Linear stability of solitary waves for the Euler-Poisson system

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Isothermal Euler-Poisson equations

The dynamics of ions in plasma is governed by:

$$\begin{aligned}
&\rho_t + \nabla \cdot (\rho u) = 0, \\
&(\rho u)_t + \nabla \cdot (\rho u \otimes u) + K \nabla \rho = \rho \nabla \phi, \\
&\varepsilon \Delta \phi = \rho - e^{-\phi}, \quad (x \in \mathbb{R}^3, \quad t \geq 0)
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- fluid (macroscopic) model arising in plasmas.
- Plasma = ionized gas consisting of positive ions and electrons
- It is often modeled as compressible gas, i.e., Euler eq. (cf. kinetic model, Vlasov-Poisson...)
- $\rho(x,t)$, $u(x,t)$ and $-\phi(x,t)$ represent the density, velocity and the electrostatic potential, respectively.
- Boltzmann-Maxwell relation $\rho_e = e^{-\phi}$
- $\sqrt{\varepsilon}$ is the (rescaled) Debye length λ_D that is, $\varepsilon \sim \lambda_D^2$ ($\lambda_D \ll 1$, lab setting $\lambda_D \sim 10^{-4}$)

Plasma solitary waves

Solitary waves for the Euler-Poisson system in 1-d

We consider the 1D Euler-Poisson system:

$$\begin{cases} \partial_t(1+n) + \partial_x((1+n)u) = 0, \\ \partial_t u + u\partial_x u + K\partial_x \log(1+n) = -\partial_x \phi, \\ \partial_x^2 \phi = e^{\phi} - (1+n), \end{cases} \tag{1.1}$$

with the boundary condition

$$n \to 0, \quad u \to 0, \quad \phi \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.2}$$

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- Seeking the solitary wave solutions by plugging the Ansatz

$$(n, u, \phi)(\tilde{x}), \quad \text{where } \tilde{x} = x - ct,$$

one has the traveling wave equations,

$$\begin{aligned} &-c(1+n)' + ((1+n)u)' = 0, \\ &-cu' + uu' + K(\log(1+n))' = -\phi', \\ &\phi'' = e^{\phi} - (1+n), \end{aligned}$$

Existence of Solitary waves for EP

Assumption

- Let $c = \sqrt{K+1} + \varepsilon$, $\varepsilon > 0$.
- $\varepsilon_0 := \sqrt{K}x_0 - \sqrt{K+1} > 0$, where x_0 is a solution of
$$x^K(K(x-1)^2 + 1) - \exp(K(x^2-1)/2) = 0$$

Theorem (Bae-K., 2019)

For $\varepsilon \in (0, \varepsilon_0)$, there is a two-parameter family of solitary waves

$$(n_c, u_c, \phi_c)(x - ct - \gamma), \qquad c \in (\sqrt{K+1}, \sqrt{K+1} + \varepsilon_0), \quad \gamma \in \mathbb{R}.$$

Moreover,

$$|(n_c, u_c, \phi_c)(x - ct)| \leq C\varepsilon \exp(-c_0 \varepsilon^{1/2}|x - ct|).$$

Remarks

- Speed $c := \sqrt{K+1} + \varepsilon$, "Relative" speed $= \varepsilon$
- Amplitude $= \mathcal{O}(\varepsilon)$, Width $= \mathcal{O}(\varepsilon^{-1/2})$
- Recall the KdV solitary wave: $u_{KdV}(x,t) = 3v \,\mathrm{sech}^2\left(\frac{\sqrt{v}}{2}(x - vt - \gamma)\right)$

Sketch the proof

Recall

$$
\begin{aligned}
& -c(1+n)' + ((1+n)u)' = 0, \\
& -cu' + uu' + K(\log(1+n))' = -\phi', \\
& \phi'' = e^{\phi} - (1+n),
\end{aligned}
$$

Introducing a new variable

$$E := -\phi', \tag{1.5}$$

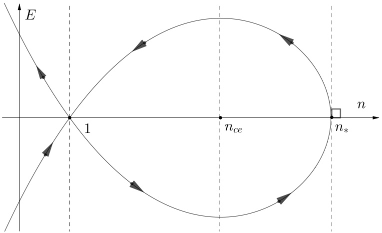
and a function

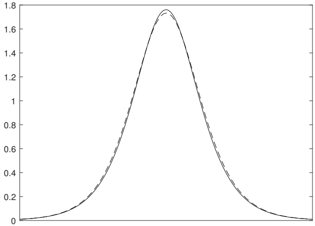
$$h(n) := \frac{dH(n)}{dn} = \frac{c^2}{(1+n)^3} - \frac{K}{1+n} \tag{1.6}$$

It can be reduced to the ODE system for n and E :

$$
\begin{aligned}
n' &= -E/h(n), \\
E' &= n - e^{H(n)}.
\end{aligned}
$$

Sketch the proof (phase plane analysis)





Small amplitude limit (KdV limit)

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Theorem (Bae-K., 2019)

Let $\xi = \varepsilon^{1/2}(x - ct)$. For all $\varepsilon \in (0, \varepsilon_0)$, there holds

$$|\phi_c(\varepsilon^{-1/2}\xi) - \varepsilon\Psi_{KdV}(\xi)| \lesssim \varepsilon^2 e^{-c|\xi|},$$

where

$$\Psi_{KdV}(\xi) = \frac{3}{V} sech^2\left(\sqrt{\frac{V}{2}}\xi\right)$$

is the solution to the KdV traveling wave equation:

$$-\Psi' + V\Psi\Psi' + (2V)^{-1}\Psi''' = 0.$$

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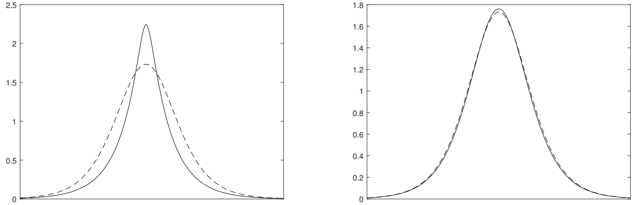
Remarks.

- Ψ_{KdV} *is the first order correction to* ϕ_c
- *Our results imply that the formal KdV expansion via GM for the EP is mathematically valid in the presence of solitary waves*
- *Gardner-Morikawa transform:* $t \to \varepsilon^{3/2}t, \quad x \to \varepsilon^{1/2}(x - Vt)$

Numerical experiments



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- The solutions of KdV are dominated by their solitary waves
- Similar stability results are expected
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Let $c := \sqrt{1+K} + \varepsilon$. Linearizing around $(1+n_c, u_c, \phi_c)$, we obtain

$$\begin{cases} \partial_t n - c\,\partial_x n + \partial_x(n_c u + u_c n) = 0, \\ \partial_t u - c\,\partial_x u + \partial_x(u_c u) + \sigma \partial_x \left(\frac{n}{1+n_c}\right) = -\partial_x \phi, \\ \partial_x^2 \phi - e^{\phi_c}\phi = -n. \end{cases}$$

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This can be written as a compact (nonlocal) form

$$\partial_t \begin{pmatrix} n \\ u \end{pmatrix} = \mathcal{L} \begin{pmatrix} n \\ u \end{pmatrix}$$

where

$$\mathcal{L} \begin{pmatrix} n \\ u \end{pmatrix} := -\partial_x \left[\begin{pmatrix} -c+u_c & 1+n_c \\ \frac{K}{1+n_c} & -c+u_c \end{pmatrix} \begin{pmatrix} n \\ u \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ (-\partial_x^2 + e^{\phi_c})^{-1} & 0 \end{pmatrix} \begin{pmatrix} n \\ u \end{pmatrix} \right]$$

Eigenvalue problem

We rewrite the eigenvalue problem $(\lambda - \mathcal{L})U = 0$ as an ODE system:

$$\begin{cases} \lambda n - c\,\partial_x n + \partial_x u + \partial_x(n_s u + u_s n) = 0, \\ \lambda u - c\,\partial_x u + \partial_x(u_s u) + \sigma\partial_x\left(\frac{n}{1+n_s}\right) + \psi = 0, \\ \phi_x - \psi = 0, \\ \psi_x - e^{\phi_s}\phi + n = 0. \end{cases}$$

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The dispersion relation for $\mathcal{L}$:

$$d(\lambda,\nu) = \det \begin{pmatrix} \lambda - c\nu & \nu & 0 & 0 \\ K\nu & \lambda - c\nu & 0 & 1 \\ 0 & 0 & \nu & -1 \\ 1 & 0 & -1 & \nu \end{pmatrix} = (\nu^2 - 1)\left[(\lambda - c\nu)^2 - K\nu^2\right] + \nu^2 = 0$$

$$\iff \lambda = \nu \left(c \pm \sqrt{\frac{1}{1-\nu^2} + \sigma} \right), \quad 1 - \nu^2 \neq 0$$

Spectrum

Let $\nu = ik$, $k \in \mathbb{R}$. Then, the dispersion relation is given by

$$\lambda(k) = ick \pm i\sqrt{\frac{k^2}{1+k^2} + K} \in i\mathbb{R}.$$

By Weyl's thoerem, one has

$$\sigma_{\text{ess}}(\mathcal{L}) \in i\mathbb{R}.$$

- $\lambda \in \mathbb{C}$ is in the *essential spectrum*, $\sigma_{\text{ess}}(L)$ if $\lambda - L$ is not Fredholm with index zero.
- $\lambda \in \mathbb{C}$ is in the *point spectrum*, $\sigma_{\text{pt}}(L)$ if $\lambda \notin \sigma_{\text{ess}}(L)$ and $\lambda - L$ is not invertible.

Weighted space

- Due to neutral spectrum in usual L^2, we consider a weighted spaces $L^2_\beta(\mathbb{R})$ where

$$L^2_\beta(\mathbb{R}) := \{f \in L^2_{\text{loc}} : e^{\beta x/2} f(x) \in L^2\}$$

for some $\beta > 0$

Note that the system is in the supersonic regime, i.e., $c > \sqrt{K+1}$

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Remarks. The weight space was successfully employed.

- Plasma sheath problems :
  - Stability [Suzuki 2011; Nishibata-Ohnawa-Suzuki 2012]
  - Quasi-neutral limit [Jung-K.-Suzuki, 2020]

- KdV: [Pego-Weinstein, 1994]; Boussinesq system : [Pego-Weinstein 1997]

Energy estimates in the weighted space

In weighted spaces $L^2_\beta(\mathbb{R}) := \{f \in L^2_{\text{loc}} : e^{\beta x/2} f(x) \in L^2\}$ for $\beta > 0$,

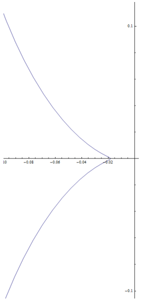
$$A_0 \begin{pmatrix} n \\ u \end{pmatrix}_t + \begin{pmatrix} -(V+\varepsilon)K & K \\ K & -(V+\varepsilon) \end{pmatrix} \begin{pmatrix} n \\ u \end{pmatrix}_x \underbrace{-\beta \begin{pmatrix} -(V+\varepsilon)K & K \\ K & -(V+\varepsilon) \end{pmatrix} \begin{pmatrix} n \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ (\phi_x - \beta\phi) \end{pmatrix}}_{\sim \mathcal{O}(\varepsilon\beta) \text{ good contribution}} = \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon^{3/2}) + \dots \quad (1.8)$$

The weighted energy method does NOT work!! it's not the "small" amplitude problem!

Spectrum on weighted space

In weighted spaces $L^2_\beta(\mathbb{R})$, the essential spectrum of $\mathcal{L}$ is

$$d(\lambda, ik-\beta) = 0 \quad \Longleftrightarrow \quad \lambda_\pm(\beta,k) = (ik-\beta)\left(c \pm \sqrt{\frac{1}{1-(ik-\beta)^2} + K}\right), \quad k \in \mathbb{R}.$$



$$\sigma_{\text{ess}}(\mathcal{L}) < -\theta(\varepsilon,\beta) < 0, \qquad \theta(\varepsilon,\beta) \sim \beta\varepsilon$$

Eigenvalue $\lambda = 0$

- By inspection, $\lambda = 0$ is the eigenvalue (point spectrum)

- the eigenfunctions which are the derivatives of the wave :

$$\mathcal{L}\partial_x(n_c, u_c)^\top = 0$$

and

$$\mathcal{L}\partial_c(n_c, u_c)^\top = -\partial_x(n_c, u_c)^\top$$

- $\lambda = 0$ is of multiplicity at least two.

- Next, we want to characterize $\lambda = 0$

Characterization of $\lambda = 0$ via Evans function

Goal: Construct the Evans function $D(\lambda)$.

- $D(\lambda)$ is analytic in λ.
- $D(\lambda) = 0 \quad \Leftrightarrow \quad \ker(\lambda - \mathcal{L}) \neq \{0\}$.
- The order of λ as a zero of $D(\lambda) =$ the algebraic multiplicity of λ as an eigenvalue of $\mathcal{L}$.

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Letting $\mathrm{y} := (n, u, \phi, \partial_x \phi)^T$, rewrite $(\lambda - \mathcal{L})(n, u)^T = 0$ as a first order ODE system

$$\left(\frac{d}{dx} - A(x, \lambda) \right) \mathrm{y} = 0.$$

Then,

$$\ker \left(\frac{d}{dx} - A(x, \lambda) \right) \neq \{0\} \quad \Leftrightarrow \quad \ker(\lambda - \mathcal{L}) \neq \{0\}.$$

1. Construct solutions, analytic in λ for each fixed x, satisfying suitable boundary conditions as $x \to \pm\infty$.
2. Need to study the matrix eigenvalues $\mu_j = \mu_j(\lambda)$ of $A_\infty(\lambda) := \lim_{|x| \to \infty} A(x, \lambda)$.

Matrix eigenvalues and Evans function

On Re $\lambda > 0$,

$$\text{Re }\mu_1 < 0 < \min_{j=2,3,4} \text{Re }\mu_j =: \mu_*$$

holds, we can construct $\mathrm{y}_i^{\pm}(x,\lambda)$ satisfying

$$\frac{d\mathrm{y}_i^{\pm}}{dx} = A(x,\lambda)\mathrm{y}_i^{\pm},$$

$$\mathrm{y}_1^{+} = O(e^{\text{Re }\mu_1 x}) \text{ as } x \to +\infty, \qquad \mathrm{y}_2^{-}, \mathrm{y}_3^{-}, \mathrm{y}_4^{-} = O(e^{\mu_* x}) \text{ as } x \to -\infty.$$

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Definition

The Evans function, $D(\lambda) := \det \left[y_1^{+}, y_2^{-}, y_3^{-}, y_4^{-} \right](0,\lambda)$.

$$\begin{aligned} D(\lambda) = 0 \quad &\Leftrightarrow \quad \text{span}\{y_1^{+}\} \cap \text{span}\{y_2^{-}, y_3^{-}, y_4^{-}\} \neq \{0\} \\ &\Leftrightarrow \quad \lambda \text{ is an eigenvalue of } \mathcal{L} \text{ in } L^2 \end{aligned}$$

- Consistent splitting holds: Re $\mu_1 + \beta < 0 < \mu_* + \beta$ for $\beta \sim \varepsilon^{1/2}$.
- $y_i^{\pm}(x,\lambda)$ can be analytically extended to the region Re $\lambda > -a$, where $a \sim \varepsilon\beta$
- $D(\lambda) = 0$ iff λ is an eigenvalue of $\mathcal{L}$ in L^2_β.

Relations between Evans functions

- $D_{KdV}(\Lambda)$: Evans function for the solitary wave of KdV
- $D_{EP}(\lambda, \varepsilon)$: Evans function for EP
- $D_*(\Lambda, \varepsilon)$: rescaled Evans function by $\Lambda = \varepsilon^{-3/2}\lambda$, i.e.,

$$D_*(\varepsilon^{-3/2}\lambda, \varepsilon) = D_{EP}(\lambda, \varepsilon)$$

Characterization of eigenvalue

- Using the eigenfunctions of $\mathcal{L}$ and $\mathcal{L}^*$ associated with $\lambda = 0$,

$$D_{EP}(0, \varepsilon) = \partial_\lambda D_{EP}(0, \varepsilon) = 0.$$

1. In general, the explicit form of the Evans function is rarely possible.
2. $D_{KdV}(\Lambda)$ vanishes only at the origin and the order is two. (Pego-Weinstein, 93,94)

Introducing $\xi = \varepsilon^{1/2} x$, $\lambda = \varepsilon^{3/2}\Lambda$, we show that

$$D_{EP}(\lambda, \varepsilon) = D_*(\Lambda, \varepsilon), \qquad D_*(\Lambda, \varepsilon) \to D_{KdV}(\Lambda) \text{ as } \varepsilon \to 0 \text{ uniformly in } \Lambda.$$

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By Rouché's theorem, we have

$\lambda = 0$ is the only zero of $D(\lambda, \varepsilon)$ and the order is two.

This implies

- $\lambda = 0$ is the only eigenvalue of $\mathcal{L}$. The algebraic multiplicity is two.

Spectral stability in the weighted space

- By inspection, 0 is the eigenvalue (point spectrum) with the derivative of the wave

- Characterize the zero eigenvalue by Evans function : algebraic multiplicity is 2 with the derivative of the wave

- Spectral stability in $L^2_\beta(\mathbb{R})$: For small amplitude waves ($\varepsilon \ll 1$),

$$\sigma^\beta_{\text{ess}}(\mathcal{L}_\varepsilon) \subset \{\lambda \in \mathbb{C} : \ \text{Re } \lambda \leq -\theta(\varepsilon, \beta) < 0\}$$
$$\sigma^\beta_{\text{pt}}(\mathcal{L}_\mu) \cap \{\lambda \in \mathbb{C} : \ \text{Re } \lambda > -\theta(\varepsilon, \beta)\} = \emptyset,$$

- the uniform resolvent estimate in $\{ \text{ Re } \lambda > -\theta(\beta, \varepsilon), |\lambda| \geq \varepsilon_0\}$:

$$\|(\lambda - \mathcal{L}_\varepsilon)^{-1}\|_{L^2_\beta \to L^2_\beta} \leq \frac{C}{\text{Re } \lambda + \beta}$$

with a pole of order $1/\lambda^2$ at $\lambda = 0$

Linear asymptotic stability of solitary waves

- Consider the operator $\mathcal{L} : H^1_\beta \times H^1_\beta \subset L^2_\beta \times L^2_\beta \to L^2_\beta \times L^2_\beta$.
- Let $c = \sqrt{1+K} + \varepsilon$ and $\beta = c_0 \varepsilon^{1/2}$, where $c_0 > 0$ is some constant.

1. $\mathcal{L}$ generates the C_0-semigroup $e^{\mathcal{L}t}$ on $L^2_\beta \times L^2_\beta$;
2. $\lambda = 0$ is an isolated eigenvalue of $\mathcal{L}$ with algebraic multiplicity two;
3. $(\lambda - \mathcal{L})^{-1}$ is uniformly bounded on Re $\lambda > 0$, outside any small neighborhood of the origin.

Theorem (Bae and K., 2021)

For any $(n_0, u_0)^T \in L^2_\beta \times L^2_\beta$ with $P(n_0, u_0)^T = 0$, there holds

$$\|e^{\mathcal{L}t}(n_0, u_0)^T\|_{L^2_\beta} \le C_\varepsilon e^{-C_\varepsilon t} \|(n_0, u_0)^T\|_{L^2_\beta} \quad \text{for all } t \ge 0.$$

Let P be the spectral projection onto the generalized eigenspace of $\mathcal{L}$ associated with $\lambda = 0$.

Linear asymptotic stability of solitary waves

The spectral projection is given by

$$P = \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda - \mathcal{L})^{-1}\, d\lambda, \quad \text{where } \Gamma_0 \text{ is a small circle centered at } \lambda = 0.$$

$$\mathrm{Ran}P = \mathrm{span}\{\partial_x (n_c, u_c)^T, \partial_c (n_c, u_c)^T\}.$$

Spectrum

$$\sigma\left(\mathcal{L}|_{\mathrm{Ran}P}\right) = \{0\} = \sigma_{\mathrm{pt}}(\mathcal{L}), \quad \sigma\left(\mathcal{L}|_{\mathrm{Ran}(I-P)}\right) = \sigma_{\mathrm{ess}}(\mathcal{L}) \subset \{\lambda : \mathrm{Re}\ \lambda < -a\}$$

Semi-group

$$e^{\mathcal{L}t}\mathrm{u}_0 = e^{\mathcal{L}t}P\mathrm{u}_0 + e^{\mathcal{L}t}(I-P)\mathrm{u}_0.$$

$$e^{\mathcal{L}t}P\mathrm{u}_0 = C_1\partial_x\Psi_c + C_2\left(\partial_c\Psi_c - t\partial_x\Psi_c\right) \quad \text{and} \quad \|e^{\mathcal{L}t}(I-P)\mathrm{u}_0\|_{L^2_\beta} \to 0 \quad \text{as } t \to +\infty.$$

Future work

- Nonlinear stability (many difficulties to overcome)

- Global existence v.s. finite-time singularity formation (C^1 blow-up)

  Note: 1. Weakly dispersive, strongly nonlinear

  2. for large amplitude data, the singularity formation is more likely

- Construction of Green function (pointwise estimate) at linear level

  • by the stationary phase method, we want to obtain the Huygens principle-like result

- Multi-D solitons (e.g., line solitons with KP-II and EP)