Defects in Liquid Crystal Flows

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Formulation

Ginzburg-Landau approximation with averaged velocity in $\Omega \subset \mathbb{R}^2$

$$\begin{cases} \partial_t v + \mathrm{u} \wedge \mathrm{curl} v + \nabla P = \mu \Delta v - \mathrm{div}(\nabla d \otimes \nabla d) \\ \partial_t d + \mathrm{u} \cdot \nabla d = \Delta d + \frac{d(1-|d|^2)}{\varepsilon^2} \\ \mathrm{div} v = 0 \\ -\alpha \Delta \mathrm{u} + \mathrm{u} = v \end{cases}$$

- $\alpha > 0$, $\varepsilon > 0$, $\mu > 0$;
- P: hydrodynamical pressure, u: velocity;
- $\mathrm{curl} v = \partial_{x_1} v_2 - \partial_2 v_1$, $\mathrm{u} \wedge \mathrm{curl} v = (\mathrm{u}_2, -\mathrm{u}_1)\mathrm{curl} v$;
- $(v, d)|_{t=0} = (v_0^\varepsilon, d_0^\varepsilon)$, $\quad (v, d)|_{\partial\Omega} = (0, g)$, $\quad \mathrm{u}|_{\partial\Omega} = 0$.

Question of Interest

Asymptotic behaviour of the vortex as the parameter $\varepsilon \to 0$.

Energy Law and Topological Defects

Energy Law

$$\frac{d}{dt}\left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + 2E(d(t))\right)$$
$$= -\left(\|\Delta u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + 2\|\partial_t d + u\cdot\nabla d\|^2_{L^2(\Omega)}\right)$$

where

$$e_\varepsilon(d) = \frac{1}{2}|\nabla d|^2 + W_\varepsilon(d), \quad W_\varepsilon(d) = \frac{1}{4\varepsilon^2}(1-|d|^2)^2$$
$$\|v_0^\varepsilon\|_{L^2(\Omega)} \le 2A, \quad E(d_0^\varepsilon) = \int_\Omega e_\varepsilon(d_0^\varepsilon)dx \le \pi N \log\frac{1}{\varepsilon} + B.$$

Topological defects

- $|g| = 1$ on $\partial\Omega$ & $\deg(g, \partial\Omega) = N > 0$

Previous results

Vortex point: $d = 0$.

- Steady Ginzburg-Landau vortices: Bethuel-Brezis-Helein (1994),Lin-Liu (2001),$\cdots$;
- Heat flow of Ginzburg-Landau vortices:
  - Two dimension: Lin (1996), Jerrard-Soner (1998);
  - Higher dimension: Lin (1998);
- Cauchy problem for heat flow of liquid crystal: Lin-Liu (1995), Lin-Lin-Wang (2010), Lin-Wang (2016), Du-Huang-Wang (2020), Kortum (2020),$\cdots$
- Other models: Lin (Wave, 1999), Lin-Xin (Schrödinger, 1999), Yu (Maxwell-Klein-Gordon, 2011), $\cdots$

Steady Ginzburg-Landau vortices: Classical results (I)

Lower bound of energy & existence of vortices

For a fixed constant $\sigma > 0$, let $0 < \varepsilon < \min\{1, \sigma\}$ and $d : B_{2\sigma} \mapsto B_1$ be a continuously differentiable function satisfying

$$|\nabla d| < \frac{K}{\varepsilon}, \quad \deg(d, \partial B_\sigma) \neq 0, \quad |d(x)| \geq \frac{1}{2} \quad \forall |x| \in [\sigma, 2\sigma].$$

Then there is a constant K, independent of ε, such that

$$\int_{B_{2\sigma}} e_\varepsilon(d) dx \geq \pi \ln\left(\frac{\sigma}{\varepsilon}\right) - K.$$

Moreover there exists $x^* \in B_\sigma$ such that $d(x^*) = 0$ and for every $\lambda \in [\varepsilon, \sigma]$

$$\int_{B_\lambda(x^*)} e_\varepsilon(d) dx \geq \pi \ln\left(\frac{\lambda}{\varepsilon}\right) - K.$$

Steady Ginzburg-Landau vortices: Classical results (II)

Asymptotic behaviour of vortices

Under the same condition as before, up to a subsequence, one has

$$\int_\Omega |\nabla |d_\varepsilon||^2 dx + \int_\Omega W_\varepsilon(d_\varepsilon) dx \leq K$$

and in the weak $\star$ topology of $C(\Omega)$

$$\frac{1}{|\ln \varepsilon|} |\nabla d_\varepsilon|^2 \to 2\pi \sum_{j=1}^{N} \delta_{a_j}$$

$$\frac{1}{4\varepsilon^2} (|d_\varepsilon|^2 - 1)^2 \to \frac{\pi}{2} \sum_{j=1}^{N} \delta_{a_j}$$

as $\varepsilon \to 0$.

The heat flow of Ginzburg-Landau vortices

- If initially $(d_0)_\varepsilon$ has isolated vortices, these vortices move with velocities of the order of $|\ln\varepsilon|^{-1}$ in the original time scaling.
- $|\ln\varepsilon|^{-1}\partial_t d_\varepsilon = \Delta d_\varepsilon + \frac{1}{\varepsilon^2} d_\varepsilon(1-|d_\varepsilon|^2)$.
- Away vortices, d_ε converges uniformly to a function d with $|d|=1$ and $e_\varepsilon(d_\varepsilon)$ converges to $\frac{1}{2}|\nabla d|^2$.

Assumption on initial data

- There is a positive constant K such that $|(d_0)_\varepsilon| \leq 1, \quad |\nabla(d_0)_\varepsilon| \leq \frac{K}{\varepsilon} \quad \text{for all} \quad x \in \mathbb{R}^2.$
- $(d_0)_\varepsilon \to \prod_{i=1}^N \frac{x-b_j}{|x-b_j|} e^{ih_0(x)}$ weakly in $H^1_{loc}(\overline{\Omega} \setminus \{b_1, \cdots, b_N\})$ for some N distinct points $b_1, \cdots, b_N$ in Ω and $N = \deg(g, \partial\Omega) > 0$;
- $\int_\Omega \rho(x)^2 \left(|\nabla(d_0)_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(|(d_0)_\varepsilon|^2 - 1)^2\right) dx \leq K$ for a constant K which is independent of ε, where

$$\rho(x) = \min\{|x - b_j|, \quad j = 1, \cdots, N\};$$

- $E((d_0)_\varepsilon) = \int_\Omega \left(|\nabla(d_0)_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(|(d_0)_\varepsilon|^2 - 1)^2\right) dx \leq \pi N|\log\varepsilon| + 2B.$

Main Result (I)

Theorem (Gan-Hu-Lin, SIMA(2022))

For the initial data $(d_0)_\varepsilon$ which satisfies estimates in Assumptions, for any $t \in \mathbb{R}^+$, as $\varepsilon \to 0$,

- *The convergence*

$$d_\varepsilon(x,t) \to d(x,t) = \prod_{j=1}^{N} \frac{x - a_j(t)}{|x - a_j(t)|} e^{ih(x,t)}$$

  holds true in $H^1_{loc}(\overline{\Omega} \setminus \{a_1(t), a_2(t), \cdots, a_N(t)\})$.

- *The functions $a_j(t) \in \Omega$, $j = 1, \cdots, N$, are Hölder continuous with Hölder exponent $\frac{3}{4}$ and satisfy the following ODE:*

$$\begin{cases} \frac{d}{dt} a_j(t) = \mathrm{u}(a_j(t), t) \\ a_j(0) = b_j. \end{cases}$$

Main Result (II)

Theorem (Continued)

- *Away from the set* $\{a_j(t) : j = 1, \cdots, N, \quad t > 0\} \subset \Omega \times \mathbb{R}^+$, *the limit function* (v, u, d) *of* $(v_\varepsilon, \mathrm{u}_\varepsilon, d_\varepsilon)$ *satisfies the system*

$$\begin{cases} \partial_t d + \mathrm{u} \cdot \nabla d = \Delta d + |\nabla d|^2 d \\ \mathrm{div} v = 0 \\ -\alpha \Delta \mathrm{u} + \mathrm{u} = v. \end{cases}$$

The limit function $h(x, t)$ *satisfies a linear parabolic equation*

$$\partial_t h(x, t) + \mathrm{u} \cdot \nabla(\Theta(x, t) + h(x, t)) + \mathcal{R}(x, t) = \Delta h(x, t)$$

in $\{(x, t) : t \in \mathbb{R}^+, x \in \Omega \setminus \{a_i(t), i = 1, \cdots, N\}\}$ *with* $\sup_{t \geq 0} \|\nabla h\|_{L^2(\Omega)} \leq K$, *where*

$$e^{i\Theta(x,t)} = \prod_{i=1}^{N} \frac{x - a_i(t)}{|x - a_i(t)|} = \prod_{i=1}^{N} \theta(x - a_i(t))$$

and

$$\mathcal{R}(x, t) = \frac{\partial \Theta}{\partial t} = -\sum_{i=1}^{N} \left(\prod_{i \neq j}^{N} \theta(x - a_j(t)) \right) \mathrm{u}(a_i(t), t) \cdot \nabla \theta(x - a_i(t)).$$

Difficulties

- The leading term $\mathrm{u} \cdot \nabla d$;
- The first and second momentums of local energy;
- The separation and evolution of vortices.

Remark

- The flow map is almost Lipschitz with respect to the space variables and the vortices also satisfy

$$|a_j(t) - a_i(t)| \leq K\gamma^{-1}|b_j - b_i|^{\gamma} \quad \text{as} \quad 1 \leq i \neq j \leq N$$

  for all $\gamma \in (0, 1)$ and $t \in \mathbb{R}^+$.
- $\|\nabla d_\varepsilon\|_{L^p(\Omega)} \leq K \quad \text{for all} \quad p \in [1, 2)$.
- $\nabla |d_\varepsilon| \in L^2(\Omega)$.

Uniform estimates

Maximal Principle$\Longrightarrow$ $|d_\varepsilon|(x,t) \le 1 \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}^+.$

Gradient estimate

For initial data $((v_0)_\varepsilon, (d_0)_\varepsilon)$ with the uniform estimates stated in Assumptions, as ε is sufficiently small, there holds true

$$|\nabla d_\varepsilon|(x,t) \le \frac{K}{\varepsilon} \quad \text{for all} \quad (x,t) \in \Omega \times \mathbb{R}^+.$$

Proof.

Direct application of the heat kernel and Hölder inequality. □

Evolution of local energy $e_\varepsilon(d)$

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Let $\eta(x)$ be a smooth function with $\eta(x) = |\nabla \eta| = 0$ for $x \in \partial\Omega$, then

$$\begin{aligned}\frac{d}{dt}\int_\Omega \eta e_\varepsilon(d)dx &= \frac{1}{2}\frac{d}{dt}\int_\Omega \eta\left(|\nabla d|^2 + \frac{(1-|d|^2)^2}{2\varepsilon^2}\right)dx \\ &= -\int_\Omega \eta|\partial_t d + \mathrm{u}\cdot\nabla d|^2 dx + \int_\Omega \left(\nabla^2\eta : \nabla d \otimes \nabla d - \Delta\eta e_\varepsilon(d)\right)dx \\ &\quad + \int_\Omega \mathrm{u}\cdot\nabla\eta e_\varepsilon(d)dx - \int_\Omega \eta\partial_j \mathrm{u}_k \partial_k d_i \partial_j d_i dx.\end{aligned}$$

- Justified by a multiplication of the equation of d by $\eta(\partial_t d + \mathrm{u}\cdot\nabla d)$;
- As $\eta = 1$ formally, the energy law is recovered with the help of the equation of u to cancel the last term.
- Used to verify the separation and evolution of vortices.

Second Momentum of $e_\varepsilon(d)$ (I)

let $\eta = \eta_\sigma : \Omega \to \mathbb{R}^+$ be a smooth function such that

$$\eta_\sigma(x) = \begin{cases} \frac{1}{2}|x - b_j|^2 & \text{if} \quad x \in B_\sigma(b_j) \\ \geq \frac{1}{4}\sigma^2 & \text{if} \quad x = \Omega_\sigma \setminus \cup_{j=1}^N B_\sigma(b_j) \\ 0 & \text{if} \quad x \in \Omega \setminus \Omega_\sigma, \end{cases}$$

where

$$\Omega_\sigma = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \sigma\}$$

and

$$0 < 16\sigma = \min_{0<\varepsilon\leq 1} \min\{\min_{i\neq j}\{|b_\varepsilon^i - b_\varepsilon^j|, \quad \min_i \mathrm{dist}(b_\varepsilon^i, \partial\Omega)\}\}.$$

Second Momentum of $e_\varepsilon(d)$ (II)

Evolution of second momentum

Suppose that for $0 \le t \le T$ and for $j, l = 1, \cdots k$, $j \neq l$, we have

$$\min\{|a_j(t) - a_l(t)|, \quad \mathrm{dist}(a_j(t), \partial\Omega)\} \ge 4\sigma.$$

Then there holds true

$$A(t) \le A(0) + K(\sigma, A, B, C)(t + t^{\frac{1}{2}}) \left(\sup_{t \ge 0} \int_\Omega e_\varepsilon(d_\varepsilon)(x,t)dx \right)$$

for all $0 \le t \le T$, where

$$A(t) = \int_\Omega \eta_\sigma(x) e_\varepsilon(d_\varepsilon(x,t))dx$$

Argument of Proof: Careful estimate of terms in the local energy law.

Gradient estimate of h

L^2 estimate of ∇h

Suppose that $|d| \geq \frac{1}{2}$ on $\tilde{\Omega}_{\sqrt{\varepsilon}} = \Omega_{\sqrt{\varepsilon}} \setminus \cup_{j=1}^{N} B_{\sqrt{\varepsilon}}(b_j)$ with $\varepsilon \leq \sigma^2$, and that there is a constant K, independent of ε, satisfying

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} e_\varepsilon(d)dx \leq \frac{1}{2}\pi N|\ln \varepsilon| + K,$$

$$\int_{\partial B_{\sqrt{\varepsilon}}(b_j)} e_\varepsilon(d)d\mathcal{H}^1(x) \leq \frac{K}{\sqrt{\varepsilon}}, \quad 1 \leq j \leq N,$$

and

$$\int_{\partial\Omega_{\sqrt{\varepsilon}}} e_\varepsilon(d)d\mathcal{H}^1(x) \leq K.$$

Then, there is a single-valued, smooth function $h(x)$ defined on $\tilde{\Omega}_{\sqrt{\varepsilon}}$ such that

$$d(x) = |d(x)| \prod_{j=1}^{N} \frac{x - b_j}{|x - b_j|} e^{ih(x)}$$

and

$$\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} |\nabla h|^2 dx \leq K.$$

Gradient estimate of h

Key facts in the proof of gradient estimate

- The orthogonality between $ie^{if(x)}$ and $e^{if(x)} \Longrightarrow$ $\int_{\tilde{\Omega}_{\sqrt{\varepsilon}}} |d|^2[\frac{1}{2}|\nabla\Theta(x)|^2 + \frac{1}{2}|\nabla h(x)|^2 + \nabla h(x)\cdot\nabla\Theta(x)]dx \leq \frac{1}{2}\pi N|\ln\varepsilon| + K$.
- The harmonic function $\Theta(x)$ is a multivalued-harmonic function on Ω so that

$$e^{i\Theta(x)} = \prod_{i=1}^{N} \frac{x-b_i}{|x-b_i|} = \prod_{i=1}^{N} \theta(x-b_i).$$

- Mean value property over small balls around each vortex and near the boundary.

Localisation of energy & existence of vortices

Localisation of energy& existence of vortices

There are constants $t_0 > 0$, K, and functions

$$a_\varepsilon^i : [0, t_0] \to B_{\sigma/2}(b_\varepsilon^i), \quad \forall i = 1, \cdots, N,$$

with $a_\varepsilon^i(0) = b_\varepsilon^i$ such that $d_\varepsilon(a_\varepsilon^i(t), t) = 0$. Moreover for any $\varepsilon \in (0, 1]$, $t \in [0, t_0]$, $\lambda \in [\varepsilon, \sigma]$, $\mu_\varepsilon(t)\Big(B_\lambda(a_\varepsilon^i(t))\Big) \geq \pi \ln\left(\frac{\lambda}{\varepsilon}\right) - K, \quad \forall i = 1, \cdots, N,$ where $\mu_\varepsilon(t) = e_\varepsilon(d_\varepsilon)(x, t)dx$.

Proof.

- Property of topological degree $\Longrightarrow \deg(d_\varepsilon(\cdot, t); \partial B_\sigma(b_\varepsilon^i)) = 1 \quad \forall i = 1, \cdots, N$. It follows the existence of vortices.
- The second momentum of the local energy d_ε.

Regularity

By a diagonalization argument, up to a subsequence, we set

$$a^i(t) := \lim_{\varepsilon\to 0} a^i_\varepsilon(t) \quad \forall i = 1, \cdots, N.$$

For $t \in \mathbb{R}^+$, $\nu_\varepsilon(t)$ converges in the sense of Radon measures

$$\nu_\varepsilon(t) = |\ln\varepsilon|^{-1} e_\varepsilon(d_\varepsilon)(x,t)\mathrm{dx} \overset{*}{\to} \pi \sum_{i=1}^{N} \delta_{a^i(t)}, \qquad t \in [0, t_0].$$

Regularity of trajectory of vortices

For every $i \in \{1, \cdots, N\}$, $a^i(\cdot)$ is a Hölder continuous function with the Hölder exponent $\frac{1}{4}$ on $[0, t_0]$. In particular $a^i_\varepsilon(t)$ converges to a^i uniformly on $[0, t_0]$.

Proof.

Apply the local energy law for d_ε for ϕ with $\mathrm{supp}\phi(x) \subset B_\sigma(b^i)$ and
$\phi(a^i(t)) = 2, \quad \phi(a^i(s)) = 1, \quad \|\nabla\phi\|_{L^\infty} = |a^i(t) - a^i(s)|^{-1}.$ □

Hopf differential

Hopf differential

$$\omega(d) = \left|\frac{\partial d}{\partial x_1}\right|^2 - \left|\frac{\partial d}{\partial x_2}\right|^2 - 2i\frac{\partial d}{\partial x_1}\cdot\frac{\partial d}{\partial x_2},$$

and the potential function

$$W(d) = \frac{1}{4\varepsilon^2}(1-|d|^2)^2.$$

A straightforward computation shows that any solution of

$$\partial_t d + \mathrm{u}\cdot\nabla d = \Delta d + \frac{1}{\varepsilon^2}(1-|d|^2)d$$

satisfies

$$\frac{\partial\omega(d)}{\partial\bar{z}} = \frac{\partial}{\partial z}\Big(2W(d)\Big) + 2\frac{\partial d}{\partial z}(\partial_t d + \mathrm{u}\cdot\nabla d),$$

where

$$\frac{\partial}{\partial\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \quad \text{and} \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

Vanishing of $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$

Vanishing of $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$

Under the same conditions as in Theorem, the scaled measure $|\ln\varepsilon|^{-1}\omega(d_\varepsilon)$ converges to 0 in the sense of Radon measures. In other words, for any $\phi \in C(\Omega \times \mathbb{R}^+)$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2\times\mathbb{R}^+} |\ln\varepsilon|^{-1}\omega(d_\varepsilon)\phi(x,t)dxdt = 0.$$

Proof.

- Separation of vortices
- Application of equation of $\omega(d_\varepsilon)$
- Higher regularity of the average velocity u

□

The first momentum of energy

Near the vortices b_i

$$\frac{d}{dt}\int_{B_r(b_i)} xe_\varepsilon(d_\varepsilon)\phi(x)dx$$
$$= -\int_{B_r(b_i)} x|\partial_t d_\varepsilon + \mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon|^2\phi(x)dx + \int_{B_r(b_i)} x(\partial_t d_\varepsilon + \mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot(\mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon)\phi(x)dx$$
$$-\int_{B_r(b_i)} \partial_t d_\varepsilon\cdot\partial_l d_\varepsilon\partial_l(x\phi(x))dx + \int_{\partial B_r(b_i)} x\frac{\partial d_\varepsilon}{\partial\vec{n}}\cdot\partial_t d_\varepsilon\phi(x)dx$$

$$\overset{cut-offfunction}{\Longrightarrow}$$

$$|\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)} xe_\varepsilon(d_\varepsilon)(x,t)\phi(x)dx - |\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)} xe_\varepsilon(d_\varepsilon)(x,0)\phi(x)dxds$$
$$= -|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)} x|\partial_t d_\varepsilon + \mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon|^2\phi(x)dxds$$
$$+|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)} x(\partial_t d_\varepsilon + \mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot(\mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon)\phi(x)dxds$$
$$-|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)} \partial_t d_\varepsilon\cdot\partial_l d_\varepsilon\partial_l x\phi(x)dxds - |\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)} \partial_t d_\varepsilon\cdot\partial_l d_\varepsilon x\partial_l\phi(x)dxds$$

ODE

- $|\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)} xe_\varepsilon(d_\varepsilon)(x,t)\phi(x)dx -$ $|\ln\varepsilon|^{-1}\int_{B_\sigma(b_i)} xe_\varepsilon(d_\varepsilon)(x,0)\phi(x)dxds \to a_i(t)-a_i(0)$
- Writing $\partial_t d$ as the difference between $\partial_t d + \mathrm{u}\cdot\nabla d$ and $\mathrm{u}\cdot\nabla d$, the most difficult part from the third term is

$$|\ln\varepsilon|^{-1}\int_0^t\int_{B_\sigma(b_i)}(\mathrm{u}_\varepsilon\cdot\nabla d_\varepsilon)\cdot\partial_l d_\varepsilon \partial_l x\phi(x)dxds$$

  which converges to zero by applying Hopf differential and Dominated Convergence Theorem.
- Since $\nabla\phi$ vanishes near vortices, all other terms will converge to zero by using uniform estimates

$$\Longrightarrow$$

$$a_i(t)-a_i(0)=\int_0^t \mathrm{u}(a_i(s),s)ds \leftrightarrow \frac{d}{dt}a_j(t)=\mathrm{u}(a_j(t),t)$$

Limit of d_ε

- there is a subsequence (still denoted by ε_m) so that $d_{\varepsilon_m}(x,t) \to d(x,t)$ weakly in $H^1_{loc}(\{(x,t) : t \in \mathbb{R}^+, x \in \Omega \setminus \{a_i(t), i = 1, \cdots, N\}\})$ and strongly in $L^2_{loc}(\Omega \times \mathbb{R}^+)$ due to the fact that $|d_\varepsilon| \leq 1$. Also $\mathrm{u} \cdot \nabla d_\varepsilon \to \mathrm{u} \cdot \nabla d$ in the sense of distributions. Moreover $|d(x,t)| = 1$ a.e. in $\Omega \times \mathbb{R}^+$.
- Due to the identity

$$d_\varepsilon \wedge (\partial_t d_\varepsilon + \mathrm{u}_\varepsilon \cdot \nabla d_\varepsilon) = \sum_{j=1}^{2} \partial_j (d_\varepsilon \wedge \partial_j d_\varepsilon) \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$

using $|d|(x,t) = 1$, it follows

$$\begin{cases} \partial_t d + \mathrm{u} \cdot \nabla d = \Delta d + |\nabla d|^2 d \quad \text{in} \quad \{(x,t) : t \in \mathbb{R}^+, x \in \Omega \setminus \{a_i(t), i = 1, \cdots, N\}\}, \\ d(x,0) = \prod_{i=1}^{N} \frac{x - b_j}{|x - b_j|} e^{ih_0(x)}, \\ d = g \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \end{cases}$$

Evolution of vorticies

- The trajectory of vortex is $\frac{1}{4}-$Holder continuity;
- Since $|d(x,t)| = 1$, the function $d(x,t)$ is smooth in $\{(x,t) : t \in \mathbb{R}^+, x \in \Omega \setminus \{a_i(t), i = 1, \cdots, N\}\}$. Moreover for any $t > 0$, the degrees $\deg(d(\cdot,t), \partial B_\sigma(b_j))$, $1 \le j \le N$ are well-defined and all equal to 1 and hence

$$d(x,t) = \prod_{j=1}^{N} \frac{x - a_j(t)}{|x - a_j(t)|} e^{ih(x,t)}.$$

- ODE $\frac{d}{dt} a_j(t) = \mathrm{u}(a_j(t), t)$ holds.

Equations of θ and h

- ODE $\Longrightarrow$

$$\partial_t\theta(x-a_i(t)) = -\partial_{x_j}\theta(x-a_i(t))\partial_t(a_i(t))_j = -\mathrm{u}_j(a_i(t),t)\partial_{x_j}\theta(x-a_i(t))$$
$$= -\mathrm{u}(a_i(t),t)\cdot\nabla\theta(x-a_i(t)),$$

- a direct computation $\Longrightarrow$ $h(x,t)$ satisfies

$$\begin{cases} \partial_t h(x,t) + \mathrm{u}\cdot\nabla(\Theta(x,t)+h(x,t)) + \mathcal{R}(x,t) = \Delta h(x,t) \\ \qquad\qquad \text{in} \quad \{(x,t) : t\in\mathbb{R}^+, x\in\Omega\setminus\{a_i(t), i=1,\cdots,N\}\}, \\ \mathcal{R}(x,t) = -\sum_{i=1}^N \left(\prod_{i\neq j}^N \theta(x-a_j(t))\right)\mathrm{u}(a_i(t),t)\cdot\nabla\theta(x-a_i(t)), \\ h(x,t) = h_0(x) \quad \text{on} \quad \partial\Omega\times\mathbb{R}^+, \\ h(x,0) = h_0(x). \end{cases}$$

Property of h

Uniform estimate of h

The function $h(x,t)$ satisfies $\sup_{t\geq 0}\|\nabla h(x,t)\|^2_{L^2(\Omega)} \leq K$. Consequently, there follows

$$d_\varepsilon(x,t) \to \prod_{i=1}^{N} \frac{x-a_j(t)}{|x-a_j(t)|} e^{ih(x,t)}$$

in $L^2_{loc}(\Omega\times\mathbb{R}^+)$ and weakly in $H^1_{loc}(\{(x,t): t\in\mathbb{R}^+, x\in\Omega\setminus\{a_i(t), i=1,\cdots,N\}\})$ as $\varepsilon\to 0$.

Remark

The function $h(x,t)$ is determined by the limit of the whole family d_ε instead of a special subsequence d_{ε_m}.

Open Questions

There are several open questions, including

- The topological degree N is negative;
- The limit as the parameter α approaches zero, that is, defects with local velocity of Ginzburg-Landau approximations;
- The asymptotic behaviour of the equation of velocity, due to the lack of uniform estimate $\|\nabla d\|_{L^2}$;
- ...

Thank you for your attention!