Singularities in the Keller-Segel system

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1 - The Keller-Segel system

The Keller-Segel equation

The Keller-Segel equation [Patlak '53], [Keller-Segel '70], [Nanjundiah '73]:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^d. \qquad \textbf{(KS)}$$

Modeling features:

- Describing the *chemotaxis* in biology, [Hillen-Painter '09]; interacting stochastic many-particles system, [Othmer-Stevens '90], [Stevens '00]), [Chavanis '08], [Hillen-Painter '08]; as a diffusion limit of a kinetic model [Chalub-Markowich-Perthame-Schmeiser '04], model of stellar dynamics under friction and fluctuations [Wolansky '92];
- Competition between diffusion of cells and aggregation;
- Rich model from mathematical point of view, [Horstman '03 & '04]. A general aggregation model is of the form

$$\partial_t u = \Delta A(u) - \nabla \cdot (B(u) \nabla \mathcal{K} * u).$$

The Keller-Segel equation:

$$\partial_t u = \Delta u + u^2 - \nabla u \cdot \nabla \Phi_u = \nabla \cdot \left(u \nabla (\ln u - \Phi_u) \right),$$

$$\Phi_u = \mathcal{K} * u, \quad \mathcal{K}(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & \text{for} \quad d = 2, \\ c_d |x|^{2-d} & \text{for} \quad d \geq 3. \end{cases}$$

- mass conservation: $M = \int_{\mathbb{R}^2} u_0(x) dx = \int_{\mathbb{R}^2} u(x,t) dx$;
- scaling invariance: $\forall \gamma > 0, \ \ u_\gamma(x,t) = \frac{1}{\gamma^2} u\left(\frac{x}{\gamma}, \frac{t}{\gamma^2}\right), \quad \|u_\gamma\|_{L^{\frac{d}{2}}} = \|u\|_{L^{\frac{d}{2}}}$

$$(L^1\text{-critical if } d = 2, \ L^1\text{-supercritical if } d \geq 3)$$

- free energy functional: $\mathcal{F}(u) = \int_{\mathbb{R}^d} u \left(\ln u - \frac{1}{2} \Phi_u \right), \quad \frac{d}{dt} \mathcal{F}(u) \leq 0$;
- stationary solution for $d = 2$: $Q_{\gamma,a}(x) = \frac{1}{\gamma^2} Q\left(\frac{x-a}{\gamma}\right)$, where

$$Q(x) = \frac{8}{(1 + |x|^2)^2}, \quad \int_{\mathbb{R}^2} Q = 8\pi.$$

Diffusion vs. Aggregation in 2D

$$\partial_t u = \nabla \cdot (\nabla u - u \nabla \Phi_u)$$

- If $M < 8\pi$: global existence + spreading, [Blanchet-Dolbeault-Perthame '06]. The proof mainly relies on the free energy functional $\mathcal{F}(u)$ and the Log HLS inequality.
- If $M = 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 u < +\infty$: **blowup in infinite time**, [Blanchet-Carrillo-Masmoudi '08]. Constructive approaches by [Ghoul-Masmoudi '18] (radial), [Davila-del Pino-Dolbeault-Musso-Wei '20] (nonradial):

$$\|u(t)\|_{L^\infty} \sim c_0 \log t \quad \text{as} \quad t \to +\infty.$$

- If $M > 8\pi$: **blowup in finite time**, [Childress-Percus '81], [Jager-Luckhaus '92], [Nagai-Senba '98], [Senba-Suzuki '03]:

$$\text{(virial identity)} \quad \frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)dx = \frac{M}{2\pi}(8\pi - M).$$

  Constructive approaches in the radial setting by [Herrero-Velázquez '96], [Raphaël-Schweyer '14]:

$$\|u(t)\|_{L^\infty} \sim C_0 \frac{e^{\sqrt{2|\log(T-t)|}}}{T-t} \quad \text{as} \quad t \to T.$$

Diffusion vs. Aggregation in high dimensions $d \geq 3$

- A critical threshold in $L^{\frac{d}{2}}$ for global existence [Calvez-Corrias-Ebde '12]

$$\|u(0)\|_{L^{\frac{d}{2}}} < \frac{8}{d} C_{GN}^{-2(1+2/d)}\big(d/2, d\big) \quad \Longrightarrow \quad \text{global existence.}$$

- Existence of self-similar (type I) blowup solutions by [Herrero-Medina-Velázquez '98]:

$$u(x,t) = \frac{1}{T-t}\varphi\left(\frac{x}{\sqrt{T-t}}\right), \quad \|u\|_{L^1} = \infty.$$

Asymptotic description of φ by [Giga-Mizoguchi-Senba '11].

- A formal derivation of non self-similar (type II) blowup solutions in the radial setting by [Herrero-Medina-Velázquez '97], [Brenner-Constantin-Kadanoff-Schenkel-Venkataramani '99]:

$$u(x,t) \sim \frac{1}{\lambda^2(t)} w\left(\frac{|x| - R(t)}{\lambda(t)}\right), \quad R(t) \sim (T-t)^{\frac{1}{d}},\ \lambda \sim R^{d-1},\ \|u\|_{L^1} < \infty.$$

- Blowup solution can be exhibited with any arbitrary mass by the scaling invariance.

Finite time blowup in the 2D Keller-Segel system

A numerical simulation of blowup for the 2D Keller-Segel system

$$\partial_t u = \Delta u - \nabla.(u\nabla\Phi_u), \quad -\Delta\Phi_u = u.$$

The spacial structure of blowup and self-similarity solution

Consider an evolution equation

$$u_t = F[u],$$

and assume that the blowup occurs at a single point (a, T). A **similarity solution** is of the form

$$u(x,t) = \frac{1}{(T-t)^\alpha} P\left(\frac{x-a}{(T-t)^\beta}\right)$$

with appropriately chosen values of α, β. Here, P is the **similarity profile** that solves

$$-\alpha P - \beta \xi \cdot \nabla P(\xi) + F[P] = 0,$$

where $F[P]$ is exactly the same expression as for $F[u]$.

- *Self-similarity of the 1st kind*: a solution P only exists for a particular pair of (α, β).
- *Self-similarity of the 2nd kind*: a solution P exists for a family of exponents (α, β) determined by extra conditions such as regularity.

An example of a self-similarity of the 2nd kind

Consider the 1D problem

$$\frac{\partial u}{\partial t} = u^2(x,t),$$

and look for a similarity blowup solution

$$u(x,t) = \frac{1}{T-t} P(\xi), \quad \xi = \frac{x-a}{(T-t)^\beta}$$

where P satisfies

$$-P - \beta \xi P_\xi + P^2 = 0, \quad P(\xi) = \frac{1}{1 + c\xi^{1/\beta}}.$$

We impose a regularity condition that P is regular for all $\xi \in \mathbb{R}$. This requires that $1/\beta$ must be a positive even integer and $c > 0$, otherwise, P develops a pole and becomes infinite,

$$\beta_i = \frac{1}{2(i+1)}, \quad i = 0, 1, \cdots$$

There are a discretely infinite series of similarity solutions of the 2nd kind

$$P_i(\xi) = \frac{1}{1 + c\xi^{2(i+1)}}, \quad c > 0.$$

Constructive approach

- Aim: **Construction** and **stability** of blowup solutions.
  - The constructed solution is asymptotically self-similarity of the 2nd kind. This is mostly the consequence of the existence of a scaling invariance, parabolic regularization and energy dissipation.
  - The stability is stated for a well-prepared class of initial data leading to blowup solutions that satisfies a prescribed asymptotic behavior.
- Constructive approach:
  - Part 1: *Constructing a good approximate solution* which formally yields the blowup rate and the blowup profile.
  - Part 2: *Reduction of the linearized problem to a finite dimensional one*, meaning that we perform certain type of estimates to control the remainder and show that it doesn't affect the leading blowup dynamics.
  - Part 3: *Solving the finite dimensional problem* (if necessary). This specifies the choice of initial data.

2.1. Single blowup in $\mathbb{R}^2$ (L^1-critical)

$$\begin{cases} \partial_t u = \Delta u + u^2 - \nabla u \cdot \nabla \Phi_u, \\ 0 = \Delta \Phi_u + u, \end{cases} \quad \text{in } \mathbb{R}^2. \qquad \textbf{(2DKS)}$$

■ Previous literature: Most of the results had been obtained at the formal level (numerical observation, formal matching asymptotic expansions) or in the radial setting to remove the nonlocal structure difficulty, i.e.

$$m_u(r,t) = \int_0^r u(\zeta, t)\zeta d\zeta, \quad \partial_t m_u = \partial_r^2 m_u - \frac{\partial_r m_u}{r} + \frac{m_u \partial_r m_u}{r}.$$

■ Objective: Developing a new framework to construct blowup solutions that is feasible to archive a classification of blowup dynamics, extension to dispersive equations.

Blowup for the 2DKS

$$\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)$$

Theorem 1 ([Collot-Ghoul-Masmoudi-Ng., '22).

- There exists a set $\mathcal{O} \subset L^1 \cap \mathcal{E}$, where $\mathcal{E} = \{u : \sum_{k=0}^{2} \|\langle x\rangle^k \nabla^k u\|_{L^2} < +\infty\}$, of initial data u_0 (not necessary radially symmetric) such that

$$u(x,t) = \frac{1}{\lambda^2(t)}\left[Q\left(\frac{x-a(t)}{\lambda(t)}\right) + \varepsilon(x,t)\right],$$

where $a(t) \to \bar{a} \in \mathbb{R}^2$ and $\sum_{k=0}^{2} \|\langle x\rangle^k \nabla^k \varepsilon(t)\|_{L^2} \to 0$ as $t \to T$, and λ is given by either

$$\lambda(t) \sim 2e^{-\frac{\gamma+2}{2}}\sqrt{T-t}\exp\left(-\frac{1}{\sqrt{2}}\sqrt{|\log(T-t)|}\right), \qquad \textbf{(C1)}$$

or

$$\lambda(t) \sim c(u_0)(T-t)^{\frac{\ell}{2}}|\log(T-t)|^{-\frac{\ell}{2(\ell-1)}}, \quad \ell \geq 2 \text{ integer}. \qquad \textbf{(C2)}$$

- Case (**C1**) is stable and Case (**C2**) is $(\ell-1)$-codimension stable.

Comments

$$\partial_t u = \nabla \cdot (\nabla u - u\nabla\Phi_u)$$

■ Existing results: formal level (numerical observation, formal matching asymptotic expansions) and in the radial setting to remove the nonlocal structure difficulty, i.e. $u(x,t) = u(r,t)$,

$$m(r) = \int_0^r u(\zeta)\zeta d\zeta, \quad \Longrightarrow \quad \boxed{\partial_t m = \partial_r^2 m - \frac{\partial_r m}{r} + \frac{\partial_r m^2}{2r}}$$

Refs: [Herrero-Velázquez '96 & '97], [Velázquez '02], [Schweyer-Raphael '14], [Dyachenko-Lushnikov-Vladimirova '13], ...

■ The new framework: nonradial setting, refined description of the stable blowup mechanism, new (unstable) blowup dynamics, a nature approach via spectral analysis/ energy-type method.

■ Perspectives: extension to dispersive equations, **classification** of the flow near the stationary state, ...

Renormalization and blowup profile

- Self-similar variables:

$$u(x,t) = \frac{1}{T-t} w(z,\tau), \quad z = \frac{x}{\sqrt{T-t}}, \quad \frac{d\tau}{dt} = \frac{1}{T-t},$$

$$\partial_\tau w = \nabla \cdot (\nabla w - w \nabla \Phi_w) - \frac{1}{2} \nabla \cdot (zw).$$

- Blowup variables: $\|w(\tau)\|_{L^\infty} \to \infty$ as $\tau \to \infty$,

$$w(z,\tau) = \frac{1}{\nu^2} v(y,\tau), \quad y = \frac{z}{\nu}, \quad \boxed{\lambda(t) = \nu(t)\sqrt{T-t}}$$

where $\nu(\tau) \to 0$ as $\tau \to \infty$ is an unknown parameter function,

$$\nu^2 \partial_\tau v = \nabla \cdot (\nabla v - v \nabla \Phi_v) + \sigma(\tau) \nabla \cdot (zv), \quad \sigma(\tau) = \mathcal{O}(\nu^2).$$

$\Longrightarrow$ The leading term in the expansion of $v \sim Q$ since $\nu \to 0$ as $\tau \to \infty$,

$$\begin{cases} 0 &= \Delta Q + Q^2 - \nabla Q \cdot \nabla \Phi_Q, \\ 0 &= \Delta \Phi_Q + Q, \end{cases} \qquad Q(x) = \frac{8}{(1+|x|^2)^2}, \quad \int_{\mathbb{R}^2} Q(x)dx = 8\pi.$$

The linearized problem

- Linearized problem: $w(z,\tau) = Q_\nu(z) + \eta(z,\tau)$, where $Q_\nu(z) = \frac{1}{\nu^2} Q\left(\frac{z}{\nu}\right)$ and η solves

$$\partial_\tau \eta = \mathscr{L}^\nu \eta + \left(\frac{\nu_\tau}{\nu} - \frac{1}{2}\right) \nabla \cdot (z Q_\nu) - \nabla \cdot \Big(\eta \Phi_\eta\Big), \qquad \nu \to 0 \text{ unknown},$$

$$\boxed{\mathscr{L}^\nu \eta = \underbrace{\nabla \cdot \big(\nabla \eta - \eta \nabla \Phi_{Q_\nu} - Q_\nu \nabla \Phi_\eta\big)}_{\equiv \mathscr{L}_0^\nu \eta} - \frac{1}{2} \nabla \cdot (z\eta)}$$

- Structure of $\mathscr{L}_0^\nu$:

$$\mathscr{L}_0^\nu \eta = \nabla \cdot \big(Q_\nu \nabla \mathscr{M}^\nu \eta\big), \quad \mathscr{M}^\nu \eta = \frac{\eta}{Q_\nu} - \Phi_\eta.$$

$\mathscr{M}^\nu$ comes from the linearization of the energy functional $\mathcal{F}(u) = \int_{\mathbb{R}^2} u\big(\ln u - \frac{1}{2}\Phi_u\big)$ around Q_ν.

$$\int_{\mathbb{R}^2} \eta \mathscr{M}^\nu \eta dz \sim \int_{\mathbb{R}^2} \frac{\eta^2}{Q_\nu} dz, \quad \textit{for } \langle \eta, 1\rangle_{L^2} = \langle \eta, \nabla.(zQ_\nu)\rangle_{L^2} = \langle \eta, \partial_j Q_\nu\rangle_{L^2} = 0.$$

A key proposition of the linear analysis in 2DKS

Proposition 2 ([Collot-Ghoul-Masmoudi-Ng., '22).

- In the radial setting and in terms of the partial mass, $\mathscr{L}^\nu$ becomes a local operator,

$$\mathrm{spec}(\mathscr{L}^\nu)|_{rad} = \left\{ \alpha_{n,\nu} = 1 - n - \frac{1}{2|\ln \nu|} + \mathcal{O}\left(\frac{1}{|\ln \nu|^2}\right), \quad n \in \mathbb{N} \right\}.$$

The analysis of eigenproblem has been done through a matched asymptotic expansions technique, where the eigenfunction $\varphi_{n,\nu}$ is built from iterative kernels of the linearized operator (think of Neumann series).
$\rightsquigarrow$ spectral analysis to control the radial part in L^2_ω.

- For the nonradial part $\rightsquigarrow$ energy methods: dissipation + coercivity

$$\int_{\mathbb{R}^2} \mathscr{L}^\nu(u\sqrt{\rho})\mathscr{M}^\nu(u\sqrt{\rho}) \leq -c_0 \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{Q_\nu}\rho, \quad \text{with} \quad \rho(z) = e^{-\frac{|z|^2}{4}},$$

up to the orthogonality condition $\langle \eta, \partial_1 Q_\nu \rangle_{L^2_{\sqrt{\rho}}} = \langle \eta, \partial_2 Q_\nu \rangle_{L^2_{\sqrt{\rho}}} = 0$.

Spectrum of $\mathscr{L}^\nu$ in the radial setting

- A singular eigenproblem: the limiting operator as $\nu \to 0$ is

$$\bar{\mathscr{L}} = \Delta + \frac{4z}{|z|^2}\cdot\nabla - \frac{1}{2}z\cdot\nabla - 1 = \Delta_{\zeta,6} - \frac{1}{2}\zeta\partial_\zeta - 1, \quad \zeta = |z|$$

with the spectrum (Hermite operator in $\mathbb{R}^6$)

$$\text{spec}(\bar{\mathscr{L}}) = \{-1-n, n \in \mathbb{N}\}.$$

- $\mathscr{L}^\nu$ acting on radially symmetric functions is transformed to

$$\mathscr{L}^\nu \varphi(\zeta) = \frac{1}{\zeta}\partial_\zeta\big(\mathscr{A}^\nu m_\varphi(\zeta)\big), \quad \mathscr{A}^\nu = \mathscr{A}_0^\nu - \frac{1}{2}\zeta\partial_\zeta,$$

$$\mathscr{A}_0^\nu = \partial_\zeta^2 - \frac{1}{\zeta}\partial_\zeta + \frac{\partial_\zeta(m_{Q_\nu}\cdot)}{\zeta} \quad \text{and} \quad m_{Q_\nu}(\zeta) = \frac{4\zeta^2}{\zeta^2+\nu^2}.$$

The eigenproblem $\mathscr{L}^\nu\varphi_{n,\nu}(\zeta) = \alpha_{n,\nu}\varphi_{n,\nu}(\zeta)$ is equivalent to

$$\Big(\mathscr{A}_0^\nu - \frac{1}{2}\zeta\partial_\zeta\Big)\phi_{n,\nu}(\zeta) = \alpha_{n,\nu}\phi_{n,\nu}(\zeta), \quad \varphi_{n,\nu}(\zeta) = \frac{\partial_\zeta\phi_{n,\nu}}{\zeta}.$$

Let $\phi_{n,\nu}(\zeta) = \nu^{-2}\phi_n(r)$ with $r = \frac{\zeta}{\nu}$, the eigenproblem becomes

$$\Big(\mathscr{A}_0 - \frac{\nu^2}{2}r\partial_r\Big)\phi_n(r) = \nu^2\alpha_{n,\nu}\phi_n(r),$$

where $\mathscr{A}_0$ is the operator $\mathscr{A}_0^\nu$ in the renormalized variable r.

Eigenfunction expansion of $\mathscr{L}^\nu$

- Iterative profiles: Let $T_{j+1}(r) = -\mathscr{A}_0^{-1} T_j(r)$, where $T_0(r) = \frac{r^2}{1+r^2}$ with $\mathscr{A}_0 T_0 = 0$, $T_j(r) \sim d_j r^{2j-2} \ln r$ for $r \gg 1$,

$$\mathscr{A}_0 = \partial_r^2 - \frac{1}{r}\partial_r + \frac{\partial_r(m_Q \cdot)}{r}, \quad m_Q(r) = \frac{4r^2}{1+r^2}.$$

Proposition 3 (Spectral properties of $\mathscr{A}^\nu$).

The operator $\mathscr{A}^\nu$: $H^2_{\omega_\nu} \to L^2_{\omega_\nu}$ is self-adjoint with compact resolvent, where $\omega_\nu(\zeta) = \frac{\nu^2}{\zeta Q_\nu} e^{-\frac{\zeta^2}{4}}$.

i) (eigenvalues) $\alpha_{n,\nu} = 1 - n - \frac{1}{2\ln\nu} - \frac{2\ln 2 - \gamma - n}{4|\ln\nu|^2} + \mathcal{O}\left(\frac{1}{|\ln\nu|^3}\right)$.

ii) (eigenfunctions) $\phi_{n,\nu}(\zeta) = \sum_{j=0}^n \frac{n!}{(n-j)!}\nu^{2j-2} T_{j,\nu}\left(\zeta/\nu\right) + \tilde{\phi}_{n,\nu}$.
$\{\phi_{n,\nu}\}_{n\in\mathbb{N}}$ forms a complete orthogonal basis in $L^2_{\omega_\nu}$.

iii) (spectral gap) For any $g \in L^2_{\omega_\nu}$ with $\langle g, \phi_{n,\nu}\rangle_{L^2_{\omega_\nu}} = 0$ for $n = \overline{0, N}$,

$$\langle g, \mathscr{A}^\nu g\rangle_{L^2_{\omega_\nu}} \le \alpha_{N+1,\nu}\|g\|^2_{L^2_{\omega_\nu}}.$$

Coercivity of $\mathscr{L}^\nu$

- for $|z| \ll 1$, the scaling term $\nabla.(z\eta)$ is considered as a small perturbation, i.e. $\mathscr{L}^\nu \approx \mathscr{L}_0^\nu$ that is symmetric under the scalar product

$$\langle u, v\rangle_{\mathscr{M}} = \langle u, \mathscr{M}^\nu v\rangle_{L^2}$$

$$\langle \mathscr{L}_0^\nu u, v\rangle_{\mathscr{M}} = \int \nabla.(Q_\nu \nabla \mathscr{M}^\nu u)\mathscr{M}^\nu v = -\int Q_\nu \nabla \mathscr{M}^\nu u \cdot \nabla \mathscr{M}^\nu v.$$

- for $|z| \gg 1$, we can ignore the term involving Φ_η, i.e. $\mathscr{L}^\nu \approx \mathscr{H}^\nu$, where

$$\mathscr{H}^\nu \eta = \nabla \cdot \big(\nabla \eta - \eta \nabla \Phi_{Q_\nu}\big) - \frac{1}{2}\nabla \cdot (z\eta) = \frac{1}{\bar{\omega}_\nu}\nabla(\bar{\omega}_\nu \nabla \eta) + (Q_\nu - 1)\eta,$$

which is symmetric in $L^2_{\bar{\omega}_\nu}$ with $\bar{\omega}_\nu = \frac{1}{Q_\nu} e^{-|z|^2/4}$.

- A global scalar product:

$$\langle u, v\rangle_* = \int u\sqrt{\rho}\mathscr{M}^\nu(v\sqrt{\rho}), \quad \rho = e^{-|z|^2/4}.$$

The coercivity of $\mathscr{L}^\nu$ with $\langle \cdot, \cdot\rangle_*$ is obtained from the coercivity of $\mathscr{M}^\nu$,

$$\langle \mathscr{L}^\nu u, u\rangle_* \approx -\int Q_\nu |\nabla \mathscr{M}^\nu(u\sqrt{\rho})|^2 \le -\delta_0 \int \frac{|\nabla u|^2}{Q_\nu}\rho + C\sum_{i=1}^{2}\langle u, \partial_i Q_\nu \sqrt{\rho}\rangle_{L^2}^2.$$

Approximate solution and the law of blowup

- The self-similar equation:

$$\partial_\tau w = \nabla \cdot \left(\nabla w - w \nabla \Phi_w\right) - \frac{1}{2}\nabla \cdot (zw), \quad w = Q_\nu + \eta.$$

- The approximate solution: for $\ell \geq 1$ integer,

$$w^{app}(z,\tau) = Q_\nu(z) + \underbrace{a_\ell(\tau)\left[\varphi_{\ell,\nu}(|z|) - \varphi_{0,\nu}(|z|)\right]}_{\text{modification driving the law of blowup}}.$$

A projection onto $\varphi_{\ell,\nu}$ and compatibility condition $a_\ell \sim -4\nu^2$:

$$(\ell = 1, \text{ stable}) \quad \frac{\nu_\tau}{\nu} = \frac{1}{4\ln\nu} + \frac{e_2}{|\ln\nu|^2} \implies \boxed{\nu = C_0 e^{-\sqrt{\frac{\tau}{2}}},}$$

$$(\ell \geq 2, \text{ unstable}) \quad \frac{\nu_\tau}{\nu} = \frac{1-\ell}{2} + \frac{\ell+1}{4\ln\nu} \implies \boxed{\nu = C_\ell e^{\frac{(1-\ell)\tau}{2}} \tau^{\frac{\ell}{2(1-\ell)}}}$$

- The linearized equation: $\varepsilon = w - w^{app}$,

$$\partial_\tau \varepsilon = \mathscr{L}^\nu \varepsilon + \mathit{Error} + \mathit{SmallLinear} + \mathit{Nonlinear}.$$

Nonlinear analysis

■ **The main issue:** The perturbation ε can be large near the origin, and the only control in L^2_ω does not allow for a use of dissipation. In particular the direction $\nabla.(z\, Q_\nu)$, which is the kernel of $\mathscr{L}_0^\nu = \nabla.\big(Q_\nu \nabla \mathscr{M}^\nu \cdot\big)$, becomes the leading part of ε in the zone $|z| \sim \nu$.

■ **The treatment:** Recall that $0 = \Delta Q_\lambda - \nabla.(Q_\lambda \nabla \Phi_{Q_\lambda})$ for any $\lambda > 0$,

$$0 = \frac{d}{d\lambda}\Big[\Delta Q_\lambda - \nabla.(Q_\lambda \nabla \Phi_{Q_\lambda})\Big]_{\lambda=\nu} \implies \mathscr{L}_0^\nu\big[\nabla.(z\, Q_\nu)\big] = 0.$$

We introduce $\tilde{\nu} \sim \nu$ and impose a local orthogonality condition to eliminate $\nabla.(z\, Q_\nu)$. It's crucial that the key proposition still holds true for the linearized operator $\mathscr{L}^{\tilde{\nu}}$ up to an admissible error, from which we are able to close the nonlinear analysis.

■ **An expectation:** Such an idea can be successfully applied to other problems in some critical regimes (NLS, nonlinear wave, ...)

2.2. Multiple collapsing blowup in $\mathbb{R}^2$ (L^1-critical)

Simulation of a multiple collapsing blowup in $\mathbb{R}^2$

Multiple collapsing blowup for the KS system in $\mathbb{R}^2$.

Known results

- [Côte-Zaag, '13]: Type I backward multisoliton solutions for the subcritical semilinear wave equation, $u_{tt} = \Delta u + |u|^{p-1}u, \quad 1 < p < \frac{d+2}{d-2}$.

- [Martel-Raphaël, '18]: infinite-time blowup solutions for the L^2-critical NLS in $\mathbb{R}^2$, $\imath u_t + \Delta u + |u|^2 u = 0$,

$$\left\| u(x,t) - \frac{e^{\imath\gamma(t)}}{\lambda(t)} \sum_{k=1}^{K} Q\left(\frac{x - x_k(t)}{\lambda(t)}\right) \right\|_{H^1} \to 0, \quad \lambda(t) \sim \frac{1}{\log t} \ \ as \ \ t \to \infty,$$

where Q is the ground state solution

$\Delta Q - Q + Q^3 = 0, \quad Q \in H^1(\mathbb{R}^2), \quad Q > 0$ radially symmetric, exponentially decay.

Applying the pseudo-conformal symmetry $v(x,t) = \frac{1}{|t|} u\left(\frac{x}{|t|}, \frac{1}{|t|}\right) e^{-\imath\frac{|x|^2}{4t}}$ yields finite-time blowup solutions.

- [Martel-Merle, '18]: soliton collision for the critical semilinear wave equation in $\mathbb{R}^5$.

Type II-multiple collapsing blowup in $\mathbb{R}^2$

[Collot-Ghoul-Masmoudi-Ng., '23]: For the case of 2 bubbles, we construct a particular example leading to a blowup solution of the form

$$u(x,t) = \frac{1}{\lambda^2(t)}\left[Q\left(\frac{x-a(t)}{\lambda(t)}\right) + Q\left(\frac{x+a(t)}{\lambda(t)}\right) + \text{correction}\right],$$

where

$$\lambda(t) = \sqrt{T-t}\,e^{-c\sqrt{|\log(T-t)|+\mathcal{O}(1)}}, \quad a(t) = a_0\sqrt{T-t}, \ \ |a_0| \sim 2.$$

In the self-similar setting, the solution becomes

$$w(z,\tau) = \frac{1}{\nu^2(\tau)}\left[Q\left(\frac{z-a_0}{\nu(\tau)}\right) + Q\left(\frac{z+a_0}{\nu(\tau)}\right) + \text{correction}\right],$$

where

$$\nu(\tau) = e^{-c\sqrt{\tau}+\mathcal{O}(1)}, \quad |a_0| \sim 2.$$

3.1. Collapsing-ring blowup in $\mathbb{R}^{d\geq 3}$ (L^1-supercritical)

Collapsing-ring blowup solutions

Introduce the profile

$$W(\xi) = \frac{1}{8}\cosh^{-2}\left(\frac{\xi}{4}\right).$$

Theorem 4 ([Collot-Ghoul-Masmoudi-Ng., arXiv]).

- There exists an open set $\mathcal{V} \in L^\infty_{rad}(\mathbb{R}^d)$ of initial data u_0 such that

$$u(x,t) = \frac{M(t)}{R(t)^{d-1}\lambda(t)}\left[W\left(\frac{|x| - R(t)}{\lambda(t)}\right) + \tilde{u}(x,t)\right],$$

where $\|\tilde{u}(t)\|_{L^\infty} \to 0$ as $t \to T$, and

$$\lambda(t) = \frac{R(t)^{d-1}}{M(t)}, \quad M(t) \to M_0, \quad R(t) = \left[(d/2)M(T-t)\right]^{\frac{1}{d}}.$$

- The constructed solution is stable under small perturbation in $\mathcal{V}$.

Traveling blowup solutions in the radial setting

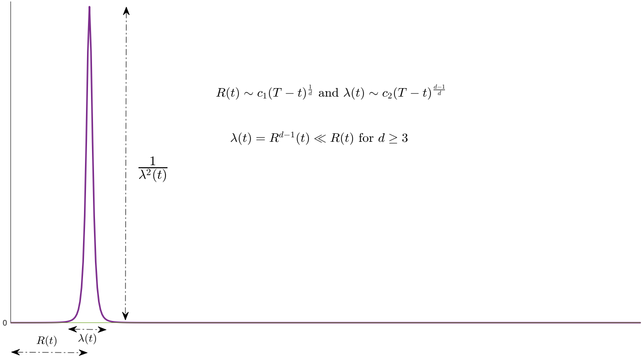


Illustration of a traveling blowup solution in the radially symmetric setting.

Traveling blowup solution in the partial mass setting

The partial mass setting $m_u(r,t) = \int_0^r u(\zeta,t)\zeta^{d-1}d\zeta$,

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

Theorem 5 ([Collot-Ghoul-Masmoudi-Ng., arXiv]).

- There exists an open set $\mathcal{O} \subset W^{1,\infty}(\mathbb{R}_+)$ of initial data $m_u(0)$ such that

$$m_u(r,t) = M(t)\left[Q\left(\frac{r-R(t)}{\lambda(t)}\right) + m_\varepsilon(r,t)\right], \quad Q(\xi) = \frac{e^{\frac{\xi}{2}}}{1+e^{\frac{\xi}{2}}},$$

where $Q'' - \frac{1}{2}Q' + QQ' = 0$, $\|m_\varepsilon(t)\|_{W^{1,\infty}} \to 0$ as $t \to T$,

$$\lambda(t) = \frac{R(t)^{d-1}}{M(t)}, \quad M(t) \to M_0, \quad R(t) = \left[(d/2)M(T-t)\right]^{\frac{1}{d}}.$$

- The constructed solution is stable under small perturbation in $\mathcal{O}$.

Traveling blowup solutions in the partial mass setting

The partial mass equation:

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}.$$

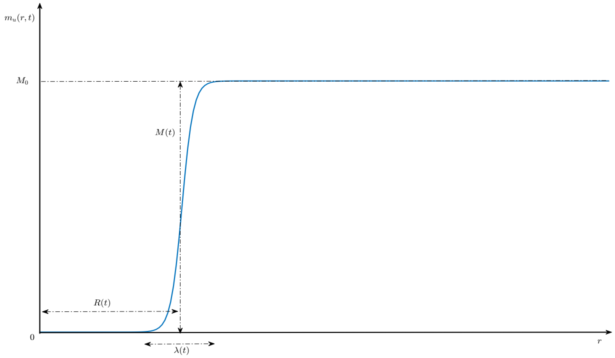


Illustration of a traveling blowup solution in the partial mass setting.

A numerical simulation for $d = 3$

Fig 5: (horizontally zoomed solution) The initial data $m_u(r,0) = MQ\left(\frac{r - M^{\frac{1}{3}}\epsilon}{1.5M^{-\frac{1}{3}}\epsilon^2}\right)$, *where* $M = 27$ *and* $\epsilon = 0.7$. *With* $\epsilon = 0.7$, *the theoretical blowup time is* $T = \epsilon^3 \approx 0.343$. *Maple solver gives an approximation of the blowup time by saying "could not compute solution for* $t > 0.32$: *Newton iteration is not converging".*

Renormalization and profile

$$\partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}$$

- Inviscid variables (fix the shock location):

$$m_u(r,t) = M(t) m_w(\zeta,\tau), \quad \zeta = \frac{r}{R(t)}, \quad \frac{d\tau}{dt} = \frac{M(t)}{R(t)^d}, \quad \text{and} \quad \nu = \frac{R^{d-1}}{M},$$

to fix the location of the shock at $\zeta = 1$,

$$\partial_\tau m_w = \left(\frac{m_w}{\zeta^{d-1}} - \frac{1}{2}\zeta\right)\partial_\zeta m_w + \nu \Delta_{\zeta,2-d}\, m_w + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\zeta \partial_\zeta m_w - \frac{M_\tau}{M} m_w.$$

- Blowup variables (zoom at the shock):

$$m_w(\zeta,\tau) = m_v(\xi,s), \quad \xi = \frac{\zeta - 1}{\nu}, \quad \frac{ds}{d\tau} = \frac{1}{\nu},$$

where m_v solves the new equation

$$\partial_s m_v = \partial_\xi^2 m_v + m_v \partial_\xi m_v - \frac{1}{2}\partial_\xi m_v + \left(\frac{R_\tau}{R} + \frac{1}{2}\right)\partial_\xi m_v - \frac{M_s}{M} m_v + l.o.t$$

- The blowup profile is connected to the traveling solution to Burgers equation:

$$Q'' - \frac{1}{2}Q' + QQ' = 0, \quad \lim_{\xi \to -\infty} Q(\xi) = 0.$$

The linearized problem $\quad \partial_t m_u = \partial_r^2 m_u - \frac{d-1}{r}\partial_r m_u + \frac{m_u \partial_r m_u}{r^{d-1}}$

- Introducing $m_q(\xi, s) = m_v(\xi, s) - Q(\xi)$ yields

$$\partial_s m_q = \mathscr{L}_0 m_q + L(m_q) + NL(m_q) + \Psi,$$

where $\mathscr{L}_0 = \partial_\xi^2 - (1/2 - Q)\partial_\xi + Q'$ is the linearied operator appearing in the study of stability of traveling wave solutions to Burgers equation.

$$\left\langle \mathscr{L}_0 g, g \right\rangle_{L^2_{\omega_0}} \leq -\delta_0 \|g\|^2_{H^1_{\omega_0}} + C\left\langle g, Q' \right\rangle^2_{L^2_{\omega_0}}, \quad \omega_0 = Q^{-1} e^{\frac{\xi}{2}}.$$

- Introducing $m_\varepsilon(\zeta, \tau) = m_w(\zeta, \tau) - Q_\nu(\zeta)$ yields

$$m_{\varepsilon,1} = \partial_\zeta m_\varepsilon, \qquad \partial_\tau m_{\varepsilon,1} = \mathscr{A}_1 m_{\varepsilon,1} + \mathcal{P} m_{\varepsilon,1} + E, \quad \zeta \geq 1,$$

where

$$\mathscr{A}_1 = -\left(\frac{d-1}{\zeta^d} + \frac{1}{2}\right) + \left(\frac{1}{\zeta^{d-1}} - \frac{1}{2}\zeta\right)\partial_\zeta + \nu \partial_\zeta^2.$$

An observation (constructive approach): $0 < \kappa \ll 1$,

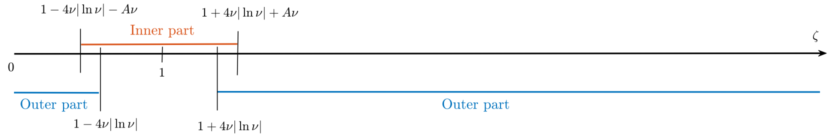
$$\phi_1 = e^{-\kappa\tau} e^{-\frac{3}{8}\left(\frac{|\zeta - 1| - 4\nu|\ln \nu|}{\nu}\right)}, \qquad \partial_\tau \phi_1 - \mathscr{A}_1 \phi_1 \geq \frac{c_0}{\nu}\phi_1, \quad \zeta \in [1, 2^{\frac{1}{d}}).$$

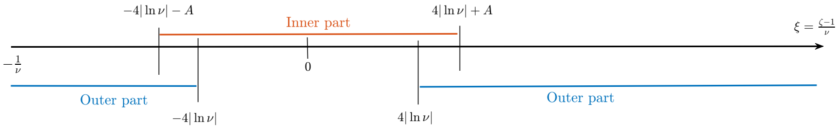
A design of the bootstrap regime

- Inner-outer estimates: $A \gg 1$, $0 < \kappa \ll 1$,

$$\|\chi_{4|\ln\nu|+A} m_q(\tau)\|_{L^2_{\omega_0}} \lesssim e^{-\kappa\tau}, \qquad \|\partial_\zeta m_\varepsilon(\tau)\|_{L^\infty(|\zeta-1|\geq 4\nu|\ln\nu|)} \lesssim e^{-\kappa\tau}$$

- The coercivity of $\mathscr{L}_0$ to control the inner norm.
- A delay estimate for a transport-type equation helps to construct $\phi_1(\zeta,\tau)$.





3.2. Type I-Log blowup in $\mathbb{R}^3$ (L^1-supercritical)

Type I-Log blowup

[Ng.-Nouaili-Zaag, '23]: We found for $d = 3$, there are finite-time blowup solutions admitting the dynamics (infinite mass) either

$$u(x,t) = \frac{1}{T-t}\left[W_{rad}\left(\frac{|x|^6}{(T-t)^3|\log(T-t)|}\right) + o_{L^\infty}(1)\right],$$

or

$$u(x,t) = \frac{1}{T-t}\left[W_{nonrad}\left(\frac{x_1^6 + x_2^6 + x_3^6}{(T-t)^3|\log(T-t)|}\right) + o_{L^\infty}(1)\right],$$

where $W_{rad}(\xi)$ and W_{nonrad} solve a first order nonlocal ODE,

$$W_{rad}(0) = W_{nonrad}(0) = 1, \quad 0 < W_{rad}(\xi), W_{nonrad}(\xi) \lesssim \xi^{-\frac{1}{3}} \quad \text{for } \xi \gg 1.$$

- Such a blowup dynamic only appears for the cases $d = 3$ and $d = 4$.
- Require computer-assistance to derive and solve a nonlinear system of ODEs.

Thank you for your attention!