Pattern Formation in Nematic Liquid Crystals

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Background of nematic liquid crystals

Nematic liquid crystals

Liquid crystal is a state of matter between liquid and solid crystal.

Nematic liquid crystals material consists of rod–like molecules. The orientational order of the molecules is retained when the liquid crystals flow.

Patterns in liquid crystals can be found when light passes through a region occupied by liquid crystals where the molecules align in some special ways.

Liquid crystals have been widely applied to the industry of display devices, photonic devices and biological sensors.

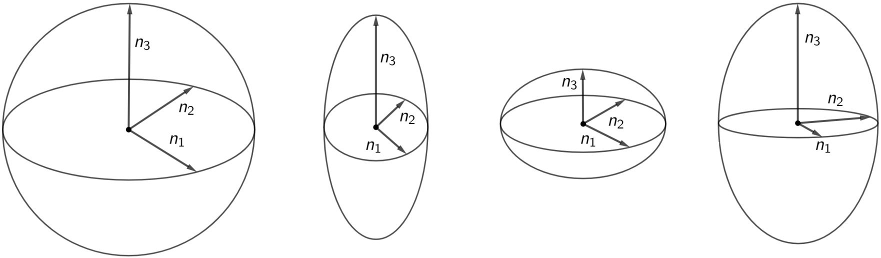
A research focus of liquid crystals : the study of its orientation.

Phases of nematic liquid crystals

Liquid crystals are optically anisotropic.

Liquid crystals refract light in different degree (refractive index) when it comes from different directions.

We use a triaxial ellipsoid to illustrate the shape of the crystalline in different phases.



$n_1 = n_2 = n_3$	$n_1 = n_2 < n_3$	$n_1 = n_2 > n_3$	$n_1 < n_2 < n_3$
isotropic	uniaxial (positive)	uniaxial (negative)	biaxial

Landau–de Gennes theory

Let

- $\mathbb{S}_0 :=$ 5–dimensional vector space consisting of real 3×3 symmetric traceless matrices;
- $B_R(x) :=$ open ball in $\mathbb{R}^3$ with center x and radius R; $\quad B_R = B_R(0)$.

The Landau–de Gennes theory (also known as the $\mathscr{Q}$–tensor theory) uses the tensor $\mathscr{Q}(x) : B_R \longrightarrow \mathbb{S}_0$ as an order parameter to describe the orientation of the equilibrium state of liquid crystals.

The normalized eigenvectors of $\mathscr{Q}(x)$ give the three principal axes of the liquid crystal at x.

The corresponding eigenvalues measure the degree of orientation.

In LdG theory, different phases of nematic liquid crystals can be characterized by quantitative relationships of the eigenvalues of $\mathscr{Q}$.

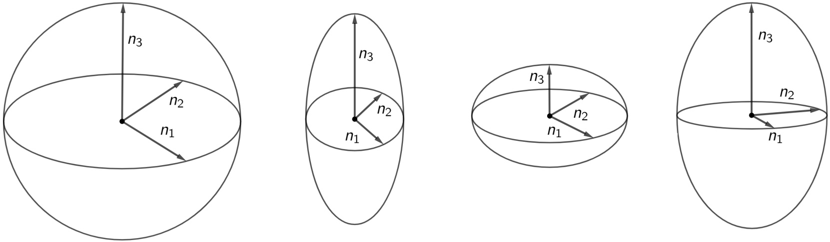
Let λ_1, λ_2 and λ_3 be the eigenvalues of $\mathscr{Q}(x)$. The liquid crystals are

(1). isotropic at x if $\lambda_1 = \lambda_2 = \lambda_3 = 0$;

(2). positively uniaxial at x if $\lambda_1 = \lambda_2 < \lambda_3$;

(3). negatively uniaxial at x if $\lambda_1 = \lambda_2 > \lambda_3$;

(4). biaxial at x if $\lambda_1 < \lambda_2 < \lambda_3$;



$n_1 = n_2 = n_3$ isotropic | $n_1 = n_2 < n_3$ uniaxial (positive) | $n_1 = n_2 > n_3$ uniaxial (negative) | $n_1 < n_2 < n_3$ biaxial

The director field $n(x)$ determined by $\mathscr{Q}(x)$ is defined to be the normalized eigenvector associated with the largest eigenvalue.

It represents the preferred orientation of the molecule at a given point.

Defects (or disclinations) are locations where the director field $n(x)$ of the molecules is discontinuous.

Note that even if $\mathscr{Q}(x)$ is continuous, its director field $n(x)$ may be discontinuous.

Patterns in liquid crystals will be found around the defects.

For an order parameter $\mathscr{Q}(x) : B_R \longrightarrow \mathbb{S}_0$, its Landau–de Gennes energy functional in the one–constant limit is:

$$\int_{B_R} \underbrace{\frac{1}{2}\left|\nabla \mathscr{Q}\right|^2}_{\text{elastic energy}} \underbrace{-\frac{a^2}{2}\left|\mathscr{Q}\right|^2 - \sqrt{6}\,\mathrm{tr}\left(\mathscr{Q}^3\right) + \frac{1}{2}\left|\mathscr{Q}\right|^4}_{\text{bulk energy}} .$$

Elastic energy : It measures the distortions of molecules.

Bulk energy : It determines the preferred phases.

The quantity $-a^2$ is the reduced temperature.

The remaining two coefficients are determined by the material.

The Euler–Lagrange equation :

$$-\Delta \mathscr{Q} = a^2 \mathscr{Q} + 3\sqrt{6}\left(\mathscr{Q}^2 - \frac{1}{3}|\mathscr{Q}|^2 \mathrm{I}_3\right) - 2|\mathscr{Q}|^2 \mathscr{Q} \quad \text{in } B_R. \tag{1}$$

In the spherical droplet problem, equation (1) is coupled with the strong anchoring hedgehog boundary condition:

$$\mathscr{Q} = \frac{\sqrt{3}}{2}\, a\, H_a \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\mathrm{I}_3\right) \text{ on } \partial B_R \text{ , where } H_a := \frac{3 + \sqrt{9 + 8a^2}}{2\sqrt{2}\, a}. \tag{2}$$

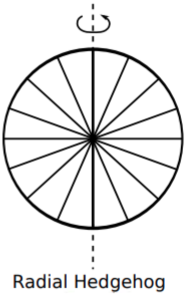
There are three fundamental defect structures of the solutions to (1) and (2).

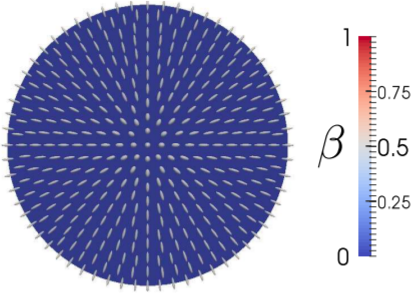
Fundamental defects (or disclinations) of the equilibrium solutions

(1). Radial hedgehog :

- it was first discovered by Schopohl and Suckin in 1988;
- the molecules are uniaxial everywhere in B_R, except the origin where the isotropic structure is found;
- the director field equals $\frac{x}{|x|}$.

Numerical simulation by Hu-Qu-Zhang 2016:



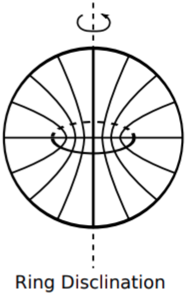


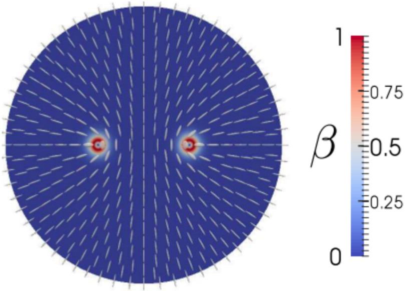
(2). Half–degree ring disclination :

For large a, it is found that radial hedgehog is not stable.

It was first discovered by Penzenstadler and Trebin in 1989 numerically.

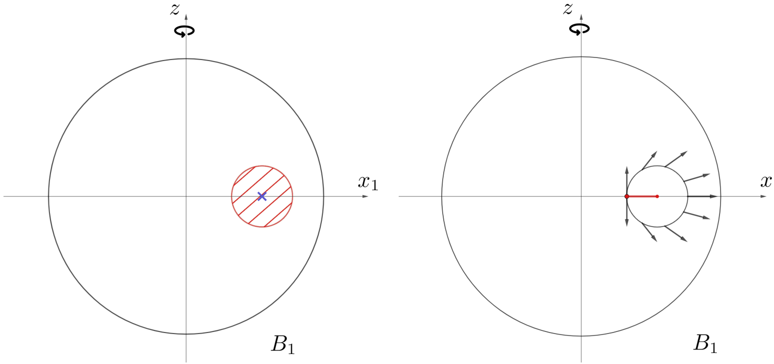
Numerical simulation by Hu-Qu-Zhang in 2016:





We can imagine that the isotropic core in the radial hedgehog broadens to the ring. (rmk 0 is no longer iso.)

- the molecules are not isotropic at the origin and are uniaxial on the ring;
- the molecules are biaxial away from and near the ring;
- the director field is undefined on the ring;
- the angle of the director field is changed by π when it travels around the ring.



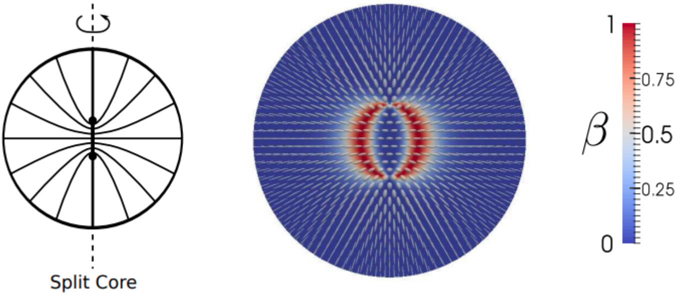
Red shaded region : biaxial
Blue cross : uniaxial (the ring)

Arrow : director field
Red line : director field not defined

(3). Split–core disclination :

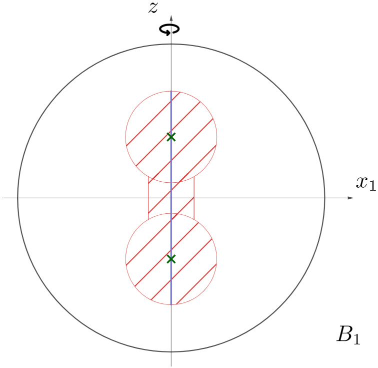
It was first discovered by Gartland and Mkaddem in 2000 numerically.

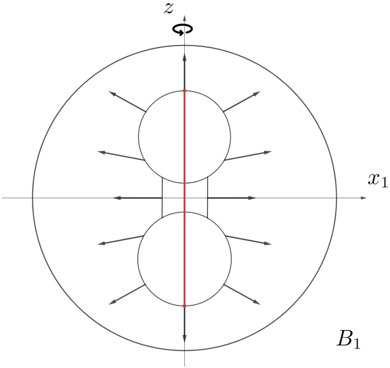
Numerical simulation by Hu-Qu-Zhang in 2016:



We can imagine that the isotropic core in the radial hedgehog is split into the core.(rmk 0 is no longer iso.)

- the molecules are uniaxial on a open line segment of z–axis;
- isotropic structure is found on the line segment;
- the molecules are biaxial for the points away from and near the line segment.
- the director field is undefined on the line segment.





Red shaded region : biaxial (the core)
Blue line : uniaxial
Green cross : isotropic

Arrow : director field
Red line : director field not defined

A rigorous proof of the existence of solutions with the half-degree ring disclination and the split-core disclination have been open for more than 20 years.

The ultimate objective :

Construct two solutions to equation (1):

$$-\Delta \mathscr{Q} = a^2 \mathscr{Q} + 3\sqrt{6}\left(\mathscr{Q}^2 - \frac{1}{3}|\mathscr{Q}|^2 \mathrm{I}_3\right) - 2|\mathscr{Q}|^2 \mathscr{Q} \quad \text{in } B_R,$$

with the hedgehog boundary condition:

$$\mathscr{Q} = \frac{\sqrt{3}}{2}\, a H_a \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3}\mathrm{I}_3\right) \text{ on } \partial B_R.$$

such that they admit half–degree ring and split–core disclinations respectively, for large enough $a > 0$.

Key and difficulty : To control the eigenvalues of $\mathscr{Q}$.

Procedure:

(1). transform the $\mathcal{Q}$–tensor equation to a 5–vector equation (E–L equation of some energy $\mathcal{E}_a$) in B_1 with unknown $w \in \mathbb{R}^5$ (with some symmetries in w);

(2). impose the Signorini problems to the energy $\mathcal{E}_a$ (minimization problems with thin obstacle) with obstacle condition on $T := \overline{B_1} \cap \{z = 0\}$. Denote the solutions to these problems by w_a;

(3). study the uniform convergence of w_a on T (as a tends to ∞ along some sequence), by using energy decay estimate;

(4). determine the sign of $w_{a,3}(0)$ by using (3) and the properties of the limiting map of w_a (up to a subsequence);

(5a). half–degree ring disclination: use the sign condition in (4) to control the eigenvalues and director field of $\mathcal{Q}$ (determined by w_a);

(5b). split–core disclination: use the sign condition in (4) and the asymptotic behavior of w_a near its zeros to control the eigenvalues and director field of $\mathcal{Q}$.

The ideas of (1) and (2) come from the numerical study in the work "Fine structure of defects in radial nematic droplets" by Gartland and Mkaddem in 2000.

They found the the third component (after the transformation from $\mathscr{Q}$ to w) of the half-degree ring solution and the split-core disclination solution are positive and negative at the origin, respectively.

(1). transform the $\mathscr{Q}$–tensor equation to a 5–vector equation (E–L equation of some energy $\mathcal{E}_a$) with unknown $w \in \mathbb{R}^5$ (with some symmetries in w);

(2). impose the Signorini problems to the energy $\mathcal{E}_a$ (minimization problems with thin obstacle) with obstacle on $T := \overline{B_1} \cap \{z = 0\}$. Denote the solutions to these problems by w_a;

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Let $x = (x_1, x_2, z)$ = rectangular coordinates, (ρ, θ, z) = cylindrical coordinates.

Let $\{L_j\}_{j=1}^5$ be the orthonormal basis spanning $\mathbb{S}_0$ and $u(\rho, z) \in \mathbb{R}^3$, we set

$$w(x) = \mathscr{L}[u(\rho, z)] := (u_1 \cos 2\theta, u_1 \sin 2\theta, u_2, u_3 \cos\theta, u_3 \sin\theta);$$

$$\begin{aligned} \mathscr{Q}(x) &= \frac{a}{\sqrt{2}}\Big\{u_1\left(\tfrac{x}{R}\right)\left(\cos 2\theta\, L_5 + \sin 2\theta\, L_2\right) + u_2\left(\tfrac{x}{R}\right) L_4 + u_3\left(\tfrac{x}{R}\right)\left(\cos\theta\, L_1 + \sin\theta\, L_3\right)\Big\} \\ &= \frac{a}{\sqrt{2}}\Big(w_1 L_5 + w_2 L_2 + w_3 L_4 + w_4 L_1 + w_5 L_3\Big)\Big|_{\frac{x}{R}}. \end{aligned} \tag{3}$$

Equation (1) : $-\Delta \mathscr{Q} = a^2 \mathscr{Q} + 3\sqrt{6}\left(\mathscr{Q}^2 - \frac{1}{3}|\mathscr{Q}|^2 \mathrm{I}_3\right) - 2|\mathscr{Q}|^2\mathscr{Q}$ in B_R

is transformed into

$$-\Delta w = \frac{3\mu}{\sqrt{2}} \nabla_w S[w] - a\mu\left(|w|^2 - 1\right) w \quad \text{and} \quad w = \mathscr{L}[u] \quad \text{in } B_1. \tag{4}$$

$\mu := aR^2 > 0$ is a fixed positive constant and $S : \mathbb{R}^5 \to \mathbb{R}$ is a degree 3 homogeneous polynomial.

Equation (4) is the Euler–Lagrange equation of

$$\mathcal{E}_a[w] := \int_{B_1} f_{a,\mu}(w) = \int_{B_1} \left|\nabla w\right|^2 + \mu\left[D_a - 3\sqrt{2}S[w] + \frac{a}{2}\left(|w|^2 - 1\right)^2\right]. \quad (5)$$

Under the ansatz in (3), the boundary condition (2) is equivalent to

$$w = \mathscr{L}[U_a^*] \qquad \text{on } \partial B_1. \quad (6)$$

Here U_a^* is the corresponding map depending only on the polar angle ϕ.

(1). transform the $\mathscr{Q}$–tensor equation to a 5–vector equation (E–L equation of some energy $\mathcal{E}_a$) with unknown $w \in \mathbb{R}^5$ (with some symmetries in w);

(2). impose the Signorini problems to the energy $\mathcal{E}_a$ (minimization problems with thin obstacle) with obstacle on $T := \overline{B_1} \cap \{z = 0\}$. Denote the solutions to these problems by w_a;

(3). study the uniform convergence of w_a on T (as a tends to ∞ along some sequence), by using energy decay estimate;

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Configuration space for the minimization problems

Definition 2.1

A 5–vector w is $\mathscr{R}$–axially symmetric on some ball B_r if

(1). $w = \mathscr{L}\left[u(\rho, z)\right]$ *on* B_r *for some* $u \in \mathbb{R}^3$*;*

(2). u_1 *and* u_2 *are even w.r.t.* z*–variable;*

(3). u_3 *is odd w.r.t.* z*–variable.*

Let the space

$$\mathscr{F}_a^s := \left\{ w \in H^1(B_1; \mathbb{R}^5) : w \text{ is } \mathscr{R}\text{–axially symmetric and } w \text{ satisfies (6)} \right\}.$$

Vectorial Signorini problem

To construct the solutions with the two desired disclinations, we hope that the third component of the solutions to the 5–vector equation (4) satisfying some sign conditions at the origin.

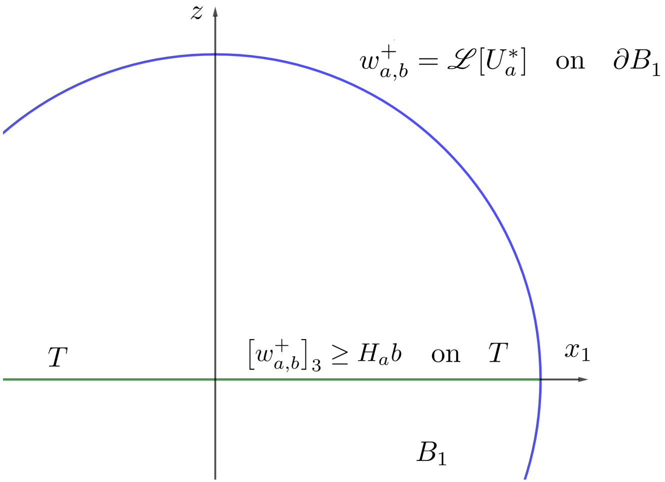
For any $b \in (-1, -1/2]$ and $c \in [-1/2, 1)$, we let

$$\mathscr{F}^{+}_{a,b} := \left\{ w \in \mathscr{F}^{s}_{a} : w_3 \geq H_a b \ \text{ on } \ T \right\}, \quad \mathscr{F}^{-}_{a,c} := \left\{ w \in \mathscr{F}^{s}_{a} : w_3 \leq H_a c \ \text{ on } \ T \right\}.$$

There are minimizers $w^{+}_{a,b}$ and $w^{-}_{a,c}$ solving the problems

$$\mathrm{Min}\left\{ \mathcal{E}_a[w] : w \in \mathscr{F}^{+}_{a,b} \right\} \qquad \text{and} \qquad \mathrm{Min}\left\{ \mathcal{E}_a[w] : w \in \mathscr{F}^{-}_{a,c} \right\}$$

respectively.



$w^+_{a,b}$ is $\mathscr{R}$–axially symmetric.

Limiting map of the minimizers

Let $a_n \longrightarrow \infty$, it can be proved that

$$w^+_{a_n,b} \longrightarrow w^+_b \quad \text{and} \quad w^-_{a_n,c} \longrightarrow w^-_c \ \text{ strongly in } H^1(B_1), \text{ up to a subsequence.}$$

The limiting maps w^+_b, $w^-_c \in H^1(B_1;\mathbb{S}^4)$ satisfy the obstacle conditions $w^+_{b,3} \geq b$ and $w^-_{c,3} \leq c$ on T respectively.

Their properties (by Yu 2020):

(1). Let $b \in (-1,-1/2]$ and $c \in [-1/2,1) \setminus \{0\}$. Except at finitely many points on z–axis (excluding the origin, north and south poles), w^+_b and w^-_c are smooth in $\overline{B_1}$.

(2). Item (1) & $\mathscr{R}$–symmetry $\Longrightarrow$

$w^+_{b,j}$ and $w^-_{c,j}$ are zero on z–axis except at their respective singularities, for $j = 1, 2, 4, 5$ ($w = \mathscr{L}[u(\rho, z)] = (u_1 \cos 2\theta, u_1 \sin 2\theta, u_2, u_3 \cos\theta, u_3 \sin\theta)$);

(3). $|w^+_b| = |w^-_c| = 1$ & item (3) & the obstacle conditions on $T \Longrightarrow$

$w^+_{b,3}(0) = 1$ and $w^-_{c,3}(0) = -1$.

Uniform convergence of $w^+_{a_n,b}$ to w^+_b on $T \Longrightarrow$ the sign of $\left[w^+_{a,b}(0)\right]_3$.

Trouble : $w^+_{a,b}$ does not solve the equation across T (a consequence of the presence of the obstacle condition on T).

(1). transform the $\mathscr{Q}$–tensor equation to a 5–vector equation (E–L equation of some energy $\mathcal{E}_a$) with unknown $w \in \mathbb{R}^5$ (with some symmetries in w);

(2). impose the Signorini problems to the energy $\mathcal{E}_a$ (minimization problems with thin obstacle) with obstacle on $T := \overline{B_1} \cap \{z = 0\}$. Denote the solutions to these problems by w_a;

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We obtain the uniform Hölder norm of $w_{a,b}^+$ by energy decay estimate.

Proposition 3.1 (Energy decay estimate, Tai-Yu 2021)

Fix $b \in (-1, -1/2]$. There exist three positive constants a_0, ϵ_0 and ν_0 with $\nu_0 \in (0, 1/2)$, such that for any $a > a_0$, if it satisfies

$$\mathcal{E}_{a;r}\left[w_{a,b}^+\right] := \frac{1}{r}\int_{B_r} f_a\left(w_{a,b}^+\right) < \epsilon_0, \tag{7}$$

then either one of the followings holds:

$$(1).\ \mathcal{E}_{a;\nu_0 r}\left[w_{a,b}^+\right] \le r^{3/2}; \qquad (2).\ \mathcal{E}_{a;\nu_0 r}\left[w_{a,b}^+\right] \le \frac{1}{2}\mathcal{E}_{a;r}\left[w_{a,b}^+\right].$$

The constants a_0, ϵ_0 and ν_0 depend on the parameter b.

Similar decay estimate holds for balls away from the origin.

By standard iteration and covering arguments, the energy decay estimates show that for any $r_0 \in (0,1)$ and any $x, y \in \overline{B_{r_0}} \cap T$, we have

$$\left| w^+_{a,b}(x) - w^+_{a,b}(y) \right| \lesssim_{r_0,b} |x-y|^{\alpha}.$$

Remark 1 : The proof of the decay estimates is done by blow-up arguments and the construction of energy comparison maps.

Remark 2 : In the proof, we let sequences $a_n \to \infty$ and $r_n \to 0$. The comparison maps are different when $a_n r_n^2$ tends to zero, positive constant and infinity. The constructions are tricky in these three cases.

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$w^{+}_{b,3}(0) = 1$ and $w^{-}_{c,3}(0) = -1$ & uniform convergence on $T \Longrightarrow$

$w^{+}_{a,b,3}(0) > 0$ and $w^{-}_{a,c,3}(0) < 0$

(1). transform the $\mathscr{Q}$–tensor equation to a 5–vector equation (E–L equation of some energy $\mathcal{E}_a$) with unknown $w \in \mathbb{R}^5$ (with some symmetries in w);

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Half–degree ring disclination is axially–symmetric, we discuss on (ρ, z)–plane.

Also denote $T := \Big\{(\rho, z) \in \mathbb{R}^2 : \rho \in [0, 1] \text{ and } z = 0\Big\}$.

(1). find $x_a = (\rho_a, 0) \in T$ such that $\mathscr{Q}$ is uniaxial (2 equal eigenvalues) at $x = x_a$;
(2). show $\mathscr{Q}$ is biaxial (3 distinct eigenvalues) around $x = x_a$;
(3). establish the π–angle change of the director field around $x = x_a$.

Consider $\mathscr{Q} = \mathscr{Q}^+_{a,b}$ with $w = w^+_{a,b} = \mathscr{L}[u_a]$ (see (3)) and let $\mathscr{Q}^* = a^{-1}\mathscr{Q}^+_{a,b}(Rx)$.

$$\mathscr{Q}^+_{a,b}(x) = \frac{a}{\sqrt{2}}\Big\{u_{a,1}\left(\tfrac{x}{R}\right)\left(\cos 2\theta\, L_5 + \sin 2\theta\, L_2\right) + u_{a,2}\left(\tfrac{x}{R}\right)\, L_4 + u_{a,3}\left(\tfrac{x}{R}\right)\left(\cos\theta\, L_1 + \sin\theta\, L_3\right)\Big\}$$

$$= \frac{a}{\sqrt{2}}\left(w^+_{a,b,1}\, L_5 + w^+_{a,b,2}\, L_2 + w^+_{a,b,3}\, L_4 + w^+_{a,b,4}\, L_1 + w^+_{a,b,5}\, L_3\right)\Bigg|_{\frac{x}{R}}.$$

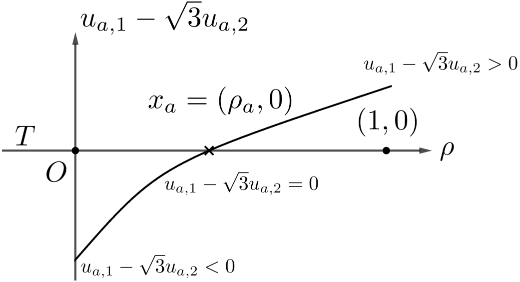
Let $w_{a,b}^{+}(x) = \mathscr{L}[u_a](\rho, z)$, we compute the eigenvalues of $\mathscr{Q}^*$:

$$\begin{cases} \lambda_1 = \dfrac{-1}{2}\left(u_{a,1} + \dfrac{1}{\sqrt{3}} u_{a,2}\right); \\ \lambda_2 = \dfrac{1}{4}\left(u_{a,1} + \dfrac{1}{\sqrt{3}} u_{a,2}\right) - \dfrac{1}{4}\sqrt{\left(u_{a,1} - \sqrt{3}\, u_{a,2}\right)^2 + 4\left(u_{a,3}\right)^2}; \\ \lambda_3 = \dfrac{1}{4}\left(u_{a,1} + \dfrac{1}{\sqrt{3}} u_{a,2}\right) + \dfrac{1}{4}\sqrt{\left(u_{a,1} - \sqrt{3}\, u_{a,2}\right)^2 + 4\left(u_{a,3}\right)^2}. \end{cases}$$

$\mathscr{Q}^*$ is uniaxial on T (recall $u_{a,3} = 0$ on T) if $u_{a,1} - \sqrt{3}u_{a,2} = 0$ ($\lambda_2 = \lambda_3$).

(1). Two equal eigenvalues (uniaxial)

Fix b close enough to -1, we consider large enough a.



At $(\rho, z) = (1, 0)$, $u_{a,1} - \sqrt{3}u_{a,2} > 0$ (by boundary value of $w^+_{a,b} = \mathscr{L}[U^*_a]$).

At $(\rho, z) = (0, 0)$, $u_{a,1} - \sqrt{3}u_{a,2} = -\sqrt{3}u_{a,2} < 0$ (by uni. conv. of u_a and the sign condition of its limiting map).

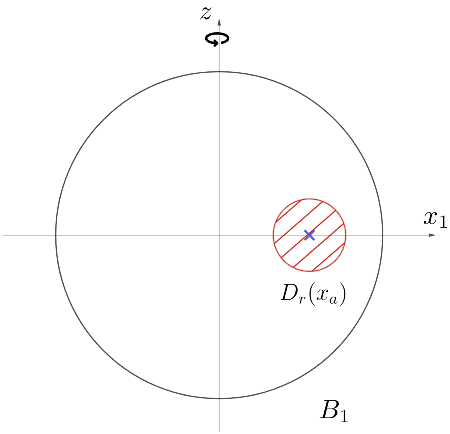
By continuity of u_a, there is $x_a := (\rho_a, 0) \in T \setminus \{0, \partial T\}$ such that $u_{a,1}(x_a) - \sqrt{3}u_{a,2}(x_a) = 0$ (uniaxial, $\lambda_3 = \lambda_2$).

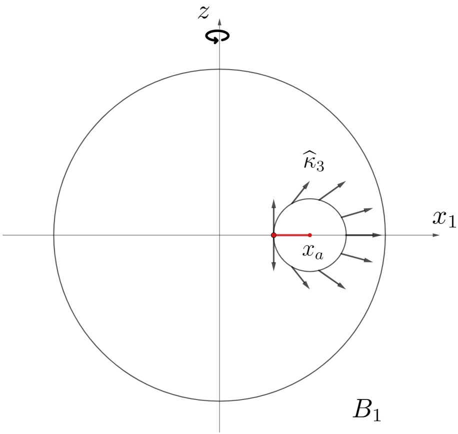
The biaxial property follows similarly.

Near x_a, the director field = the normalized eigenvector corresponding to λ_3 (the largest one)

$$\widehat{\kappa}_3 = \frac{\sqrt{2}}{2}\left(1 + \frac{u_{a,1} - \sqrt{3}\,u_{a,2}}{\sqrt{\left(u_{a,1} - \sqrt{3}\,u_{a,2}\right)^2 + 4\left(u_{a,3}\right)^2}}\right)^{1/2} e_\rho + \frac{\sqrt{2}u_{a,3}}{\sqrt{\left(u_{a,1} - \sqrt{3}\,u_{a,2}\right)^2 + 4\left(u_{a,3}\right)^2}}\left(1 + \frac{u_{a,1} - \sqrt{3}\,u_{a,2}}{\sqrt{\left(u_{a,1} - \sqrt{3}\,u_{a,2}\right)^2 + 4\left(u_{a,3}\right)^2}}\right)^{-1/2} e_z \tag{8}$$

The director field $\widehat{\kappa}_3$ is undefined at $x = x_a$.





Red shaded region : biaxial
Blue cross : uniaxial (the ring)

Arrow : director field
Red line : director field not defined

(1). transform the $\mathscr{Q}$–tensor equation to a 5–vector equation (E–L equation of some energy $\mathcal{E}_a$) with unknown $w \in \mathbb{R}^5$ (with some symmetries in w);

(2). impose the Signorini problems to the energy $\mathcal{E}_a$ (minimization problems with thin obstacle) with obstacle on $T := \overline{B_1} \cap \{z = 0\}$. Denote the solutions to these problems by w_a;

(3). study the uniform convergence of w_a on T (as a tends to ∞ along some sequence), by using energy decay estimate;

(4). determine the sign of $w_{a,3}(0)$ by using (3) and the properties of the limiting map of w_a (up to a subsequence);

(5a). half–degree ring disclination: use the sign condition in (4) to control the eigenvalues and director field of $\mathscr{Q}$ (given by w_a);

(5b). split–core disclination: use the sign condition in (4) and the asymptotic behavior of w_a near its zeros to control the eigenvalues and director field of $\mathscr{Q}$.

Split–core disclination is also axially–symmetric about z–axis, we discuss on (x_1, z)–plane instead of $\mathbb{R}^3$.

Fix c close to 1. For large enough a, we let $\mathscr{Q} = \mathscr{Q}^-_{a,c}$ with $w = w^-_{a,c} = \mathscr{L}[u^-_{a,c}]$. Set $\mathscr{Q}^* = a^{-1}\mathscr{Q}^-_{a,c}(Rx)$.

(1). find 2 points (zero points of $w^-_{a,c}$) on z–axis such that $\mathscr{Q}^*$ is isotropic;

$\left[u^-_{a,c}\right]_j = 0$ on z–axis for $j = 1, 3$;
$\left[u^-_{a,c}\right]_2 > 0$ at the north pole (by boundary value of $u^-_{a,c}$);
$\left[u^-_{a,c}\right]_2 < 0$ at the origin (by uni. conv. of $u^-_{a,c}$ and the sign condition of its limiting map).

(2). show that $\mathscr{Q}^*$ is uniaxial on the open line segment joining that two points;

$\left[u^-_{a,c}\right]_j = 0$ on z–axis for $j = 1, 3$.

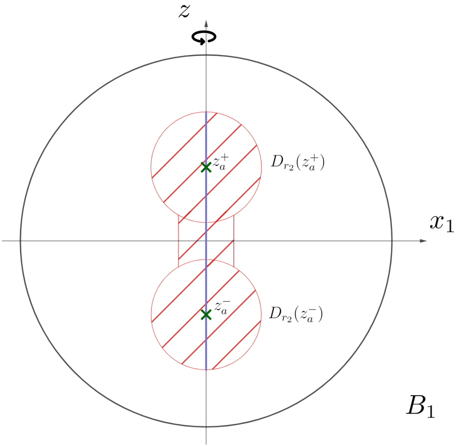
(3). show that $\mathscr{Q}^*$ is biaxial away from and near the line.

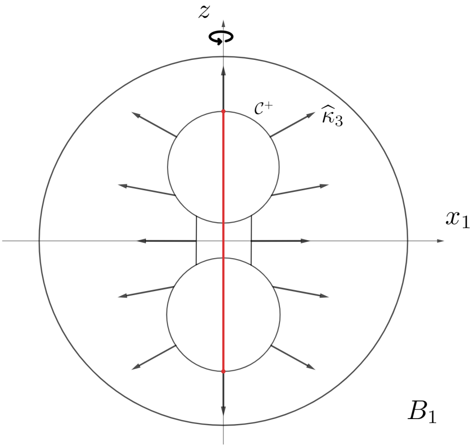
Suppose z_a^+ is the lowest zero of $u_{a,c}^-$ on the positive z–axis and (r, ψ, θ) are the spherical coordinates with respect to center z_a^+.

(3) (biaxiality) requires the asymptotic behavior:

$$\lim_{(a^{-1},r)\to(0,0)} \sum_{j=0}^{2} \left\| \partial_\psi^j \Big(\Pi_{\mathbb{S}^2}[u_{a,c}^-(r,\psi)] \Big) - \partial_\psi^j (0, \cos\psi, \sin\psi)^\top \right\|_{\infty;[0,\pi]} = 0.$$

Using the limit, $\mathscr{Q}$ is biaxial near and away from the line segment. $(\lambda_3 > \lambda_1 > \lambda_2)$





Red shaded region : biaxial (the core)
Blue line : uniaxial
Green cross : isotropic

Arrow : director field
Red line : director field not defined

Some results about the minimizers of Ginzburg–Landau type functional

Goal: establish the limit :

$$\lim_{(a^{-1},r)\to(0,0)} \sum_{j=0}^{2} r^j \left\| \nabla^j \left(\Pi_{\mathbb{S}^4}[w_{a,c}^-] \right) - \nabla^j \Lambda(\cdot - z_a) \right\|_{\infty;\partial B_r(z_a)} = 0.$$

Here z_a is a zero of $w_{a,c}^-$ and Λ is the tangent map of w_c^- at its singularity. It equals either

$$\Lambda_+(\zeta) = \frac{1}{|\zeta|}\left(0, 0, \zeta_3, \zeta_1, \zeta_2\right) \quad \text{or} \quad \Lambda_-(\zeta) = \frac{1}{|\zeta|}\left(0, 0, -\zeta_3, \zeta_1, \zeta_2\right).$$

Tangent map of w_c^- at singularity $x = s_k := \lim_{r_n \to 0} w_c^-(s_k + r_n\xi) = \Lambda(\zeta)$

By Yu 2020, it shows that $\lim_{r\to 0} \sum_{j=0}^{2} r^j \left\| \nabla^j w_c^- - \nabla^j \Lambda(\cdot - s_k) \right\|_{\infty;\partial B_r(s_k)} = 0.$

It seems that there is no literature discussing the convergence of the Ginzburg–Landau minimizers to its limiting harmonic map near the singularity.

$$\lim_{(a^{-1},r)\to(0,0)} \sum_{j=0}^{2} r^j \left\| \nabla^j \left(\Pi_{\mathbb{S}^4}[w_{a,c}^-] \right) - \nabla^j \Lambda(\cdot - z_a) \right\|_{\infty;\partial B_r(z_a)} = 0.$$

Remark 1 : The limit exists and equals 0 as (a^{-1}, r) approaching $(0, 0)$ along any path.

Remark 2 : The convergence result in the limit is up to the singularity of w_c^-. (z_a converges to the singularity of w_c^- up to a subsequence)

Compare with [Bethuel, Brezis and Orlandi 2001] and [Rivière 1999].

They only provide the convergence results of the Ginzburg-Landau minimizer away from the singularities of its limiting map (the limiting map is obtained up to a subsequence).

The limit is proved by contradiction.

Let $a_n \to \infty$, $r_n \to 0$ and z_n be a zero of $w^-_{a_n,c}$ and consider the rescaled map $w^{(n)}(\zeta) := w^-_{a_n,c}(z_n + r_n\zeta)$.

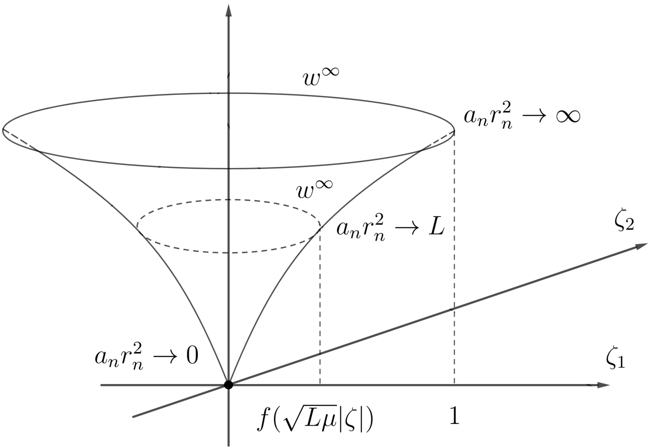
Exterior core $(a_n r_n^2 \to \infty)$: $w^{(n)} \longrightarrow w^\infty = \Lambda$ strongly in $H^1_{\text{loc}}(\mathbb{R}^3)$.

Interior core $(a_n r_n^2 \to L$ for some $L \geq 0)$: $w^{(n)} \longrightarrow w^\infty$ in $C^2_{\text{loc}}(\mathbb{R}^3)$.

$w^\infty = f(\sqrt{L\mu}|\zeta|)\Lambda$. Here $f(r) : [0,\infty) \longrightarrow [0,1)$ is smooth with $f(0) = 0$, $f(\infty) = 1$ and $f'(r) > 0$ for all $r \in [0,\infty)$.

Let $\zeta = (\zeta_1, \zeta_2, 0) \in \mathbb{S}^2 \cap T$, we can think that $\Lambda = (0, 0, 0, \zeta_1, \zeta_2)$ is also on $\mathbb{S}^2 \cap T$.

The limiting map w^∞ can be concluded by: (vertical axis representing the limiting of $a_n r_n^2$)



Main result

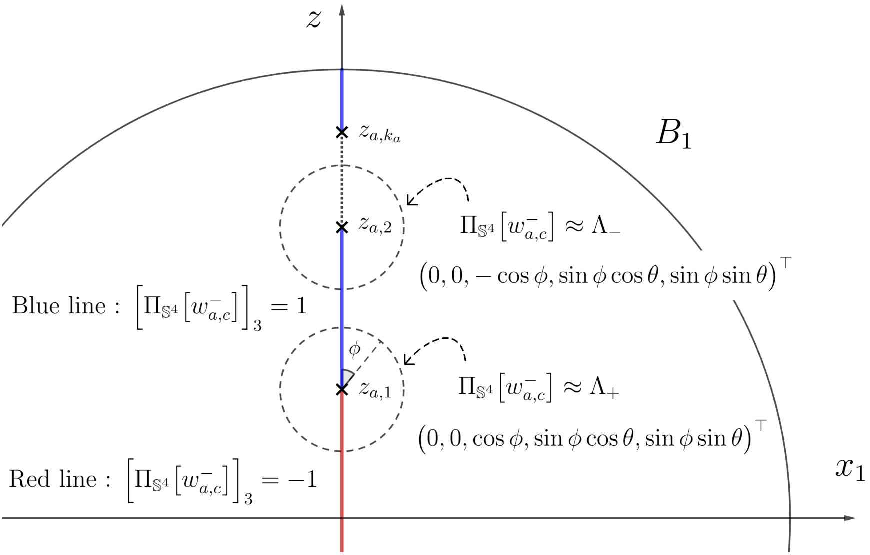
Proposition 6.1 (Tai-Yu 2021)

Let the zeros of $w^-_{a,c}$ on the positive z–axis be $z_{a,1}$, ..., z_{a,k_a}, where k_a is the total number of zeros.

It satisfies

$$\lim_{(a^{-1},r)\to(0,0)} \max_{k=1,\dots,k_a} \sum_{j=0}^{2} r^j \left\| \nabla^j \Pi_{\mathbb{S}^4}\left[w^-_{a,c}\right] - \nabla^j \left[\Lambda_k\left(\cdot - z_{a,k}\right)\right] \right\|_{\infty;\partial B_r(z_{a,k})} = 0.$$

Moreover, if these zeros are ordered so that the third coordinate of $z_{a,k}$ is increasing as j runs from 1 to k_a, then $\Lambda_k = \Lambda_+$ if k is odd, $\Lambda_k = \Lambda_-$ if k is even.



z
z_{a,k_a}
B_1
$z_{a,2}$
$\Pi_{\mathbb{S}^4}\left[w_{a,c}^-\right] \approx \Lambda_-$
$(0, 0, -\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta)^\top$
Blue line : $\left[\Pi_{\mathbb{S}^4}\left[w_{a,c}^-\right]\right]_3 = 1$
ϕ
$z_{a,1}$
$\Pi_{\mathbb{S}^4}\left[w_{a,c}^-\right] \approx \Lambda_+$
$(0, 0, \cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta)^\top$
x_1
Red line : $\left[\Pi_{\mathbb{S}^4}\left[w_{a,c}^-\right]\right]_3 = -1$

Thanks for Your Attention !