On global dynamics of zero-speed and superluminal solutions to Benjamin-Bona-Mahony equation

Chulkwang Kwak

Ewha Womans University

May 11, 2022

Analysis and PDE Seminar

This talk is based on the joint works

"Extended decay properties for generalized BBM equations"

with C. Muñoz

and

"On the Dynamics of Zero-Speed Solutions for Camassa–Holm-Type Equations"

with M. A. Alejo, M. F. Cortez, and C. Muñoz

M. A. Alejo, M. F. Cortez, C. Kwak and C. Muñoz, *On the Dynamics of Zero-Speed Solutions for Camassa–Holm-Type Equations*, Int. Math. Res. Not. IMRN 2021, no. 9, 6543–6585.

C. Kwak and C. Muñoz, *Extended Decay Properties for Generalized BBM Equation*. In: Miller P., Perry P., Saut JC., Sulem C. (eds) Nonlinear Dispersive Partial Differential Equations and Inverse Scattering. Fields Institute Communications, vol 83. Springer, New York, NY (2019) https://doi.org/10.1007/978-1-4939-9806-7_8

BBM equation

The **one dimensional Benjamin-Bona-Mahony (BBM)** equation

$$\text{(BBM)} \quad (1-\partial_x^2)\partial_t u + \partial_x\left(u+u^2\right) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

- BBM equation describes a model for the unidirectional propagation of long-crested, surface water waves [1]
- BBM equation is a regularized version of the KdV equation via the standard "Boussinesq trick" (replacing u_x by $-u_t$).

$$\text{(KdV)} \quad \partial_t u + \partial_x\left(\partial_x^2 u - u + u^2\right) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}.$$

- Simpler well-posedness and comparable qualitative behavior compared with the original KdV equation, but BBM is not integrable, unlike KdV [2, 3]

T. B. Benjamin, J. L. Bona, and J. J. Mahony, (1972), *Model Equations for Long Waves in Nonlinear Dispersive Systems*, Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences, 272 (1220): 47–78.

J. L. Bona, W. G. Pritchard, and L. R. Scott, *Solitary wave interaction*, Physics of Fluids 23, 438 (1980).

Y. Martel, F. Merle, and T. Mizumachi, *Description of the inelastic collision of two solitary waves for the BBM equation*, Arch. Rat. Mech. Anal. May 2010, Volume 196, Issue 2, pp 517–574.

Generalized BBM equation

The following equation generalizes BBM equation to higher degree nonlinear cases

$$\text{(gBBM)} \quad (1-\partial_x^2)\partial_t u + \partial_x (u + u^p) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \quad p = 2, 3, \cdots .$$

- gBBM equation has a Hamiltonian structure

$$u_t = -\mathcal{D}\frac{\delta H}{\delta u}, \quad H(u) = \int_{\mathbb{R}} \frac{1}{2}u^2 + \frac{1}{p+1}u^{p+1}, \quad \mathcal{D} = \partial_x(1-\partial_x^2)^{-1}.$$

- gBBM equation allows the following *energy conservation law*

$$E[u](t) = \int_{\mathbb{R}} (u^2 + u_x^2)(t,x)\,dx = \int_{\mathbb{R}} (u^2 + u_x^2)(0,x)\,dx = E[u](0).$$

Solitary waves

- gBBM possesses solitary wave solutions of the form : for any $c > 1$

$$\varphi_c(t,x) := (c-1)^{\frac{1}{p-1}} Q\left(\sqrt{\frac{c-1}{c}}(x-ct)\right),$$

$$Q(s) := \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}s)}\right)^{\frac{1}{p-1}},$$

is a solitary wave solution of (gBBM), moving to the right with speed $c > 1$.

- For $c^*(p) = 1$, when $p = 2,3,4,5$, and $c^*(p) = \dfrac{(p-1)(2+\sqrt{2(p+3)})}{4(p+1)}$, when $p \geq 6$, it is known that
  - when $p \geq 2$ and $c > c^*(p)$, $\varphi_c(t,x)$ is stable [3, 1]
  - when $p \geq 6$ and $1 < c < c^*(p)$, $\varphi_c(c-ct)$ is unstable [2]

K. El Dika, and Y. Martel, *Stability of N solitary waves for the generalized BBM equations*, Dyn. Partial Differ. Equ. 1 (2004), no. 4, 401–437.

P. E. Souganidis, W. A. Strauss, *Instability of a class of dispersive solitary waves.* Proc. Roy. Soc. Edinburgh Sect. A 114 (1990), no. 3-4, 195–212.

M. I. Weinstein, *Existence and dynamic stability of solitary wave solutions of equations arising in long wave propagation.* Comm. Partial Differential Equations 12 (1987), no. 10, 1133–1173.

Solitary waves

- gBBM equation has solitary waves with negative speed: for $c > 0$ and p even,

$$\varphi_c(t,x) := -(c+1)^{1/(p-1)} Q\left(\sqrt{\frac{c+1}{c}}(x+ct)\right),$$

  is solitary wave for (gBBM) [1],
- It is never small in the energy space.
- No solitary waves with $0 < c < 1$ exists for any p, and with $c < 0$ for any odd p.

H. Kalisch, N. T. Nguyen, *Stability of negative solitary waves*, Electron. J. Differential Equations 2009, No. 158, 20 pp.

Main results

Decay of small global energy solutions in the exterior regions, i.e., $(-\infty, -at) \cup (bt, \infty)$, $a > \dfrac{1}{8}$ and $b > 1$. To be more precise, let $I_{ext}(t)$ be given by

$$I_{ext}(t) := (-\infty, -at) \cup (bt, \infty), \quad t > 0.$$

Then,

Theorem 1 - Decay in exterior region, K.-Muñoz (2019)

Let $a > \frac{1}{8}$ and $b > 1$ be given. Let $u_0 \in H^1$ be such that, for some $\varepsilon = \varepsilon(a, b) > 0$ small, one has

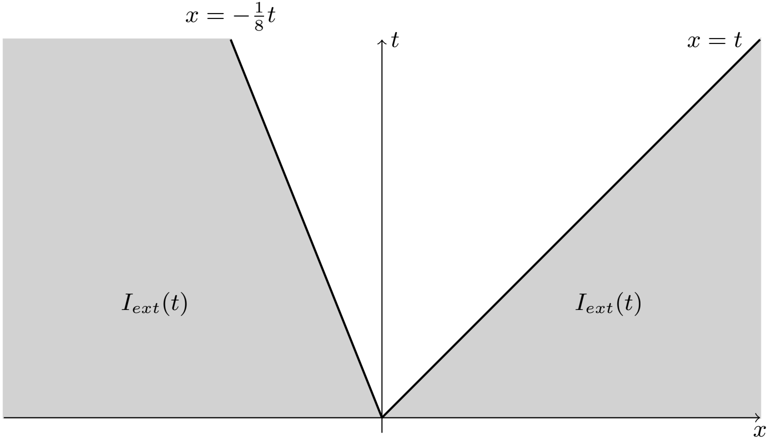
$$\|u_0\|_{H^1} < \varepsilon.$$

Let $u \in C(\mathbb{R}, H^1)$ be the corresponding **global (small) solution** of (gBBM) with initial data $u(t = 0) = u_0$. Then, there is strong decay to zero:

$$\lim_{t \to \infty} \|u(t)\|_{H^1(I_{ext}(t))} = 0.$$

C. Kwak and C. Muñoz, *Extended Decay Properties for Generalized BBM Equation.* In: Miller P., Perry P., Saut JC., Sulem C. (eds) Nonlinear Dispersive Partial Differential Equations and Inverse Scattering. Fields Institute Communications, vol 83. Springer, New York, NY (2019) https://doi.org/10.1007/978-1-4939-9806-7_8

Exterior region



Comments

- The proof of Theorem provides the mild rate of decay

$$\int_2^\infty \int e^{-c_0|x+\sigma t|} \left(u^2 + u_x^2\right)(t,x) dx\, dt \lesssim_{c_0} \varepsilon^2,$$

  where σ is fixed and such that $\sigma = a$ or $\sigma = -b$.

- Theorem also considers the cases $p = 2$ and $p = 3$, which are difficult to attain using standard scattering techniques because of very weak linear decay estimates of order $O(t^{-1/3})$ [1], and the presence of long range nonlinearities.

- The case of decay inside the interval $(-\infty, -at)$ is in strong contrast with the similar decay problem for the KdV equation on the left, which has not been rigorously proved yet.

- Theorem is in concordance with the existence of small solitary waves for gBBM near $x \sim t$.

J. Albert, *On the Decay of Solutions of the Generalized Benjamin-Bona-Mahony Equation*, J. Math. Anal. Appl. 141, 527–537 (1989)

Plane wave

For the **linear flow** associated to gBBM equation, recall the well-known notion of **plane wave**.

Definition - Plane wave and plane wave region

Let $k, w \in \mathbb{R}$. We say that

$$u_{pw} = e^{i(kx-wt)}$$

is a **plane wave** if there exists $w = w(k)$ such that u_{pw} solves the linear gBBM equation.

The quantity $w'(k)$ (if exists) is called the *group velocity*, and the set of rays

$$\{(t, w'(k)t) \in [1,\infty) \times \mathbb{R} \ : \ k \in \mathbb{R}\},$$

is denoted by PW, or **plane wave region**.

vs. KdV

A direct computation gives

$$w'_{BBM}(k) = \frac{1-k^2}{(1+k^2)^2} \quad \text{and} \quad w'_{KdV}(k) = 1-3k^2.$$

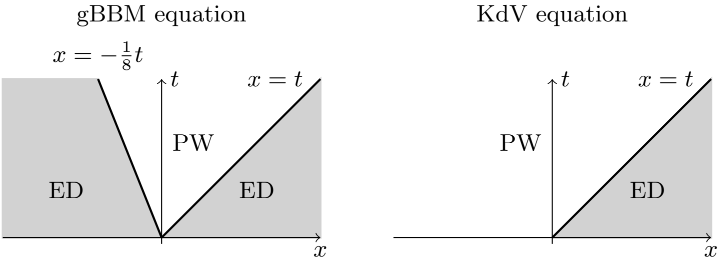


Figure: *Left.* Decay of BBM solution holds true in the exterior region outside a cone, while all linear plane waves (PW) stay in the interior of the cone. *Right.* Decay of KdV solution holds true in the exterior region outside a cone, while all linear plane waves (PW) stay in the left portion of space-time plane.

Remarks

- There are no **small solitary waves** of speed greater than b and less than $-a$.
- Small energy of solutions seems be concentrated on PW, as $t \to \infty$.
- (modified) Scattering is expected to occur in PW, if exists. See **Germain-Pusateri-Rousset(2016)** and **Harrop-Griffiths(2016)**

P. Germain, F. Pusateri and F. Rousset, *Asymptotic stability of solitons for mKdV*, Advances in Math. 299 (2016) 272–330.

B. Harrop-Griffiths, *Long time behavior of solutions to the mKdV*, Comm. Partial Differential Equations 41 (2016) no. 2 282–317.

Speed one and breather solutions

Speed one and breather solutions

We shall say that a *nontrivial* global strong solution $u = u(t,x)$ is a *zero-speed solution* if it satisfies

$$\limsup_{t\to+\infty} \|u(t)\|_{L^2(I)} > 0, \tag{1}$$

for any compact set $I \subset \mathbb{R}$. On the other hand, if u is *periodic* in time, we shall say that u is a *breather* solution. We shall say that a *nontrivial* global strong solution $u = u(t,x)$ is a *speed one solution* if $u(t, \cdot - t)$ is a zero speed solution.

- A breather is always a zero-speed solution, but the opposite is not true in general
- (BBM) solitary waves are not speed one solutions

Goal

No speed one (non-decaying) solutions for (BBM)

Main theorems

Let $b \in (0,1)$, and $J_b(t)$ be an interval in space defined by

$$J_b(t) := \left(t - \frac{t^b}{\log t}, t + \frac{t^b}{\log t} \right), \qquad |t| \geq 2$$

for any $C > 0$, $0 \leq b < 1$ and $t > 2$.

- $J_b(t)$ is centered around the line $x = t$

Theorem 2 (Alejo-Cortez-K.-Muñoz 21')

Let $u \in C(\mathbb{R}, H^1) \cap L^\infty(\mathbb{R}, L^1)$ be a solution of (BBM), **no size condition** required. Then

$$\lim_{t \to \infty} \|u(t)\|_{H^1(J_{1/2}(t))} = 0. \tag{2}$$

In particular, u cannot be a speed one solution.

M. A. Alejo, M. F. Cortez, C. Kwak and C. Muñoz, *On the Dynamics of Zero-Speed Solutions for Camassa–Holm-Type Equations*, Int. Math. Res. Not. IMRN 2021, no. 9, 6543–6585.

Linearly dominated region

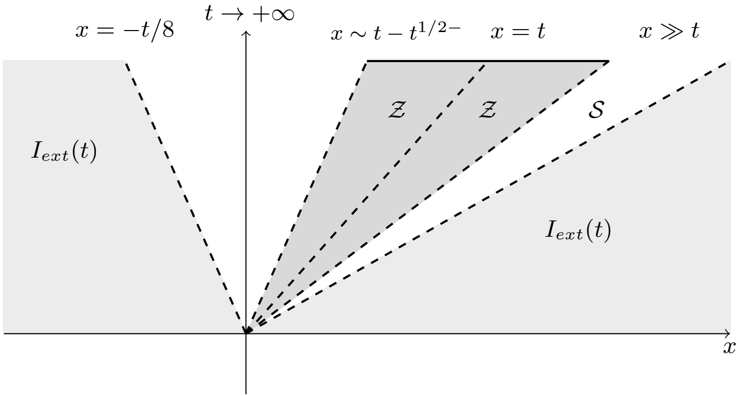


Figure: Graphic description of Theorems. The set $\mathcal{Z}$ represents the space region of size $\sim t^{1/2-}$ where the solution converges to zero in H^1 norm (Theorem 2). The thick line above $\mathcal{Z}$ represents $J_{1/2}(t)$. $\mathcal{S}$ stands for the solitonic region, where solitons belong (small solitons have speeds $c \sim 1^+$). Under a small data condition, the exterior region $I_{ext}(t)$ has no mass at infinity in time (Theorem 1). Large solitons may exist in the white region $x < 0$, moving to the left, in the cases p even.

Non-existence of breathers

Nonexistence of breathers for (gBBM) with $p = 2k$, $k = 1, 2, \dots$ (even power nonlinearities), around $x = t$, under the condition $xu(t, \cdot - t) \in L^1$

- Assume that $v(t, x) := u(t, \cdot - t)$ is periodic in time and nontrivial.
- v solves $\partial_t v + \partial_x (1 - \partial_x^2)^{-1} \big(\partial_x^2 v + v^{2k}\big)$, $k = 1, 2, \dots$
- A direct computation gives

$$\frac{d}{dt} \int xv = \frac{d}{dt} \int x(v - v_{xx}) = \int v^{2k} > 0.$$

- v cannot be periodic in time due to the non-triviality of v

Non-existence of breathers

Nonexistence of breathers for (BBM), around $x = 0$, under some particular condition of initial data. Assume either:

(1) Positive mean condition:

$$\int u_0 \geq 0,$$

or

(2) Smallness and restricted negative mean condition:

$$-E[u_0] \leq \int u_0 \leq 0,$$

and

$$\sup_{t \in \mathbb{R}} \|u(t)\|_{H^1} = \epsilon_0 < \frac{3}{2C_3},$$

where C_3 is the Gagliardo-Nirenberg constant ($H^1 \hookrightarrow L^3$), that is, C_3 satisfies

$$\|u\|_{L^3}^3 \leq C_3 \|u\|_{L^2}^{\frac{5}{2}} \|u_x\|_{L^2}^{\frac{1}{2}}.$$

and where $E[u_0] = E[u](t = 0)$

About the proofs: Decay via Virial functionals

The proof of previous theorems is based in the introduction of suitable **virial identities** [1, 2, 3].

Y. Martel and F. Merle, *A Liouville theorem for the critical generalized Korteweg-de Vries equation*, J. Math. Pures Appl. (9) **79** (2000), no. 4, 339–425.

Y. Martel and F. Merle, *Asymptotic stability of solitons for subcritical generalized KdV equations*, Arch. Ration. Mech. Anal. **157** (2001), no. 3, 219–254.

F. Merle and P. Raphaël, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Ann. of Math. (2) **161** (2005), no. 1, 157–222.

Decay (in exterior region) via Virial functionals

The main virial term is given by the functional

$$\mathcal{I}(t) := \frac{1}{2}\int \varphi\left(\frac{x+\sigma t}{L}\right)\left(u^2+u_x^2\right)(t,x)dx, \quad L \gg 1,$$

where $\varphi = \varphi(x)$ is a smooth and bounded function. Clearly, the functional $\mathcal{I}(t)$ is well-defined for H^1 functions.

Theorems follow if there is a parameter $\alpha \in \mathbb{R}$ and $d_0 > 0$ such that

$$\frac{d}{dt}\mathcal{I}(t) \gtrsim_{d_0} \int e^{-\frac{d_0}{L}|x+\sigma t|}(u^2+u_x^2)(t,x)dx.$$

Localized energy estimates complete the proof:

$$\mathcal{I}_{t_0}(t) := \frac{1}{2}\int \varphi\left(\frac{x+\sigma t_0-\tilde{\sigma}(t_0-t)}{L}\right)\left(u^2+u_x^2\right)(t,x)dx$$

implies

$$\limsup_{t\to\infty}\int \varphi\left(\frac{x-\sigma t}{L}\right)\left(u^2+u_x^2\right)(t,x)dx = 0.$$

Thank You

for Your Attention!!