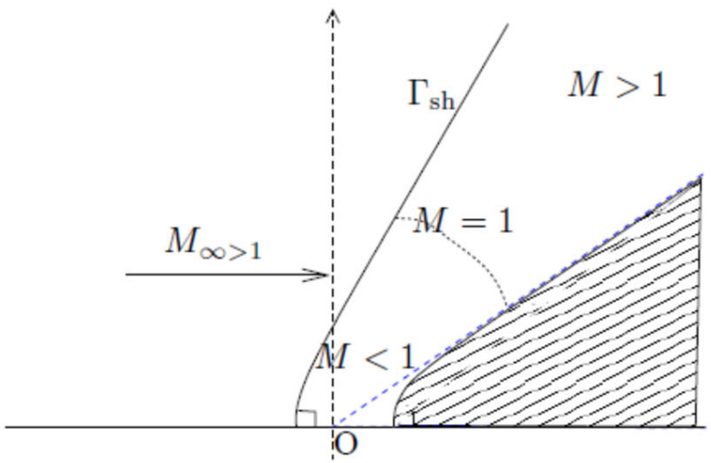
& Detached shock solution of full Euler system?

Q2. Given incoming supersonic state, can we determine the optimal shock detached distance $d_0^* > 0$?

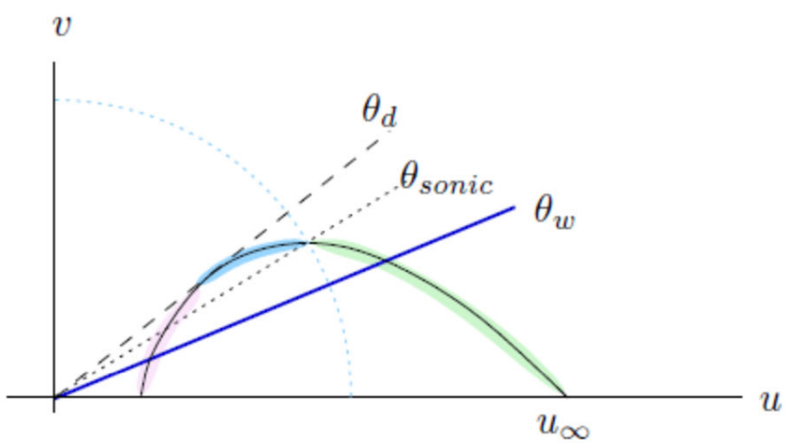
Further questions

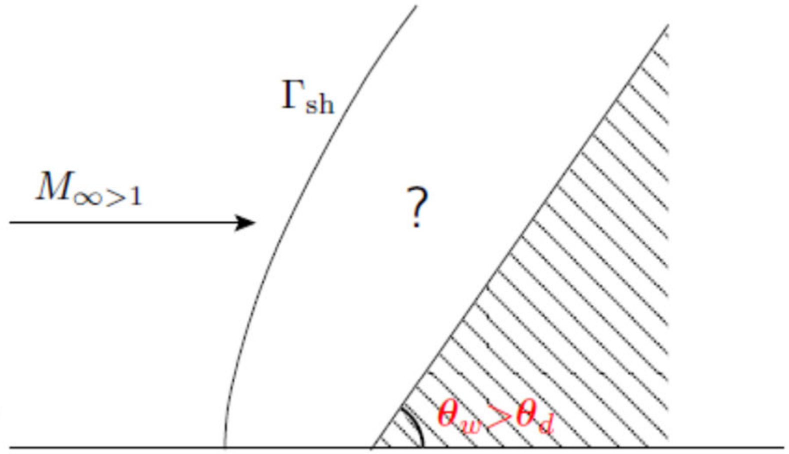
Q3. Is it possible for a detached shock solution to converge to the weak shock solution at $|\mathrm{x}| = \infty$?





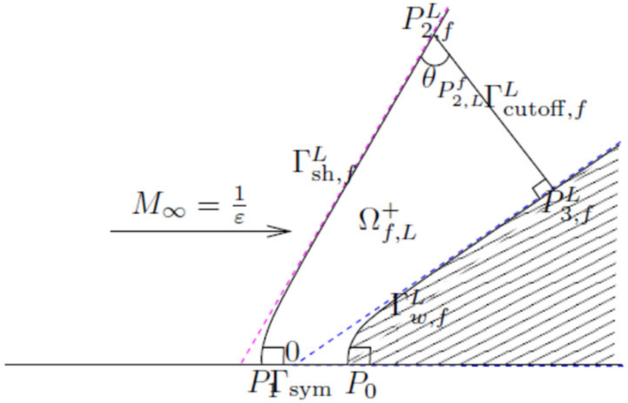
Q4. Detached shock solution past a wedge of the half-angle $\theta_w > \theta_{\text{det}}$ with the straight boundary?





THANK YOU!!

How to prove the convexity of $x_1 = f_{sh}(x_2)$?



(1) Take $L_n = 4(n + L_*)$.

(2) For each n, find $x_1 = f_{sh}^{(n)}(x_2)$.

(3) sgn $\frac{d^2}{dx_2^2} f_{sh}^{(n)}(x_2) = sgn \frac{d}{dx_2} \left|(u_1, u_2)\left(f_{sh}^{(n)}(x_2), x_2\right)\right|$

(4)* Show that $\left|(u_1, u_2)\left(f_{sh}^{(n)}(x_2), x_2\right)\right|$ monotonically increases.

(Here, the convexity of the blunt body is heavily used.)