Sharp interface limits
of scalar and vectorial Allen-Cahn equations

Analysis and PDE Seminar
jointly by CUHK, HKU and UNIST

Yuning Liu NYU Shanghai

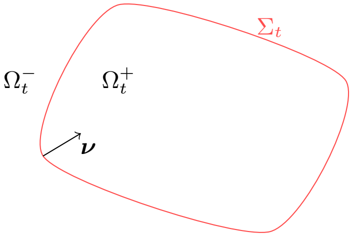
Table of contents

Consider the MCF $\{\Sigma_t\}_{t>0}$ parametrized by $\boldsymbol{\varphi}_t(s)$ with s being the local coordinate:

$$\partial_t \boldsymbol{\varphi}_t(s) = \kappa(\boldsymbol{\varphi}_t(s), t)\boldsymbol{\nu}(\boldsymbol{\varphi}_t(s), t)$$

Let $r = \mathrm{d}_\Sigma(x,t)$ be the signed distance function (positive inside). Differentiating the identity $\mathrm{d}_\Sigma(\boldsymbol{\varphi}_t(s) + r\boldsymbol{\nu}(s,t), t) \equiv r$ leads to

$$\nabla \mathrm{d}_\Sigma = \boldsymbol{\nu}, \quad \partial_t \mathrm{d}_\Sigma = -\partial_t \boldsymbol{\varphi}_t(s) \cdot \boldsymbol{\nu}$$



For fixed t, let $\pi(x)$ be the projection of x on Σ_t and $\{\kappa_i\}_{1\leqslant i\leqslant d-1}$ are the principal curvatures,

$$\Delta \mathrm{d}_\Sigma(x) = \sum_{i=1}^{d-1} \frac{-\kappa_i(\pi(x))}{1-\kappa_i(\pi(x))\mathrm{d}_\Sigma} = -\sum_{i=1}^{d-1}\kappa_i - \mathrm{d}_\Sigma \sum_{i=1}^{d-1}\kappa_i^2 + o(\mathrm{d}_\Sigma)$$

Barrier formulation : $\quad (\partial_t - \Delta)\mathrm{d}_\Sigma = \mathrm{d}_\Sigma |A|^2 + o(\mathrm{d}_\Sigma)$

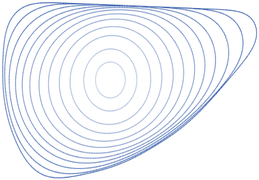
Example: let $\Sigma_t = S_{R(t)}$ be the sphere of radius $R(t)$ centered at 0 with $R(0) = R_0$,

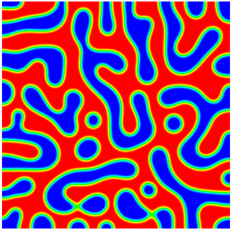
$$\mathrm{d}_\Sigma = R(t) - |x|, \ \boldsymbol{\nu} = \nabla \mathrm{d}_\Sigma = -x/|x|, \ \operatorname{div} \boldsymbol{\nu} = \Delta \mathrm{d}_\Sigma = -(d-1)/|x|.$$

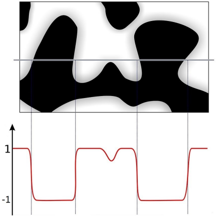
If Σ_t evolves by MCF, then $R(t) = \sqrt{R_0^2 - 2(d-1)t}$.

Phase–field approximation

The **phase–field models** are widely adopted in the description of the evolution of **interfaces** in continuum mechanics. They can be constructed to purposely reproduce a given **sharp interface model** when the thickness of their diffused interface, usually denoted by ε, trends to 0.







Other implicit representations: thresholding scheme, level–set method
Typical phase models: Allen–Cahn (L^2 gradient flow) $\to$ MCF, Cahn–Hilliard (H^1 gradient flow) $\to$ Hele–Shaw, 4-th order Allen–Cahn $\to$ Willmore flow.

Ginzburg-Landau equation

Consider Ginzburg-Landau equation under diffusive scaling $(x,t) \to (\varepsilon x, \varepsilon^2 t)$:

$$\partial_t c_\varepsilon = \Delta c_\varepsilon - \varepsilon^{-2} W'(c_\varepsilon) \tag{GL}$$

where $W(c) = (c^2-1)^2$. It is the gradient flow of $\int \frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon} W(c_\varepsilon) dx =: d\mu_t^\varepsilon$:

$$\frac{d}{dt}\int d\mu_t^\varepsilon = -\int \varepsilon |\partial_t c_\varepsilon|^2 \, dx \tag{1}$$

Major challenge: ∇c_ε is not bounded in any L^p space due to concentration.

- Modica-Mortola '77, Bronsard-Kohn '91: $\mu_t^\varepsilon \xrightarrow{\varepsilon \to 0} \sigma \mathcal{H}^{d-1} \llcorner \Sigma_t$ under Γ-convergence. $\psi_\varepsilon(x,t) = \int_0^{c_\varepsilon(x,t)} \sqrt{2W(z)}\, dz$ is bounded in BV class.
- De Mottoni-Schatzman '95: local, asymptotic expansion $c_\varepsilon \approx c_A = \theta(\frac{\mathrm{d}_\Sigma}{\varepsilon}) + o(\varepsilon^2)$.
- Evans, Soner, Souganidis '92: global convergence to viscosity sol. to $\frac{\partial_t u}{|\nabla u|} = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$: $(\partial_t - \Delta)\mathrm{d}_\Sigma > 0$ in $\{x \in \Omega : \mathrm{d}_\Sigma(x,t) > 0\}$ as viscosity solu.
- Ilmanen '93, Chen '96, Röger-Schätzle '06: global, convergence to varifold solution Brakke '78: convergence of the localized energy law:

$$\frac{d}{dt}\int \phi(x) d\mu_t^\varepsilon = \int \varepsilon \left(-\Delta c_\varepsilon + \frac{1}{\varepsilon^2}W'(c_\varepsilon)\right) \nabla c_\varepsilon \cdot \nabla \phi - \int \varepsilon \left(\Delta c_\varepsilon - \frac{1}{\varepsilon^2}W'(c_\varepsilon)\right)^2 \phi$$

Modulated energy method: Barrier

Recall $d\mu_t^\varepsilon = \left(\frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}W(c_\varepsilon)\right)\,dx$ and the energy stress tensor

$$\mathrm{T} = \left(\frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}W(c_\varepsilon)\right)\mathrm{I}_d - \varepsilon\nabla c_\varepsilon \otimes \nabla c_\varepsilon$$

For any test function $\phi(x,t)$, one can derive

$$\frac{d}{dt}\int \phi(x,t)\,d\mu_t^\varepsilon = \int \partial_t\phi\,d\mu_t^\varepsilon + \int \underbrace{\varepsilon\left(-\Delta c_\varepsilon + \tfrac{1}{\varepsilon^2}W'(c_\varepsilon)\right)\nabla c_\varepsilon}_{\mathrm{div}\,\mathrm{T}}\cdot\nabla\phi\,dx$$

$$-\int \varepsilon\left(\Delta c_\varepsilon - \tfrac{1}{\varepsilon^2}W'(c_\varepsilon)\right)^2\phi\,dx$$

Motivated by Ilmanen '93 (co-dim=1) and Lin '96, Jerrard–Soner '98 (co-dim=2), we choose $\phi = \frac{1}{2}\mathrm{d}_\Sigma(x,t)^2$ in $\Sigma_t(\delta)$ and constant outside $\Sigma_t(2\delta)$:

$$\partial_t\phi - \Delta\phi + 1 \lesssim \phi \text{ in } \Sigma_t(2\delta), \tag{2}$$

This function will cut-off the singularity at Σ_t and lead to

$$\frac{d}{dt}\int \underbrace{\phi\left(\tfrac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \tfrac{W(c_\varepsilon)}{\varepsilon}\right)dx}_{d\mu_t^\varepsilon} \lesssim \int \phi\,d\mu_t^\varepsilon + \varepsilon\int \underbrace{\left(\tfrac{1}{2}|\nabla c_\varepsilon|^2 - \tfrac{W(c_\varepsilon)}{\varepsilon^2}\right)dx}_{d\zeta_t^\varepsilon}.$$

Modica '85, Ilmanen '93 use MP to show that the discrepancy $d\zeta_t^\varepsilon$ preserves **negativity**. This implies the strong convergence of c_ε in $L^2_{loc}(\mathbb{R}^d\backslash\Sigma_t)$.

Inner solution is the expansion of c_ε near the interface in a stretched variable $z = \mathrm{d}_\Sigma(x,t)/\varepsilon$, which is introduced to relax the sharp transition of c_ε near the interface. Outer solution determines the boundary condition of the inner solution at $z = \pm\infty$. We use the Ansatz

$$c_A(x,t) = c_0(\tfrac{\mathrm{d}_\Sigma}{\varepsilon}, x, t) + \varepsilon c_1 + \cdots \text{ near } \Sigma_t \qquad \text{(inner solution)}$$

and look for c_A solving Allen-Cahn equations up to a tail:

$$\underbrace{\frac{\partial_t \mathrm{d}_\Sigma}{\varepsilon}\partial_z c_0 + \partial_t c_0}_{=\partial_t c_A} \approx \underbrace{\frac{\Delta \mathrm{d}_\Sigma}{\varepsilon}\partial_z c_0 + \frac{1}{\varepsilon^2}\partial_z^2 c_0 + \Delta c_0 + \cdots - \frac{1}{\varepsilon^2}W'(c_0)}_{=\Delta c_A - \frac{1}{\varepsilon^2}W'(c_A)}$$

- $O(\varepsilon^{-2})$: we choose $c_0 = \theta(\frac{\mathrm{d}_\Sigma}{\varepsilon})$, the optimal profile:

$$\left.\begin{aligned} -\theta''(z) + W'(\theta(z)) &= 0, \forall z \in \mathbb{R}, \\ \theta(0) = 0, \theta(\pm\infty) &= \pm 1. \end{aligned}\right\} \Rightarrow \theta(z) = \tanh(\frac{z}{\sqrt{2}})$$

- $O(\varepsilon^{-1})$: $(\partial_t - \Delta)\mathrm{d}_\Sigma = 0$ on Σ_t leads to MCF.

$$(\partial_t - \Delta)c_0 + \frac{1}{\varepsilon^2}W'(c_0) = (\partial_t \mathrm{d}_\Sigma - \Delta \mathrm{d}_\Sigma)\,\theta'(\tfrac{\mathrm{d}_\Sigma}{\varepsilon})$$

$$= \frac{\partial_t \mathrm{d}_\Sigma - \Delta \mathrm{d}_\Sigma}{\mathrm{d}_\Sigma}\frac{\mathrm{d}_\Sigma}{\varepsilon}\theta'\left(\frac{\mathrm{d}_\Sigma}{\varepsilon}\right) \approx \sqrt{\varepsilon} \quad \text{in} \quad L^2(\Sigma_t(\delta))$$

Brakke's inequality

Let $\{\Sigma_t\}_{t>0}$ be a family of $(d-1)$-closed surface in $\mathbb{R}^d$ with normal velocity V, then

$$\frac{d}{dt}\int_{\Sigma_t} f(x)\,\mathrm{d}\mathcal{H}^{d-1} = \int_{\Sigma_t} (V\cdot\nabla^{\perp} f - fV\cdot\mathbf{H})\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3}$$

Brakke's lemma

Assume V be normal, then $V=\mathbf{H}$ is equivalent to the Brakke's inequality:

$$\frac{d}{dt}\int_{\Sigma_t} f(x)\,\mathrm{d}\mathcal{H}^{d-1} \leqslant \int_{\Sigma_t} (\mathbf{H}\cdot\nabla f - f\mathbf{H}\cdot\mathbf{H})\,\mathrm{d}\mathcal{H}^{d-1} \tag{4}$$

Subtracting (3) from (4): $$0\leqslant \int_{\Sigma_t} \nabla f\cdot(\mathbf{H}-V) - f\mathbf{H}\cdot(\mathbf{H}-V) \tag{5}$$

To blow-up at $x_0\in\Sigma_t$, we set $w=\mathbf{H}-V$ and replace f by $\lambda^{2-d} f(\frac{x-x_0}{\lambda})$.

$$0\leqslant \lambda^{1-d}\int_{\Sigma_t} (\nabla f)(\tfrac{x-x_0}{\lambda})\cdot w(x)dx + O(\lambda)$$

$$=\int_{\frac{\Sigma_t - x_0}{\lambda}} \nabla f(y)\cdot w(x_0+\lambda y)dy + O(\lambda) \xrightarrow{\lambda\to 0} \int_{T_{x_0}\Sigma_t} (\nabla_\Sigma f + \mathbf{n}\partial_r f)\cdot w(x_0).$$

Since $T_{x_0}\Sigma_t \cong \mathbb{R}^{d-1}$ and w is normal, this implies $\mathbf{H}=V$. The book of Tonegawa' 2019 gives a friendly introduction of Brakke flow.

Phase-field approximation of Brakke's inequality

Writing $\mu_t = \mathcal{H}^{d-1} \llcorner \Sigma_t$ and $\mu_t^\varepsilon(A) \triangleq \int_A \left(\varepsilon|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}W(c_\varepsilon)\right) dx$.
The Brakke's inequality for MCF writes

$$\frac{d}{dt}\int f\,d\mu_t \leqslant \int \mathbf{H}\cdot\nabla^\perp f\,d\mu_t - \int |\mathbf{H}|^2 f\,d\mu_t,$$

$$\frac{d}{dt}\int f\,d\mu_t^\varepsilon = \int \varepsilon\left(-\Delta c_\varepsilon + \frac{1}{\varepsilon^2}W'(c_\varepsilon)\right)\nabla c_\varepsilon\cdot\nabla f\,dx - \int \varepsilon\left(\Delta c_\varepsilon - \frac{1}{\varepsilon^2}W'(c_\varepsilon)\right)^2 f\,dx$$

Based on Ilmanen '93, Chen '96, finally Röger-Schätzle '06 proved the following property under $d = 2, 3$: If $\mu_t^\varepsilon \to \mu_t$ weakly as Radon measure, then

- the lower-semicontinuity property holds:

$$\int f|\mathbf{H}|^2\,d\mu_t \leqslant \liminf_{\varepsilon\to 0}\int \varepsilon\left(\Delta c_\varepsilon - \frac{1}{\varepsilon^2}W'(c_\varepsilon)\right)^2 f\,dx$$

  for f conti. Here a localization is needed to insert f in the inequality.
- μ_t is $(d-1)$-integral, i.e. $\mu_t = \theta_t\mathcal{H}^{d-1} \llcorner \Sigma_t$ with Σ_t being a countably $(d-1)$-rectifiable, $\mathcal{H}^{d-1}$-measurable set and $\theta \in L^1_{loc}(\mathcal{H}^{d-1} \llcorner \Sigma_t)$.

Table of contents

Keller–Rubinstein–Sternberg problem

We consider the vectorial (or matrix-valued) Ginzburg–Landau equation with potentials of higher dimensional wells.

$$\partial_t Q_\varepsilon = \Delta Q_\varepsilon - \varepsilon^{-2}\partial W(Q_\varepsilon) \qquad Q_\varepsilon : \Omega \subset \mathbb{R}^d \mapsto \mathbb{R}^n \tag{GL}$$

where Argmin $W(Q) = \mathbb{M} \triangleq \mathbb{M}^+ \sqcup \mathbb{M}^- \subset \mathbb{R}^n$ are two disjoint submanifolds.

Conjecture of Rubinstein–Sternberg–Keller '89

For well-prepared initial data, the gradient of Q_ε will be concentrated on a moving interface Σ_t governed by MCF and as $\varepsilon \downarrow 0$,

$$\int_\Omega \left(\frac{\varepsilon}{2}|\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon}W(Q_\varepsilon)\right) dx \approx \sigma \mathcal{H}^{d-1}(\Sigma_t) + \varepsilon \sum_{\pm} \sigma^\pm \int_{\Omega_t^\pm} |\nabla Q^\pm|^2 \, dx \tag{6}$$

Moreover $Q^\pm : \Omega_t^\pm \times (0,T) \mapsto \mathbb{M}^\pm$ satisfy harmonic map heat flow.

1. Lin–Pan–Wang '13: Establishing (6) in the steady case, $\mathbb{M}^\pm$ hypersurfaces, $\Sigma_t = \Sigma$ unique smooth stable minimum surface. $Q^\pm|_{\partial\Omega_t^\pm}$ form a minimal pair.
2. Fei–Wang–Zhang-Zhang '18: dynamic case with $\mathbb{M} = \mathbb{CP}^2 \sqcup \{0^5\} \subset \mathbb{R}^5$ by Hilbert expansion. Laux–L. '21 by modulated energy.
3. Fei–Lin–Wang–Zhang ' arxiv : dynamic case with $\mathbb{M} = \mathbb{O}(n)$ by Hilbert expansion.

Isotropic–Nematic phase transition

Consider the matrix-valued GL equation

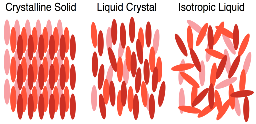
$$\partial_t Q_\varepsilon = \Delta Q_\varepsilon - \frac{1}{\varepsilon^2}\partial W(Q_\varepsilon)$$

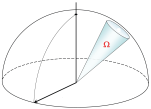
where $Q_\varepsilon : \Omega \to \mathbb{Q}$ where

$$\mathbb{Q} \triangleq \{Q \in \mathbb{R}^{3\times 3} : Q = Q^T, \operatorname{tr} Q = 0\},$$

$$W(Q) = \frac{a}{2}\operatorname{tr}(Q^2) - \frac{b}{3}\operatorname{tr}(Q^3) + \frac{c}{4}\left(\operatorname{tr}(Q^2)\right)^2$$

is the Landau's expansion of the (molecular) free energy and $a, b, c > 0$ are temperature dependent constants. At the critical temperature $b^2 = 27ac$, $s_- = 0$, $s_+ = \frac{b+\sqrt{b^2-24ac}}{4c}$.





$$\text{Argmin } W(Q) = \mathbb{M} := \{s(\mathbf{n}\otimes\mathbf{n} - \tfrac{1}{3}\mathrm{I}_3) : \mathbf{n} \in \mathbb{S}^2, s = s_\pm\}.$$

$$\text{where} \quad Q(x) = \int_{\mathbb{S}^2} f(x,p)(p\otimes p - \tfrac{1}{3}I)\, d\mathcal{H}^2(p)$$

is the normalized second moment of the distribution $f(x,\cdot)$ for fixed $x \in \Omega$.

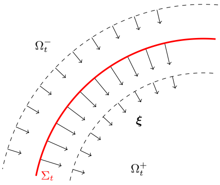
The quasi-distance is define by [1]

$$\mathrm{d}^W(Q) \triangleq \inf_{\gamma} \int_0^1 \sqrt{2W(\gamma(t))}|\gamma'(t)|dt \qquad (7)$$

where γ is any curve satisfying $\gamma(1) = Q$ and

$$\gamma(0) \in \mathbb{M}^+ = \{s^+(\mathbf{n}\otimes\mathbf{n} - \mathrm{I}_3/3) : \mathbf{n} \in \mathbb{S}^2\}$$

Moreover, we define $\psi_\varepsilon(x,t) \triangleq \mathrm{d}^W_\varepsilon \circ Q_\varepsilon(x,t)$
where $\mathrm{d}^W_\varepsilon(Q) \triangleq (\phi_\varepsilon *_{\mathbb{Q}} \mathrm{d}^W)(Q)$, mollified by an isotropic function ϕ_ε in $\mathbb{Q}$. Motivated by Fischer–Laux–Simon '21 on the scalar case, we introduce



$$E_\varepsilon[Q_\varepsilon|\Sigma](t) \triangleq \int_\Omega \left(\frac{\varepsilon}{2}|\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon}W(Q_\varepsilon) + \varepsilon^5 - \boldsymbol{\xi}\cdot\nabla\psi_\varepsilon\right) dx,$$

where $\boldsymbol{\xi}$ is an extension of the normal of Σ_t whose modulus decay in d^2_Σ.
Note that $|\partial \mathrm{d}^W(\cdot)| = \sqrt{2W(\cdot)}$ a.e. in $\mathbb{Q}$. So $(\mathbb{Q}, 2W)/\mathbb{M}$ is a (quotient) metric space with distance function (7).

[1] In the scalar case $\mathrm{d}^W(z) = \int_0^z \sqrt{2W(s)}\,ds$.

Modulated energy method: Calibration

Recall $s_- = 0, s_+ = \frac{b+\sqrt{b^2-24ac}}{4c}$, $\psi_\varepsilon(x,t) \triangleq \mathrm{d}_\varepsilon^W \circ Q_\varepsilon(x,t)$ and the quasi-distance

$$E_\varepsilon[Q_\varepsilon|\Sigma](t) \triangleq \int_\Omega \left(\frac{\varepsilon}{2} |\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon} W(Q_\varepsilon) + \varepsilon^5 - \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \right) dx$$

Laux–L. ' 21

Let Σ_t be a smooth MCF on $[0,T]$ and $\varepsilon \|Q_\varepsilon^{in}\|_{L^\infty(\Omega)} + E_\varepsilon[Q_\varepsilon|\Sigma](0) \lesssim \varepsilon$,

then we have $\quad E_\varepsilon[Q_\varepsilon|\Sigma](t) \lesssim \varepsilon$ for any $t \in [0,T]$.

Moreover, up to the extraction of a subsequence $\varepsilon = \varepsilon_k$,

$$Q_{\varepsilon_k} \xrightarrow{k\to\infty} Q = s_\pm \left(\mathbf{n} \otimes \mathbf{n} - \tfrac{1}{3} \mathrm{I}_3 \right) \text{ weak-star in } L_t^\infty(H_{loc}^1(\Omega_t^\pm))$$

where $\mathbf{n} : \Omega_t^+ \mapsto \mathbb{S}^2$ is a weak solution of harmonic heat flow with 0-Neumann boundary condition on $\partial\Omega_t^+ = \Sigma_t$.

By L.' preprint, one can derive, for some $\beta > 0$, a quantitative estimate

$$\|\psi_\varepsilon - \sigma \mathbf{1}_{\Omega_t^+}\|_{L^1(\Omega)} \lesssim \varepsilon^\beta$$

under an assumption on the initial data. This follows from a Gröwall inequality, rather than the BV compactness of ψ_ε.

Calibration and discrepancy: scalar case

In the scalar case $\psi_\varepsilon(x,t) = \int_0^{c_\varepsilon(x,t)} \sqrt{2W(z)}\,dz$, $\mathrm{n}_\varepsilon = \frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}$, and $\boldsymbol{\xi}$ satisfies

$$\partial_t \boldsymbol{\xi} + (\mathbf{H}\cdot\nabla)\boldsymbol{\xi} + (\nabla\mathbf{H})^T\boldsymbol{\xi} = O(\mathrm{d}_\Sigma).$$

We can write the modulated energy by:

$$\begin{aligned} E_\varepsilon[c_\varepsilon|\Sigma](t) &= \int_\Omega \frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}W(c_\varepsilon) - \overbrace{\boldsymbol{\xi}\cdot\frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}|\nabla c_\varepsilon|\sqrt{2W(c_\varepsilon)}}^{\boldsymbol{\xi}\cdot\nabla\psi_\varepsilon} \\ &= \int_\Omega \underbrace{\frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}W(c_\varepsilon) - |\nabla c_\varepsilon|\sqrt{2W(c_\varepsilon)}}_{\text{discrepancy}^2} + \int_\Omega \underbrace{(1-\boldsymbol{\xi}\cdot\mathrm{n}_\varepsilon)}_{\gtrsim \min(\mathrm{d}_\Sigma^2,1)}|\nabla\psi_\varepsilon| \end{aligned}$$

By $$\varepsilon|\nabla c_\varepsilon|^2 = |\nabla\psi_\varepsilon| + \sqrt{\varepsilon}|\nabla c_\varepsilon|\underbrace{\left(\sqrt{\varepsilon}|\nabla c_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(c_\varepsilon)}\right)}_{\text{discrepancy}}$$

and the Cauchy–Schwarz inequality, one can show

$$E_\varepsilon[c_\varepsilon|\Sigma](t) \gtrsim \int_\Omega \varepsilon|\nabla c_\varepsilon|^2 \min(\mathrm{d}_\Sigma^2, 1)$$

One can show $E_\varepsilon[c_\varepsilon|\Sigma](t)$ satisfies a Grönwall inequality and thus with appropriate initial data we have $E_\varepsilon[c_\varepsilon|\Sigma](t) \lesssim \varepsilon$. This recovers the estimate of Lin '96 when co-dim $= 1$ without employing MP.

Calibration and discrepancy: vectorial case

In the vectorial case, we shall derive a Grönwall inequality for $E_\varepsilon[Q_\varepsilon|\Sigma](t)$: Introducing an orthogonal projection in the direction $\partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)$:

$$\Pi_{Q_\varepsilon}\partial_{x_i}Q_\varepsilon = \left(\partial_{x_i}Q_\varepsilon : \frac{\partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)}{|\partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)|}\right)\frac{\partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)}{|\partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)|}, \text{ with } x_0 \triangleq t.$$

By Grönwall's inequality
$$\int_\Omega \left(\frac{1}{2}|\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(Q_\varepsilon)\right)(1-\boldsymbol{\xi}\cdot \mathrm{n}_\varepsilon) + \sum_{i=1}^d \int_\Omega |\partial_{x_i}Q_\varepsilon - \Pi_{Q_\varepsilon}\partial_{x_i}Q_\varepsilon|^2 \lesssim \varepsilon^{-1}E_\varepsilon[Q_\varepsilon|\Sigma](t) \lesssim 1.$$

This leads to the modulated estimate of ∇Q_ε without employing MP:

$$\int_\Omega |\nabla Q_\varepsilon|^2 \min(\mathrm{d}_\Sigma^2, 1) \lesssim \varepsilon^{-1}E_\varepsilon[Q_\varepsilon|\Sigma](t)$$

Define the phase-field analogy of the mean curvature vector by $\mathbf{H}_\varepsilon = -\left(\varepsilon\Delta Q_\varepsilon - \varepsilon^{-1}\partial F(Q_\varepsilon)\right) : \frac{\nabla Q_\varepsilon}{|\nabla Q_\varepsilon|}$, one can derive a differential inequality:

$$\frac{d}{dt}E_\varepsilon[Q_\varepsilon|I] + \frac{1}{2\varepsilon}\int \left(\varepsilon^2|\partial_t Q_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2 + \left|\mathbf{H}_\varepsilon - \varepsilon|\nabla Q_\varepsilon|\mathbf{H}\right|^2\right)dx + \frac{1}{2\varepsilon}\int \left|\varepsilon\partial_t Q_\varepsilon - (\operatorname{div}\boldsymbol{\xi})\partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)\right|^2 dx \lesssim E_\varepsilon[Q_\varepsilon|I].$$

To derive the compactness of $\partial_t Q_\varepsilon$, we use $\int_\Omega |\nabla Q_\varepsilon|^2 \min(\mathrm{d}_\Sigma^2, 1) \lesssim 1$ and

$$\int_{\Omega_T} |\partial_t Q_\varepsilon + \mathbf{H} \cdot \nabla Q_\varepsilon|^2$$
$$\leqslant \varepsilon^{-2} \int_{\Omega_T} \varepsilon^2 |\partial_t Q_\varepsilon|^2 - |\mathbf{H}_\varepsilon|^2 + \Big|\mathbf{H}_\varepsilon - \varepsilon |\nabla Q_\varepsilon| \mathbf{H}\Big|^2 \lesssim 1$$

Formal asymptotic expansion shows $\partial_t Q_\varepsilon + \mathbf{H} \cdot \nabla Q_\varepsilon = \varepsilon^{-1} (\partial_t \mathrm{d}_\Sigma + \mathbf{H} \cdot \nabla \mathrm{d}_\Sigma) + O(1)$. Later we show $[\Pi_{Q_\varepsilon} \partial_{x_i} Q_\varepsilon, Q_\varepsilon] = 0$ for $0 \leqslant i \leqslant d$. So we have

$$[\partial_i Q_\varepsilon, Q_\varepsilon] = [\partial_i Q_\varepsilon - \Pi_{Q_\varepsilon} \partial_i Q_\varepsilon, Q_\varepsilon] \xrightarrow{k\to\infty} \bar{S}_i(x,t) \text{ weakly in } L_t^2 L_x^2(\Omega)$$

By Chen's formulation of GL, (under summation convention)

$$[\partial_t Q_\varepsilon - \Pi_{Q_\varepsilon} \partial_t Q_\varepsilon, Q_\varepsilon] = \partial_{x_i} [\partial_{x_i} Q_\varepsilon - \Pi_{Q_\varepsilon} \partial_{x_i} Q_\varepsilon, Q_\varepsilon].$$

Converge locally $Q_\varepsilon \xrightarrow{\varepsilon\to 0} Q = s_\pm (\mathbf{n} \otimes \mathbf{n} - \frac{1}{3} \mathrm{I}_3) \in L_t^2 L_{loc}^2 (\Omega_t^\pm)$ and $\mathbf{n}$ is a weak solution of harmonic heat flow into $\mathbb{S}^2$ with 0 Neumann bdy condition on Σ_t:

$$\int_0^T \int_{\Omega_t^+} \partial_t \mathbf{n} \wedge \mathbf{n} \cdot \boldsymbol{\varphi} + \int_0^T \int_{\Omega_t^+} (\partial_j \mathbf{n} \wedge \mathbf{n}) \cdot \partial_j \boldsymbol{\varphi} = 0, \quad \forall \boldsymbol{\varphi} \in C_c^1(\Omega)$$

Note that if $\mathbf{n}$ is regular enough, we have

$$\partial_t \mathbf{n} - \Delta \mathbf{n} = |\nabla \mathbf{n}|^2 \mathbf{n} \text{ in } \Omega_t^+, \quad \partial_{\boldsymbol{\nu}} \mathbf{n}|_{\Sigma_t} = 0$$

A commutator identity

For any $0 \leqslant i \leqslant d$, we have $[\Pi_{Q_\varepsilon}\partial_{x_i}Q_\varepsilon, Q_\varepsilon] = 0, \quad \forall x \in \Omega$.

Recall the orthogonal projection $\Pi_{Q_\varepsilon}\partial_{x_i}Q_\varepsilon \parallel \partial \mathrm{d}_\varepsilon^W(Q_\varepsilon)$. As $\mathrm{d}_\varepsilon^W(Q)$ is isotropic (i.e. only depends on the eigenvalues of Q), there exists a smooth symmetric function $g_\varepsilon(\lambda_1, \lambda_2, \lambda_3)$ such that

$$\mathrm{d}_\varepsilon^W(Q) = g_\varepsilon(\lambda_1(Q), \lambda_2(Q), \lambda_3(Q))$$

Let Q_0 be a matrix having distinct eigenvalues, then $\lambda_i(Q)$ and the eigenvectors $\mathbf{n}_i(Q)$ are real-analytic functions of Q near Q_0, and then by chain rule

$$\partial \mathrm{d}_\varepsilon^W(Q) = \sum_{k=1}^{3} \frac{\partial g_\varepsilon}{\partial \lambda_k}\frac{\partial \lambda_k}{\partial Q} = \sum_{k=1}^{3} \frac{\partial g_\varepsilon}{\partial \lambda_k}\mathbf{n}_k(Q) \otimes \mathbf{n}_k(Q)$$

in a neighborhood of Q_0. We also have

$$Q = \sum_{k=1}^{3} \lambda_k(Q)\mathbf{n}_k(Q) \otimes \mathbf{n}_k(Q)$$

So we have $\left[\partial \mathrm{d}_\varepsilon^W(Q), Q\right] = 0$, holds in a neighborhood of Q_0 having distinct eigenvalues, and thus for a general Q by continuity of $\partial \mathrm{d}_\varepsilon^W(Q)$.

Boundary conditions on the free boundary

For any vector field $\boldsymbol{\varphi} \in C_c^1(\Omega, \mathbb{R}^3)$, let Φ be the corresponding anti-symmetric matrix-valued function. Apply the anti-symmetric product $[\cdot, Q_\varepsilon]$ to the Q_ε equ:

$$\int_0^T \int_\Omega [\partial_t Q_\varepsilon, Q_\varepsilon] : \Phi + \int_0^T \int_\Omega [\partial_j Q_\varepsilon, Q_\varepsilon] : \partial_j \Phi = 0$$

We denote $\Sigma_t(\delta)$ the $\delta-$neighborhood of Σ_t. Equivalently,

$$\sum_\pm \int_0^T \int_{\Omega_t^\pm \setminus \Sigma_t(\delta)} [\partial_t Q_\varepsilon, Q_\varepsilon] : \Phi + [\partial_j Q_\varepsilon, Q_\varepsilon] : \partial_j \Phi$$

$$+ \int_0^T \int_{\Sigma_t(\delta)} [\partial_t Q_\varepsilon, Q_\varepsilon] : \Phi + [\partial_j Q_\varepsilon, Q_\varepsilon] : \partial_j \Phi = 0$$

The convergences $\lim_{\varepsilon \to 0} Q_\varepsilon$ are not up to the boundary Σ_t but $\nabla_{x,t} Q \in L_t^2 L^2(\Omega_t^+)$:

$$\int_0^T \int_{\Omega_t^+ \setminus \Sigma_t(\delta)} [\partial_t Q, Q] : \Phi + [\partial_j Q, Q] : \partial_j \Phi + \int_0^T \int_{\Sigma_t(\delta)} \bar{S}_0 : \Phi + \bar{S}_j : \partial_j \Phi = 0$$

for some $\bar{S}_i \in L^2_{x,t}$ by modulated estimate. Take $Q = s_\pm(\mathbf{n} \otimes \mathbf{n} - \frac{1}{3}\mathrm{I}_3) \in L_t^2 L_x^2(\Omega_t^\pm)$:

$$\int_0^T \int_{\Omega_t^+} \partial_t \mathbf{n} \wedge \mathbf{n} \cdot \boldsymbol{\varphi} + \int_0^T \int_{\Omega_t^+} (\partial_j \mathbf{n} \wedge \mathbf{n}) \cdot \partial_j \boldsymbol{\varphi} = 0, \quad \forall \boldsymbol{\varphi} \in C_c^1(\Omega)$$

Convergence estimate of ψ_ε

We give a heuristic proof basic on the following result:

$$\partial_{x_i} f^+ = \partial_{x_i} f \mathbf{1}_{\{x\in\Omega: f(x)>0\}}, \qquad \forall f \in W^{1,1}(\Omega).$$

Recall $\psi_\varepsilon(x,t) = \mathrm{d}_\varepsilon^W \circ Q_\varepsilon(x,t)$. Differentiating $\int (\psi_\varepsilon - \sigma \mathbf{1}_{\Omega^+})^+ \, dx$ leads to delta mass $\nabla \mathbf{1}_{\Omega^+}$. To avoid it, we introduce the weighted energy

$$g_\varepsilon(t) := \int (\psi_\varepsilon - \sigma \mathbf{1}_{\Omega^+})^+ \zeta(\mathrm{d}_\Sigma) \, dx$$

for a cut-off ζ and let $U_\varepsilon = \{x \in \Omega : \psi_\varepsilon > \sigma \mathbf{1}_{\Omega^+}\}$. Using $(\partial_t + \mathbf{H}\cdot\nabla)\, \zeta(\mathrm{d}_\Sigma) = O(\mathrm{d}_\Sigma)$,

$$\begin{aligned} g_\varepsilon'(t) =& \int_{U_\varepsilon} \Big(\partial_t Q_\varepsilon + (\mathbf{H}\cdot\nabla) Q_\varepsilon\Big) : \partial \mathrm{d}_\varepsilon^W (Q_\varepsilon)\, \zeta(\mathrm{d}_\Sigma) \\ & - \underbrace{\int_{U_\varepsilon} \mathbf{H}\cdot\nabla\psi_\varepsilon \zeta(\mathrm{d}_\Sigma)}_{=\int \mathbf{H}\cdot\nabla(\psi_\varepsilon - \sigma\mathbf{1}_{\Omega^+})^+\zeta(\mathrm{d}_\Sigma)} + \int (\psi_\varepsilon - \sigma\mathbf{1}_{\Omega^+})^+ \underbrace{\partial_t \zeta(\mathrm{d}_\Sigma)}_{\approx -\mathbf{H}\cdot\nabla\zeta(\mathrm{d}_\Sigma)} \end{aligned}$$

So we can derive a Grönwall inequality of $g_\varepsilon(t) \lesssim \varepsilon$. To get the desired estimate of ψ_ε, one simply employ standard trick to remove the weight $\zeta(\mathrm{d}_\Sigma)$ at the price of a weaker convergence rate ε^β.

Thank you for your attention !

Related works which (might) be (re)proved by the modulated energy method:

1. Allen–Cahn/(3D)Navier–Stokes (Constant mobility) to MCF/Navier–Stokes, Hensel–L. 'preprint. Variable mobility case which converges to two-phase NSE, is open.
2. Cahn–Hilliard to Hele-Shaw. New proof of Alikakos–Bates–Chen '96 using modulated method ?
3. Allen–Cahn to triple junction dynamics. Fischer–Marveggio' preprint
4. Contact angle model. Abels–Moser '20, Moser–Hensel ' preprint
5. Anisotropic models (with anchoring boundary conditions): Lin–Wang '21 for Ericksen's models (static), L. 'preprint for a simplified Landau-De Gennes dynamics (planar case).
6. Conjecture of Rubinstein–Sternberg–Keller for (two) closed hypersurfaces.