Long-Time Solutions of the Kähler-Ricci Flow

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Ricci Flow

Let (M, g_0) be a Riemannian manifold. Ricci flow equation is given by:

$$\frac{\partial}{\partial t} g(t) = -\mathrm{Ric}(g(t)), \quad g(0) = g_0$$

- A nonlinear, weakly parabolic system of PDEs.
- Introduced by Hamilton in 1982.
- A major tool in geometric analysis for solving many important conjectures, such as Poincaré Conjecture (Hamilton, Perelman, et. al), Differentiable Sphere Theorem (Brendle-Schoen), etc.

This talk will mainly focus on the **Kähler-Ricci Flow** on compact Kähler manifolds.

Kähler manifolds

Let (X, J) be a complex manifold, $J^2 = -id$ and g is a Hermitian metric (i.e. $g(JY, JZ) = g(Y, Z)$). Denote

$$\begin{aligned} \omega(\cdot,\cdot) &= g(J\cdot,\cdot) \\ &= \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \qquad \text{(locally)} \end{aligned}$$

g is a Kähler metric if and only if $d\omega = 0$.

The Ricci curvature tensor/form is given by

$$\begin{aligned} R_{i\bar{j}} &= -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g) \qquad \text{(Kähler manifolds)} \\ \mathrm{Ric} &= \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j \end{aligned}$$

Kähler-Ricci Flow

Let (X^n, J, ω_0) be a compact Kähler manifold. The Kähler-Ricci flow equation is defined as:

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) \qquad \text{(unnormalized)}$$

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) \pm \omega_t \qquad \text{(normalized)}$$

- Introduced by H.D. Cao in 1985.
- J is fixed along the flow.
- Kähler condition is preserved along the flow.

Kähler class

Each ω_t is a closed 2-form, hence represents a class $[\omega_t] \in H^2_{\mathrm{dR}}(X)$.
Under the unnormalized Kähler-Ricci flow, $[\omega_t] \in H^2_{\mathrm{dR}}(X)$ evolves by

$$\frac{d}{dt}[\omega_t] = -c_1(X) \implies [\omega_t] = [\omega_0] - tc_1(X)$$

The geometric behavior of KRF is predicted by $[\omega_0]$ and $c_1(X)$.

Theorem (Cao, Tsuji, Tian-Z.Zhang)

The maximal existence time of the Kähler-Ricci Flow is given by

$$T = \sup\{t : [\omega_t] = [\omega_0] - tc_1(X) > 0\}.$$

Corollary:

(unnormalized) Kahler-Ricci flow on X has long-time solution iff the canonical line bundle K_X is numerically effective (nef).

Main argument of the theorem is similar to standard continuation method:

- Kähler-Ricci flow $\frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega)$ is equivalent to the following complex Monge-Amperé equation

$$\frac{\partial \varphi_t}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n}{\Omega} - \varphi_t$$

  in a sense that, $\omega_t = \hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$ solves the KRF if and only if φ_t solves the complex Monge-Amperé equation. Here $\hat{\omega}_t \in [\omega_0] - tc_1(X)$ is an explicit reference metric.
- Linearization of the RHS in the complex Monge-Amperé is:

$$\Delta_{\omega_t} - \mathrm{id}$$

- As long as Δ_{ω_t} remains elliptic on $[0, \tau]$, one can apply standard analysis as in Yau's proof of the Calabi conjecture to prove smooth convergence of φ_t as $t \to \tau^-$, and short-time existence after τ can be obtained by standard parabolic PDE technique.

Canonical Cases

$$T = \sup\{t : [\omega_t] = [\omega_0] - tc_1(X) > 0\}.$$

If $c_1(X) = 0$, then $[\omega_t] \equiv [\omega_0]$ and $T = \infty$.
(Cao, 1985) $\omega_t \to \omega_{\mathsf{CY}}$ smoothly as $t \to \infty$.

If $c_1(X) < 0$ and $[\omega_0] = -c_1(X)$, then $[\omega_t] = (1+t)[\omega_0]$ and $T = \infty$.
(Cao, 1985) $\frac{1}{1+t}\omega_t \to \omega_{\mathsf{KE}}$ smoothly as $t \to \infty$.

If $c_1(X) > 0$ and $[\omega_0] = c_1(X)$, then $[\omega_t] = (1-t)[\omega_0]$ and $T = 1$.

- $\omega_t \to \cdot$ in Gromov-Hausdorff sense as $t \to 1$ (Perelman).
- If there exists a Kähler-Einstein metric or more generally a Kähler-Ricci soliton on X, then ω_t converges to that KE/KRS after rescaling (Tian-Zhu, Dervan-Szekelyhidi).

Semi-ample canonical line bundles

How about somewhere between $c_1(X) = 0$ and $c_1(X) < 0$?

K_X is said to be semi-ample if there exists a large integer k such that the sections in $H^0(X, K_X^{\otimes k})$ give a holomorphic map $f : X \to \Sigma \subset \mathbb{CP}^N$ into a complex projective space with normal projective variety image Σ called the canonical model of X.

Furthermore, $c_1(X) = -[f^*\omega_\Sigma]$ for some Kähler metric ω_Σ on Σ, and $\dim(\Sigma)$ is called the Kodaira dimension of X.

For each $p \in \Sigma$, we call $X_p := f^{-1}(p)$ a fiber based at p.

- If p is a regular value of f, we call X_p a regular fiber, and it is a Calabi-Yau manifold.
- Otherwise, if p is a singular value of f, we call X_p a singular fiber.

Semi-ample canonical line bundles, con't

Under the Kähler-Ricci flows

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) \tag{1}$$

$$\frac{\partial}{\partial t}\omega_t = -\mathrm{Ric}(\omega_t) - \omega_t \tag{2}$$

Suppose $\dim(X) = n$, and denote $S \subset \Sigma$ the set of singular values of $f : X \to \Sigma$. Then,

- If $\mathrm{Kod}(X) = 0$, then X is Calabi-Yau and by Cao the KRF (1) converges to a Ricci-flat metric.
- If $\mathrm{Kod}(X) = n$ and $S = \emptyset$, then K_X is ample and as proven by Cao in 1985 that the KRF (2) converges to a negative Kähler-Einstein metric.
- If $\mathrm{Kod}(X) = n$ but $S \neq \emptyset$, i.e. K_X is nef and big, not ample, then by Tsuji, Tian-Z.Zhang the KRF (2) converges to a positive current representing $-c_1(X)$, and the convergence is smooth away from $f^{-1}(S)$.

Intermediate cases: $0 < \text{Kod}(X) < \dim(X)$

Known convergence results:

- Song-Tian (2006, 2008) proved convergence in the sense of current to a positive current $f^*\omega_\Sigma \in -c_1(X)$, and away from S, ω_Σ satisfies
$$\text{Ric}(\omega_\Sigma) = -\omega_\Sigma + \omega_{\text{WP}}.$$
They conjectured that the convergence can be improved to Gromov-Hausdroff topology on X, and C^∞-topology away from $f^{-1}(S)$.
- When $X = \Sigma \times E$, where $c_1(\Sigma) < 0$ and E admits flat metrics, then M. Gill (2012) proved that the convergence is smooth.
- When X is a fibered by complex tori, and $[\omega_0]$ is rational, then F.-Z.Zhang (2012) proved that the convergence is smooth away from $f^{-1}(S)$.
- The rationality assumption was later removed by Hein-Tosatti (2014).
- Tosatti-Weinkove-Yang (2015) proved that in general the convergence is in C^0 (wrt the metric) away from $f^{-1}(S)$.

Intermediate cases: $0 < \text{Kod}(X) < \dim(X)$

More recent convergence results:

- F.-M.C.Lee (2020) proved that if the regular fibers are biholomorphic to each other, then the convergence is smooth away from $f^{-1}(S)$.
- J.Chu-M.C.Lee (2021) proved $C^{0,\alpha}_{\text{loc}}$-convergence away from $f^{-1}(S)$.

It is conjectured (by Song-Tian) that:

- convergence is smooth away from $f^{-1}(S)$, and
- globally, (M, ω_t) converges to $(\Sigma, \omega_{\text{GKE}})$ in Gromov-Hausdorff sense.

Torus fibrations

Theorem (F-Z.Zhang, Crelle 2012)

Under the normalized KRF on X with semi-ample K_X with the fiber map $f : X \to \Sigma$ such that $f^{-1}(p)$ is a complex tori for each regular value $p \in \Sigma$, then we have

1. $\omega_t \to f^*\omega_\Sigma$ *in* $C^\infty_{\mathrm{loc}}(f^{-1}(\Sigma \backslash S))$*-topology;*
2. $\|\mathrm{Rm}\|_{\omega_t}$ *is uniformly bounded on every compact subset* $K \subset f^{-1}(\Sigma \backslash S)$.

Remarks:

- If $S = \emptyset$, then the above result implies Gromov-Hausdorff convergence. Furthermore by (2) above, the Riemann curvature will be uniformly bounded on $X \times [0, \infty)$, i.e. the Ricci flow solution is of Type III.

Idea of Proof

The key idea is a parabolic analogue of Gross-Tosatti-Y.G.Zhang's work [GTZ] on collapsing of Ricci-flat metrics. Pick a local trivialization $B \times \mathbb{T}_z$ and define $\lambda_t : B \times \mathbb{T}_z \to B \times \mathbb{T}_z$ by $(z, \xi) \mapsto (z, e^t \xi)$. In [GTZ], a special reference metric was constructed on $B \times \mathbb{T}_z$, and this reference metric has good rescaling properties so that one can rewrite the KRF solution locally on $B \times \mathbb{T}_z$ as:

$$\lambda_t^* \omega_t = \omega_{SF} + \pi^* \omega_\Sigma + \sqrt{-1}\partial\bar{\partial} u_t$$

where ω_{SF} is an explicit fixed metric, and u_t is a modified Kähler potential. The KRF equation can be rewritten locally as:

$$\log \det(\omega_{SF} + \pi^* \omega_\Sigma + \sqrt{-1}\partial\bar{\partial} u_t) = \lambda_t^* \left(\frac{\partial \varphi_t}{\partial t} + \varphi_t \right) + \log \Omega.$$

$\omega_{SF} + \pi^* \omega_\Sigma \simeq \delta$ hence the equation is elliptic. Evans-Krylov's theory implies $u_t \in C^{2,\alpha}$.

Idea of Proof - bootstrapping

Two bootstrapping equations:

$$\begin{aligned}\Delta_{\lambda_t^*\omega_t}(Du_t) = &-\mathrm{Tr}_{\lambda_t^*\omega_t} D(\omega_{SF} + \pi^*\omega_\Sigma) \\ &+ D\left\{\lambda_t^*\left(\frac{\partial \varphi_t}{\partial t} + \varphi_t\right)\right\} + D\log\Omega.\end{aligned}$$

$H := \lambda_t^*\left(\frac{\partial \varphi_t}{\partial t} + \varphi_t\right)$ satisfies:

$$\left(\frac{\partial}{\partial t} - \Delta_{\lambda_t^*\omega_t}\right)H = \langle \partial_{\mathbb{E}} H, \partial_\xi + \bar{\partial}_\xi \rangle_\delta - n + r + \mathrm{Tr}_{\lambda_t^*\omega_t}\pi^*\omega_\Sigma$$

If $u_t \in C^{2,\alpha}$, then $\lambda_t^*\omega_t = \omega_{SF} + \pi^*\omega_\Sigma + \sqrt{-1}\partial\bar{\partial}u_t \in C^{0,\alpha}$. Schauder's estimate on these two equations implies $Du_t \in C^{2,\alpha}$ and hence $u_t \in C^{3,\alpha}$. Then, keep going and so C^∞-convergence away from singular fibers.

Uniform Curvature Bounds

Another interesting question to investigate is the uniform boundedness of curvatures along normalized KRF:

Riemann curvature: For any compact subset $K \subset \Sigma \backslash S$, we know that

- If fibers are complex tori, then Rm is uniformly bounded on $K \times [0, \infty)$ along the normalized KRF (F.-Z.Zhang).
- If fibers are not complex tori, then Rm blows up on $K \times [0, \infty)$, so that the Ricci flow solution is of Type IIb (Tosatti-Y.G.Zhang).
- Tosatti-Y.G.Zhang also proved that the blow-up rate of $\|\text{Rm}\|$ in the non-torus case is at least e^t, i.e. $\|\text{Rm}\| \geq \frac{1}{C} e^t$
- F.-Y.S.Zhang (2019) proved that the rate e^t is sharp, i.e. $\|\text{Rm}\| \leq Ce^t$.

Uniform Curvature Bounds, con't

Scalar curvature:
It is also interesting to know that scalar curvature must be uniformly bounded along normalized KRF, regardless of whether $S = \emptyset$ and the topology of fibers (Song-Tian).

It was shown by W.J.-Jian (2018) that the scalar curvature converges to $-\mathrm{Kod}$ away from singular fibers.

Uniform Curvature Bounds, con't

Ricci curvature:
The uniform bound of the Ricci curvature is an important hypothesis in works about Gromov-Hausdroff's convergence. For example, when $\mathrm{Kod}(X) = 1$, Song-Tian-Z.L.Zhang (2019) shows that uniform bound on Ric on some $f^{-1}(U) \times [0, \infty)$, $U \cap S = \emptyset$, would imply Gromov-Hausdorff convergence.

Jian-Song (2021) proved that $\|\mathrm{Ric}\|$ is uniformly bounded on any compact set K away from singular fibers when $\mathrm{Kod} = 1$.

C.Y.Fung (2021) further proved that $\|\mathrm{Ric}\| \to 1$ away from singular fibers when $\mathrm{Kod} = 1$.

Uniform bound on Ric

We have recently established the following higher-order regularity result:

Theorem (F.-Lee, J. Funct. Anal. 2021)

Let X be a compact Kähler manifold with semi-ample K_X and $0 < \mathrm{Kod}(X) < \dim X$. Suppose further that the regular fibers are biholomorphic to each other, then the normalized Kähler-Ricci flow converges in C^∞_{loc}-topology away from singular fibers.

Corollary

In particular, under the same assumption of the theorem, the Ricci curvature of any neighborhood $f^{-1}(U)$, where $U \subset\subset \Sigma \backslash S$, is uniformly bounded along the flow.

Higer-Order Regularity

Some remarks:

- The theorem is proved by a delicate blow-up analysis adopted from Hein-Tosatti's work about the collapsing of Ricci-flat metrics (an elliptic problem).
- In order to apply their argument (which is elliptic) on the Kähler-Ricci flow (which is parabolic), we have established the parabolic version of the cylindrical Schauder's estimates.
- We can obtain the corollary about the Ricci curvature because the "model" solution of the flow that we compare with is the time-dependent product metric $g_P(t) = g_{\mathbb{C}^m} + e^{-t} g_{RF}$, where g_{RF} is a Ricci-flat metric of a regular fiber. The metric $g_P(t)$ has uniform Ricci curvature bound independent of t.

Still unknown

- Higher-order regularity of the flow when the regular fibers are not biholomorphic?
- Ricci curvature uniform bound along the flow on $f^{-1}(U)$, $U \subset\subset \Sigma \backslash S$, when the regular fibers are not biholomorphic?

Thank you!