On steady states for the Vlasov-Schrödinger-Poisson system

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Physical background

a large number of electrons (collisonless)

• microscopic level $\to$ ex) Hartree-Fock, Dirac-Fock, ...

• mesoscopic level $\to$ electron gases   ex) Vlasov-Poisson, Vlasov-Maxwell, ...

• macroscopic level $\to$ electron fluids   ex) Euler-Poisson, ...

Physical background

In some physical settings, electrons are confined strongly in certain direction(s).

$\Rightarrow$ kinetic+quantum hybrid models

Physics of semiconductors

∘ nanowires

∘ **two-dimensional electron gases (2DEGs)**
- at the heterojunction between two semiconductors
  modulation-doped field-effect transistor (MODFET)
  a high-electron-mobility-transistor (HEMT)
- a graphene

• Ben Abdallah-Méhats 2005: Rigorous derivation (via partial semi-classical limit)

Domain

Space domain

$$\Omega = \omega \times (0,1) \ni (y,z) = x$$

- ω is a smooth bounded domain in $\mathbb{R}^2$.

- $y \in \omega$: "unconfined" direction $\quad \leftrightarrow \quad$ $z \in (0,1)$: "confined" direction

• Motivation: partial semi-classical limit

$$\Omega_\epsilon = \omega \times (0,\epsilon) \quad \underset{\text{scaling}}{\longrightarrow} \quad \Omega = \omega \times (0,1) \qquad\qquad \Omega = \omega \times (0,1)$$

$$-\frac{\epsilon^2}{2}\Delta_y - \frac{\epsilon^2}{2}\partial_z^2 \qquad\qquad -\frac{\epsilon^2}{2}\Delta_y - \frac{1}{2}\partial_z^2 \quad \underset{\text{formally}}{\longrightarrow} \quad \frac{|v|^2}{2} - \frac{1}{2}\partial_z^2$$

2D phase space

$$(y,v) \in \omega \times \mathbb{R}^2$$

Q) Minimize the energy of N non-interacting electrons in a bounded region

(1) kinetic description (mesoscopic scaling)

- electron gases $\rightsquigarrow$ $f(x, v) : \Omega \times \mathbb{R}^3 \to [0, 1]$ (Pauli's exclusion principle)

- Under the mass constraint $\|f\|_{L^1(\Omega\times\mathbb{R}^3)} = N$, minimize the energy

$$\mathcal{E}(f) = \iint_{\Omega\times\mathbb{R}^3} \big(\tfrac{|v|^2}{2} + V(x)\big) f(x, v) dxdv.$$

$$\underset{\text{minimizer}}{\Rightarrow} \quad \boxed{f^*(x, v) = \mathbb{1}_{(\frac{|v|^2}{2} + V(x) \leq \mu)}}$$

for some $\mu > 0$ such that $\|f^*\|_{L^1(\Omega\times\mathbb{R}^3)} = N$.

(2) quantum model (microscopic scaling)

• many electrons $\rightsquigarrow$ self-adjoint $\gamma : L^2(\Omega) \to L^2(\Omega)$ with $0 \leq \gamma \leq 1$ (Pauli's exclusion principle)

• Under the mass(=particle number) constraint $\mathrm{Tr}(\gamma) = N \in \mathbb{N}$, minimize the energy

$$\mathcal{E}(\gamma) = \mathrm{Tr}\big((-\tfrac{1}{2}\Delta_x + V)\gamma\big).$$

$$\underset{\text{minimizer}}{\Rightarrow} \quad \boxed{\mathbb{1}_{(-\frac{1}{2}\Delta_x + V \leq \mu)} = \sum_{j=1}^{N} |\psi_j^*\rangle\langle\psi_j^*|}$$

for $\mu > 0$ such that $-\frac{1}{2}\Delta + V$ has N normalized eigenfunctions $\{\psi_j^*\}_{j=1}^N$ whose eigenvalues $\leq \mu$.

Notation: $|\psi\rangle\langle\psi|$ is a one-particle projection onto a unit vector ψ.

• operator $\mathbb{1}_{(-\frac{1}{2}\Delta_x + V \leq \mu)} \quad \leftrightarrow \quad$ orthonormal set $\{\psi_j^*\}_{j=1}^N \subset L^2(\Omega)$

(3) kinetic-quantum hybrid model

$$\underset{\text{minimizer?}}{\Rightarrow} \quad \boxed{\mathbb{1}_{\left(\frac{|v|^2}{2}-\frac{1}{2}\partial_z^2+V(y,z)\leq\mu\right)}}$$

• For each $y \in \omega$, 1D Schrödinger operator $-\frac{1}{2}\partial_z^2 + V(y,\cdot)$ acting on $L^2(0,1)$ with zero boundary.

- eigenvalues: $\lambda_1^*(y) < \lambda_2^*(y) < \lambda_3^*(y) < \cdots \to \infty$
- normalized eigenfunctions: $\{\chi_j^*(y,\cdot)\}_{j=1}^\infty$.

• The above object means

$$\mathbb{1}_{\left(\frac{|v|^2}{2}-\frac{1}{2}\partial_z^2+V(y,z)\leq\mu\right)} = \sum_{j=1}^\infty f_j^*(y,v)|\chi_j^*(y,\cdot)\rangle\langle\chi_j^*(y,\cdot)|,$$

where $f_j^* = \mathbb{1}_{\left(\frac{|v|^2}{2}+\lambda_j^*(y)\leq\mu\right)}$.

It is equivalent to the sequences $\mathbf{f}^* = \{f_j^*\}_{j=1}^\infty$ and $\boldsymbol{\chi}^* = \{\chi_j^*\}_{j=1}^\infty$.

(3) kinetic-quantum hybrid model (continued)

By "partial" semi-classical limits (see [Ben Abdallah-Méhats 2005]),

- $\mathbf{f} = \{f_j\}_{j=1}^{\infty}$, $0 \le f_j(y,v) \le 1$ (distribution on the 2D phase space)
- $\boldsymbol{\chi} = \{\chi_j\}_{j=1}^{\infty}$, $\langle \chi_j(y,\cdot), \chi_k(y,\cdot)\rangle_{L^2(0,1)} = \delta_{jk}$ (quantum states)

$$\mathcal{M}(\mathbf{f}) = \sum_{j=1}^{\infty} \iint_{\omega\times\mathbb{R}^3} f_j dydv$$

$$\begin{aligned}\mathcal{E}(\mathbf{f},\boldsymbol{\chi}) &= \sum_{j=1}^{\infty} \iint_{\omega\times\mathbb{R}^3} \left(\tfrac{|v|^2}{2} + \tfrac{1}{2}\|\partial_z\chi_j\|_{L^2(0,1)}^2\right) f_j(y,v)dydv \\ &\quad + \sum_{j=1}^{\infty} \iint_{\Omega\times\mathbb{R}^3} V(x)|\chi_j(x)|^2 f_j(y,v)dxdv\end{aligned}$$

$\Rightarrow$ $(\mathbf{f}^*, \boldsymbol{\chi}^*)$ minimizes the energy under the mass constraint $\mathcal{M}(\mathbf{f}) = M$.

Vlasov-Schrödinger-Poisson model

We follow the setup of Ben Abdallah-Méhats 2004.

kinetic distribution sequence

$$\mathbf{f} = \{f_j\}_{j=1}^{\infty} \quad \text{with } f_j = f_j(y, v) : \omega \times \mathbb{R}^2 \to [0, \infty).$$

The kinetic admissible class $\mathcal{A}_{\text{c.m.}}$ is the collection of kinetic distribution sequences $\mathbf{f} \in \ell^1(\mathbb{N}; L^1(\omega \times \mathbb{R}^2))$ such that for all $j \in \mathbb{N}$ and all $(y, v) \in \omega \times \mathbb{R}^2$,

$$0 \leq f_j(y, v) \leq 1 \quad \text{(Pauli exclusion principle).}$$

quantum state sequence

$$\boldsymbol{\chi} = \{\chi_j\}_{j=1}^{\infty} \quad \text{with } \chi_j = \chi_j(y, z) : \omega \times (0, 1) \to \mathbb{C}.$$

The quantum admissible class $\mathcal{A}_{\text{q.m.}}$ is defined as the collection of quantum state sequences $\boldsymbol{\chi}$ such that for all $y \in \omega$, $\chi_j(y, \cdot) \in H_0^1(0, 1)$ and

$$\langle \chi_j(y, \cdot), \chi_k(y, \cdot) \rangle_{L^2(0,1)} = \delta_{jk} \text{ for all } j, k \in \mathbb{N} \quad \text{(partial orthonormality).}$$

A pair $(\mathbf{f}, \boldsymbol{\chi}) \in \mathcal{A}_{\text{c.m.}} \times \mathcal{A}_{\text{q.m.}}$ is called admissible.

Vlasov-Schrödinger-Poisson model

total density

$$\rho_{(\mathbf{f},\chi)}(x) := \sum_{j=1}^{\infty} \rho_{f_j}(y)|\chi_j(x)|^2 = \sum_{j=1}^{\infty} \left(\int_{\mathbb{R}^2} f_j(y,v)dv \right) |\chi_j(x)|^2$$

mass

$$\mathcal{M}(\mathbf{f}) = \sum_{j=1}^{\infty} \iint_{\omega\times\mathbb{R}^3} f_j dydv$$

energy

$$\mathcal{E}(\mathbf{f},\chi) = \sum_{j=1}^{\infty} \iint_{\omega\times\mathbb{R}^3} \left(\frac{|v|^2}{2} + \left\langle \left(-\frac{1}{2}\partial_z^2 + V_{\text{ext}}\right)\chi_j, \chi_j \right\rangle_{L^2((0,1))} \right) f_j dxdv$$
$$+\frac{1}{2}\int_{\Omega} |\nabla U_{\rho_{(\mathbf{f},\chi)}}(x)|^2 dx,$$

where and U_ρ solves the Poisson equation

$$\begin{cases} -\Delta U_\rho = \rho & \text{in } \Omega, \\ U_\rho = 0 & \text{on } \partial\omega \times (0,1), \\ \partial_z U_\rho = 0 & \text{on } \omega \times \{0,1\}. \end{cases}$$

$\Rightarrow$ The associated time evolution equation is the **Vlasov-Schrödinger-Poisson system**.

(1) A kinetic distribution sequence $\mathbf{f}(t) = \{f_j(t, y, v)\}_{j=1}^{\infty}$ obeys the 2D Vlasov equation

$$\begin{cases} \partial_t f_j + v \cdot \nabla_y f_j - \lambda_j(t, y) \cdot \nabla_v f_j = 0 & \text{on } (0, T) \times \omega \times \mathbb{R}^2, \\ f_j(0, y, v) = f_{j,0}(y, v) & \text{on } \omega \times \mathbb{R}^2, \\ f_j(t, y, v) = g_j(t, y, v) & \text{on } (0, T) \times \Sigma_-, \end{cases}$$

where the outgoing set is given by

$$\Sigma_- = \Big\{(y, v) \in \partial\omega \times \mathbb{R}^2 : v \cdot n_y < 0\Big\}$$

and n_y is the outgoing normal vector at $y \in \partial\omega$.

Vlasov-Schrödinger-Poisson model

(2) For each $(t,y)\in[0,T)\times\omega$, a partially orthonormal quantum state sequence $\boldsymbol{\chi}(t,y,\cdot)=\{\chi_j(t,y,\cdot)\}_{j=1}^{\infty}$ solves the quasistatic 1D Schrödinger equation with respect to the z-variable,

$$\begin{cases}\left(-\frac{\partial_z^2}{2}+(U+V_{\text{ext}})(t,y,\cdot)\right)\chi_j(t,y,\cdot)=\lambda_j(t,y)\chi_j(t,y,\cdot) & \text{on } (0,1),\\ \chi_j(t,y,z)=0 & \text{if } z=0,1,\end{cases}$$

where $\lambda_j(t,y)$ is the j-th eigenvalue of the operator $-\frac{\partial_z^2}{2}+(U+V_{\text{ext}})(t,y,\cdot)$.

(3) For each $t\in[0,T)$, the self-consistent electronic potential $U(t,\cdot)$ satisfies the Poisson equation

$$\begin{cases}-\Delta U=\rho & \text{in } \Omega,\\ U=0 & \text{on } \partial\omega\times(0,1),\\ \partial_z U=0 & \text{on } \omega\times\{0,1\}.\end{cases}$$

with $\rho=\rho_{(\mathbf{f}(t),\boldsymbol{\chi}(t))}$.

Ben Abdallah-Méhats 2004: Existence of a global solution.

Vlasov-Schrödinger-Poisson model: Main Result

Assume that $V_{\text{ext}} \in C(\overline{\Omega}) \cap C^1(\Omega)$ and $V_{\text{ext}} \geq 0$.

We consider the mass-constraint energy minimization problem

$$\mathcal{E}_{\min}(M) := \inf \Big\{ \mathcal{E}(\mathbf{f}, \chi) : (\mathbf{f}, \chi) \text{ is admissible and } \mathcal{M}(\mathbf{f}) = M \Big\}.$$

1D spectral theory

Given a potential function $U \in L^2(0,1)$, let

$$H[U] = -\tfrac{1}{2}\partial_z^2 + U(z)$$

be the Schrödinger operator acting on $L^2(0,1)$ with zero boundary condition.

- $H[U]$ has only countably many simple eigenvalues

$$\lambda_1[U] < \lambda_2[U] < \lambda_3[U] < \cdots < \lambda_j[U] < \cdots \to \infty.$$

- j-th $L^2(0,1)$-normalized eigenfunction is given by

$$\chi_j[U] \in H_0^1(0,1).$$

Vlasov-Schrödinger-Poisson model: Main Result

Theorem (Minimization of the free energy; H'-Jin, Preprint 2022)

1. *(Existence) The variational problem* $\mathcal{E}_{\min}(M)$ *has a minimizer* $(\mathbf{f}^*, \chi^*)$.
2. *(Uniqueness) The minimizer* $(\mathbf{f}^*, \chi^*)$ *is unique in the sense that if* $(\tilde{\mathbf{f}}^*, \tilde{\chi}^*)$ *is another minimizer, then* $U_{\rho_{(\mathbf{f}^*,\chi^*)}} = U_{\rho_{(\tilde{\mathbf{f}}^*,\tilde{\chi}^*)}}$.
3. *(Self-consistent equation) For some* $\mu > 0$,

$$f_j^*(y,v) = \mathbb{1}_{\left(\frac{|v|^2}{2}+\lambda_j^*(y)\leq\mu\right)}, \quad \lambda_j^*(y) = \lambda_j[(U_{\rho_{(\mathbf{f}^*,\chi^*)}} + V_{\text{ext}})(y,\cdot)]$$

   and $\chi_j^* = \chi_j^*[(U_{\rho_{(\mathbf{f}^*,\chi^*)}} + V_{\text{ext}})(y,\cdot)]$.
4. *(Monotone finite subband structure) For a.e.* $(y,v) \in \omega \times \mathbb{R}^2$, $f_j^*(y,v)$ *is strictly decreasing in* j *and* $f_j^*(y,v) \equiv 0$ *for* $j \geq \frac{\sqrt{3\mu}}{\pi}$.

Remarks

- Ben Abdallah-Méhats 2004: Existence of a stationary solution.
- Structural information.
- A larger class of steady states are constructed (free energy minimizers).

Vlasov-Schrödinger-Poisson model: Main Result

Theorem (Conditional dynamical stability; H'-Jin, Preprint 2022)

For $M > 0$, a unique minimizer $(\mathbf{f}^*, \chi^*)$ for the problem $\mathcal{E}_{\min}(M)$ constructed in the previous theorem is a stable solution to the Vlaosv-Schrödinger-Poisson system in the following sense: Given $\epsilon > 0$, there exists $\delta > 0$ such that the following hold. We assume that

1. $(\mathbf{f}_0, \chi_0) \in \mathcal{A}_{\text{c.m.}} \times \mathcal{A}_{\text{q.m.}}$, $|\mathcal{M}(\mathbf{f}_0) - M| \leq \delta$ *and* $|\mathcal{E}(\mathbf{f}_0, \chi_0) - \mathcal{E}(\mathbf{f}^*, \chi^*)| \leq \delta$.
2. $(\mathbf{f}(t), \chi(t))$ *is a unique global weak solution to the Vlasov-Schrödinger-Poisson system with initial data* $(\mathbf{f}_0, \chi_0)$, *and* $\mathrm{f}(t)$ *satisfies the the specular-reflection boundary condition. Moreover,* $\mathcal{M}(\mathbf{f}(t), \chi(t)) = \mathcal{M}(\mathbf{f}_0, \chi_0)$ *and* $\mathcal{E}(\mathbf{f}(t), \chi(t)) = \mathcal{E}(\mathbf{f}_0, \chi_0)$ *for all* $t \in \mathbb{R}$.

Then,

$$\sup_{t\in\mathbb{R}} \|\nabla(U_{\rho_{(\mathrm{f}(t),\chi(t))}} - U_{\rho_{(\mathrm{f}^*,\chi^*)}})\|_{L^2(\Omega)}^2 \leq \epsilon.$$

Remarks

◦ It is conditional, because well-posedness is not known for the VSP model.

◦ first stability result for the kinetic-quantum hybrid model.

Difficulties in construction of an energy minimizer

Sketch of the proof: existence of an energy minimizer

◦ By concentration-compactness principle.

Let $\{(\mathbf{f}^{(n)}, \chi^{(n)})\}_{n=1}^{\infty}$ be a minimizing sequence. Taking the $n \to \infty$ limit, we will obtain a minimizer.

◦ Some new ideas are needed.

Difficulties in construction of an energy minimizer

1. Lack of compactness with respect to the j-index

All j-summed quantities are invariant under translation in j.

2. Compactness of the quantum state part

Not obvious to obtain compactness for the quantum states $\{\boldsymbol{\chi}^{(n)}\}_{n=1}^{\infty}$.

Can we use the quantity

$$\sum_{j=1}^{\infty}\iint_{\omega\times\mathbb{R}^2}\|\partial_z\chi_j^{(n)}(y,\cdot)\|_{L^2(0,1)}^2 f_j^{(n)}(y,v)dydv?$$

- How to get a uniform bound for $\|\partial_z\chi_j^{(n)}(y,\cdot)\|_{L^2(0,1)}$.
- Even they are bounded, the choice of convergent sub-sequence depends on y.

Ideas to prove existence of a minimizer: 3-Step refinement

Step 1: Rearrangement

◦ The mass and the energy are invariant under the rearrangement by

$$\big(f_j^\sigma(y,v), \chi_j^\sigma(x)\big) = \big(f_{\sigma(j;y,v)}(y,v), \chi_{\sigma(j;y,v)}(x)\big) \quad \text{for each } (y,v) \in \omega \times \mathbb{R}^2.$$

⇒ Rearranging $(\mathbf{f}^{(n)}, \boldsymbol{\chi}^{(n)})$, we assume that $\|\partial_z \chi_j^{(n)}(y,\cdot)\|_{L^2(0,1)}$ is non-decreasing in j.

◦ By the min-max principle (+orthonormality), $\|\partial_z \chi_j^{(n)}(y,\cdot)\|_{L^2(0,1)} \geq \frac{\pi j}{\sqrt{3}}$.

$$\begin{aligned} \Rightarrow \mathcal{E}_{\min}(M) \leftarrow \mathcal{E}(\mathbf{f}^{(n)}, \boldsymbol{\chi}^{(n)}) &\geq \frac{1}{2}\sum_{j=1}^\infty \iint_{\omega\times\mathbb{R}^3} \|\partial_z \chi_j^{(n)}(y,\cdot)\|_{L^2(0,1)}^2 f_j(y,v)\,dxdv \\ &\geq \frac{\pi^2}{6}\sum_{j=1}^\infty j^2 \|f_j^{(n)}\|_{L^1(\omega\times\mathbb{R}^2)}. \end{aligned}$$

The rearranged minimizing sequence obeys the uniform weighted summation bound

$$\sup_{n\geq 1}\left\{\sum_{j=1}^\infty j^2 \|f_j^{(n)}\|_{L^1(\omega\times\mathbb{R}^2)}\right\} \leq \frac{6}{\pi^2}\mathcal{E}_{\min}(M) + o_n(1).$$

Step 2: Partial minimization with fixed quantum states

Fix a quantum state $\chi^{(n)}$ in a minimizing sequence in Step 1, and minimize the energy.

$$\mathcal{E}_{\min}(M;\chi^{(n)}) := \inf\Big\{\mathcal{E}(\mathbf{f},\chi^{(n)}) : \mathbf{f} \text{ is admissible and } \mathcal{M}(\mathbf{f}) = M\Big\}.$$

◦ By standard concentration-compactness principle, we prove existence of a minimizer.

Replace $\mathbf{f}^{(n)}$ by the above minimizer $\Rightarrow$ refined minimizing sequence $(\mathbf{f}^{(n)},\chi^{(n)})$

◦ Each $\mathbf{f}^{(n)}$ solves the self-consistent equation (or Euler-Lagrange equation)

$$f_j^{(n)} = \mathbb{1}_{\left(\frac{|v|^2}{2}+h_j^{(n)}(y)\le\mu^{(n)}\right)}$$

where $h_j^{(n)}(y) := \langle(-\frac{\partial_z^2}{2} + U_{\rho_{(\mathbf{f}^{(n)},\chi^{(n)})}} + V_{\text{ext}})\chi_j^{(n)}, \chi_j^{(n)}\rangle_{L^2(0,1)}$.

Step 2: Partial minimization with fixed quantum states (continued)

∘ Rearranging, we may assume that $h_j^{(n)}(y)$ is non-increasing $\Rightarrow h_j^{(n)}(y) \geq \frac{\pi j}{\sqrt{3}}$.

∘ $\mu^{(n)}$ is bounded uniformly in n.

$$\Rightarrow \begin{cases} \bullet \text{ (monotonicity) } f_j^{(n)}(y,v) \text{ is non-increasing in } j \text{ for each } (y,v) \\ \bullet \text{ (finite subbands) } f_j^{(n)} \equiv 0 \text{ for all } j \geq J+1 \text{ (} J \text{ is independent of } n\text{).} \end{cases}$$

Step 3: Partial minimization with fixed kinetic distributions

Fix $\mathbf{f}^{(n)}$ in a minimizing sequence in Step 2, and minimize the energy.

$$\mathcal{E}_{\min}(M;\mathbf{f}^{(n)}) := \inf\left\{\mathcal{E}(\mathbf{f}^{(n)},\boldsymbol{\chi}) : \boldsymbol{\chi} \text{ is admissible}\right\}$$

Theorem (Ben Abdallah-Méhats 2004)

For $q > \frac{4}{3}$, if $V_{\text{ext}} \in L^{q'}(\omega; L^\infty(0,1))$, $\|\{\rho_{f_j}\}_{j=1}^\infty\|_{\ell^1(\mathbb{N};L^q(\omega))} < \infty$ and $\{\rho_{f_j}(y)\}_{j=1}^\infty$ is non-increasing for each $y \in \omega$, then the Schrödinger-Poisson equation

$$\begin{cases} -\Delta V = \rho_{(\mathrm{f},\boldsymbol{\chi}[V+V_{\text{ext}}])} & \text{in } \Omega, \\ V = 0 & \text{on } \partial\omega \times (0,1), \\ \partial_z V = 0 & \text{on } \omega \times \{0,1\}, \end{cases}$$

where $\boldsymbol{\chi}[V+V_{\text{ext}}] = \{\chi_j[V+V_{\text{ext}}]\}_{j=1}^\infty$, has a unique solution $U^* \in H^1(\omega)$.

$\Rightarrow$ For $\mathbf{f}^{(n)}$, we obtain $V^{(n)} \rightsquigarrow \boldsymbol{\chi}^{(n)} = \{\chi_j[V^{(n)} + V_{\text{ext}}]\}_{j=1}^\infty$.

We claim that $\boldsymbol{\chi}^{(n)}$ minimizes the energy $\mathcal{E}(\mathbf{f}^{(n)},\boldsymbol{\chi})$.

Ideas to prove existence of a minimizer: 3-Step refinement

Step 3: Partial minimization with fixed kinetic distributions (continued)

Lemma (Coercivity of the free energy)

Suppose that f *is non-increasing and has only finitely many non-zero subbands and* $(\mathrm{f}, \boldsymbol{\chi}^*) \in \mathcal{A}_{\text{c.m.}} \times \mathcal{A}_{\text{q.m.}}$ *with*

$$\boldsymbol{\chi}^* = \boldsymbol{\chi}[U^* + V_{\text{ext}}] \text{ and } \mathcal{E}(\mathrm{f}, \boldsymbol{\chi}^*) < \infty,$$

where U^* *is the solution to the Schrodinger-Poisson equation. Then, for any* $(\tilde{\mathrm{f}}, \tilde{\boldsymbol{\chi}}) \in \mathcal{A}^{\downarrow}_{\text{c.m.}} \times \mathcal{A}_{\text{q.m.}}$ *with* $\mathcal{M}(\tilde{\mathrm{f}}) = \mathcal{M}(\mathrm{f})$*, we have*

$$\mathcal{E}(\tilde{\mathrm{f}}, \tilde{\boldsymbol{\chi}}) - \mathcal{E}(\mathrm{f}, \boldsymbol{\chi}^*) \geq \frac{1}{2}\|\nabla(U_{\rho_{(\tilde{\mathrm{f}}, \tilde{\boldsymbol{\chi}})}} - U_{\rho_{(\mathrm{f}, \boldsymbol{\chi}^*)}})\|^2_{L^2(\Omega)}.$$

$\Rightarrow$ refined minimizing sequence $\{(\mathbf{f}^{(n)}, \boldsymbol{\chi}^{(n)})\}_{n=1}^{\infty}$.

Remark By refinements, we obtain a minimizing sequence such that
(1) $f_j^{(n)} \equiv 0$ for all $j \geq J+1$ and $f_j^{(n)}(y, v)$ is non-increasing in j.
(2) $\chi_j^{(n)}$ is the j-th function of $-\frac{1}{2}\partial_z^2 + U_{\rho_{(\mathbf{f}^{(n)}, \boldsymbol{\chi}^{(n)})}} + V_{\text{ext}}$.

Ideas to prove existence of a minimizer: 3-Step refinement

Last step: $n \to \infty$

Lemma (Stability for 1D Schrödinger operator)

If $U, V \in L^2(0,1)$, then there exists $C > 0$, independent of U, V and j, such that

$$|\lambda_j[U] - \lambda_j[V]| + \|\chi_j[U] - \chi_j[V]\|_{L^\infty(0,1)} \le Ce^{C(\|U\|_{L^2(0,1)} + \|V\|_{L^2(0,1)})}\|U - V\|_{L^1(0,1)}.$$

elliptic regularity (Poisson equation) $\Rightarrow \{U_{\rho_{(\mathbf{f}^{(n)}, \boldsymbol{\chi}^{(n)})}}\}_{n=1}^\infty$ is uniformly bounded and equi-continuous in Ω. $\Rightarrow$ So is $\{\chi_j^{(n)}(x)\}_{n=1}^\infty$ in Ω (only finite j's are meaningful).

∘ By Arzelà-Ascoli, $\chi_j^{(n)}(x) \to \chi_j^*$ in $C(\Omega)$ passing to a subsequence.

∘ $f_j^{(n)} \rightharpoonup f_j^*$ in $L^2(\omega \times \mathbb{R}^2)$ and $\rho_{f_j^{(n)}} \rightharpoonup \rho_{f_j^*}$ in $L^{\frac{6}{5}}(\omega)$ passing to a subsequence.

Then, we can show that $(\mathbf{f}^*, \boldsymbol{\chi}^*)$ is a desired minimizer (by lower semi-continuity).

- coercivity estimate $\Rightarrow$ uniqueness and stability.

Thank you for your attention!