Relativistic BGK model for multi-component particle system

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09.15. 2022

Joint Analysis and PDE Seminar
(CUHK, HKU and UNIST)

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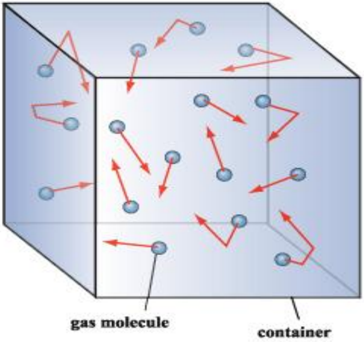
Boltzmann equation

Interacting Particle Systems

- Interacting particle systems: gas, plasma, Universe, society, active particles,...



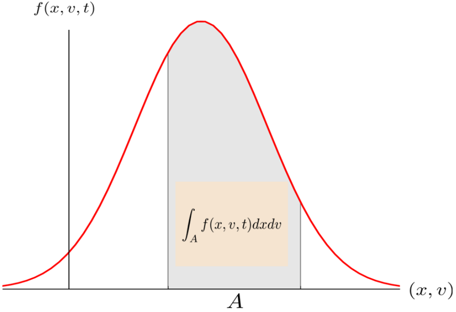
Velocity distribution function: Ludwig Boltzmann(1844∼1906)

- How particles are distributed in phase space (x, v) at time t.

  $\Rightarrow f(x, v, t)$: Velocity Distribution Function.

Velocity Distribution Function

- How particles are distributed in the phase space?
- $\int_A f(x, v, t)dxdv = \#$ of particles such that $(x, v) \in A$ at time t

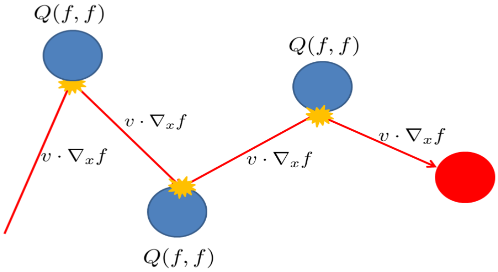


The Boltzmann equation

- For non-ionized monatomic rarefied gas (1872):

$$\partial_t f + v \cdot \nabla_x f = Q(f, f),$$

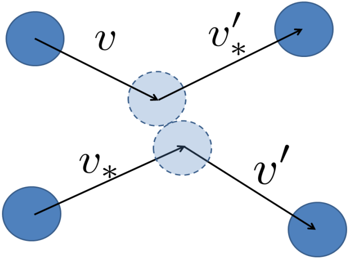
- Transport+collision



Collision Operator

$$Q(f,f)(v) \equiv \int_{\mathbb{R}^3\times\mathbb{S}^2_+} B(v-v_*,\omega)(f(v')f(v'_*)-f(v)f(v_*))d\omega dv_*.$$

$$v' = v - [(v-v_*)\cdot\omega]\omega, \quad v'_* = v_* + [(v-v_*)\cdot\omega]\omega.$$



Q satisfies

- Conservation Laws

$$\int_{\mathbb{R}^3} Q(f,f)(1,v,|v|^2)dv = 0$$

- H-theorem

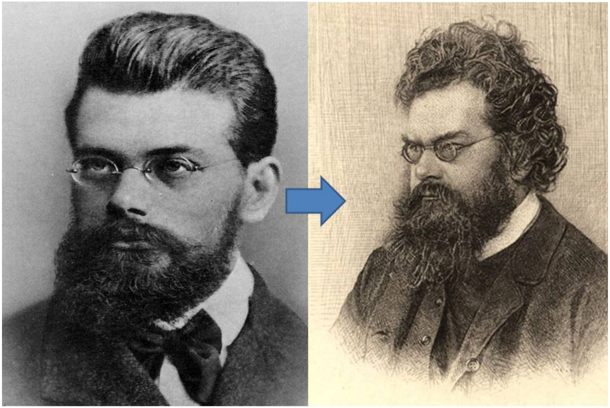
$$\int_{\mathbb{R}^3} Q(f,f)\ln f dv \leq 0$$

- Equilibrium

$$Q(f,f) = 0 \ \Leftrightarrow \ f = \mathcal{M} = Ce^{a|v|^2+b\cdot v+c}.$$

But...

- His achievements were never fully appreciated during his life time.
- He suffered severely from criticism - irreversibility, atomic theory and so on.



The importance of stress management

Local Maxwellian

- Recall

$$Q(\mathcal{M},\mathcal{M}) = 0 \ \Leftrightarrow\ f = \mathcal{M} = Ce^{a|v|^2+b\cdot v+c}$$

- Due to the conservation laws, $\mathcal{M}$ has to be

$$\mathcal{M}(f)(x,v,t) = \frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^3}} \exp\Big(-\frac{|v-U(x,t)|^2}{2T(x,t)}\Big).$$

where

$$\rho(x,t) = \int_{\mathbb{R}^3} f(x,v,t)dv$$
$$\rho(x,t)U(x,t) = \int_{\mathbb{R}^3} f(x,v,t)vdv$$
$$\rho(x,t)T(x,t) = \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2dv.$$

BGK model

Collision Operator

- Intricate structure

$$Q(f,f) \equiv \underbrace{\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B(v-v_*,\omega)(f(v')f(v'_*)-f(v)f(v_*))d\omega dv_*.}_{\textit{Nonlinear}}$$

Collision Operator

- Intricate structure

$$Q(f,f) \equiv \int_{\mathbb{R}^3\times\mathbb{S}^2_+} \underbrace{B(v-v_*,\omega)(f(v')f(v'_*)-f(v)f(v_*))}_{\textit{Convolution}} d\omega dv_*.$$

Collision Operator

- Intricate structure

$$Q(f,f) \equiv \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} B(v - v_*, \omega) \underbrace{\left(f(v')f(v'_*) - f(v)f(v_*)\right)}_{\textit{Cancellation}} d\omega dv_*.$$

Collision Operator

- Intricate structure

$$Q(f,f) \equiv \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} B(v - v_*, \omega)(f(v')f(v'_*) - f(v)f(v_*)) \underbrace{d\omega dv_*}_{5-fold}.$$

BE: fundamental but not practical

- Hard to develop fast & efficient numerical methods.
- Most difficulties and costs arise in the computation of Q.

The Boltzmann-BGK model

- BGK Model (Bhatnagar-Gross-Krook [1954]):

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\tau}(\mathcal{M}(f) - f)$$

- τ: Relaxation time.

$$\frac{1}{\tau} = \frac{1}{\kappa}\rho^{\alpha} T^{\beta}.$$

- Most popular choices

$$\alpha = \beta = 0, \quad \text{or} \quad \alpha = 1, \beta = 0.$$

$\mathcal{M}(f)$?

- $\mathcal{M}$: Local Maxwellian

$$\mathcal{M}(f)(x,v,t) = \frac{\rho(x,t)}{\sqrt{(2\pi T(x,t))^3}} \exp\Big(- \frac{|v-U(x,t)|^2}{2T(x,t)} \Big).$$

where

$$\begin{aligned} \rho(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)dv \\ \rho(x,t)U(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)vdv \\ \rho(x,t)T(x,t) &= \int_{\mathbb{R}^3} f(x,v,t)|v-U(x,t)|^2 dv. \end{aligned}$$

Recall Properties of Q

- Conservation Laws

$$\int_{\mathbb{R}^3} Q(f,f)(1, v, |v|^2)dv = 0.$$

- Entropy dissipation

$$\int_{\mathbb{R}^3} Q(f,f) \ln f dv \leq 0.$$

- Equilibrium

$$Q(f,f) = 0 \Leftrightarrow f = \mathcal{M}.$$

Property of $\mathcal{M} - f$

- Conservation Laws

$$\int_{\mathbb{R}^3} \{\mathcal{M}(f) - f\}(1, v, |v|^2)dv = 0.$$

- Entropy dissipation

$$\int_{\mathbb{R}^3} \{\mathcal{M}(f) - f\} \ln f dv \leq 0.$$

- Equilibrium

$$\mathcal{M}(f) - f = 0 \Leftrightarrow f = \mathcal{M}.$$

- Collision process of BE $\Rightarrow$ Relaxation process

- Much lower computational cost compared to BE

- Still shares important features with BE:
  - Conservation laws
  - H-theorem
  - Relaxation to equilibrium.
  - Correct Euler Limit

- Very popular model for numerical experiments in kinetic theory (citation 9900)

Correct Euler limit but...

- Collision process of BE $\Rightarrow$ Relaxation process

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Correct Euler limit but...

- Collision process of BE $\Rightarrow$ Relaxation process
- Much lower computational cost compared to BE
- Still shares important features with BE:
  - Conservation laws
  - H-theorem
  - Relaxation to equilibrium.
  - Correct Euler Limit
  - Navier-Stokes Limit ?
- Very popular model for numerical experiments in kinetic theory (citation 9900)

BGK model for multi-component gases

Boltzmann equation for multi-component gases

- Boltzmann equation for gas mixtures

$$\partial_t f_i + v \cdot \nabla_x f_i = Q_i := \sum_{k=1}^{N} Q_{ik}(f_i, f_k)$$

$$Q_{ij}(f_i, f_k) := \int_{\mathbb{R}^3 \times \mathbb{S}^2_+} B_{ik}(v - v_*, \omega)(f_i(v')f_k(v'_*) - f_i(v)f_k(v_*))d\omega dv_*$$

Boltzmann equation for 2-component gases

- Boltzmann equation for gas mixtures

$$\partial_t f_1 + v\cdot\nabla_x f_1 = Q_{11}(f_1,f_1) + Q_{12}(f_1,f_2),$$
$$\partial_t f_2 + v\cdot\nabla_x f_2 = Q_{21}(f_2,f_1) + Q_{22}(f_2,f_2)$$

$$Q_{ij}(f_i,f_k) := \int_{\mathbb{R}^3\times\mathbb{S}^2_+} B_{ik}(v-v_*,\omega)(f_i(v')f_k(v'_*) - f_i(v)f_k(v_*))d\omega dv_*$$

Conservation laws, Entropy, Equilibrium

- Boltzmann collision operator for gas mixture satisfies

1. Conservation laws:

$$\int_{\mathbb{R}^3} Q_i dv = 0, \qquad \sum_{i=1}^{N} \int_{\mathbb{R}^3} (v, |v|^2) Q_i dv = 0.$$

2. H-theorem:

$$\sum_{i=1}^{N} \int_{\mathbb{R}^3} Q_i \ln f_i dv \leq 0$$

3. Equilibrium:

$$\forall i \quad Q_i \equiv 0 \implies \quad f_i = n_i \left(\frac{m_i}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{m_i |v - U|^2}{2\pi T}\right)$$

BGK models for multi-component gases

- Many BGK-type relaxation operators $\mathcal{R}_i$ have been suggested to mimic Q_i

$$\partial_t f_i + v \cdot \nabla_x f_i = \mathcal{R}_i$$

- Two types of BGK models for gas mixtures:

1. single BGK operator (unlike the BE collision operator)

$$\mathcal{R}_i = \nu_i(\mathcal{M}_i - f_i)$$

2. bi-species BGK operators (As the BE collision operator)

$$\mathcal{R}_i = \sum_{k=1}^{N} \mathcal{R}_{ij} = \sum_{k=1}^{N} \nu_{ik}(\mathcal{M}_{ij} - f_i)$$

BGK models for multi-component gases

- Many BGK-type relaxation operators $\mathcal{R}_i$ have been suggested to mimic Q_i

$$\partial_t f_i + v \cdot \nabla_x f_i = \mathcal{R}_i$$

- Two types of BGK models for gas mixtures:

1. single BGK operator

$$\begin{aligned} \partial_t f_1 + v \cdot \nabla f_1 &= \nu_1(\mathcal{M}_1 - f_1) \\ \partial_t f_2 + v \cdot \nabla f_2 &= \nu_2(\mathcal{M}_2 - f_2) \end{aligned}$$

2. bi-species BGK operators

$$\begin{aligned} \partial_t f_1 + v \cdot \nabla f_1 &= \nu_{11}(\mathcal{M}_{11} - f_1) + \nu_{12}(\mathcal{M}_{12} - f_1) \\ \partial_t f_2 + v \cdot \nabla f_2 &= \nu_{21}(\mathcal{M}_{21} - f_2) + \nu_{22}(\mathcal{M}_{22} - f_2) \end{aligned}$$

Boltzmann equation and BGK model

- Many BGK-type relaxation operators have been suggested to mimic Q_i

$$\partial_t f_i + v \cdot \nabla_x f_i = \mathcal{R}_i$$

- Two types of BGK models for gas mixtures:

1. single BGK operator
   : Andries, Aoki, and Perthame (2002), Groppi, Rjasanow and Spiga (2009), Brull, Pavan, and Schneider (2012)

2. bi-species BGK operators
   : Gross and Krook (1956), Sirovich (1962) Morse(1964) Hamel (1965), Goldman and Sirovich (1967), Greene (1973), Garzo (1989), Haack, Hauck, and Murillo (2017), Klingenberg, Pirner, and Puppo (2017), Bobylev, Bisi, Groppi, Spiga, and Potapenko (2018)

Boltzmann equation and BGK model

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$$\partial_t f_i + v \cdot \nabla_x f_i = \mathcal{R}_i$$

- Two types of BGK models for gas mixtures:
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Efforts to derive BGK model with bi-species BGK operator

- 1956∼1989:

$$\widetilde{Q}_i = \nu_{ii}\left(\mathcal{M}_{ii}(n_i, U_i, T_i) - f_i\right) + \nu_{ij}\left(\mathcal{M}_{ij}(n_i, U_{ij}, T_{ij}) - f_i\right)$$

- The parameters n_{ij}, U_{ij}, T_{ij} are determined by conservation laws $+\ \alpha$

- H-theorem (?)

Why do we bother to look like BE? $\rightarrow$ Single relaxation operator

• AAP model (2002)

$$\widetilde{Q}_i = \nu_i \Big(\mathcal{M}_i(\widetilde{n}_i, \widetilde{U}_i, \widetilde{T}_i) - f_i \Big)$$

• The parameters $\widetilde{n}_i$, $\widetilde{U}_i$, $\widetilde{T}_i$ are determined in a way that the exchange rates of mass, momentum, and energy coincide with the Boltzmann's one.

$$\int_{\mathbb{R}^3} (1, v, |v|^2) Q_i dv = \int_{\mathbb{R}^3} (1, v, |v|^2) \widetilde{Q}_i dv$$

• H-theorem holds

Relativistic BGK model for multi-component gases

Relativistic BGK-type models for gas mixtures

- Relativistic Boltzmann equation for gas mixtures:

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = Q_i = \sum_{k=1}^{N} Q_{ik}(f_i, f_k)$$

$f_i(x^\mu, p_i^\mu)$: the momentum distribution representing the number density of i-th species at the phase point (x^μ, p_i^μ).

$$Q_{ij} := \frac{1}{p_i^0} \int (f_i' f_j' - f_i f_j) F_{ij} \sigma_{ij} \, d\Omega \frac{dp_j}{p_j^0},$$

- The relativistic collision operator for gas mixtures satisfies that

1. conservation laws

$$\int_{\mathbb{R}^3} Q_i dp_i = 0, \quad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^\mu Q_i dp_i = 0.$$

2. H-theorem

$$\sum_{i=1}^{N} \int_{\mathbb{R}^3} Q_i \ln f_i dp_i \leq 0.$$

3. Equilibrium:

$$\forall i \quad Q_i \equiv 0 \implies \quad f_i = \frac{g_s}{h^3} \exp\left(\beta \mu_{E_i}\right) \exp\left(-\beta U^\mu p_\mu\right)$$

Relativistic BGK models for gas mixtures

- Relativistic BGK model for gas mixtures:

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = \widetilde{Q}_i.$$

- We want to replace the Boltzmann collision operator by a BGK-type operator.

Relativistic BGK models for a single component gas: Many choices

- At least 3 different types of BGK model exists.
- Depending on the interpretation of macroscopic four velocity:
  - Marle model (1965): Eckart frame
  - Anderson-Witting model (1974): Landau-Lipschitz frame
  - Ruggeri-Pennisi model (2017): Eckart frame

Choices

- single relaxation operator vs bi-species relaxation operator?
- Marle vs Anderson-Witting vs Ruggeri-Pennisi?

Choices

- single relaxation operator vs bi-species relaxation operator?
- Marle vs Anderson-Witting vs Ruggeri-Pennisi?

Relativistic BGK models for a single component gas

• Marle model (1965) : A **relativistic** BGK-type model for single species gas

$$\partial_t f + \frac{p}{p^0} \cdot \nabla_x f = \frac{cm}{\tau p^0}(\mathcal{J}(f) - f)$$

where the attractor $\mathcal{J}(f)$ is given by a Jüttner distribution:

$$\mathcal{J}(f) = \frac{g_s}{h^3} \exp(\beta \mu_E) \exp(-\beta U^\mu p_\mu) .$$

• Anderson-Witting model (1974), Ruggeri-Pennisi model (2017)

Main Result: A new multi-component relativistic BGK model

Our relativistic BGK for gas mixtures

- We suggest a relativistic multi-component BGK model with single BGK operator:

$$\partial_t f_i + \frac{cp_i}{p_i^0} \cdot \nabla_x f_i = \mathcal{R}_i = \frac{cm_i}{\tau_i p^0}(\mathcal{J}_i - f_i).$$

where the attractor $\mathcal{J}_i$ is given by

$$\mathcal{J}_i = \frac{g_{s_i}}{h^3} \exp\left(\widetilde{\beta}\mu_{E_i}\right) \exp\left(-\widetilde{\beta}\widetilde{U}^\mu p_{i\mu}\right).$$

Main issue: Determination of Equilibrium coefficients

- Classical equilibrium

$$\frac{\rho}{\sqrt{(2\pi T)^3}} e^{-\frac{|v-U|^2}{2T}}$$

$\rightarrow$ Equilibrium coefficients ρ, U, T must be determined by f.

- Relativistic equilibrium

$$\frac{g_s}{h^3} \exp\left(\beta \mu_{E_i}\right) \exp\left(-\beta U^\mu p_\mu\right)$$

$\rightarrow$ $\frac{g_s}{h^3} \exp\left(\beta \mu_{E_i}\right)$, β, $\tilde{U}^\mu$ must be determined by f.

Not available yet

- Literature on relativistic BGK models for gas mixture is very limited.
- Results on Well-definedness of these equilibrium coefficients are missing in the literature.
- The computation of transport coefficients is possible even without knowing how equilibrium coefficients is defined.
- but wihtout such precise difinition, investigation of the dynamics of relativistic gases at the kinetic level using the model is not possible.

Main goal

- We want to determine equilibrium coefficients in such a way that

1. Conservation laws hold:

$$\int_{\mathbb{R}^3} \mathcal{R}_i dp_i = 0, \quad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^{\mu} \mathcal{R}_i dp_i = 0.$$

2. H-theorem hold:

$$\sum_{i=1}^{N} \int_{\mathbb{R}^3} \mathcal{R}_i \ln f_i dp \leq 0.$$

3. Equilibrium is correctly captured:

$$\forall i \quad \mathcal{R}_i \equiv 0 \implies \quad f_i = \frac{g_{s_i}}{h^3} \exp\left(\widetilde{\beta} \mu_{E_i}\right) \exp\left(-\widetilde{\beta} \widetilde{U}^{\mu} p_{i\mu}\right).$$

Minimum requirement: the conservation laws

- The system of equations

$$\int_{\mathbb{R}^3} \widetilde{Q}_i dp_i = 0, \quad \sum_{i=1}^{N} \int_{\mathbb{R}^3} p_i^{\mu} \widetilde{Q}_i dp_i = 0.$$

is rewritten as

$$\frac{g_{si}}{h^3} e^{\widetilde{\beta}\mu_{Ei}} \int_{\mathbb{R}^3} e^{-\widetilde{\beta}\tilde{U}^{\mu} p_{i\mu}} \frac{dp_i}{p_i^0} = \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0},$$

$$\sum_{i=1}^{N} \frac{m_i}{\tau_i} \frac{g_{si}}{h^3} e^{\widetilde{\beta}\mu_{Ei}} \int_{\mathbb{R}^3} p_i^{\mu} e^{-\widetilde{\beta}\tilde{U}^{\mu} p_{i\mu}} \frac{dp_i}{p_i^0} = \sum_{i=1}^{N} \frac{m_i}{c\tau_i} n_i U_i^{\mu}$$

where

$$\int_{\mathbb{R}^3} p_i^{\mu} f_i \frac{dp_i}{p_i^0} = N_i^{\mu} = n_i U^{\mu}, \text{ and } U^{\mu} U_{\mu} = c^2.$$

The relaxation operator of our model is well-defined

Theorem (Hwang, Lee, and Yun (2022))

1. *The presentation of the head part of the Jüttner distribution:*

$$\frac{g_{si}}{h^3}\exp\left(\tilde{\beta}\mu_{Ei}\right) = \frac{\int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0}}{\int_{\mathbb{R}^3} e^{-\tilde{\beta}\tilde{U}^\mu p_{i\mu}} \frac{dp_i}{p_i^0}}.$$

2. *The relation satisfied by $\tilde{\beta}$:*

$$\sum_{i=1}^N \frac{m_i}{\tau_i}\frac{\int_{\mathbb{R}^3} e^{-c\tilde{\beta}p_i^0}\,dp_i}{\int_{\mathbb{R}^3} e^{-c\tilde{\beta}p_i^0}\frac{dp_i}{p_i^0}}\int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c}\left[\left(\sum_{i=1}^N \frac{m_i}{\tau_i} n_i U_i^\mu\right)\left(\sum_{j=1}^N \frac{m_j}{\tau_j} n_j U_{j\mu}\right)\right]^{\frac{1}{2}}.$$

3. *The presentation of the auxiliary parameter $\tilde{U}^\mu$:*

$$\tilde{U}^\mu = c\frac{\sum_{i=1}^N \frac{m_i}{\tau_i} n_i U_i^\mu}{\left[\left(\sum_{i=1}^N \frac{m_i}{\tau_i} n_i U_i^\mu\right)\left(\sum_{j=1}^N \frac{m_j}{\tau_j} n_j U_{j\mu}\right)\right]^{\frac{1}{2}}}.$$

Unique Determination of $\tilde{\beta}$

The nonlinear relation:

$$\sum_{i=1}^{N} \frac{m_i}{\tau_i} \frac{\int_{\mathbb{R}^3} e^{-c\tilde{\beta}p_i^0}\, dp_i}{\int_{\mathbb{R}^3} e^{-c\tilde{\beta}p_i^0} \frac{dp_i}{p_i^0}} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c}\left[\left(\sum_{i=1}^{N} \frac{m_i}{\tau_i} n_i U_i^{\mu}\right)\left(\sum_{j=1}^{N} \frac{m_j}{\tau_j} n_j U_{j\mu}\right)\right]^{\frac{1}{2}}.$$

admits a unique $\tilde{\beta}$

The relaxation operator of our model is well-defined

Theorem (Hwang, Lee, and Yun (2022))

The conservation laws uniquely determine the Jüttner distribution $\mathcal{J}_i$ *by*

$$\mathcal{J}_i = \frac{\int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0}}{\int_{\mathbb{R}^3} e^{-c\tilde{\beta} p_i^0} \frac{dp_i}{p_i^0}} \exp\left(-\tilde{\beta} \widetilde{U}^\mu p_{i\mu}\right), \quad \text{for} \quad i = 1, ..., N.$$

where $\widetilde{U}^\mu$ *is given by*

$$\widetilde{U}^\mu = c \frac{\sum_{i=1}^N \frac{m_i}{\tau_i} n_i U_i^\mu}{\left[\left(\sum_{i=1}^N \frac{m_i}{\tau_i} n_i U_i^\mu\right)\left(\sum_{j=1}^N \frac{m_j}{\tau_j} n_j U_{j\mu}\right)\right]^{1/2}}$$

and $\widetilde{\beta}$ *is defined as the unique solution to*

$$\sum_{i=1}^N \frac{m_i}{\tau_i} \frac{\int_{\mathbb{R}^3} e^{-c\tilde{\beta} p_i^0} dp_i}{\int_{\mathbb{R}^3} e^{-c\tilde{\beta} p_i^0} \frac{dp_i}{p_i^0}} \int_{\mathbb{R}^3} f_i \frac{dp_i}{p_i^0} = \frac{1}{c}\left[\left(\sum_{i=1}^N \frac{m_i}{\tau_i} n_i U_i^\mu\right)\left(\sum_{j=1}^N \frac{m_j}{\tau_j} n_j U_{j\mu}\right)\right]^{\frac{1}{2}}.$$

Our model also satisfies

1. (Non-negativitiy) $f_i \geq 0$.
2. Conservation laws
$$\frac{\partial N_i^{\mu}}{\partial x^{\mu}} = 0, \quad \frac{\partial T^{\mu\nu}}{\partial x^{\nu}} = 0.$$
3. (H-theorem)
$$sum_{i=1}^{N} \int_{\mathbb{R}^3} f_i \ln \widetilde{Q}_i dp_i \leq 0.$$
4. (Equilibrium) The equilibrium is given by the Jüttner distribution.
5. (Indifferentiability principle) All species have the same mass m and same relaxation time τm $f := \sum_{i=1}^{N} f_i$ satisfies the Malre model for single species.

Also Newtonian limit to CRS model

- Dimensionless version:

$$\frac{\partial}{\partial \bar{t}}\bar{f}_i + \frac{1}{\sqrt{1+|\epsilon v|^2}} v\cdot\nabla_{\bar{x}}\bar{f}_i = \frac{\nu_i}{\sqrt{1+|\epsilon v|^2}}\left(\bar{\mathcal{J}}_i(\bar{f}_i) - \bar{f}_i\right)$$

where

$$t = \bar{t}s, \quad x = \bar{x}L, \quad p_i = v\mu_i, \quad \mu_i = \frac{m_i L}{s}, \quad \epsilon = \frac{\mu_i}{cm_i}, \quad \text{and} \quad \nu_i = \frac{s}{\tau_i}.$$

Theorem (Hwang, Lee, and Yun (2002))

The dimensionless attractor $\bar{\mathcal{J}}_i(\bar{f}_i)$ approaches the attractor $\mathcal{M}_i$ of GRS model in Newtonian limit sense ($\epsilon \to 0$).

Future

- Anderson-Witting type, Ruggeri-Pennisi type.
- bi-species type relaxation operator.
- Existence, Hydrodynamic limit, Newtonian limit.

Thank you for your interest!