On a class of active scalar equations with applications on MG equations

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Outline

Introduction

Active scalar equations

We are interested on a type of active scalar equations on $\mathbb{T}^d \times [0, \infty)$ (or $\mathbb{R}^d \times [0, \infty)$):

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = -\kappa(-\Delta)^\gamma \theta + S, \\ u = M_\nu[\theta], \theta(x, 0) = \theta_0(x) \end{cases} \tag{1.1}$$

where the vector field u is related to θ by some operator $M_\nu[\cdot]$. Here S, θ_0 are some given functions.

Set-up

- Unknown functions: $(u, \theta) = (u, \theta)(x, t)$
- u is a vector and θ is a scalar
- S, θ_0 are given
- d: Dimension (mainly $d = 2$ or 3)
- $\nu, \kappa \in [0, \infty)$: "diffusive" constants
- $\gamma \in (0, 1]$
- M_ν: A Fourier multiplier operator which relates u and θ, and M_ν depends on ν

What are M_ν's?

M_ν behaves drastically different for cases $\nu > 0$ and $\nu = 0$:

- $\widehat{M_\nu[\theta]}(k) \approx |k|^{-2}\hat{\theta}(k)$ when $\nu > 0$;
- $\widehat{M_{\nu=0}[\theta]}(k) \approx |k|\hat{\theta}(k)$ when $\nu = 0$.

Hence M_ν is smoothing of order 2 for $\nu > 0$, but M_0 becomes singular of order 1.

Some more assumptions

We further impose:

- $\mathrm{div}(u) = 0$ (divergence-free for u)
- θ_0, S are mean-zero on $\mathbb{T}^d$

Motivation

Active scalar equations have many applications on fluid mechanics:

- vorticity formulation of the 2D Euler equation (equation for ω);
- surface quasi-geostrophic (SQG) equation (for $u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\theta$);
- *magneto-geostrophic (MG) equation (comes from incompressible MHD);
- incompressible porous media (IPM) equation, etc.

MG equations

One important example comes from MG equations. The original model is given by (θ is the buoyancy field)

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta + S \tag{1.2}$$

The constitutive law is obtained from the linear system

$$e_3 \times u = -\nabla P + e_2 \cdot \nabla b + \theta e_3 + \nu \Delta u, \tag{1.3}$$

$$0 = e_2 \cdot \nabla u + \Delta b, \tag{1.4}$$

$$\nabla \cdot u = 0, \nabla \cdot b = 0, \tag{1.5}$$

where (e_1, e_2, e_3) is the standard Cartesian unit vectors, P is the pressure and b is the magnetic field.

MG equations (cont'd)

Vector manipulations of (1.3)-(1.5) give the following expression

$$\begin{aligned} &\{[\nu\Delta^2 - (e_2 \cdot \nabla)^2]^2 + (e_3 \cdot \nabla)^2 \Delta\} u \\ &= -[\nu\Delta^2 - (e_2 \cdot \nabla)^2] \nabla \times (e_3 \times \nabla\theta) \\ &\quad + (e_3 \cdot \nabla)\Delta(e_3 \times \nabla\theta). \end{aligned} \tag{1.6}$$

Hence by "solving" u in terms of θ from (1.6), we can write $u = M_\nu[\theta]$ and the explicit expression M_ν are given by...

MG equations (cont'd)

$$\widehat{(M_\nu)}_1(k) = [k_2 k_3 |k|^2 - k_1 k_3 (k_2^2 + \nu |k|^4)] D(k)^{-1},$$
$$\widehat{(M_\nu)}_2(k) = [-k_1 k_3 |k|^2 - k_2 k_3 (k_2^2 + \nu |k|^4)] D(k)^{-1},$$
$$\widehat{(M_\nu)}_3(k) = [(k_1^2 + k_2^2)(k_2^2 + \nu |k|^4)] D(k)^{-1},$$

where

$$D(k) = |k|^2 k_3^2 + (k_2^2 + \nu |k|^4)^2.$$

Directions of study

In view of the class of active scalar equations (1.1), we mainly discuss the cases when:

- (Diffusive) $\kappa > 0$, $\nu \geq 0$ and $\gamma = 1$
- (Non-diffusive) $\kappa = 0$ and $\nu \geq 0$

Diffusive case

Diffusive case

In this section, we focus on the following class of active scalar equations (we take $\kappa = 1$):

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = \Delta \theta + S, \\ u = M_\nu[\theta], \theta(x,0) = \theta_0(x), \end{cases} \tag{2.1}$$

where $\nu \geq 0$. The results on (2.1) can be summarised in Theorem 2.1 and references can be found in [FS18, FS21, FS22, FV11a].

Results on Diffusive case

Theorem 2.1

Let θ_0 and S be given with $\theta_0 \in L^2$ and $S \in L^2 \cap L^\infty$. Then for any $\nu \geq 0$, a Leray-Hopf weak solution θ^ν to (2.1) evolving from θ_0 is a classical solution.

Remarks on Theorem 2.1

For the case of MG equations, we can also address the convergence of θ when $\nu \to 0$. In [FS18], the authors showed that given $\tau > 0$, for all $s \geq 0$, we have

$$\lim_{\nu \to 0} \|(\theta^\nu - \theta)(t, \cdot)\|_{H^s} = 0$$

whenever $t \geq \tau$.

Proof of Theorem 2.1

We focus on the singular case when $\nu = 0$ and write $\theta = \theta^\nu$. Theorem 2.1 can be proved in the following steps:

- consider the linear problem when u is replaced by a given drift velocity v;
- show that a weak solutions of the linear drift-diffusion equation instantly become Hölder continuous in space-time;
- by "bootstraping", one can show higher regularity, i.e. $\theta \in C^\infty$;
- come back to the non-linear problem.

Linear problem

Consider

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta = \Delta \theta + S, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{2.2}$$

where v is a given divergence-free vector field with $v_j = \partial_{x_i} V_{ij}$ and

$$V_{ij} \in L^\infty(0,\infty; BMO) \cap L^2(0,\infty; H^1).$$

Why we need $V_{ij} \in BMO$?

Note that (2.2) is invariant under the scaling transformation

$$\theta(x,t) \to \theta^{(\lambda)}(x,t) = \theta(\lambda x, \lambda^2 t),$$

hence L^∞ is the critical Lebesgue space with respect to the natural scaling of the equation.

Why we need $V_{ij} \in BMO$? (cont'd)

- In the original non-linear problem, if we write $u_j = \partial_{x_i} V_{ij}[\theta]$, then we can see that V_{ij} becomes Calderón-Zygmund zero-order operator, which maps L^∞ to BMO.
- Hence for the linear problem, we assume $V_{ij} \in BMO$ as well.

Linear problem (cont'd)

What we need:

(a) weak solutions θ to (2.2) are bounded above for positive time;

(b) weak solutions θ to (2.2) are Hölder continuous in positive time.

From L^2 to L^∞

To prove (a), by using the De Giorgi method,

- we make use of a recurrence nonlinear relation between consecutive truncations of θ at an increasing sequence of levels;
- we also make use of the energy inequality (which holds for weak solutions) that controls $|\nabla\theta|$ by θ, and the opposite effect of the Sobolev inequality that controls θ by $|\nabla\theta|$.

From L^2 to L^∞ (cont'd)

More precisely, for $h > 0$, we have the following level set energy inequality for the truncated function $(\theta - h)_+$:

$$\int |(\theta(t_2, x) - h)_+|^2 dx + 2 \int_{t_1}^{t_2} \int |\nabla(\theta - h)|^2$$
$$\leq \int |(\theta(t_1, x) - h)_+|^2 dx + 2 \int_{t_1}^{t_2} \int |S(\theta - h)_+|$$

for all $0 < t_1 < t_2 < \infty$.

From L^2 to L^∞ (cont'd)

Fix $t_0 > 0$ and define

$$c_n = \sup_{t_n \le t \le t_0} \int |\theta_n|^2 + 2\int_{t_n}^{\infty}\int |\nabla\theta_n|^2,$$

where $\theta_n = (\theta(t,\cdot) - h_n)_+$, $t_n = t_0 - \frac{t_0}{2^n}$, $h_n = H - \frac{H}{2^n}$

From L^2 to L^∞ (cont'd)

With the help of Sobolev inequality, we can show that

$$c_n \leq C_d\Big(\frac{1}{t}+1\Big)\Big(\frac{2^{n(\frac{d+3}{d-1})}}{H^{\frac{4}{d-1}}}\Big)c_{n-1}^{\frac{d+1}{d-1}}.$$

for some constant $C_d > 0$ which depends only on d.

From L^2 to L^∞ (cont'd)

By choosing $H > 0$ large enough, we have $c_n \to 0$ and θ is bounded above by H. The outcome is that for all $t \in (0, 1]$, we have

$$\|\theta(\cdot, t)\|_{L^\infty} \leq C_d\left[\left(1 + \frac{1}{t}\right)^{\frac{d-1}{4}}\left(\|\theta_0\|_{L^2} + \|S\|_{L^2}\right) + \|S\|_{L^\infty}\right].$$

Proving Hölder regularity

Next to prove (b), let Q_ρ be the parabolic cylinder

$$Q_\rho = B_\rho(x_0) \times [t_0, t_0 + \delta\rho^2]$$

for $\delta \in (0, 1)$, $\rho > 0$ and $(x_0, t_0) \in \mathbb{T}^d \times (0, \infty)$. We want to show that the oscillation of θ in Q_ρ decays at some fixed rate as ρ decreases.

Proving Hölder regularity (cont'd)

More precisely, we claim that

$$\left(\sup_{Q_1}\theta - \inf_{Q_1}\theta\right) \leq \alpha\left(\sup_{Q_2}\theta - \inf_{Q_2}\theta\right),$$

for some $\alpha \in (0,1)$ independent of R, where $Q_1 = B_r \times [t_1, t_1 + \delta_0 r^2]$ and $Q_2 = B_R \times [t_1, t_1 + \delta_0 R^2]$ with $R > r$.

Proving Hölder regularity (cont'd)

Assume that $h_0 \leq \sup_{Q_{r_0}} \theta$ where $r_0 > 0$ is arbitrary. Using the de Giorgi type estimates, one can show that

$$\sup_{Q_{r_0/2}} \theta - h_0 \leq C\left(\frac{|\{\theta > h_0\} \cap Q_{r_0}|^{\frac{1}{d+2}}}{r_0}\right)^{\frac{1}{2}} \left(\sup_{Q_{r_0}} \theta - h_0\right), \qquad (2.3)$$

where $C = C(d, \|V_{ij}\|_{L^\infty_t BMO})$ is a positive constant.

Proving Hölder regularity (cont'd)

The key application of (2.3) is to pick an increasing sequence H_n such that

$$H_n := \sup_{Q_2} \theta - \frac{(\sup_{Q_2} \theta - \inf_{Q_2} \theta)}{2^n},$$

then $H_n \to \sup_{Q_2} \theta$ and for sufficiently large n_1 (needed to be independent of R), we have

$$C\left(\frac{|\{\theta > H_{n_1}\} \cap Q_2|^{\frac{1}{d+2}}}{R}\right)^{\frac{1}{2}} < \frac{1}{2}.$$

Proving Hölder regularity (cont'd)

Then we have

$$\sup_{Q_1} \theta \leq H_{n_1} + \frac{1}{2}(\sup_{Q_2} \theta - H_{n_1}),$$

and hence

$$\left(\sup_{Q_1} \theta - \inf_{Q_1} \theta\right) \leq \left(1 - \frac{1}{2^{n_1+2}}\right)\left(\sup_{Q_2} \theta - \inf_{Q_2} \theta\right).$$

This means that the oscillation of θ decays at a fixed rate with respect to distance, which implies Hölder continuity of θ at the arbitrary point (x_0, t_1).

Non-diffusive case

Non-diffusive case

In this section, we focus on the following class of active scalar equations:

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = S, \\ u = M_\nu[\theta], \theta(x,0) = \theta_0(x), \end{cases} \tag{3.1}$$

where $\nu \geq 0$. We subdivide the cases by

(i) when $\nu = 0$;

(ii) when $\nu > 0$.

For simplicity, we fix $d = 3$ (which is consistent with the MG equations).

Case (i) when $\nu = 0$

- When $\nu = 0$, the equation (3.1) is locally wellposed in analytic class of functions; see [FV11b].
- The idea is that when $\nu = 0$, there is at most one derivative loss in x.

Gevrey-space

Fix $r > 3$. A function $\theta \in C^\infty(\mathbb{R}^3)$ belongs to the Gevrey class G^s where $s \geq 1$, if there exists $\tau > 0$, known as the Gevrey-class radius, such that the G^s_τ-norm is finite, i.e.

$$\|\theta\|^2_{G^s_\tau} = \|\Lambda^r e^{\tau \Lambda^{\frac{1}{s}}} \theta\|^2_{L^2} = \sum_{k \in \mathbb{Z}^3_*} |k|^{2r} e^{2\tau |k|^{\frac{1}{s}}} |\widehat{\theta}(k)|^2.$$

Gevrey-space (cont'd)

- the exponent $r > 3$ gives us some "freedom" in the analysis.
- Since we are working in the mean-free setting, we take $\mathbb{Z}^3_* = \{k \in \mathbb{Z}^3 : |k| \neq 0\}$ and we define $G^s := \cup_{\tau>0} G^s_\tau$.
- We point out that for the case when $s = 1$, G^s gives the space of analytic functions.

Local-in-time existence in analytic space

We take L^2-inner product of (3.1) with $\Lambda^{2r}e^{2\tau\Lambda}\theta$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{G^s_\tau} - \dot{\tau}\|\Lambda^{\frac{1}{2}}\theta\|^2_{G^s_\tau} = \langle u\cdot\nabla\theta, \Lambda^{2r}e^{2\tau\Lambda}\theta\rangle + \langle S, e^{2\tau\Lambda^{\frac{1}{s}}}\theta\rangle \tag{3.2}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$ and $\tau = \tau(t) > 0$ is the radius of convergence.

Local-in-time existence in analytic space (cont'd)

Treating u as $\nabla\theta$, we can show that

$$|\langle u \cdot \nabla\theta, \Lambda^{2r} e^{2\tau\Lambda}\theta\rangle| \leq C\|\Lambda^{\frac{1}{2}}\theta\|^2_{G^s_\tau}\|\theta\|_{G^s_\tau},$$

hence if τ is chosen to be decreasing sufficiently fast, one can obtain from (3.2) that $\frac{d}{dt}\|\theta\|^2_{G^s_\tau} < 0$ in short time, which gives local existence of θ in analytic space.

Illposedness for MG equations

- In general, the equation (3.1) may not be wellposed in Sobolev space.
- In the case of MG equation, the equation is in fact illposed in Sobolev spaces; see [FV11b].
- This is due to the "evenness" of the operator $M_{\nu=0}$ (for the Fourier multiplier) which prevents a commutator-type cancellation.
- More precisely, we lost control on the term (when taking $u \approx \nabla\theta$)

$$\int [(-\Delta)^{\frac{s}{2}}(u \cdot \nabla\theta)][(-\Delta)^{\frac{s}{2}}\theta].$$

Hadamard Illposedness

A Cauchy problem for a certain partial differential equation is called wellposed in X in the sense of Hadamard, if

- for any initial data in X, the problem has a unique solution in $L^\infty(0, T; X)$, with T depending only on the initial data in the X -norm, and
- the solution map $Y \to L^\infty(0, T; X)$ satisfies some continuity properties, for a suitable space $Y \subset X$.

Illposedness for MG equations (cont'd)

To show illposedness,

- we construct a steady state and a sequence of eigen-functions, with arbitrarily large eigenvalues, for the linearized MG equation around this steady state.
- by a perturbative argument, one can further show that linear Illposedness implies the Lipschitz Illposedness for the nonlinear problem.
- the results are summarised in Theorem 3.1 which was proved in [FV11b].

Illposedness for MG equations (cont'd)

Theorem 3.1

The non-diffusive MG equations are locally Lipschitz (X, Y) ill-posed in Sobolev spaces $Y \subset X$ embedded in $W^{1,4}(\mathbb{T}^3)$.

Case (ii) when $\nu > 0$

- Given $\nu > 0$ and initial datum $\theta_0 \in W^{s,3}$ with $s > 0$, we can prove that (3.1) has a unique global-in-time solution $\theta \in W^{s,3}$; see [FS19].
- The main reason for the global-in-time existence is that, the 2-order smoothing effect from M_ν provides us better control on $\|u\|_{L^2_t L^\infty_x}$.
- The results are summarised in Theorem 3.2.

Global-in-time existence of $W^{s,3}$ solutions

Theorem 3.2

Let $\nu > 0$ and $\theta_0 \in W^{s,3}$ for $s > 0$, and let $S = S(x)$ be a C^∞-smooth source term. Then we have

$$\|\theta(\cdot,t)\|_{W^{s,3}} \leq C\|\theta_0\|_{W^{s,3}} \exp\left(C\int_0^t \|\nabla u(\cdot,\tilde{t})\|_{L^\infty} d\tilde{t} + Ct\|S\|_{W^{s,3}} \right).$$

Here $C > 0$ is a dimensional constant.

Global-in-time existence in Gevrey space

- When the initial datum θ_0 and forcing term S are in some Gevrey-class G^s for $s \geq 1$, we can prove that there exists global-in-time Gevrey-class G^s solution to (3.1).
- The results are summarised in Theorem 3.3 which was proved in [FS19].

Global-in-time existence in Gevrey space (cont'd)

Theorem 3.3

Fix $\nu > 0$ and $s \geq 1$. Let θ_0 and S be of Gevrey-class s with radius of convergence $\tau_0 > 0$. There exists a unique Gevrey-class s solution θ to (3.1) on $\mathbb{T}^3 \times [0, \infty)$ with radius of convergence at least $\tau = \tau(t)$ for all $t \in [0, \infty)$, where τ is a decreasing function satisfying

$$\tau(t) \geq \tau_0 e^{-C\left(\|e^{\tau_0 \Lambda^{\frac{1}{s}}} \theta_0\|_{L^2} + 2\|e^{\tau_0 \Lambda^{\frac{1}{s}}} S\|_{L^2}\right)t}. \tag{3.3}$$

Here $C > 0$ is a constant which depends on ν but independent of t.

Proof of Theorem 3.3

We take L^2-inner product of (3.1) with $e^{2\tau\Lambda^{\frac{1}{s}}}\theta$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}^2 - \dot{\tau}\|\Lambda^{\frac{1}{2s}}e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}^2$$
$$\leq \left| - \langle u \cdot \nabla\theta, e^{2\tau\Lambda^{\frac{1}{s}}}\theta\rangle \right| + \|e^{\tau\Lambda^{\frac{1}{s}}}S\|_{L^2}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}. \tag{3.4}$$

Proof of Theorem 3.3 (cont'd)

The key is to show that

$$\left| -\langle e^{\tau\Lambda^{\frac{1}{s}}}(u\cdot\nabla\theta), e^{\tau\Lambda^{\frac{1}{s}}}\theta\rangle \right| \le C\tau \|e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2} \|\Lambda^{\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}^2, \quad (3.5)$$

but how can we get that "extra" τ?
We make use of the divergence-free property of u, which gives

$$\langle u\cdot\nabla\Lambda^r e^{\tau\Lambda^{\frac{1}{s}}}\theta, \Lambda^r e^{\tau\Lambda^{\frac{1}{s}}}\theta\rangle = 0.$$

Proof of Theorem 3.3 (cont'd)

We then insert the above term to the non-linear term in question and get

$$\begin{aligned}
&\left|\langle u\cdot\nabla\theta, e^{2\tau\Lambda^{\frac{1}{s}}}\theta\rangle\right| \\
&= \left|\langle u\cdot\nabla\theta, e^{2\tau\Lambda^{\frac{1}{s}}}\theta\rangle - \langle u\cdot\nabla e^{\tau\Lambda^{\frac{1}{s}}}\theta, e^{\tau\Lambda^{\frac{1}{s}}}\theta\rangle\right| \\
&= \left| i(2\pi)^3 \sum_{j+k=l} (\hat{u}(j)\cdot k)(\hat{\theta}(k)\cdot\bar{\hat{\theta}}(l))e^{\tau|l|^{\frac{1}{s}}}(e^{\tau|l|^{\frac{1}{s}}} - e^{\tau|k|^{\frac{1}{s}}})\right|,
\end{aligned}$$

where $j,k,l\in\mathbb{Z}^3_*$ are the Fourier frequencies.

Proof of Theorem 3.3 (cont'd)

We make use of the inequality $e^x - 1 \leq xe^x$ for $x \geq 0$ and the triangle inequality $|k+j|^{\frac{1}{s}} \leq |k|^{\frac{1}{s}} + |j|^{\frac{1}{s}}$ to obtain

$$\left| e^{\tau |l|^{\frac{1}{s}}} - e^{\tau |k|^{\frac{1}{s}}} \right| \leq C\tau \frac{|j|}{|k|^{1-\frac{1}{s}} + |l|^{1-\frac{1}{s}}} e^{\tau |l|^{\frac{1}{s}}} e^{\tau |k|^{\frac{1}{s}}}.$$

This will give us the τ we need!

Proof of Theorem 3.3 (cont'd)

We also need the 2-order smoothing assumption on M_ν to get

$$\|\Lambda^{2+\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}} u\|_{L^2} \leq C_\nu \|\Lambda^{\frac{1}{2s}} e^{\tau\Lambda^{\frac{1}{s}}} \theta\|_{L^2}$$

and

$$\|\Lambda^2 e^{\tau\Lambda^{\frac{1}{s}}} u\|_{L^2} \leq C_\nu \|e^{\tau\Lambda^{\frac{1}{s}}} \theta\|_{L^2},$$

which will be important for showing (3.5).

Proof of Theorem 3.3 (cont'd)

Hence we obtain from (3.4) that

$$\frac{1}{2}\frac{d}{dt}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}^2 - \dot{\tau}\|\Lambda^{\frac{1}{2s}}e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}^2$$
$$\leq C\tau\|e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}\|\Lambda^{\frac{1}{2s}}e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}^2 + \|e^{\tau\Lambda^{\frac{1}{s}}}S\|_{L^2}\|e^{\tau\Lambda^{\frac{1}{s}}}\theta\|_{L^2}.$$

Proof of Theorem 3.3 (cont'd)

Choose $\tau > 0$ such that

$$\dot{\tau} + C\tau \| e^{\tau \Lambda^{\frac{1}{s}}} \theta \|_{L^2} = 0,$$

then we have

$$\frac{1}{2}\frac{d}{dt} \| e^{\tau \Lambda^{\frac{1}{s}}} \theta \|_{L^2}^2 \leq \| e^{\tau \Lambda^{\frac{1}{s}}} S \|_{L^2} \| e^{\tau \Lambda^{\frac{1}{s}}} \theta \|_{L^2},$$

which gives

$$\| e^{\tau(t) \Lambda^{\frac{1}{s}}} \theta(t) \|_{L^2} \leq \| e^{\tau_0 \Lambda^{\frac{1}{s}}} \theta_0 \|_{L^2} + 2 \| e^{\tau_0 \Lambda^{\frac{1}{s}}} S \|_{L^2}$$

and τ satisfies the lower bound (3.3).

Other directions

Other directions

Due to limited time, there are still some other directions of study which haven't been discussed, for example:

- we can study the fractionally diffusive case for the equation (1.1) when $\kappa > 0$, $\nu \geq 0$ and $\gamma \in (0, 1)$.
- the cases for $\nu = 0$ and $\nu > 0$ are very different.

Case when $\nu = 0$

In [FRV12], the authors showed that the nonlocal operator $(-\Delta)^\gamma$ produces a sharp dichotomy across the value $\gamma = \frac{1}{2}$:

- when $\gamma \in (\frac{1}{2}, 1)$, the equation (1.1) is locally wellposed in Sobolev spaces;
- when $\gamma \in (0, \frac{1}{2})$, the equation (1.1) becomes Lipschitz ill-posed;
- when $\gamma = \frac{1}{2}$, the problem is globally well-posed for $\kappa \gg 1$, respectively ill-posed for $\kappa \ll 1$.

Case when $\nu > 0$

In [FS21,FS22], if S, $\theta_0 \in H^1$, the authors proved that the equation (1.1) has a global-in-time solution $\theta \in H^1$. Furthermore, one has the following theorem about global attractor:

Theorem 4.1

Let $S \in L^\infty \cap H^1$. For $\nu, \kappa > 0$ and $\gamma \in (0, 1]$, the solution map $\pi^\nu(t) : H^1 \to H^1$ associated to (1.1) possesses a unique global attractor $\mathcal{G}^\nu \subset H^1$.

Global attractor

A compact set $\mathcal{G}^\nu \subset H^1$ is a global attractor for $\pi^\nu(\cdot)$ if

- $\pi^\nu(t)\mathcal{G}^\nu = \mathcal{G}^\nu$ for all $t \in \mathbb{R}$;
- for any bounded set K, $dist(\pi^\nu(t)K, \mathcal{G}^\nu) \to 0$ as $t \to \infty$, where $dist(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_{H^1}$.

Remarks on the global attractor

We can obtain some additional properties on the attractors $\mathcal{G}^\nu$ under the assumption that $\gamma \in [\frac{1}{2}, 1]$, which gives

- $\mathcal{G}^\nu$ is fully invariant, namely

$$\pi^\nu(t)\mathcal{G}^\nu = \mathcal{G}^\nu, \qquad \forall t \geq 0.$$

- $\mathcal{G}^\nu$ is maximal in the class of H^1-bounded invariant sets.
- $\mathcal{G}^\nu$ has finite fractal dimension.

Remarks on the global attractor (cont'd)

The reasons why we need to restrict to $\gamma \in [\frac{1}{2}, 1]$:

- one need a bound for $\|\nabla u\|_{L^\infty}$ in terms of $\|(-\Delta)^{\frac{1}{2}+\frac{\gamma}{4}}\theta\|_{L^2}$;
- which can only be true when $\gamma \geq 1$ (and $d \leq 3$).

Some more directions

There are some future directions of study, for example:

- Address the average behaviour of the energy dissipation rate by considering the limits of

$$\kappa\langle|\nabla\theta^{\kappa,\nu}|^2\rangle = \kappa \lim_{T\to\infty} \frac{1}{T}\int_0^T \int |\nabla\theta^{\kappa,\nu}|^2 dxds$$

  as κ, $\nu \to 0$;
- Study how "odd" or "evenness" for the constitutive laws M_ν can affect the solutions of the abstract system (1.1).

Thank You! :-)

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Acknowledgement

S. Friedlander is supported by NSF DMS-1613135 and A. Suen is supported by Hong Kong General Research Fund (GRF) grant project number 18300720 and 18300821.