Regular solutions to L_p Minkowski problem

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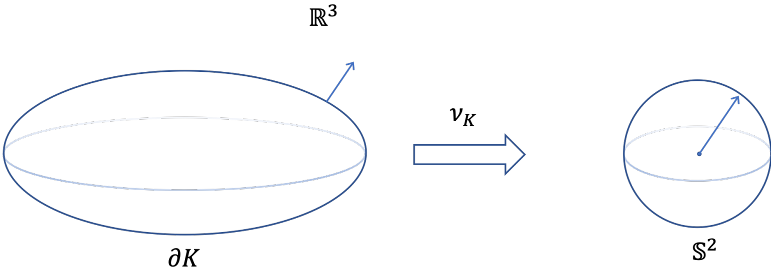
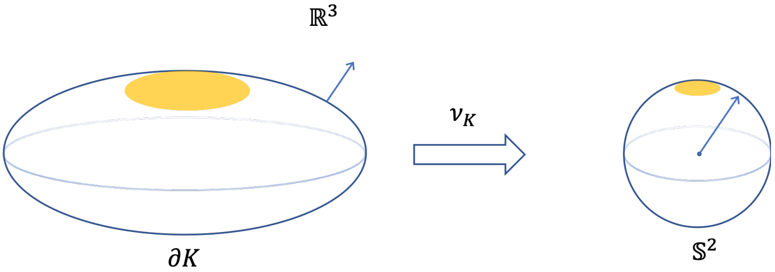
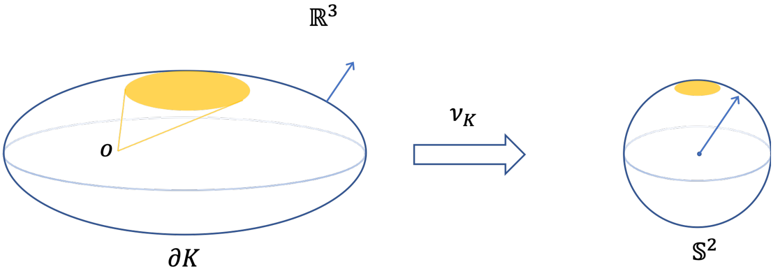
joint work with Kyeongsu Choi and Minhyun Kim



- $K \mapsto \mu_K$



- $K \mapsto \mu_K$
- For example, surface area



- $K \mapsto \mu_K$
- For example, surface area or cone volume.
- Can we characterize geometric measures μ_K?
  (Equivalently, what is the image of the mapping $K \mapsto \mu_K$)

Surface area measure

- Let K be a convex body in $\mathbb{R}^{n+1}$ (compact convex set with nonempty interior), and let $\nu_K : \partial K \to \mathbb{S}^n$ be the outward unit normal vector.
- Any convex body defines the so called *surface area measure* on $\mathbb{S}^n$: The surface area measure $S(K, \cdot)$ of K is defined on a Borel set $\omega \subset \mathbb{S}^n$ by

  $$S(K, \omega) = |\nu_K^{-1}(\omega)|,$$

  where $|\cdot|$ denotes the surface area.
- Total measure: $S(K, \mathbb{S}^n) = |\nu_K^{-1}(\mathbb{S}^n)| = |\partial K|$.

- Observation: if μ is a surface area measure, then
  1. Surface area measure has centroid at origin:

$$\int_{\mathbb{S}^n} z \, \mathrm{d}\mu(z) = \int_{\partial K} \nu(x) \mathrm{d}\mathcal{H}^n(x) = o.$$

  2. Surface area measure is not concentrated on a great subsphere:

$$\mu(E) \neq \mu(\mathbb{S}^n) \quad \text{for all great subsphere } E \subset \mathbb{S}^n.$$

- Can we characterize the surface area measure?
- **Minkowski problem:** What are the necessary and sufficient conditions on a nonzero finite Borel measure μ on $\mathbb{S}^n$ to be $\mu = S(K, \cdot)$ of a convex body K? (Minkowski, 1903)
- Minkowski problem is completely solved by Minkowski (discrete case) and Alexandrov (general case).
- $\mu = S(K, \cdot)$ for a convex body $K \iff$ 1. and 2. hold for μ.

- Uniqueness? The convex body is unique up to translation.
- Brunn–Minkowski inequality: If K and L are convex bodies, then $V^{1/n}$ is concave, i.e., for $0 \le t \le 1$,

$$V^{1/n}(tK + (1-t)L) \ge tV^{1/n}(K) + (1-t)V^{1/n}(L).$$

  Moreover, equality holds if and only if K and L are homothetic.
- If $S(K,\cdot) = S(L,\cdot)$ for convex bodies K and L, then

$$m(t) = V(tK + (1-t)L)^{1/n}$$

  is concave, which implies $m(1) \ge m(0)$. Similarly, $m(1) \le m(0)$ and so $V(K) = V(L)$. By the equality condition, $K = L$ (up to translation).

- Regularity?
- In smooth category (measure with density),

$$\mathrm{d}S(K,\cdot) = \frac{1}{\mathcal{K}}\mathrm{d}\sigma_{\mathbb{S}^n}, \quad \mu = f\mathrm{d}\sigma_{\mathbb{S}^n},$$

  and thus the Minkowski problem becomes solving the following Monge–Ampère type PDE on $\mathbb{S}^n$:

$$\det(\nabla_i\nabla_j h + h\delta_{ij}) = \frac{1}{\mathcal{K}} = f \quad \text{on } \mathbb{S}^n,$$

  where $\mathcal{K}$ is the Gauss curvature and h is the support function of K.
- In terms of local graph function u,

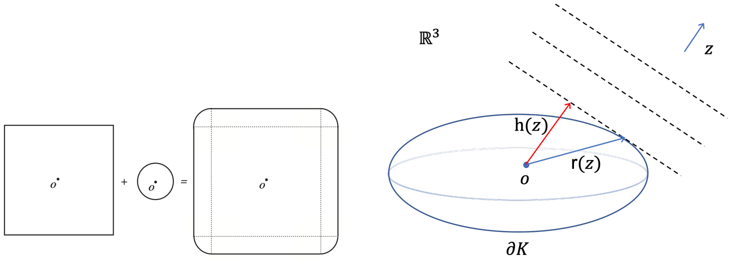
$$\frac{\det(D^2u)}{(1+|Du|^2)^{\frac{1}{n+2}}} = f.$$

- If $f \in C^\infty$, then $\partial K \in C^\infty$. (C^∞ regularity by Pogorelov, Nirenberg, Cheng–Yau, and $C^{2,\alpha}$ regularity by Caffarelli)
- Summary: the surface area measures are characterized by 1. and 2. In which case, the solution convex body is well understood.

Variational point of view

- $\mathrm{Vol}(K + tL)$
- Let $K + L = \{x + y : x \in K, y \in L\}$ be the Minkowski sum, and let $h_L : \mathbb{S}^n \to \mathbb{R}$ be the support function of L defined by

$$h_L(z) = \max\{z \cdot x : x \in L\}.$$



- Aleksandrov variational formula:

$$\left.\frac{\mathrm{d}\,\mathrm{Vol}(K + tL)}{\mathrm{d}t}\right|_{t=0^+} = \int_{\mathbb{S}^n} h_L(z)\,\mathrm{d}S(K, z)$$

L_p surface area measure

- Firey's p-linear combination $K +_p L$ of K and L ($p \geq 1$):

$$h_{K+_pL} = (h_K^p + h_L^p)^{1/p}, \quad h_{t\cdot_p L} = t^{1/p} h_K$$

- There exists a Borel measure $S_p(K, \cdot)$ on $\mathbb{S}^n$ such that

$$\left.\frac{\mathrm{d}\,\mathrm{Vol}(K +_p t \cdot_p L)}{\mathrm{d}t}\right|_{t=0^+} = \frac{1}{p}\int_{\mathbb{S}^{n-1}} h_L^p(z)\,\mathrm{d}S_p(K, z).$$

- The measure $S_p(K, \cdot)$ is called as the L_p surface area measure.
- It turns out that for $p \geq 1$,

$$\mathrm{d}S_p(K, \cdot) = h_K^{1-p}\mathrm{d}S(K, \cdot).$$

- The L_p surface area measure can be defined for all $p \in \mathbb{R}$ through the relation above.

L_p Minkowski problem

- L_p **Minkowski problem:** What are the necessary and sufficient conditions on a nonzero finite Borel measure μ on $\mathbb{S}^n$ to be $\mu = S_p(K, \cdot)$ of a convex body K? (Lutwak '93)
- PDE: for a density function f,

$$\det(\nabla_i \nabla_j h + h\delta_{ij}) = \frac{1}{\mathcal{K}} = h^{p-1} f \quad \text{on } \mathbb{S}^n.$$

- Examples: classical case ($p = 1$), logarithmic case ($p = 0$), affine case ($p = -n - 1$).
- Soliton of (anisotropic) α-Gauss curvature flow through the relation $\alpha = 1/(1 - p)$.
- If $p \neq 1$, the location of the origin is important!

Logarithmic Minkowski problem ($p = 0$)

- In particular, $p = 0$, corresponds to the logarithmic Minkowski problem. This is related to the cone volume:

$$\frac{1}{n+1} \mathrm{d}S_0(K, \cdot) = \frac{1}{n+1} h_K \mathrm{d}S(K, \cdot), \quad \frac{1}{n+1} S_0(K, \mathbb{S}^n) = \mathrm{Vol}(K)$$

- In 2013, Böröczky–Lutwak–Yang–Zhang solved the logarithmic case under even assumption ($\mu(E) = \mu(-E)$):

$$\mu = S_0(K, \cdot) \iff 1. \quad \frac{\mu(\xi \cap \mathbb{S}^n)}{\mu(\mathbb{S}^n)} \leq \frac{\dim(\xi)}{n+1}, \quad \xi \leq \mathbb{R}^{n+1}$$

2. some extra condition when equality holds

- Non-symmetric case is open.
- For other $p \neq 0, 1$, some sufficient conditions have been provided, but the L_p Minkowski problem is still open for symmetric or non-symmetric, except for the lower dimensional case ($n = 1$).

L_p Minkowski problem (measure with density)

- Existence of solutions is guaranteed for sufficiently smooth, positive f. Regularity?
- Recall the PDE: for a density function f,

$$\det(\nabla_i \nabla_j h + h\delta_{ij}) = \frac{1}{\mathcal{K}} = h^{p-1} f \quad \text{on } \mathbb{S}^n$$

- C^0 estimate or diameter estimate is important.
  Blaschke selection theorem (compactness): Let $\{K_n\}$ be a sequence of convex bodies contained in fixed bounded set. Then there is convex set K such that (up to subsequence)

$$K_i \to K \quad \text{in Hausdorff distance.}$$

- Positive lower bound on h is crucial for regularity. (whether the origin lies in the interior or not)

Overview for various range of p

- $p > n+1$: Smooth unique solution.
  At the maximum point of h, it follows from the PDE that

$$h_{\max}^{1-p+n} \geq f_{\min}, \quad h_{\max} \leq \frac{1}{f_{\min}^{1/(p-n-1)}}, \quad h_{\min} \geq \frac{1}{f_{\max}^{1/(p-n-1)}}.$$

- $p \leq -n+1$: No diameter estimate. The origin lies in the interior. Smooth solutions.
- $-n+1 < p < n+1$ $(p \neq 1)$: Example of a convex body with the origin on its boundary. Weak solution. Regularity for even case. Logarithmic case in $\mathbb{R}^3 (p = 0,\ n = 2)$.

Known examples with origin on the boundary

- When $-n+1 < p < n+1$ $(p \neq 1)$, the example of convex body with the origin on its boundary is given by (locally)

$$g(x) = (|x|-1)_+^q, \quad |x| < 2,$$

where

$$q = \frac{p+n}{p+n-1}.$$

- If $p > -n+2$, then $q < 2$. Not $C^{1,1}$ class (unbounded curvature).
- If $p \leq -n+2$, then $q \geq 2$. At least $C^{1,1}$ class (bounded curvature).
- **For $-n+1 < p \leq -n+2$, can we find a regular ($C^{1,1}$) solution for any smooth positive f?**

Result. Existence of regular solution ($p = 0$, $n = 2$)

Theorem (Choi–Kim–L.)

Let $f > 0$ be a function in $C^2(\mathbb{S}^2)$. Then there exists a solution Σ to the logarithmic Minkowski problem such that Σ is a closed convex hypersurface of class $C^{1,1}$ with bounded mean curvature.

- This is the optimal regularity.
- The proof is based on curvature flow approach.

Idea of proof

1. Observe that the equation for the logarithmic Minkowski problem is

$$\frac{h}{\mathcal{K}} = f \iff 0 = h - f\mathcal{K}.$$

2. Consider the following normalized anisotropic Gauss curvature flow

$$X_t = X - f(\nu)\mathcal{K}\nu \quad (h_t = h - f\mathcal{K})$$

3. (**Goal**) Convergence: $h_t \to 0$ (subsequencially) with bounded curvatures.

Idea of proof (continued)

4. Entropy:

$$E(\Omega) = \sup_{z \in \Omega} E(\Omega, z)$$

where ($h_z = \langle X - z, \nu \rangle$)

$$E(\Omega, z) = \int_{\mathbb{S}^n} (\log h_z) f.$$

If Ω_t is a solution to the flow, then $E(\Omega_t)$ decreases in time and is nonnegative.
5. Prove diameter estimate $|X| \leq R$ and existence of inner ball $B(z_0, \rho)$.
6. Principal curvature estimate $0 < \lambda_1 \leq \lambda_2 \leq C$: apply the maximum principle to the following quantities

$$\frac{f\mathcal{K}}{h_{z_0} - \rho/2}, \quad \frac{f\lambda_2}{2R^2 - |X|^2}$$

Idea of proof (continued)

7. (For example) For $F = f\mathcal{K}$,

$$\partial_\tau F = \mathcal{L}F - 2F + F^2 H.$$

For $P = \frac{f\mathcal{K}}{h_{z_0} - \rho/2}$,

$$\partial_\tau P = \mathcal{L}P + \langle \nabla P, \cdots \rangle + Q,$$

where $Q \simeq C - H$. Note that

$$Q \leq 0 \quad \text{if} \quad \mathcal{K} \gg 1.$$

Thank you!