Entropic Dispersion and the Heat Death of the Universe

Analysis & PDE Seminar, Hong Kong–Korea

Diogo Arsénio

Division of Science, New York University Abu Dhabi, United Arab Emirates

9 May 2025

Dispersion: What is it?

- Take a physical system made up of particles or waves.
- Dispersion occurs when the building blocks of the system move away from each other, i.e., they disperse.
- Dispersion leads to:
  - Local and global decay, mass goes to infinity.
  - Strichartz estimates.
  - Gain of integrability.
  - Local smoothing estimates (Kato effect, averaging lemmas).
  - Hypoellipticity, mixing property.
- In kinetic theory, the transport operator $(\partial_t + v \cdot \nabla_x)$ is responsible for dispersion. The Castella–Perthame estimates capture well the effects of dispersion in the whole space.

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Consider the equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$.

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Consider the equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$.

$$\boxed{f(t,x,v) = f_0(x - tv, v) + \int_0^t g(s, x - (t-s)v, v) dv}$$

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Consider the equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$.

$$\boxed{f(t,x,v) = f_0(x - tv, v) + \int_0^t g(s, x-(t-s)v, v) dv}$$

Theorem: [Castella, Perthame] Take $1 \le r \le p \le \infty$. Then,

$$\boxed{\|f(x-tv,v)\|_{L^p_x L^r_v} \le |t|^{-d(\frac{1}{r}-\frac{1}{p})} \|f(x,v)\|_{L^r_x L^p_v}}$$

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Consider the equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$.

$$\boxed{f(t,x,v) = f_0(x - tv, v) + \int_0^t g(s, x-(t-s)v, v)\,dv}$$

Theorem: [Castella, Perthame] Take $1 \le r \le p \le \infty$. Then,

$$\boxed{\|f(x-tv,v)\|_{L^p_x L^r_v} \le |t|^{-d(\frac{1}{r}-\frac{1}{p})} \|f(x,v)\|_{L^r_x L^p_v}}$$

Proof:

$$\begin{aligned} \|f(x-tv,v)\|_{L^p_x L^r_v} &= \|f(y, \frac{x-y}{t})\|_{L^p_x L^r_y} |t|^{-\frac{d}{r}} \\ &\le \|f(y, \frac{x-y}{t})\|_{L^r_y L^p_x} |t|^{-\frac{d}{r}} = \|f(y,x)\|_{L^r_y L^p_x} |t|^{-d(\frac{1}{r}-\frac{1}{p})} \end{aligned}$$

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Consider the equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$.

$$\boxed{f(t,x,v) = f_0(x - tv, v) + \int_0^t g(s, x - (t-s)v, v)\, dv}$$

Theorem: [Castella, Perthame] Take $1 \leq r \leq p \leq \infty$. Then,

$$\boxed{\|f(x - tv, v)\|_{L_x^p L_v^r} \leq |t|^{-d\left(\frac{1}{r} - \frac{1}{p}\right)} \|f(x,v)\|_{L_x^r L_v^p}}$$

Castella–Perthame Estimates

[Castella, Perthame; CRAS, 1996]

Consider the equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f(t,x,v) = g(t,x,v) \\ f(0,x,v) = f_0(x,v) \end{cases}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$.

$$\boxed{f(t,x,v) = f_0(x - tv, v) + \int_0^t g(s, x-(t-s)v, v) dv}$$

Theorem: [Castella, Perthame] Take $1 \leq r \leq p \leq \infty$. Then,

$$\boxed{\|f(x-tv,v)\|_{L^p_x L^r_v} \leq |t|^{-d(\frac{1}{r}-\frac{1}{p})} \|f(x,v)\|_{L^r_x L^p_v}}$$

Remark: The free transport equation conserves all $L^p_{x,v}$ norms.
$\Rightarrow$ The endpoint case $p = r$ is trivial (equality).

Castella-Perthame Estimates (continued)

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq a, p, q, r \leq \infty$ such that $a < q$,

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad \frac{2}{a} = \frac{1}{p} + \frac{1}{r}$$

Then,

$$\boxed{\|f(x - tv, v)\|_{L_t^q L_x^p L_v^r} \leq C\|f(x, v)\|_{L_{x,v}^a}}$$

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq a, p, q, r \leq \infty$ such that $a < q$,

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad \frac{2}{a} = \frac{1}{p} + \frac{1}{r}$$

Then,

$$\boxed{\|f(x - tv, v)\|_{L^q_t L^p_x L^r_v} \leq C\|f(x, v)\|_{L^a_{x,v}}}$$

Idea of proof:

1. Show a dual estimate employing a TT^* method and the classical Hardy–Littlewood–Sobolev (HLS) inequality.
2. Apply the resulting estimate to $|f|^\alpha, \forall \alpha > 0$ to recover the full range of parameters.

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq a, p, q, r \leq \infty$ such that $a < q$,

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad \frac{2}{a} = \frac{1}{p} + \frac{1}{r}$$

Then,

$$\boxed{\|f(x - tv, v)\|_{L^q_t L^p_x L^r_v} \leq C\|f(x, v)\|_{L^a_{x,v}}}$$

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq a, p, q, r \leq \infty$ such that $a < q$,

$$\frac{2}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad \frac{2}{a} = \frac{1}{p} + \frac{1}{r}$$

Then,

$$\boxed{\|f(x - tv, v)\|_{L^q_t L^p_x L^r_v} \leq C\|f(x, v)\|_{L^a_{x,v}}}$$

Remarks

- An explicit constant C can be obtained from optimal constants for the HLS inequality [Lieb; Annals of Math., 1983].
- The endpoint case $q = \infty$ and $a = p = r$ is trivial (equality).
- Taking $r = 1$ gives

  $$\boxed{\|f(x - tv, v)\|_{L_t^{\frac{2p'}{d}} L^p_x L^1_v} \leq C_p\|f(x, v)\|_{L^{\frac{2p}{p+1}}_{x,v}}}$$

  with $\lim_{p \to 1} C_p = 1$.
- One can show $C_p \leq K^{p-1}$ for some $K > 0$ and p close to 1.

Castella-Perthame Estimates (continued)

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq k, l, p, q, r \leq \infty$ such that $1 < l < k < \infty$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)ds \right\|_{L_t^k L_x^p L_v^r} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}}$$

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq k, l, p, q, r \leq \infty$ such that $1 < l < k < \infty$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)ds \right\|_{L_t^k L_x^p L_v^r} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}}$$

Idea of proof: Use estimate for initial data and the HLS inequality.

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq k, l, p, q, r \leq \infty$ such that $1 < l < k < \infty$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x - (t-s)v, v) ds \right\|_{L_t^k L_x^p L_v^r} \leq C \| g(t,x,v) \|_{L_t^l L_x^r L_v^p}}$$

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq k, l, p, q, r \leq \infty$ such that $1 < l < k < \infty$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x - (t-s)v, v) ds \right\|_{L_t^k L_x^p L_v^r} \leq C \| g(t,x,v) \|_{L_t^l L_x^r L_v^p}}$$

Remarks:

- An explicit constant C can be obtained from optimal constants for the HLS inequality [Lieb; Annals of Math., 1983].
- The endpoint case $q = k = \infty$, $l = 1$ and $p = r$ is trivial (equality if $g \geq 0$).

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq k, l, p, q, r \leq \infty$ such that $1 < l < k < \infty$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)ds \right\|_{L_t^k L_x^p L_v^r} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}}$$

Remarks:

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq k, l, p, q, r \leq \infty$ such that $1 < l < k < \infty$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 + \frac{1}{k} = \frac{1}{q} + \frac{1}{l}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)ds \right\|_{L_t^k L_x^p L_v^r} \leq C \| g(t,x,v) \|_{L_t^l L_x^r L_v^p}}$$

Remarks:

- Taking $r = 1$ and $l = k'$ gives

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)ds \right\|_{L_t^{\frac{2p'}{d}} L_x^p L_v^1} \leq C_p \| g(t,x,v) \|_{L_t^{\frac{2p}{d-(d-2)p}} L_x^1 L_v^p}}$$

  with $\lim_{p\to 1} C_p = 1$.
- One can show $C_p \leq K^{p-1}$ for some $K > 0$ and p close to 1.

Castella-Perthame Estimates (continued)

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq l, p, q, r \leq \infty$ such that $q > 1$, $\frac{1}{p} + \frac{1}{r} \leq 1$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 = \frac{1}{l} + \frac{1}{2q}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)\, ds \right\|_{L_t^\infty L_{x,v}^{\frac{2pr}{p+r}}} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}}$$

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq l, p, q, r \leq \infty$ such that $q > 1$, $\frac{1}{p} + \frac{1}{r} \leq 1$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 = \frac{1}{l} + \frac{1}{2q}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x - (t-s)v, v)\,ds \right\|_{L_t^\infty L_{x,v}^{\frac{2pr}{p+r}}} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}}$$

Idea of proof: Write $\|h\|_{L_{x,v}^{\frac{2pr}{p+r}}} = \|h^2\|_{L_{x,v}^{\frac{pr}{p+r}}}^{\frac{1}{2}}$ to perform a kind of TT^* estimate, and use the HLS inequality.

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq l, p, q, r \leq \infty$ such that $q > 1$, $\frac{1}{p} + \frac{1}{r} \leq 1$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 = \frac{1}{l} + \frac{1}{2q}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)\,ds \right\|_{L_t^\infty L_{x,v}^{\frac{2pr}{p+r}}} \leq C \|g(t,x,v)\|_{L_t^l L_x^r L_v^p}}$$

Castella-Perthame Estimates (continued)

Theorem: [Castella, Perthame] Take $1 \leq l, p, q, r \leq \infty$ such that $q > 1$, $\frac{1}{p} + \frac{1}{r} \leq 1$,

$$\frac{1}{q} = d\left(\frac{1}{r} - \frac{1}{p}\right) \qquad \text{and} \qquad 1 = \frac{1}{l} + \frac{1}{2q}$$

Then,

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)\,ds \right\|_{L_t^\infty L_{x,v}^{\frac{2pr}{p+r}}} \leq C \| g(t,x,v) \|_{L_t^l L_x^r L_v^p}}$$

Remarks:

- As before, an explicit constant can be obtained and the case $p = r$, $q = \infty$ and $l = 1$ is trivial.
- However, we cannot reach the case $p = r = 1$.
- The adjoint operator $\int_t^T h(s, x-(t-s)v, v)ds$ satisfies similar estimates. A duality argument gives that

$$\boxed{\left\| \int_0^t g(s, x-(t-s)v, v)\,ds \right\|_{L_t^{\frac{2p'}{d}} L_x^p L_v^1} \leq C \| g(t,x,v) \|_{L_t^1 L_{x,v}^{\frac{2p}{p+1}}}} \quad \forall 1 \leq p < \frac{d}{d-1}$$

Application to the Boltzmann Equation

Application to the Boltzmann Equation

Theorem [A.; 2009; CMP, 2011]
Let $d = 2, 3$ and $f_0(x, v) \in L^d_{x,v}(R^d \times R^d)$. Consider the Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = Q(f, f) \\ f(t = 0) = f_0 \end{cases}$$

for a suitable cross-section $b(v - v_*, \sigma)$.
If $\|f_0\|_{L^d_{x,v}}$ is small enough, there exists a global solution

$$f \in L^\infty_t L^d_{x,v} \cap L^\lambda_t L_x^{d\frac{\lambda}{\lambda-1}} L_v^{d\frac{\lambda}{\lambda+1}}$$

for some $d < \lambda < \infty$.

Application to the Boltzmann Equation

Theorem [A.; 2009; CMP, 2011]
Let $d = 2, 3$ and $f_0(x, v) \in L^d_{x,v}(R^d \times R^d)$. Consider the Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = Q(f, f) \\ f(t = 0) = f_0 \end{cases}$$

for a suitable cross-section $b(v - v_*, \sigma)$.
If $\|f_0\|_{L^d_{x,v}}$ is small enough, there exists a global solution

$$f \in L^\infty_t L^d_{x,v} \cap L^\lambda_t L_x^{d\frac{\lambda}{\lambda-1}} L_v^{d\frac{\lambda}{\lambda+1}}$$

for some $d < \lambda < \infty$.

Idea of proof: Use the Castella–Perthame dispersive estimates in combination with convolution inequalities for Q^+.

Application to the Boltzmann Equation

Theorem [A.; 2009; CMP, 2011]
Let $d = 2, 3$ and $f_0(x, v) \in L^d_{x,v}(R^d \times R^d)$. Consider the Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = Q(f, f) \\ f(t = 0) = f_0 \end{cases}$$

for a suitable cross-section $b(v - v_*, \sigma)$.
If $\|f_0\|_{L^d_{x,v}}$ is small enough, there exists a global solution

$$f \in L^\infty_t L^d_{x,v} \cap L^\lambda_t L_x^{d\frac{\lambda}{\lambda-1}} L_v^{d\frac{\lambda}{\lambda+1}}$$

for some $d < \lambda < \infty$.

Application to the Boltzmann Equation

Theorem [A.; 2009; CMP, 2011]
Let $d = 2, 3$ and $f_0(x, v) \in L^d_{x,v}(R^d \times R^d)$. Consider the Boltzmann equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = Q(f, f) \\ f(t = 0) = f_0 \end{cases}$$

for a suitable cross-section $b(v - v_*, \sigma)$.
If $\|f_0\|_{L^d_{x,v}}$ is small enough, there exists a global solution

$$f \in L^\infty_t L^d_{x,v} \cap L^\lambda_t L_x^{d\frac{\lambda}{\lambda-1}} L_v^{d\frac{\lambda}{\lambda+1}}$$

for some $d < \lambda < \infty$.

Remarks:

- The convolutions inequalities are related to the regularizing properties of Q^+. They do not hold for Q^-.
- Other results on the Boltzmann equation based on dispersion:
  - Solutions near global Maxwellians [Bardos, Gamba, Golse, Levermore; CMP, 2016].
  - Solutions with Gaussian weights near vacuum [Illner, Shinbrot; CMP, 1984]

Question

Question

How can we measure the dispersion of solutions to the Boltzmann equation which satisfy natural conservation laws only?

More generally, the same question applies to any kinetic system $f(t,x,v)$ such that the mass is conserved, and the energy and the entropy decay.

Question

How can we measure the dispersion of solutions to the Boltzmann equation which satisfy natural conservation laws only?

More generally, the same question applies to any kinetic system $f(t, x, v)$ such that the mass is conserved, and the energy and the entropy decay.

Difficulty: Dispersion is not captured by the $L^1_{x,v}$ norm.

Question

How can we measure the dispersion of solutions to the Boltzmann equation which satisfy natural conservation laws only?

More generally, the same question applies to any kinetic system $f(t,x,v)$ such that the mass is conserved, and the energy and the entropy decay.

Difficulty: Dispersion is not captured by the $L^1_{x,v}$ norm.

How can we measure dispersion in the $L^1_{x,v}$ setting?

Question

How can we measure the dispersion of solutions to the Boltzmann equation which satisfy natural conservation laws only?

More generally, the same question applies to any kinetic system $f(t,x,v)$ such that the mass is conserved, and the energy and the entropy decay.

Difficulty: Dispersion is not captured by the $L^1_{x,v}$ norm.

How can we measure dispersion in the $L^1_{x,v}$ setting?

- Local decay? $\int_{B(0,R)\times\mathbb{R}^d} f(t,x,v)dxdv \to 0$ as $t\to\infty$? $(\forall 0<R<\infty)$

Question

How can we measure the dispersion of solutions to the Boltzmann equation which satisfy natural conservation laws only?

More generally, the same question applies to any kinetic system $f(t,x,v)$ such that the mass is conserved, and the energy and the entropy decay.

Difficulty: Dispersion is not captured by the $L^1_{x,v}$ norm.

How can we measure dispersion in the $L^1_{x,v}$ setting?

- Local decay? $\int_{B(0,R)\times\mathbb{R}^d} f(t,x,v)dxdv \to 0$ as $t\to\infty$? $(\forall 0<R<\infty)$
- Mass escaping to infinity? $\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x|^\alpha dxdv \to \infty$ as $t\to\infty$? $(\forall \alpha>0)$

Question

How can we measure the dispersion of solutions to the Boltzmann equation which satisfy natural conservation laws only?

More generally, the same question applies to any kinetic system $f(t,x,v)$ such that the mass is conserved, and the energy and the entropy decay.

Difficulty: Dispersion is not captured by the $L^1_{x,v}$ norm.

How can we measure dispersion in the $L^1_{x,v}$ setting?

- Local decay? $\int_{B(0,R)\times\mathbb{R}^d} f(t,x,v)dxdv \to 0$ as $t \to \infty$? $(\forall 0 < R < \infty)$
- Mass escaping to infinity? $\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x|^\alpha dxdv \to \infty$ as $t \to \infty$? $(\forall \alpha > 0)$
- Is there any other appropriate measure of dispersion? Yes! (the entropy)

Important Remark

Important Remark

Solutions to collisional kinetic equations satisfy formally that

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x-tv|^2 dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)(x-tv)\cdot v dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|v|^2 dxdv = 0$$

Important Remark

Solutions to collisional kinetic equations satisfy formally that

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x-tv|^2 dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)(x-tv)\cdot v dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|v|^2 dxdv = 0$$

$$\begin{aligned}\Rightarrow \int f(t,x,v)|x|^2 dxdv &= \int f(t,x,v)\left(|x-tv|^2 + 2t(x-tv)\cdot v + t^2|v|^2\right) dxdv \\ &= \int f_0|x|^2 dxdv + 2t\int f_0 x\cdot v dxdv + t^2\int f_0|v|^2 dxdv\end{aligned}$$

Important Remark

Solutions to collisional kinetic equations satisfy formally that

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x-tv|^2 dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)(x-tv)\cdot v dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|v|^2 dxdv = 0$$

$$\Rightarrow \int f(t,x,v)|x|^2 dxdv = \int f(t,x,v)\left(|x-tv|^2 + 2t(x-tv)\cdot v + t^2|v|^2\right) dxdv$$

$$= \int f_0|x|^2 dxdv + 2t\int f_0 x\cdot v dxdv + t^2\int f_0|v|^2 dxdv$$

$\Rightarrow$ Formally, one can expect $\int f(t,x,v)|x|^2 dxdv$ to grow quadratically as $t\to\infty$.

Important Remark

Solutions to collisional kinetic equations satisfy formally that

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x-tv|^2 dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)(x-tv)\cdot v dxdv = 0$$

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|v|^2 dxdv = 0$$

$$\begin{aligned}\Rightarrow \int f(t,x,v)|x|^2 dxdv &= \int f(t,x,v)\left(|x-tv|^2 + 2t(x-tv)\cdot v + t^2|v|^2\right) dxdv \\ &= \int f_0|x|^2 dxdv + 2t\int f_0 x\cdot v dxdv + t^2\int f_0|v|^2 dxdv\end{aligned}$$

$\Rightarrow$ Formally, one can expect $\int f(t,x,v)|x|^2 dxdv$ to grow quadratically as $t\to\infty$.

However, rigorously, one only has weak sequential lower semi-continuity of convex functionals and, therefore, we only know that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)|x|^2 dxdv \lesssim t^2$$

Achieving Nontrivial Endpoint Estimates

Achieving Nontrivial Endpoint Estimates

Basic idea (classical):
Suppose $h_1, h_2 \in C^1([0,\infty),\mathbb{R})$ are such that

$$\boxed{h_1(\lambda) \leq h_2(\lambda) \ (\forall \lambda > 0) \quad \text{and} \quad h_1(0) = h_2(0)}$$

Then,

$$\boxed{h_1'(0) \leq h_2'(0)}$$

Achieving Nontrivial Endpoint Estimates

Basic idea (classical):
Suppose $h_1, h_2 \in C^1([0,\infty),\mathbb{R})$ are such that

$$\boxed{h_1(\lambda) \leq h_2(\lambda)\ (\forall \lambda > 0) \quad \text{and} \quad h_1(0) = h_2(0)}$$

Then,

$$\boxed{h_1'(0) \leq h_2'(0)}$$

Examples:

- The logarithmic HLS inequality
- The entropic uncertainty principle

Example: The Logarithmic HLS Inequality

[Carlen, Loss; Geom. and Funct. An., 1992] [Beckner; Annals of Math., 1993]

Example: The Logarithmic HLS Inequality

[Carlen, Loss; Geom. and Funct. An., 1992] [Beckner; Annals of Math., 1993]

The HLS inequality:

$$\boxed{\int_{\mathbb{R}^d \times \mathbb{R}^d} h(x)|x-y|^{-\lambda} h(y) dxdy \leq C(d,\lambda) \|h\|_{L^p}^2}$$

with $p = \frac{2d}{2d-\lambda}$, $0 \leq \lambda < d$, and $C(d,\lambda)$ the optimal constant from [Lieb; Ann. of Math., 1983].

Example: The Logarithmic HLS Inequality

[Carlen, Loss; Geom. and Funct. An., 1992] [Beckner; Annals of Math., 1993]

The HLS inequality:

$$\boxed{\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x)|x-y|^{-\lambda}h(y)dxdy \leq C(d,\lambda)\|h\|_{L^p}^2}$$

with $p=\frac{2d}{2d-\lambda}$, $0\leq\lambda<d$, and $C(d,\lambda)$ the optimal constant from [Lieb; Ann. of Math., 1983].

For $\lambda=0$, one has an equality. $\Rightarrow$ Take derivative $\frac{d}{d\lambda}$ and then set $\lambda=0$ to deduce

$$\boxed{\begin{aligned}-\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x)\log|x-y|h(y)dxdy \leq {} & \left(\frac{d}{d\lambda}C(d,\lambda)|_{\lambda=0}\right)\|h\|_{L^1}^2 \\ & + \frac{\|h\|_{L^1}}{d}\int_{\mathbb{R}^d} h(x)\log\left(\frac{h(x)}{\|h\|_{L^1}}\right)dx\end{aligned}}$$

for all suitable $h\geq 0$.

(See also logarithmic Sobolev inequalities [Gross; Am. J. Math., 1975])

Example: The Entropic Uncertainty Principle

[Beckner; Annals of Math., 1975]

Example: The Entropic Uncertainty Principle

[Beckner; Annals of Math., 1975]

Hausdorff–Young inequality:

$$\boxed{\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \ (\forall 1 \leq p \leq 2) \text{ with } \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} f(x) dx}$$

With a sharp constant:

$$\|\hat{f}\|_{L^{p'}} \leq \left(p^{\frac{1}{p}} / p'^{\frac{1}{p'}} \right)^{\frac{1}{2}} \|f\|_{L^p}$$

Example: The Entropic Uncertainty Principle

[Beckner; Annals of Math., 1975]

Hausdorff–Young inequality:

$$\boxed{\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \ (\forall 1 \leq p \leq 2) \text{ with } \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi \cdot x} f(x) dx}$$

With a sharp constant:

$$\|\hat{f}\|_{L^{p'}} \leq \left(p^{\frac{1}{p}} / p'^{\frac{1}{p'}} \right)^{\frac{1}{2}} \|f\|_{L^p}$$

For $p = 2$, one has an equality. $\Rightarrow$ Normalize $\|f\|_{L^2} = \|\hat{f}\|_{L^2} = 1$, take derivative $\frac{d}{dp}$ and set $p = 2$ to deduce the entropic uncertainty principle:

$$\boxed{\int_{\mathbb{R}} f^2 \log(f^2) dx + \int_{\mathbb{R}} \hat{f}^2 \log(\hat{f}^2) dx \leq \log \frac{2}{e}}$$

Example: The Entropic Uncertainty Principle

[Beckner; Annals of Math., 1975]

Example: The Entropic Uncertainty Principle

[Beckner; Annals of Math., 1975]

Recall that the entropy is minimized by Maxwellians (i.e., Gaussian distributions):

$$\Rightarrow \int_{\mathbb{R}} f^2 \log(f^2) dx \geq -\frac{1}{2} \log\left(2\pi e \int_{\mathbb{R}} f(x)^2 (x - x_0)^2 dx\right)$$

$$\int_{\mathbb{R}} \hat{f}^2 \log(\hat{f}^2) d\xi \geq -\frac{1}{2} \log\left(2\pi e \int_{\mathbb{R}} \hat{f}(\xi)^2 (\xi - \xi_0)^2 d\xi\right)$$

Example: The Entropic Uncertainty Principle

[Beckner; Annals of Math., 1975]

Recall that the entropy is minimized by Maxwellians (i.e., Gaussian distributions):

$$\Rightarrow \int_{\mathbb{R}} f^2 \log(f^2) dx \geq -\frac{1}{2} \log \left(2\pi e \int_{\mathbb{R}} f(x)^2 (x - x_0)^2 dx \right)$$

$$\int_{\mathbb{R}} \hat{f}^2 \log(\hat{f}^2) d\xi \geq -\frac{1}{2} \log \left(2\pi e \int_{\mathbb{R}} \hat{f}(\xi)^2 (\xi - \xi_0)^2 d\xi \right)$$

Combining this with the entropic uncertainty principle recovers the Heisenberg–Weyl uncertainty principle:

$$\boxed{\frac{1}{4\pi} \leq \left(\int_{\mathbb{R}} f(x)^2 (x - x_0)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \hat{f}(\xi)^2 (\xi - \xi_0)^2 d\xi \right)^{\frac{1}{2}}}$$

Entropic Dispersion

Entropic Dispersion

We apply the previous logarithmic endpoint method to the Castella–Perthame estimates, for suitable densities $f_0(x,v) \geq 0$, which gives

$$\boxed{\begin{aligned}\int \left(\int f_0(x-tv,v)dv \right) \log \left(\frac{\int f_0(x-tv,v)dv}{\iint f_0(y,w)dydw} \right) dx + d \log|t| \iint f_0(x,v)dxdv \\ \leq \iint f_0(x,v) \log \left(\frac{f_0(x,v)}{\int f_0(x,w)dw} \right) dxdv\end{aligned}}$$

Entropic Dispersion

We apply the previous logarithmic endpoint method to the Castella–Perthame estimates, for suitable densities $f_0(x,v) \geq 0$, which gives

$$\boxed{\begin{aligned} \int \left(\int f_0(x-tv,v)dv \right) \log \left(\frac{\int f_0(x-tv,v)dv}{\iint f_0(y,w)dydw} \right) dx + d \log|t| \iint f_0(x,v)dxdv \\ \leq \iint f_0(x,v) \log \left(\frac{f_0(x,v)}{\int f_0(x,w)dw} \right) dxdv \end{aligned}}$$

and, if $\iint f_0(x,v)dxdv = 1$,

$$\boxed{\begin{aligned} \int \exp \left(\frac{2}{d} \int \left(\int f_0(x-tv,v)dv \right) \log \left(\int f_0(x-tv,v)dv \right) dx \right) dt \\ \leq C_* \exp \left(\frac{1}{d} \iint f_0(x,v) \log f_0(x,v)dxdv \right) \end{aligned}}$$

Entropic Dispersion

Entropic Dispersion

Remarks:

- C_* can be computed but likely not optimal.
- Similar estimates can be obtained for the source $\int_0^t g(s, x-(t-s)v, v)ds$.
- Major obstacle: These estimates require nonnegative densities and they are not monotonic (because $0 \leq f \leq h \not\Rightarrow f \log f \leq h \log h$).
- It is possible to incorporate loss terms of collision operators into dispersive estimates as a damping, which makes all densities nonnegative. (This idea can be applied to the BGK equation, for instance.)

Kinetic Uncertainty Principle

Kinetic Uncertainty Principle

Take $f(x,y) \geq 0$ and write $\Phi(z) = z \log z$. Applying Jensen's inequality with the probability measure $\frac{\int f(x,y)dx}{\iint f(x,y)dxdy} dy$ gives

$$\iint f(x,y)dxdy\Phi\left(\frac{\int f(x,y)dy}{\iint f(x,y)dxdy}\right) \leq \int \left(\int f(x,y)dx\right) \Phi\left(\frac{f(x,y)}{\int f(x,y)dx}\right) dy$$

which, supposing $\iint f(x,y)dxdy = 1$, for simplicity, can be rewritten into the kinetic uncertainty principle:

$$\boxed{\begin{aligned} &\int \left(\int f(x,y)dy\right) \log\left(\int f(x,y)dy\right) dx \\ &\quad + \int \left(\int f(x,y)dx\right) \log\left(\int f(x,y)dx\right) dy \\ &\quad \leq \iint f(x,y) \log f(x,y) dxdy \end{aligned}}$$

Kinetic Uncertainty Principle (continued)

Kinetic Uncertainty Principle (continued)

⇒ Take $f(x,v) \geq 0$ and write

$$m_0 = \int f dx dv \qquad m_{2,x} = \int f(x,v)|x - x_0|^2 dx dv$$

$$m_{2,v} = \int f(x,v)|v - v_0|^2 dx dv \qquad H = \int f(x,v) \log f(x,v) dx dv$$

⇒ Recall that entropies are minimized by Maxwellians. The kinetic uncertainty principle gives

$$\boxed{(m_{2,x} m_{2,v})^{\frac{1}{2}} \geq \frac{d}{2\pi e} m_0^{1+\frac{1}{d}} \exp\left(-\frac{H}{d m_0}\right)}$$

Kinetic Uncertainty Principle (continued)

$\Rightarrow$ Take $f(x,v) \geq 0$ and write

$$m_0 = \int f dx dv \qquad m_{2,x} = \int f(x,v)|x - x_0|^2 dx dv$$

$$m_{2,v} = \int f(x,v)|v - v_0|^2 dx dv \qquad H = \int f(x,v) \log f(x,v) dx dv$$

$\Rightarrow$ Recall that entropies are minimized by Maxwellians. The kinetic uncertainty principle gives

$$\boxed{(m_{2,x} m_{2,v})^{\frac{1}{2}} \geq \frac{d}{2\pi e} m_0^{1+\frac{1}{d}} \exp\left(-\frac{H}{d m_0}\right)}$$

Remarks:

- This kinetic uncertainty principle is purely statistical!
- No quantum particle/wave interpretation.
- However, if $f(x,v)$ is not allowed to concentrate on very small scales (smaller than Planck's constant), then the entropy will be bounded by a constant.
- If $f(x,v)$ is the Husimi transform of some wave function, then H is Wehrl's entropy and is bounded by a constant [Lieb; CMP, 1978].

Kinetic Uncertainty Principle (continued)

Kinetic Uncertainty Principle (continued)

Next, suppose that $f(t,x,v) \geq 0$ satisfies that

1. $f(0,x,v) = f_0(x,v)$
2. $m_0 = \int f(t,x,v)dxdv = \int f_0(x,v)dxdv$
3. $\int f(t,x,v)|v|^2 dxdv \leq \int f_0(x,v)|v|^2 dxdv$
4. $\int f(t,x,v)|x-tv|^2 dxdv \leq \int f_0(x,v)|x|^2 dxdv$
5. $H(t) = \int f \log f(t,x,v)dxdv \leq \int f_0 \log f_0(x,v)dxdv = H_0$

Kinetic Uncertainty Principle (continued)

Next, suppose that $f(t,x,v) \geq 0$ satisfies that

1. $f(0,x,v) = f_0(x,v)$
2. $m_0 = \int f(t,x,v)dxdv = \int f_0(x,v)dxdv$
3. $\int f(t,x,v)|v|^2 dxdv \leq \int f_0(x,v)|v|^2 dxdv$
4. $\int f(t,x,v)|x-tv|^2 dxdv \leq \int f_0(x,v)|x|^2 dxdv$
5. $H(t) = \int f \log f(t,x,v)dxdv \leq \int f_0 \log f_0(x,v)dxdv = H_0$

Writing $\rho(t,x) = \int f(t,x,v)dv$, applying the kinetic uncertainty principle to $f(t,y,\frac{y-x}{t})$, and employing that entropies are optimized by Maxwellians, one can show that

$$\boxed{\begin{aligned} \int \rho \log \rho(t,x)dx + dm_0 \log|t| + \frac{d}{2} m_0 \log\left(\frac{d}{2\pi e} m_0\right) \\ \leq H_0 + \frac{d}{2} m_0 \log\left(\iint f_0(x,v)|x|^2 dxdv\right) \end{aligned}}$$

Kinetic Uncertainty Principle (continued)

Kinetic Uncertainty Principle (continued)

$$\boxed{\begin{aligned}\int \rho \log \rho(t,x)dx + dm_0 \log|t| + \frac{d}{2} m_0 \log\left(\frac{d}{2\pi e} m_0\right) \\ \leq H_0 + \frac{d}{2} m_0 \log\left(\iint f_0(x,v)|x|^2 dxdv\right)\end{aligned}}$$

Kinetic Uncertainty Principle (continued)

$$\boxed{\begin{aligned}\int \rho \log \rho(t,x)dx + dm_0 \log|t| + \frac{d}{2} m_0 \log\left(\frac{d}{2\pi e} m_0\right)& \\ \leq H_0 + \frac{d}{2} m_0 \log\left(\iint f_0(x,v)|x|^2 dxdv\right)&\end{aligned}}$$

Remarks:

- This estimate is stable with respect to weak convergence.
- It shows the decay of $\int \rho \log \rho(t,x)dx$ as $t \to \infty$. It's a measure of dispersion.
- It works for renormalized solutions of the Boltzmann equation.
- It is uniform in the Knudsen number.
- It applies to solutions of the compressible Euler system. (Apply it to the Maxwellian $f = M_f$.)

Kinetic Uncertainty Principle (continued)

Kinetic Uncertainty Principle (continued)

Using, again, that the entropy $\int \rho \log \rho(t,x)dx$ is minimized by a Maxwellian gives

$$\boxed{\begin{aligned}&\log(t^2) + \left(2+\frac{2}{d}\right)\log m_0 + 2\log\left(\frac{d}{2\pi e}\right)\\ &\qquad \leq H_0 + \log\left(\iint f_0(x,v)|x|^2 dxdv\right) + \log\left(\iint f(t,x,v)|x|^2 dxdv\right)\end{aligned}}$$

Kinetic Uncertainty Principle (continued)

Using, again, that the entropy $\int \rho \log \rho(t,x)dx$ is minimized by a Maxwellian gives

$$\boxed{\begin{aligned}&\log(t^2) + \left(2+\frac{2}{d}\right)\log m_0 + 2\log\left(\frac{d}{2\pi e}\right)\\ &\qquad \le H_0 + \log\left(\iint f_0(x,v)|x|^2 dxdv\right) + \log\left(\iint f(t,x,v)|x|^2 dxdv\right)\end{aligned}}$$

Theorem: Renormalized solutions of the Boltzmann equation satisfy that

$$\boxed{\int f(t,x,v)|x|^2 dxdv \sim t^2 \text{ as } t\to\infty.}$$

Kinetic Uncertainty Principle (continued)

Using, again, that the entropy $\int \rho \log \rho(t,x)dx$ is minimized by a Maxwellian gives

$$\boxed{\begin{aligned}&\log(t^2) + \left(2 + \frac{2}{d}\right)\log m_0 + 2\log\left(\frac{d}{2\pi e}\right)\\ &\qquad \leq H_0 + \log\left(\iint f_0(x,v)|x|^2 dxdv\right) + \log\left(\iint f(t,x,v)|x|^2 dxdv\right)\end{aligned}}$$

Theorem: Renormalized solutions of the Boltzmann equation satisfy that

$$\boxed{\int f(t,x,v)|x|^2 dxdv \sim t^2 \text{ as } t \to \infty.}$$

Theorem: (The preceding estimates can be localized.) $\forall R > 0$, $\exists C_0 > 0$ such that

$$\boxed{\int_{B(0,R)} \rho(t,x)dx \leq \frac{C_0}{\log t}, \ \forall t > 0.}$$

The Heat Death of the Universe

The Heat Death of the Universe

Theorem: Consider $\widetilde{\rho}(t,x) = t^d \rho(t,tx)$ and $\widetilde{f}(t,x,v) = t^d f(t,tx,v)$. Up to a Galilean transformation, the family $\{\widetilde{\rho}(t,x)\}_{t>0}$ is relatively compact in the weak topology of $L^1(dx)$, whereas $\{\widetilde{f}(t,x,v)\}_{t>0}$ forms a tight family of bounded measures. Let $\chi \in L^1(dx)$ be a weak limit point $\widetilde{\rho}(t) \rightharpoonup \chi$ in L^1, as $t \to \infty$ (for a subsequence). Up to extraction, it holds that

$$\widetilde{f} \rightharpoonup^* \chi(x) \otimes \delta_{v=x}$$

in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$, as $t \to \infty$.

The Heat Death of the Universe

Theorem: Consider $\widetilde{\rho}(t,x) = t^d \rho(t,tx)$ and $\widetilde{f}(t,x,v) = t^d f(t,tx,v)$. Up to a Galilean transformation, the family $\{\widetilde{\rho}(t,x)\}_{t>0}$ is relatively compact in the weak topology of $L^1(dx)$, whereas $\{\widetilde{f}(t,x,v)\}_{t>0}$ forms a tight family of bounded measures. Let $\chi \in L^1(dx)$ be a weak limit point $\widetilde{\rho}(t) \rightharpoonup \chi$ in L^1, as $t \to \infty$ (for a subsequence). Up to extraction, it holds that

$$\widetilde{f} \rightharpoonup^* \chi(x) \otimes \delta_{v=x}$$

in $\mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d)$, as $t \to \infty$.

Interpretation: As $t \to \infty$, the gas reaches a state with no internal energy left to sustain thermodynamic process. The energy has been fully converted into kinetic energy and all particles move away from each other.