Asymptotic Analysis of the Charge-Conserving Poisson–Boltzmann Equation with Isolated Singularities

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Charge-Conserving Poisson–Boltzmann Equation

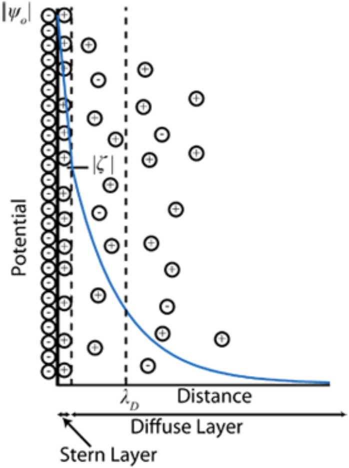
In this talk, we consider the boundary value problem of Poisson–Boltzmann model:

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\displaystyle\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\displaystyle\int_\Omega e^{-u}} & \text{in } \Omega \backslash \{P_1, ..., P_N\}, \\ u = u_0 & \text{on } \partial\Omega. \end{cases} \qquad \text{(PB)}$$

- $\Omega \subset \mathbb{R}^n$, $n \geq 2$: bounded, simply connected domain with smooth boundary.
- $P_1, ..., P_N$: N locations in Ω.
- α, β, ϵ: positive constants.
- u_0: smooth function on $\partial\Omega$.

Background

Electric Double Layer[1]



- Helmholtz (1853)
- Gouy (1910)
- Chapman (1913)
- Debye–Hückel (1923)
- Stern (1924)

[1] http://web.mit.edu/lemi/rsc_electrokinetics.html

Gouy–Chapman Theory

- Gauss's law:

$$\nabla \cdot E = \frac{\rho}{\varepsilon}.$$

  E: electric field; ρ: charge density; ε: dielectric constant.

- electric potential u:

$$E = -\nabla u.$$

- charge density $\rho = c_p - c_n$.
  c_p: charge density of positive ions;
  c_n: charge density of negative ions.

- Poisson equation:

$$\varepsilon \Delta u = c_n - c_p.$$

Gouy–Chapman Theory, continued

- Gouy–Chapman's prediction: Boltzmann distribution

$$c_n = c_{n,0} e^{u}, \quad c_p = c_{p,0} e^{-u}.$$

  $c_{n,0}$, $c_{p,0}$: the reference charge densities of negative and positive ions taken at zero potential.

- For a bounded domain, we have the charge conservation:

$$\int_\Omega c_n = \alpha, \quad \int_\Omega c_p = \beta \quad \Rightarrow \quad c_{n,0} = \frac{\alpha}{\int_\Omega e^{u}}, \quad c_{p,0} = \frac{\beta}{\int_\Omega e^{-u}}.$$

  α, β: total charges of negative and positive ions.

- dielectric constant $\varepsilon = \varepsilon_0 \varepsilon_r$.
  $\varepsilon_0 \sim 8.85 \times 10^{-12}\ F \cdot m^{-1}$: vacuum permittivity;
  ε_r: relative dielectric constant ($\varepsilon_r \sim 80$ for water).

Charge-conserving Poisson–Boltzmann equation:

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\displaystyle\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\displaystyle\int_\Omega e^{-u}} & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

α, β: total charges of negative and positive ions

u_0: surface potential

ϵ^2: dielectric constant

Poisson–Nernst–Planck Equation

The solution to the charge-conserving Poisson–Boltzmann equation can be regarded as the steady-state electric potential u in the Poisson–Nernst–Planck (PNP) equation for ionic transport

$$\begin{cases} \partial_t c_n = \nabla \cdot (\nabla c_n - c_n \nabla u)\,, \\ \partial_t c_p = \nabla \cdot (\nabla c_p + c_p \nabla u)\,, \\ \epsilon^2 \Delta u = c_n - c_p, \end{cases}$$

with no-flux boundary conditions for c_n and c_p.

- Applications in semiconductors, ion channels, batteries, ...
- Well-posedness, long-time behavior:
  Biler–Hebisch–Nadzieja (1994), Biler–Dolbeault (2000), Hsieh–Lin (2015), ...
- With fixed point charges:
  Biler–Nadzieja (1997), Hsieh–Yu (2021), ...

Poisson–Boltzmann Model with Fixed Point Charges

- Rubinstein (1986), Friedman–Tintarev (1987), Krzywicki–Nadzieja (1990): Radially symmetric solutions on annuli.
- Point charges described by Dirac measures:

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\displaystyle\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\displaystyle\int_\Omega e^{-u}} + \displaystyle\sum_{j=1}^{N} a_j \delta_{P_j} & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

  for a_j's $\in \mathbb{R} \backslash \{0\}$.
- More generally, we consider

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\displaystyle\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\displaystyle\int_\Omega e^{-u}} & \text{in } \Omega \backslash \left\{P_1, ..., P_N\right\}, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Known Results

Charge-conserving Poisson–Boltzmann equation:

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\displaystyle\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\displaystyle\int_\Omega e^{-u}} & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

- Lee (2014): Existence and uniqueness.
- Lee (2016): If $\alpha = \beta$, given K compact subset of Ω,

$$\max_{x_1, x_2 \in K} |u(x_1) - u(x_2)| \longrightarrow 0 \quad \text{exponentially as} \quad \epsilon \to 0.$$

- Lee–Lee–Hyon–Lin–Liu (2011): If $\alpha = \beta$, $n = 1$, $\Omega = (a, b)$,

$$\lim_{\epsilon \to 0} u(x) = \frac{1}{2}(u_0(a) + u_0(b)),$$

for all $x \in (a, b)$.

Questions

1. With electroneutrality $\alpha = \beta$, is there an limit for the solution u as ϵ goes to 0? If so, what is the limit?

   - We only have
   $$\max_{x_1, x_2 \in K} |u(x_1) - u(x_2)| \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0$$
   for compact subsets $K \subset \Omega$.

   - The equation
   $$\epsilon^2 \Delta u = \frac{\alpha e^u}{\int_\Omega e^u} - \frac{\beta e^{-u}}{\int_\Omega e^{-u}}$$
   is invariant under adding a constant.

Questions

2. Existence, Uniqueness and Asymptotic behavior with fixed point charges.

- Solutions $u \notin H^1$.
- No maximum principle.
- Remark: Without singularity and $\alpha = \beta$,

$$\min_{\partial\Omega} u_0 \le u \le \max_{\partial\Omega} u_0.$$

Proof: If u is not constant and attains its minimum at $x_0 \in \Omega$,

$$0 \le \frac{\epsilon^2}{\alpha} \Delta u \bigg|_{x_0} = \frac{e^{u(x_0)}}{\int_\Omega e^u} - \frac{e^{-u(x_0)}}{\int_\Omega e^{-u}} < 0.$$

Existence and Uniqueness

Elliptic Equations with Isolated Singularities

Consider a general elliptic equation

$$\Delta u = g(u) \quad \text{in } \Omega \backslash \{0\}.$$

Ω: domain in $\mathbb{R}^n$ containing 0
g: smooth, non-decreasing function
For $n = 2$, Vazquez–Veron (1984): If

$$\lim_{|s| \to +\infty} g(s) s^{-3} = +\infty,$$

then the limit

$$c = \lim_{x \to 0} \frac{u(x)}{\log 1/|x|}$$

exists in $\mathbb{R} \cup \{+\infty, -\infty\}$. In addition, the limit c satisfies $-2/a_g^- \leq c \leq 2/a_g^+$, where

$$a_g^+ := \sup \left\{ a > 0 : \lim_{s \to +\infty} e^{-as} g(s) = +\infty \right\},$$
$$a_g^- := \sup \left\{ a > 0 : \lim_{s \to -\infty} e^{as} g(s) = -\infty \right\}.$$

Consider a general elliptic equation

$$\Delta u = g(u) \quad \text{in } \Omega \backslash \{0\}.$$

Ω: domain in $\mathbb{R}^n$ containing 0
g: smooth, non-decreasing function

For $n \geq 3$, Brezis–Veron (1980): If g satisfies

$$\liminf_{s \to +\infty} \frac{g(s)}{s^{n/(n-2)}} > 0 \quad \text{and} \quad \limsup_{s \to -\infty} \frac{g(s)}{|s|^{n/(n-2)}} < 0,$$

then there exists a C^2 function on Ω which coincides almost everywhere with u.

Necessary Conditions

Let $u \in C^2(\Omega \backslash \{P_1, ..., P_N\})$ be a solution of

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\displaystyle\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\displaystyle\int_\Omega e^{-u}} & \text{in } \Omega \backslash \left\{P_1, ..., P_N\right\}, \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

- If $n = 2$, one of the following alternatives holds
  - (i) u can be smoothly extended across P_j;
  - (ii) u has order $\alpha_j \in [-2, 2] \backslash \{0\}$ at P_j.

- If $n \geq 3$, all the singularities are removable.

Main Results, Existence and Uniqueness

Theorem (Hsieh–Yu, ARMA (2020))

For $n = 2$, given $\alpha_1, ..., \alpha_N \in [-2, 2]$, there exists a unique solution u to the boundary value problem (PB) such that the order of u at P_j equals to α_j for each j. In addition, $u \in C^\infty(\overline{\Omega}\backslash\{P_1, ..., P_N\})$. And locally near P_j, if $\alpha_j \neq 0$, u can be represented by

$$\begin{cases} u = -\alpha_j \log|x - P_j| + u_j & \text{if } \alpha_j \in (-2, 2)\backslash\{0\}, \\ u = -\alpha_j \log|x - P_j| - \alpha_j \log\log|x - P_j|^{-1} + u_j & \text{if } \alpha_j \in \{-2, 2\}, \end{cases}$$

for some continuous function u_j. Moreover, ∇u_j is continuous if $|\alpha_j| < 1$, and

$$\nabla u_j = \begin{cases} O(1) & \text{if } \ |\alpha_j| = 1, \\ O\left(|x - P_j|^{1-|\alpha_j|}\right) & \text{if } \ 1 < |\alpha_j| < 2, \\ O\left(|x - P_j|^{-1}\log^{-2}|x - P_j|^{-1}\right) & \text{if } \ |\alpha_j| = 2, \end{cases}$$

as $x \to P_j$, for $j = 1, ..., N$.

Main Results, Existence and Uniqueness, continued

Theorem (Hsieh–Yu, ARMA (2020))

If $n \geq 3$, there exists a unique solution u to the problem (PB). Moreover, u can be smoothly extended across each P_j so that $u \in C^\infty(\overline{\Omega})$.

Remark

In two-dimensional case, with an introduction of the Dirac measures as the fixed charges, u is the unique weak solution of

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{\alpha e^u}{\int_\Omega e^u} - \dfrac{\beta e^{-u}}{\int_\Omega e^{-u}} - 2\pi\epsilon^2 \sum_{j=1}^{N} \alpha_j \delta_{P_j} & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

where δ_y is the Dirac measure centered at y.

Ideas of the Proof, Existence and Uniqueness

- When there is no singularity, the energy functional for classical PB:

$$I_0[u] := \frac{\epsilon^2}{2}\int_\Omega |\nabla u|^2 + \alpha \log \int_\Omega e^u + \beta \log \int_\Omega e^{-u}.$$

  Remark: It holds for the case $n \geq 3$.

- For $n = 2$, we use the ansatz

$$u = v - \sum_{j=1}^{N} \alpha_j \log |x - P_j|, \qquad v \in H^1(\Omega).$$

- Elliptic regularity theory.
- Local representation of u similar to Gaussian curvature equation (Kraus–Roth (2008)).

Asymptotics

Normal Coordinates

In order to describe the asymptotics of u^ϵ near P_j's and $\partial\Omega$

- Near P_j, we define

$$\sigma_* := \frac{1}{4}\min\{1, |P_i - P_j|, \mathrm{dist}(P_k, \partial\Omega) : i, j, k = 1, ..., N \text{ and } i \neq j\}$$

  and consider $B_{\sigma_*}(P_j)$, $j = 1, ..., N$.

- Near $\partial\Omega$, we define

$$\Omega_\eta := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \eta\}.$$

  we can choose a constant $0 < \eta_* < \dfrac{1}{2}\min\{\mathrm{dist}(P_j, \partial\Omega) : j = 1, ..., N\}$ small enough such that Ω_{η_*} admits the normal coordinate system (d, π) such that

$$x = \pi(x) - d(x)\nu(x)$$

  for each $x \in \Omega_{\eta_*}$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance between x and $\partial\Omega$, $\pi(x)$ is the projection of x on $\partial\Omega$, and $\nu(x)$ is the outer normal vector of $\partial\Omega$ at $\pi(x)$. Moreover, d and π are smooth on $\overline{\Omega_{\eta_*}}$.

Theorem (Hsieh–Yu, ARMA (2020))

Assume that $\alpha = \beta = 1$. *When* $n = 2$, *let* u^ϵ *be the solution of (PB), which has order* α_j *at* P_j *with* $\alpha_j \in [-2, 2] \backslash \{0\}$, $j = 1, ..., N$. *We have*

(i) *For any compact subset* K *of* $\Omega \backslash \{P_1, ..., P_N\}$, $\max_{x_1, x_2 \in K} |u^\epsilon(x_1) - u^\epsilon(x_2)|$ *tends to zero exponentially as* $\epsilon \to 0$.

(ii) $\|u^\epsilon - U^\epsilon\|_{L^\infty(\Omega)} = O\left(\epsilon \log \frac{1}{\epsilon}\right)$ *as* $\epsilon \to 0$, *where*

$$U^\epsilon(x) := \begin{cases} \bar{\mu} + V_j^\epsilon(x) & \text{for } x \in B_{\sigma_*}(P_j), j = 1, ..., N, \\ \bar{\mu} + V_0^\epsilon(x) & \text{for } x \in \Omega_{\eta_*}, \\ \bar{\mu}, & \text{otherwise.} \end{cases}$$

Main Results, Asymptotics

Theorem (continued)

Here $\bar{\mu}$ is a constant defined by

$$\bar{\mu} := \log \frac{\displaystyle\int_{\partial\Omega} e^{\frac{u_0(y)}{2}}}{\displaystyle\int_{\partial\Omega} e^{-\frac{u_0(y)}{2}}}.$$

For $j = 1, ..., N$, $V_j^\epsilon(x)$ is the solution of

$$\begin{cases} \epsilon^2 \Delta V_j^\epsilon = \dfrac{1}{|\Omega|}\left(e^{V_j^\epsilon} - e^{-V_j^\epsilon}\right) & \text{in } B_{\sigma_*}(P_j)\backslash\{P_j\}, \\ V_j^\epsilon = 0 & \text{on } \partial B_{\sigma_*}(P_j), \\ \lim_{x\to P_j} \dfrac{V_j^\epsilon(x)}{\log|x-P_j|^{-1}} = \alpha_j. & \end{cases}$$

And V_0^ϵ is defined by

$$V_0^\epsilon(x) := 2\log\left(\frac{1 + a(\pi(x))e^{-\sqrt{\frac{2}{|\Omega|}}\frac{d(x)}{\epsilon}}}{1 - a(\pi(x))e^{-\sqrt{\frac{2}{|\Omega|}}\frac{d(x)}{\epsilon}}}\right), \quad \text{with} \quad a(y) = \frac{e^{\frac{u_0(y)-\bar{\mu}}{2}} - 1}{e^{\frac{u_0(y)-\bar{\mu}}{2}} + 1},$$

by employing the normal coordinates in Ω_{η_*}.

Main Results, Asymptotics

Theorem (Hsieh–Yu, ARMA (2020))

If $n \geq 3$, then u^ϵ, the solution of (PB), satisfies

(i) *For any compact subset K of Ω,* $\max_{x_1,x_2\in K} |u^\epsilon(x_1) - u^\epsilon(x_2)|$ *tends to zero exponentially as $\epsilon \to 0$.*

(ii) $\|u^\epsilon - U^\epsilon\|_{L^\infty(\Omega)} = O\left(\epsilon \log \frac{1}{\epsilon}\right)$ *as $\epsilon \to 0$, where*

$$U^\epsilon(x) := \begin{cases} \bar{\mu} + V_0^\epsilon(x) & \text{for } x \in \Omega_{\eta_*}, \\ \bar{\mu}, & \text{otherwise.} \end{cases}$$

Here $\bar{\mu}$ and V_0^ϵ are defined in the same way as in the previous Theorem.

Definitions of $M[u^\epsilon]$, $\mu[u^\epsilon]$ and v^ϵ

To deal with the non-local coefficients in (PB)

$$\begin{cases} \epsilon^2 \Delta u = \dfrac{e^u}{\displaystyle\int_\Omega e^u} - \dfrac{e^{-u}}{\displaystyle\int_\Omega e^{-u}} & \text{in } \Omega \backslash \{P_1, ..., P_N\}, \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

we let

$$M[u^\epsilon] := \left(\int_\Omega e^{u^\epsilon} \int_\Omega e^{-u^\epsilon} \right)^{1/2} \quad \text{and} \quad \mu[u^\epsilon] := \frac{1}{2} \log \frac{\displaystyle\int_\Omega e^{u^\epsilon}}{\displaystyle\int_\Omega e^{-u^\epsilon}}.$$

Then the function $v^\epsilon := u^\epsilon - \mu[u^\epsilon]$ satisfies

$$\begin{cases} \epsilon^2 \Delta v^\epsilon = M[u^\epsilon]^{-1} \left[e^{v^\epsilon} - e^{-v^\epsilon} \right] & \text{in } \Omega \backslash \{P_1, ..., P_N\}, \\ v^\epsilon = u_0 - \mu[u^\epsilon] & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$M[u^\epsilon] = \int_\Omega e^{v^\epsilon} = \int_\Omega e^{-v^\epsilon}.$$

Sketch of the Proof

- Uniform bounds of $M[u^\epsilon]$ and $\mu[u^\epsilon]$:
  - By Hölder's inequality,

$$M[u^\epsilon] = \left(\int_\Omega e^{u^\epsilon} \int_\Omega e^{-u^\epsilon}\right)^{1/2} \geq |\Omega|$$

  - Assume that $M[u^{\epsilon_k}] \to \infty$ as $k \to \infty$.
  - Without loss of generality, we can assume $\mu[u^{\epsilon_k}] \geq b \in \mathbb{R}$.
  - Construct $w = -\sum_{\alpha_j > 0} \alpha_j \log|x - P_j| - 2 \sum_{\alpha_j = 2} \log\log|x - P_j|^{-1} + w_0$ such that $w \geq v_k := u^{\epsilon_k} - \mu[u^{\epsilon_k}]$ with $\int_\Omega e^w < \infty$. Then we have

$$M[u^{\epsilon_k}] = \int_\Omega e^{v_k} \leq \int_\Omega e^w < \infty, \qquad \text{a contradiction.}$$

  - For $\mu[u^\epsilon]$, by comparison principle ...

Sketch of the Proof, continued

- Asymptotic behavior of v^ϵ:

$$\begin{cases} M\left[u^\epsilon\right]\epsilon^2\Delta v^\epsilon = e^{v^\epsilon} - e^{-v^\epsilon} & \text{in } \Omega\backslash\left\{P_1, ..., P_N\right\} \text{ (or } \Omega \text{ for } n \geq 3), \\ v^\epsilon = u_0 - \mu\left[u^\epsilon\right] & \text{on } \partial\Omega. \end{cases}$$

  - $\|v^\epsilon\|_{L^\infty(K)}$ exponentially decays to 0 as $\epsilon \to 0$ for any given compact subset K of $\Omega\backslash\{P_1, ...P_N\}$ $(n = 2)$ or Ω $(n \geq 3)$.

$$\epsilon^2\Delta(v^\epsilon)^2 = 2\epsilon^2\left(v^\epsilon\Delta v^\epsilon + |\nabla v^\epsilon|^2\right) \geq \frac{4}{M\left[u^\epsilon\right]}(v^\epsilon)^2.$$

  - Note: v^ϵ goes to zero exponentially on $\partial B_{\sigma_*}(P_j)$'s and $\partial\Omega_{\eta_\star}\backslash\partial\Omega$.
  - $\|v^\epsilon - \overline{V}_j^\epsilon\|_{L^\infty(B_{\sigma_*}(P_j))}$ exponentially decays to 0 as $\epsilon \to 0$ for $n = 2$.
  - $\|v^\epsilon - \overline{V}_0^\epsilon\|_{L^\infty(\Omega_{\eta_*})} = O(\epsilon)$ as $\epsilon \to 0$.

Sketch of the Proof, continued

- Definitions of $\overline{V}_j^\epsilon$'s:
  For $j = 1, ..., N$, $\overline{V}_j^\epsilon(x)$ is the solution of

$$\begin{cases} \epsilon^2 \Delta \overline{V}_j^\epsilon = \dfrac{1}{M_\epsilon}\left(e^{\overline{V}_j^\epsilon} - e^{-\overline{V}_j^\epsilon}\right) & \text{in } B_{\sigma_*}(P_j)\backslash\{P_j\}, \\ \overline{V}_j^\epsilon = 0 & \text{on } \partial B_{\sigma_*}(P_j), \\ \lim_{x\to P_j} \dfrac{\overline{V}_j^\epsilon(x)}{\log|x-P_j|^{-1}} = \alpha_j. & \end{cases}$$

  And $\overline{V}_0^\epsilon$ is defined by

$$\overline{V}_0^\epsilon(x) := 2\log\left(\frac{1 + a_\epsilon(\pi(x)) e^{-\sqrt{\frac{2}{M_\epsilon}}\frac{d(x)}{\epsilon}}}{1 - a_\epsilon(\pi(x)) e^{-\sqrt{\frac{2}{M_\epsilon}}\frac{d(x)}{\epsilon}}}\right) \quad \text{with} \quad a_\epsilon(y) = \frac{e^{\frac{u_0(y)-\mu_\epsilon}{2}} - 1}{e^{\frac{u_0(y)-\mu_\epsilon}{2}} + 1},$$

  by employing the normal coordinates in Ω_{η_*}, where $M_\epsilon = M[u^\epsilon]$ and $\mu_\epsilon = \mu[u^\epsilon]$.

- Note: $\overline{V}_0^\epsilon$ solves one-dimensional Poisson–Boltzmann equation without non-local coefficients in the normal direction.

Sketch of the Proof, continued

- Definitions of $\overline{V}_j^{\epsilon}$'s:
  For $j = 1, ..., N$, $\overline{V}_j^{\epsilon}(x)$ is the solution of

$$\begin{cases} \epsilon^2 \Delta \overline{V}_j^{\epsilon} = \frac{1}{M_\epsilon}\left(e^{\overline{V}_j^{\epsilon}} - e^{-\overline{V}_j^{\epsilon}}\right) & \text{in } B_{\sigma_*}(P_j)\backslash\{P_j\}, \\ \overline{V}_j^{\epsilon} = 0 & \text{on } \partial B_{\sigma_*}(P_j), \\ \lim_{x \to P_j} \frac{\overline{V}_j^{\epsilon}(x)}{\log|x - P_j|^{-1}} = \alpha_j. & \end{cases}$$

  And $\overline{V}_0^{\epsilon}$ is defined by

$$\overline{V}_0^{\epsilon}(x) := 2\log\left(\frac{1 + a_\epsilon(\pi(x)) e^{-\sqrt{\frac{2}{M_\epsilon}}\frac{d(x)}{\epsilon}}}{1 - a_\epsilon(\pi(x)) e^{-\sqrt{\frac{2}{M_\epsilon}}\frac{d(x)}{\epsilon}}}\right) \quad \text{with} \quad a_\epsilon(y) = \frac{e^{\frac{u_0(y)-\mu_\epsilon}{2}} - 1}{e^{\frac{u_0(y)-\mu_\epsilon}{2}} + 1},$$

  by employing the normal coordinates in Ω_{η_*}, where $M_\epsilon = M[u^\epsilon]$ and $\mu_\epsilon = \mu[u^\epsilon]$.

- Note: $\overline{V}_0^{\epsilon}$ solves one-dimensional Poisson–Boltzmann equation without non-local coefficients in the normal direction.

Sketch of the Proof, continued

- Recall the definitions of V_j^ϵ:
  For $j = 1, ..., N$, $V_j^\epsilon(x)$ is the solution of

$$\begin{cases} \epsilon^2 \Delta V_j^\epsilon = \dfrac{1}{|\Omega|}\left(e^{V_j^\epsilon} - e^{-V_j^\epsilon}\right) & \text{in } B_{\sigma_*}(P_j)\backslash\{P_j\}, \\ V_j^\epsilon = 0 & \text{on } \partial B_{\sigma_*}(P_j), \\ \lim_{x\to P_j} \dfrac{V_j^\epsilon(x)}{\log|x-P_j|^{-1}} = \alpha_j. & \end{cases}$$

  And V_0^ϵ is defined by

$$V_0^\epsilon(x) := 2\log\left(\frac{1 + a(\pi(x))e^{-\sqrt{\frac{2}{|\Omega|}}\frac{d(x)}{\epsilon}}}{1 - a(\pi(x))e^{-\sqrt{\frac{2}{|\Omega|}}\frac{d(x)}{\epsilon}}}\right) \quad \text{with} \quad a(y) = \frac{e^{\frac{u_0(y)-\bar{\mu}}{2}} - 1}{e^{\frac{u_0(y)-\bar{\mu}}{2}} + 1}.$$

- Need to show the convergence $M[u^\epsilon] \to |\Omega|$ and $\mu[u^\epsilon] \to \bar{\mu}$

Sketch of the Proof, continued

- Convergence of $M[u^\epsilon]$: $|M[u^\epsilon] - |\Omega|| = O(\epsilon)$
  - $M[u^\epsilon] = \int_\Omega e^{v^\epsilon} = \int_\Omega e^{-v^\epsilon}$.
  - We can rewrite it as
    $$M[u^\epsilon] = \frac{1}{2}\int_\Omega \left(e^{v^\epsilon} + e^{-v^\epsilon}\right) = |\Omega| + \frac{1}{2}\int_\Omega \left(e^{v^\epsilon} + e^{-v^\epsilon} - 2\right).$$
    We have
    $$\int_\Omega \left(e^{v^\epsilon} + e^{-v^\epsilon} - 2\right) = \underbrace{\sum_{j=1}^{N}\int_{B_{\sigma*}(P_j)}}_{O(\epsilon^2)} + \underbrace{\int_{\Omega\setminus\cup_{j=1}^N B_{\sigma*}(P_j)\cup\Omega_{\eta*}}}_{\text{exponentially decay}} + \underbrace{\int_{\Omega_{\eta*}}}_{O(\epsilon)}.$$
  - Recall: (1) $\|v^\epsilon\|_{L^\infty(K)}$ exponentially decays to 0 as $\epsilon \to 0$ for any given compact subset K of $\Omega\setminus\{P_1, ...P_N\}$.
    (2) $\|v^\epsilon - \overline{V}_j^\epsilon\|_{L^\infty(B_{\sigma*}(P_j))}$ exponentially decays to 0 as $\epsilon \to 0$.
    (3) $\|v^\epsilon - \overline{V}_0^\epsilon\|_{L^\infty(\Omega_{\eta*})} = O(\epsilon)$ as $\epsilon \to 0$.

Sketch of the Proof, continued

- Convergence of $\mu[u^\epsilon]$: $|\mu[u^\epsilon] - \bar{\mu}| = O\left(\epsilon \log \frac{1}{\epsilon}\right)$

Recall that $\int_\Omega \left(e^{v^\epsilon} - e^{-v^\epsilon}\right) = 0$.

$$\int_\Omega \left(e^{v^\epsilon} - e^{-v^\epsilon}\right) = \underbrace{\sum_{j=1}^N \int_{B_{\sigma*}(P_j)}}_{O(\epsilon^2)} + \underbrace{\int_{\Omega \setminus \cup_{j=1}^N B_{\sigma*}(P_j) \cup \Omega_{\eta*}}}_{\text{exponentially decay}} + \underbrace{\int_{\Omega_{\eta*} \setminus \Omega_{-N_* \epsilon \log \epsilon}}}_{O(\epsilon^{q*})} + \int_{\Omega_{-N_* \epsilon \log \epsilon}}$$

By choosing N_* sufficiently large (independent of ϵ), we have $q_* \geq 2$.

$$\int_{\Omega_{-N_* \epsilon \log \epsilon}} \left[\left(e^{v^\epsilon} - e^{-v^\epsilon}\right) - \left(e^{\overline{V}_0^\epsilon} - e^{-\overline{V}_0^\epsilon}\right)\right] = O\left(\epsilon^2 \log \frac{1}{\epsilon}\right)$$

$$\int_{\Omega_{-N_* \epsilon \log \epsilon}} \left(e^{\overline{V}_0^\epsilon} - e^{-\overline{V}_0^\epsilon}\right) = \epsilon \sqrt{2M_\epsilon} \int_{\partial\Omega} \left(e^{\frac{u_0 - \mu_\epsilon}{2}} - e^{-\frac{u_0 - \mu_\epsilon}{2}}\right) + O(\epsilon^2)$$

Consequently,

$$\int_{\partial\Omega} \left(e^{\frac{u_0 - \mu_\epsilon}{2}} - e^{-\frac{u_0 - \mu_\epsilon}{2}}\right) = O\left(\epsilon \log \frac{1}{\epsilon}\right).$$

Thank you!