Inertial Ericksen-Leslie system for nematic liquid crystals

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Outline of the presentation

- Inertial Ericksen-Leslie system for nematic liquid crystals ;
- Well-posedness for incompressible and compressible models ($\lambda_1 < 0$ and $\lambda_1 = 0$) ;
- Inertia limit ($\lambda_1 < 0$) ;
- Low Mach number limit ($\lambda_1 < 0$) ;
- Dissipative solutions (simplified Ginzburg-Landau approximation) .

Part 0: Introduction to the models

Ericksen-Leslie system for nematic liquid crystals

- Hydrodynamic theory of liquid crystals: (Leslie: "Some constitutive equations for liquid crystals". Arch. Rational Mech. Anal. 28 (1968), no. 4, 265-283.)

$$\begin{cases} \rho\dot{\mathrm{u}} = \rho\mathrm{F} + \mathrm{div}\hat{\sigma}\,, \mathrm{div}\mathrm{u} = 0\,, & \text{(Balance of momentum)} \\ \rho_1\ddot{\mathrm{d}} = \rho_1\mathrm{G} + \hat{\mathrm{g}} + \mathrm{div}\pi\,. & \text{(Balance of angular momentum)} \end{cases} \tag{0.1}$$

- Unknown functions:
  - u: flow velocity.
  - d: direction field of the liquid molecules, $|\mathrm{d}| = 1$.
- ρ: fluid density (constant, for simplicity).
- $\rho_1 \geq 0$: the inertial density.
- $\hat{\mathrm{g}}$: intrinsic force associated with d.
- π: director stress.
- F and G: external body force and external director body force.

- $\mathrm{A} = \frac{1}{2}(\nabla\mathrm{u} + \nabla^\top\mathrm{u})$: rate of strain tensor.
- $\mathrm{B} = \frac{1}{2}(\nabla\mathrm{u} - \nabla^\top\mathrm{u})$: skew-symmetric part of the strain rate.
- $\omega = \dot{\mathrm{d}} = \partial_t\mathrm{d} + (\mathrm{u}\cdot\nabla)\mathrm{d}$: material derivative of d.
- $\mathbb{N} = \omega + \mathrm{Bd}$: rigid rotation part of director changing rate by fluid vorticity. ("Oldroyd derivative")
- The constitutive relations for $\hat{\sigma}$, π and $\hat{\mathrm{g}}$ are given by:

$$\begin{aligned} \hat{\sigma}_{ij} &= -p\delta_{ij} + \sigma_{ij} - \rho\frac{\partial W}{\partial \mathrm{d}_{k,i}}\mathrm{d}_{k,j}\,, \quad \pi_{ij} = \beta_i\mathrm{d}_j + \rho\frac{\partial W}{\partial \mathrm{d}_{j,i}}\,, \\ \hat{\mathrm{g}}_{ij} &= \gamma\mathrm{d}_i - \beta_j\mathrm{d}_{i,j} - \rho\frac{\partial W}{\partial \mathrm{d}_i} + \mathrm{g}_i\,. \end{aligned} \tag{0.2}$$

- p: the pressure, vector $\beta = (\beta_1\,, \cdots, \beta_n)^\top$ and the scalar function γ: Lagrangian multipliers for the constraint $|\mathrm{d}| = 1$.

Ericksen-Leslie's system for liquid crystals

- W: the Oseen-Frank energy functional

$$\begin{aligned} 2W =& k_1(\mathrm{div}\mathrm{d})^2 + k_2|\mathrm{d}\cdot(\nabla\times\mathrm{d})|^2 + k_3|\mathrm{d}\times(\nabla\times\mathrm{d})|^2 \\ &+(k_2+k_4)\left[\mathrm{tr}(\nabla\mathrm{d})^2 - (\mathrm{div}\mathrm{d})^2\right]. \end{aligned}$$

- g: the kinematic transport $\mathrm{g}_i = \lambda_1\mathbb{N}_i + \lambda_2\mathrm{d}_j\mathrm{A}_{ij}$: the effect of the macroscopic flow field on the microscopic structure. 1st term: rigid rotation of molecule; 2nd term: stretching of the molecule by the flow.
- σ: stress tensor $\sigma_{ij} = \mu_1\mathrm{d}_k\mathrm{A}_{kp}\mathrm{d}_p\mathrm{d}_i\mathrm{d}_j + \mu_2\mathbb{N}_i\mathrm{d}_j + \mu_3\mathrm{d}_i\mathbb{N}_j$ $+\mu_4\mathrm{A}_{ij} + \mu_5\mathrm{A}_{ik}\mathrm{d}_k\mathrm{d}_j + \mu_6\mathrm{d}_i\mathrm{A}_{jk}\mathrm{d}_k$. The coefficients μ_i are called Leslie coefficients, $\mathrm{div}\sigma = \frac{1}{2}\mu_4\Delta\mathrm{u} + \mathrm{div}\tilde{\sigma}$.
- $\lambda_1 = \mu_2 - \mu_3\,, \quad \lambda_2 = \mu_5 - \mu_6\,.$
- $\mu_2 + \mu_3 = \mu_6 - \mu_5$: *Parodi's relation.* (Wu-Xu-Liu, ARMA 2013)
- For simplicity, assume $\rho = 1$, $\mathrm{F} = 0$, $\mathrm{G} = 0$. We also take $k_1 = k_2 = k_3 = 1$, $k_4 = 0$.

The dynamic theory and inertia

- The derivation of the dynamic equations was proposed by J. L. Ericksen and then adjusted by F. Leslie and started from an energy balance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \{\tfrac{1}{2}|u|^2 + \rho_1|\dot{d}|^2 + W\}\,\mathrm{d}v$$

$$= \int_V (-\Delta)\,\mathrm{d}v + \text{boundary terms and “harmless terms”}$$

  where Δ dissipation function and W the Oseen-Frank energy.

- Concerning the inertial constant ρ_1, F. Leslie in *Adv. in Physics, v. 4, 1978* stated:

  “*the term involving σ represent rotational kinetic energy of the material element and therefore σ is an inertial constant. While this contribution to the kinetic energy is undoubtely negligible in most circumstances we retain it in the general form which follows, since it could conceivably play a nontrivial rote when the anisotropic axis is subjected to large accelerations...*”

The physical experiments on inertial waves

Based on the theoretical analysis of Ericksen-Leslie in 1968 and 1979, in particular on twisted waves, a few Chinese physicist did some experiments to verified the existence of solutions to the inertia Ericksen-Leslie system in early 1980's.

- "Experiments on Director Waves in Nematic Liquid Crystals " (*PHYSICAL REVIEW LETTERS* **49** (No.18), 1982)
- by **Guozhen Zhu**, Department of Fundamental Courses, Tsinghua University.

Ericksen-Leslie system

- Ericksen-Leslie's hyperbolic liquid crystal model reduces to the following form:

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u}\cdot\nabla\mathrm{u} - \frac{1}{2}\mu_4\Delta\mathrm{u} + \nabla p = -\mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d}) + \mathrm{div}\tilde{\sigma}\,, \\ \mathrm{div}\mathrm{u} = 0\,, \\ \rho_1\ddot{\mathrm{d}} = \Delta\mathrm{d} + \gamma\mathrm{d} + \lambda_1(\dot{\mathrm{d}} - \mathrm{B}\mathrm{d}) + \lambda_2\mathrm{A}\mathrm{d}\,, \end{cases} \tag{0.3}$$

on $\mathbb{R}^n \times \mathbb{R}^+$ with the constraint $|\mathrm{d}| = 1$, where the Lagrangian multiplier γ is given by

$$\gamma \equiv \gamma(\mathrm{u},\mathrm{d},\dot{\mathrm{d}}) = -\rho_1|\dot{\mathrm{d}}|^2 + |\nabla\mathrm{d}|^2 - \lambda_2\mathrm{d}^\top\mathrm{A}\mathrm{d}\,. \tag{0.4}$$

- Initial data:

$$\mathrm{u}|_{t=0} = \mathrm{u}^{in}(x),\ \dot{\mathrm{d}}|_{t=0} = \tilde{\mathrm{d}}^{in}(x),\ \mathrm{d}|_{t=0} = \mathrm{d}^{in}(x)\,, \tag{0.5}$$

where d^{in} and $\tilde{\mathrm{d}}^{in}$ satisfy the constraint and compatibility condition:

$$|\mathrm{d}^{in}| = 1\,, \quad \tilde{\mathrm{d}}^{in}\cdot\mathrm{d}^{in} = 0\,. \tag{0.6}$$

Special case: Navier-Stokes coupled with wave map

A particularly important special case of inertial Ericksen-Leslie system is that the term $\mathrm{div}\tilde{\sigma}$ vanishes. Namely, the coefficients μ_i's, $(1 \le i \le 6, i \ne 4)$ of $\mathrm{div}\tilde{\sigma}$ are chosen as 0, which immediately implies $\lambda_1 = \lambda_2 = 0$. Consequently, the system (0.3) reduces to a model which is Navier-Stokes equations coupled with a (damping) wave map from $\mathbb{R}^n$ to $\mathbb{S}^2$:

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u} \cdot \nabla \mathrm{u} + \nabla p = \frac{1}{2}\mu_4 \Delta \mathrm{u} - \mathrm{div}(\nabla \mathrm{d} \odot \nabla \mathrm{d})\,, \\ \mathrm{div}\mathrm{u} = 0\,, \\ \rho_1 \ddot{\mathrm{d}} = \Delta \mathrm{d} + (-\rho_1 |\dot{\mathrm{d}}|^2 + |\nabla \mathrm{d}|^2)\mathrm{d}\,. \end{cases} \tag{0.7}$$

- if $\mathrm{d} =$ constant vector, then u satisfies the standard Navier-Stokes equations.
- if $\mathrm{u} = 0$, then d is called "*twisted wave*" (proposed by J.L. Ericksen, 1968) (Note! this does NOT mean the "u-equations" vanish!)

$\rho_1 = 0, \lambda_1 = -1$, parabolic model

- If $\rho_1 = 0$ and $\lambda_1 = -1$ in the 3rd equation of (0.3), the system reduces to the parabolic Ericksen-Leslie's system.
- The static analogue of the parabolic Ericksen-Leslie's system is Oseen-Frank model (Hardt-Kinderlehrer-Lin, 80's).
- Ginzburg-Landau approximation : partial regularity and regularity (Lin-Liu, 90's and early 00's).
- Global weak solutions with at most a finite number of singular times (2D, Lin-Lin-Wang, 2010, 3D, Lin-Wang, 2015).

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u}\cdot\nabla\mathrm{u} + \nabla p = \Delta\mathrm{u} - \mathrm{div}(\nabla\mathrm{d}\odot\nabla\mathrm{d}), \\ \mathrm{div}\mathrm{u} = 0, \\ \partial_t\mathrm{d} + \mathrm{u}\cdot\nabla\mathrm{d} = \Delta\mathrm{d} + |\nabla\mathrm{d}|^2\mathrm{d}, \quad |\mathrm{d}| = 1, \end{cases} \tag{0.8}$$

- More general parabolic Ericksen-Leslie's system: many people $\cdots$

$\rho_1 > 0$, hyperbolic model (inertial Ericksen-Leslie model)

- Very few analytical works for general model, except for very *special* case: $\mathrm{u} = 0$ and 1-D, even this simplest case is very subtle.
  - Singularities of variational wave equation. (Saxton 1989, Glassey-Hunter-Zheng 1996, Chen-Huang-Liu 2015 $\cdots$)
  - Orientational waves: splay and twist waves. (Ali-Hunter 2007)
  - Dissipative and energy conservative solution to variational wave equations. (Zheng-Zhang, Bressan, $\cdots$)

$$\mathrm{u}_{tt} - c(\mathrm{u})(c(\mathrm{u})\mathrm{u}_x)_x = 0\,.$$

- Difficulties for *general* model:
  - Double material derivative on d is a nonlinear operator on d

$$\ddot{\mathrm{d}} = \partial_t^2\mathrm{d} + 2\mathrm{u}\cdot\nabla\partial_t\mathrm{d} + \mathrm{u}\nabla\mathrm{u}\nabla\mathrm{d} + \mathrm{u}(\nabla^2\mathrm{d})\mathrm{u}\,.$$

  - The most troublesome part is the Lagrange multiplier γ that generates high derivatives. In particular, if add the unit lengh constraint, the situation gets even worse.

Inertial model in Q-tensor framework

Qian-Sheng model (Physical Review E 1998):

$$\dot{\mathrm{u}} + \nabla p - \frac{\beta_4}{2}\Delta \mathrm{u} = \mathrm{div}\left(-L\nabla Q \odot \nabla Q + \beta_1 Q\mathrm{tr}\{QA\} + \beta_5 AQ + \beta_6 QA\right)$$
$$+\mathrm{div}\left(\frac{\mu_2}{2}(\dot{Q} - [\Omega, Q] + \mu_1[Q, \dot{Q} - [\Omega, Q]])\right),$$
$$\mathrm{div}\mathrm{u} = 0\,,$$
$$J\ddot{Q} + \mu_1\dot{Q} = L\Delta Q - aQ + b\left(Q^2 - \frac{1}{d}|Q|^2 I_d\right) - cQ|Q|^2 + \frac{\tilde{\mu}_2}{2}A$$
$$+\mu_1[\Omega, Q]\,,$$

where $[A, B] = AB - BA$, $A_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ and $(\nabla Q \odot \nabla Q)_{ij} = Q_{kl,i}Q_{kl,j}$.

Mathematical works for inertial Qian-Sheng model

- Global existence and uniqueness for small initial data (De Anna-Zarnescu 2016).
  - key assumption 1: Newtonian viscosity β_4 is large enough.
  - key assumption 2: the coefficient $a > 0$ which gives "damping".
  - technical trick: higher-order commutator estimate.
- A class of global "twist waves" (solutions of the coupled system for which *the flow vanishes for all time*) (De Anna-Zarnescu 2016).
- A global existence of the *dissipative solution* which is inspired from that of incompressible Euler equation inspired by P-L. Lions (Feireisl-Rocca-Schimperna-Zarnescu 2016).
- First results involving *second order material derivative*, many open questions left. (My student Yangjun Ma made some progress in this direction. JHDE 2021)
- From Qian-Sheng to E-L (Li-Wang, SIMA 2020).

Inertial Ericksen-Leslie liquid crystal model for a compressible flow:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathrm{u}) = 0\,, \\ \partial_t(\rho \mathrm{u}) + \operatorname{div}(\rho \mathrm{u} \otimes \mathrm{u}) + \nabla p = \operatorname{div}(\Sigma_1 + \Sigma_2 + \Sigma_3)\,, \\ \rho \ddot{\mathrm{d}} = \Delta \mathrm{d} + \Gamma \mathrm{d} + \lambda_1(\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}) + \lambda_2 \mathrm{A}\mathrm{d}\,, \end{cases} \tag{0.9}$$

on $\mathbb{R}^N \times \mathbb{R}^+$, $(N = 2, 3)$ with the geometric constraint $|\mathrm{d}| = 1$.
$p(\rho) = a\rho^\gamma$ with $\gamma \geq 1$, $a > 0$, and

$$\Sigma_1 := \frac{1}{2}\mu_4(\nabla \mathrm{u} + \nabla^\top \mathrm{u}) + \xi \operatorname{div} \mathrm{u} I\,,$$

$$\Sigma_2 := \frac{1}{2}|\nabla \mathrm{d}|^2 I - \nabla \mathrm{d} \odot \nabla \mathrm{d}\,,$$

$$\Sigma_3 := \tilde{\sigma}\,.$$

Remarks:

- In fact, this is the most general form appeared in Leslie's 1968 ARMA paper.
- For the even more general form including *temperature* equation, see Chun Liu's recent ARMA paper.

Our goals

- Well-posedness in the framework of classical solutions;
  - Incompressible model, $\lambda_1 < 0$, (J-Luo, *SIMA* 2019)
  - Incompressible model, $\lambda_1 = 0$, (Huang-J-Luo-Zhao, *SIMA* 2020)
  - Compressible model, $\lambda_1 < 0$, (J-Luo-Tang, *M3AS* 2019)
  - Compressible model, $\lambda_1 = 0$, (Huang-J-Luo-Zhao, preprint 2019)
- Zero inertia limit ($\rho_1 \to 0$) to the corresponding *parabolic* Ericksen-Leslie model.
  - Incompressible model, $\lambda_1 < 0$, $\mathrm{u} = 0$ (J-Luo-Tang-Zarnescu, *CMS* 2019)
  - Incompressible model, $\lambda_1 < 0$, (J-Luo, *JFA* 2021)

General principles: 1. $\lambda_1 < 0$ case is relatively easier, since it gives some "damping" effect. $\lambda_1 = 0$ case is essentially related to the wave map type equations with target manifold $\mathbb{S}^2$. 2. All above results, we need some restrictions on the coefficients.

Part I: Well-posedness

Proposition

(Basic energy-dissipation law) If (u, d) is a smooth solution to the system (0.3) with initial conditions (0.5), then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(|\mathrm{u}|_{L^2}^2 + \rho_1|\dot{\mathrm{d}}|_{L^2}^2 + |\nabla \mathrm{d}|_{L^2}^2\Big) + \frac{1}{2}\mu_4|\nabla \mathrm{u}|_{L^2}^2 + \mu_1|\mathrm{d}^\top \mathrm{A}\mathrm{d}|_{L^2}^2$$
$$-\lambda_1|\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}|_{L^2}^2 - 2\lambda_2\langle\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}, \mathrm{A}\mathrm{d}\rangle + (\mu_5 + \mu_6)|\mathrm{A}\mathrm{d}|_{L^2}^2 = 0\,.$$

Moreover, the above basic energy law is *dissipated* if the Leslie coefficients satisfy that either

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 < 0\,, \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0\,,$$

or

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 = 0\,, (1-\delta)\mu_4(\mu_5 + \mu_6) \geq 2|\lambda_2|^2$$

for some $\delta \in (0, 1)$.

Lagrangian multiplier γ and constraint $|\mathrm{d}| = 1$

Lemma: Assume (u, d) is a classical solution to the Ericksen-Leslie system (0.3)-(0.5) satisfying $\mathrm{u} \in L^\infty_T H^s_x \cap L^2_T H^{s+1}_x$, $\nabla \mathrm{d} \in L^\infty_T H^s_x$, $\dot{\mathrm{d}} \in L^\infty_T H^s_x$ and $|\mathrm{d}|_{L^\infty_{T,x}} < \infty$ for some $T \in (0, \infty)$, where $s > \frac{n}{2} + 1$. If $|\mathrm{d}| = 1$, then the Lagrangian multiplier γ is

$$\gamma = -\rho_1 |\dot{\mathrm{d}}|^2 + |\nabla \mathrm{d}|^2 - \lambda_2 \mathrm{d}^\top \mathrm{A} \mathrm{d} \,. \tag{0.10}$$

Conversely, if we give the form of γ as (0.10) and d satisfies the initial data conditions $\tilde{\mathrm{d}}^{in} \cdot \mathrm{d}^{in} = 0$, $|\mathrm{d}^{in}| = 1$, then $|\mathrm{d}| = 1$.
Proof: $h = |\mathrm{d}|^2 - 1$ solves for a given smooth vector field u:

$$\begin{cases} \rho_1 \ddot{h} - \lambda_1 \dot{h} - \Delta h = 2\gamma h \,, \\ \dot{h}|_{t=0} = 0 \,, \quad h|_{t=0} = 0 \,. \end{cases} \tag{0.11}$$

Our goal is to verify $h(t, x) = 0$ for later time $0 < t < T$.

Equations for $\mathrm{d}(t,x)$ with a given velocity $\mathrm{u}(t,x)$

For a given velocity field $\mathrm{u}(t,x)$, consider the equation of $\mathrm{d} \in \mathbb{S}^2$:

$$\begin{cases} \rho_1 \ddot{\mathrm{d}} = \Delta \mathrm{d} + \gamma(\mathrm{u}, \mathrm{d}, \dot{\mathrm{d}})\mathrm{d} + \lambda_1(\dot{\mathrm{d}} - \mathrm{B}\mathrm{d}) + \lambda_2 \mathrm{A}\mathrm{d}\,, \\ \mathrm{d}(0,x) = \mathrm{d}_0(x)\,,\ \dot{\mathrm{d}}(0,x) = \tilde{\mathrm{d}}_0(x)\,. \end{cases} \tag{0.12}$$

Proposition

For $s > \frac{n}{2} + 1$ and $T_0 > 0$, $(\mathrm{d}^{in}, \tilde{\mathrm{d}}^{in}, \mathrm{u}) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \times \mathbb{R}^n$ satisfy $\nabla \mathrm{d}^{in} \in H^s$, $\tilde{\mathrm{d}}^{in} \in H^s$ and $\mathrm{u} \in C(0, T_0; H^s) \cap L^1(0, T_0; H^{s+1})$. Then there exists $0 < T \leq T_0$, depending only on d^{in}, $\tilde{\mathrm{d}}^{in}$ and u, such that the system (0.12) has a unique classical solution d satisfying $\nabla \mathrm{d}\,,\ \dot{\mathrm{d}} \in C(0, T; H^s)$. Moreover, there is a positive constant C_2, depending only on d^{in}, $\tilde{\mathrm{d}}^{in}$ and u, such that the solution d satisfies the bound

$$\rho_1 |\dot{\mathrm{d}}|^2_{L^\infty(0,T;H^s)} + |\nabla \mathrm{d}|^2_{L^\infty(0,T;H^s)} \leq C_2\,.$$

Proof of the Proposition

Construct the approximate system

$$\begin{cases} \rho_1 \partial_t \dot{d}^\epsilon = -\rho_1 \mathcal{J}_\epsilon(u\nabla \cdot \dot{d}^\epsilon) + \Delta d^\epsilon + \mathcal{J}_\epsilon\Big(\gamma(u, d^\epsilon, \dot{d}^\epsilon) d^\epsilon\Big) \\ \qquad\qquad + \lambda_1 \dot{d}^\epsilon - \lambda_1 \mathcal{J}_\epsilon(Bd^\epsilon) + \lambda_2 \mathcal{J}_\epsilon(Ad^\epsilon)\,, \\ \partial_t d^\epsilon = \dot{d}^\epsilon - \mathcal{J}_\epsilon(u \cdot \nabla d^\epsilon)\,, \\ (d^\epsilon, \dot{d}^\epsilon)|_{t=0} = (\mathcal{J}_\epsilon d_0, \mathcal{J}_\epsilon \tilde{d}_0)\,. \end{cases} \tag{0.13}$$

Define the energy functional $E_\epsilon(t)$

$$E_\epsilon(t) = \rho_1 |\dot{d}^\epsilon|^2_{H^s} + |\nabla d^\epsilon|^2_{H^s} + |d^\epsilon - \mathcal{J}_\epsilon d_0|^2_{L^2}\,,$$

$$\tfrac{d}{dt} E_\epsilon(t) \le C_1 (1 + |u|_{H^{s+1}})[1 + E_\epsilon(t)]^2 \tag{0.14}$$

holds for all $t \in [0, T_\epsilon)$. We obtain the uniform energy bound

$$\rho_1 |\dot{d}^\epsilon|^2_{H^s} + |\nabla d^\epsilon|^2_{H^s} + |d^\epsilon - \mathcal{J}_\epsilon d_0|^2_{L^2} \le 2E^{in}\,, \tag{0.15}$$

for all $\epsilon > 0$ and $t \in [0, T]$. Pass to the limit $\epsilon \to 0$, and use the previous lemma to prove the limit satisfies $|d| = 1$.

Let $s > \frac{n}{2} + 1$, $\mathrm{u}^{in},\ \tilde{\mathrm{d}}^{in} \in H^s(\mathbb{R}^n),\ \nabla \mathrm{d}^{in} \in H^s(\mathbb{R}^n)$, $|\mathrm{d}^{in}| = 1$, $\tilde{\mathrm{d}}^{in} \cdot \mathrm{d}^{in} = 0$ and $E^{in} \equiv |\mathrm{u}^{in}|^2_{H^s} + \rho_1 |\tilde{\mathrm{d}}^{in}|^2_{H^s} + |\nabla \mathrm{d}^{in}|^2_{H^s}$. If the Leslie coefficients satisfy either

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 < 0\,, \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0\,, \quad \text{or} \tag{0.16}$$

$$\mu_1 \geq 0\,, \mu_4 > 0\,, \lambda_1 = 0\,, (1-\delta)\mu_4(\mu_5 + \mu_6) \geq 2|\lambda_2|^2 \tag{0.17}$$

for some $\delta \in (0, 1)$, then:

(I). If $E^{in} < \infty$, then there exists $T > 0$, depending only on E^{in} and Leslie coefficients, such that the system (0.3)-(0.5) admits a unique solution $\mathrm{u} \in L^\infty_T H^s_x \cap L^2_T H^{s+1}_x$, $\nabla \mathrm{d} \in L^\infty_T H^s_x$ and $\dot{\mathrm{d}} \in L^\infty_T H^s_x$. Moreover, the solution (u, d) satisfies

$$\sup_{0 \leq t \leq T} \Big(|\mathrm{u}|^2_{H^s} + \rho_1 |\dot{\mathrm{d}}|^2_{H^s} + |\nabla \mathrm{d}|^2_{H^s} \Big)(t) + \tfrac{1}{2}\mu_4 \int_0^T |\nabla \mathrm{u}|^2_{H^s}(\tau) \mathrm{d}\tau \leq C_0\,,$$

where C_0 depends only on E^{in}, Leslie coefficients and T.

Main Theorem (II)

(II). If in addition, $\lambda_1 < 0$, then there exists $\epsilon_0 > 0$, depending only on Leslie coefficients, such that if $E^{in} \leq \epsilon_0$, then system (0.3)-(0.5) has a unique global solution $\mathrm{u} \in L^\infty(0,\infty; H^s_x) \cap L^2(0,\infty; H^{s+1}_x)$, $\nabla \mathrm{d} \in L^\infty(0,\infty; H^s_x)$ and $\dot{\mathrm{d}} \in L^\infty(0,\infty; H^s_x)$. Moreover, the solution (u, d) satisfies

$$\sup_{t\geq 0}\left(|\mathrm{u}|^2_{H^s} + \rho_1|\dot{\mathrm{d}}|^2_{H^s} + |\nabla \mathrm{d}|^2_{H^s}\right) + \frac{1}{2}\mu_4 \int_0^\infty |\nabla \mathrm{u}|^2_{H^s} \mathrm{d}t \leq C_1 E^{in},$$

where the constant $C_1 > 0$ depends upon the Leslie coefficients and inertia constant ρ_1.

The iterating approximate system: for all integer $k \geq 0$,

$$\begin{cases} \partial_t \mathrm{u}^{k+1} + \mathrm{u}^k \cdot \nabla \mathrm{u}^k - \frac{1}{2}\Delta \mathrm{u}^{k+1} + \nabla p^{k+1} = \\ -\mathrm{div}(\nabla \mathrm{d}^k \odot \nabla \mathrm{d}^k) + \mathrm{div}\tilde{\sigma}(\mathrm{u}^{k+1}, \mathrm{d}^k, \dot{\mathrm{d}}^k)\,, \\ \mathrm{div}\mathrm{u}^{k+1} = 0\,, \\ \rho_1 \partial_t \dot{\mathrm{d}}^{k+1} + \rho_1 \mathrm{u}^k \cdot \nabla \dot{\mathrm{d}}^{k+1} = \Delta \mathrm{d}^{k+1} + \gamma(\mathrm{u}^k, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1})\mathrm{d}^{k+1} \\ \qquad + \lambda_1(\dot{\mathrm{d}}^{k+1} + \mathrm{B}^k \mathrm{d}^{k+1}) + \lambda_2 \mathrm{A}^k \mathrm{d}^{k+1}\,, \\ (\mathrm{u}^{k+1}, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1})\big|_{t=0} = (\mathrm{u}^{in}(x), \mathrm{d}^{in}(x), \tilde{\mathrm{d}}^{in}(x)) \in \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n\,, \end{cases} \tag{0.18}$$

where $\dot{\mathrm{d}}^{k+1} = \partial_t \mathrm{d}^{k+1} + \mathrm{u}^k \cdot \nabla \mathrm{d}^{k+1}$ is the iterating approximate material derivatives, the iterating approximate Lagrangian multiplier $\gamma(\mathrm{u}^k, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1})$ is

$$\gamma(\mathrm{u}^k, \mathrm{d}^{k+1}, \dot{\mathrm{d}}^{k+1}) = -\rho_1 |\dot{\mathrm{d}}^{k+1}|^2 + |\nabla \mathrm{d}^{k+1}|^2 - \lambda_2 (\mathrm{d}^{k+1})^\top \mathrm{A}^k \mathrm{d}^{k+1}\,.$$

Uniform energy estimate

- The energy functional $E(t) = |\mathrm{u}|_{H^s}^2 + \rho_1|\dot{\mathrm{d}}|_{H^s}^2 + |\nabla \mathrm{d}|_{H^s}^2$.
- The energy-dissipation functional $D(t)$ for $\lambda_1 < 0$:

$$D(t) = \tfrac{1}{2}\mu_4|\nabla \mathrm{u}|_{H^s}^2 + \mu_1|\mathrm{d}^\top(\nabla^k \mathrm{A})\mathrm{d}|_{L^2}^2 - \lambda_1|\nabla^k \dot{\mathrm{d}} + (\nabla^k \mathrm{B})\mathrm{d} + \tfrac{\lambda_2}{\lambda_1}(\nabla^k \mathrm{A})\mathrm{d}|_{L^2}^2 + (\mu_5 + \mu_6 + \tfrac{\lambda_2^2}{\lambda_1})|(\nabla^k \mathrm{A})\mathrm{d}|_{L^2}^2 ,$$

- The energy dissipation functional $D(t)$ for $\lambda_1 = 0$:

$$D(t) = \tfrac{1}{4}\delta\mu_4|\nabla \mathrm{u}|_{H^s}^2 + \mu_1|\mathrm{d}^\top(\nabla^k \mathrm{A})\mathrm{d}|_{L^2}^2 + (\mu_5 + \mu_6 - \tfrac{2\lambda_2^2}{(1-\delta)\mu_4})|(\nabla^k \mathrm{A})\mathrm{d}|_{L^2}^2 + \tfrac{1}{2}(1-\delta)\mu_4\Big(|\nabla^{k+1}\mathrm{u}|_{L^2} - \tfrac{2|\lambda_2|}{(1-\delta)\mu_4}|(\nabla^k \mathrm{A})\mathrm{d}|_{L^2}\Big)^2 .$$

Lemma

A priori estimate*:*

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}E(t) + D(t) \le CE^{\frac{3}{2}}(t) + C\sum_{p=1}^{4} E^{\frac{p+1}{2}}(t)D^{\frac{1}{2}}(t) .$$

A lemma

$$E_{k+1}(t) = |\mathrm{u}^{k+1}|^2_{H^s} + \rho_1|\dot{\mathrm{d}}^{k+1}|^2_{H^s} + |\nabla \mathrm{d}^{k+1}|^2_{H^s}\,,$$
$$D_{k+1}(t) = \tfrac{1}{2}\mu_4|\nabla \mathrm{u}^{k+1}|^2_{H^s}\,.$$

Lemma

Assume that $(\mathrm{u}^{k+1}, \mathrm{d}^{k+1})$ *is the solution to the iterating approximate system* (0.18) *and we define*

$$T_{k+1} \equiv \left\{\tau \in [0, T^*_{k+1});\ \sup_{t\in[0,\tau]} E_{k+1}(t) + \int_0^\tau D_{k+1}(t)\mathrm{d}t \le M\right\},$$

where $T^*_{k+1} > 0$ *is the existence time of the iterating approximate system* (0.18)*. Then for any fixed* $M > E^{in}$ *there is a constant* $T > 0$*, depending only on Leslie coefficients,* M *and* E^{in}*, such that*

$$T_{k+1} \ge T > 0\,.$$

Proof of Theorem (I)

Note $E_\epsilon(0) \leq E^{in}$. If $E^{in} < \min\left\{1, \frac{\rho_1\beta^2}{[96C(\sqrt{\rho_1}(\mu_1+\mu_6)+|\lambda_1|-\lambda_2)]^2}\right\}$, we have $\mathbf{Q}(E_\epsilon(0)) \leq \frac{1}{8}\beta$. Define

$$T_\epsilon^* = \sup\left\{T > 0; E_\epsilon(t) \leq 2 \text{ and } \mathbf{Q}(E_\epsilon(t)) \leq \tfrac{1}{4}\beta \text{ hold for all } t \in [0, T]\right\}.$$

Lemma **??** implies that $E_\epsilon(t)$ is continuous, thus $T_\epsilon^* > 0$. So for any fixed $\epsilon > 0$ and for all $t \in [0, T_\epsilon^*]$

$$\frac{1}{2}\frac{\partial}{\partial t}E_\epsilon(t) + \left[\tfrac{1}{4}\beta - \mathbf{Q}(E_\epsilon(t))\right]F_\epsilon(t) \leq 12C_1 E_\epsilon(t),$$

which implies $E_\epsilon(t) \leq E^{in}e^{24C_1 t}$ holds for all $\epsilon > 0$ and $t \in [0, T_\epsilon^*]$. Thus let $0 < T \leq \frac{1}{48C_1}\ln\frac{1}{E^{in}}$ such that for all $t \in [0, \min\{T, T_\epsilon^*\}]$

$$E_\epsilon(t) \leq E^{in}e^{24C_1 T} \leq \sqrt{E^{in}} < 1,$$

which consequently implies that for all $t \in [0, \min\{T, T_\epsilon^*\}]$

$$\mathbf{Q}(E_\epsilon(t)) \leq 12C\Big(\mu_1 + \mu_6 + \frac{|\lambda_1|-\lambda_2}{\sqrt{\rho_1}}\Big)(E^{in})^{\frac{1}{4}}.$$

Proof of Theorem (I)– continued

Let $\epsilon_0 = \min\left\{1, \frac{\rho_1\beta^2}{[96C(\sqrt{\rho_1}(\mu_1+\mu_6)+|\lambda_1|-\lambda_2)]^2}, \frac{\rho_1^2\beta^4}{[96C(\sqrt{\rho_1}(\mu_1+\mu_6)+|\lambda_1|-\lambda_2)]^4}\right\}$, s.t. if $E^{in} < \epsilon_0$, then for all $t \in [0, \min\{T, T_\epsilon^*\}]$, $E_\epsilon(t) \leq \sqrt{E^{in}} < 1$, and $\mathbf{Q}(E_\epsilon(t)) \leq \frac{1}{8}\beta$. Thus, by the continuity of $E_\epsilon(t)$, $T_\epsilon^* \geq T$. As a consequence, for all $t \in [0, T]$,

$$E^\epsilon(t) + \frac{1}{4}\beta \int_0^t |\nabla \mathrm{u}^\epsilon|_{H^s}^2(\tau)\partial\tau \leq \widetilde{C}_1(E^{in}, T), \qquad (0.19)$$

where $\widetilde{C}_1(E^{in}, T) = E^{in} + 12C_1 T\sqrt{E^{in}} > 0$. Thus the energy estimate is closed. The rest convergence proof is tedious but standard.

Proof of Theorem (III)–Global solution

Need a new *a priori* estimate: For $1 \le k \le s$, take ∇^k in 3rd equation of (0.3), multiply by $\nabla^k \mathrm{d}$. (we did it for $\dot{\mathrm{d}}$ in local existence.)

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left(\rho_1|\dot{\mathrm{d}}+\mathrm{d}|^2_{\dot{H}^s} + (|\lambda_1|-\rho_1)|\mathrm{d}|^2_{\dot{H}^s} - \rho_1|\dot{\mathrm{d}}|^2_{\dot{H}^s}\right) - \rho_1|\dot{\mathrm{d}}|^2_{\dot{H}^s} + |\nabla \mathrm{d}|^2_{\dot{H}^s} \\ \le&(|\lambda_1|-7\lambda_2)|\nabla \mathrm{u}|_{H^s}|\nabla \mathrm{d}|_{\dot{H}^s} + C\rho_1|\dot{\mathrm{d}}|_{H^s}|\nabla \mathrm{u}|_{H^s}|\nabla \mathrm{d}|_{H^s} \\ &+C(1+\rho_1)(|\dot{\mathrm{d}}|^2_{H^s} + |\nabla \mathrm{d}|^2_{\dot{H}^s})(|\nabla \mathrm{d}|_{H^s} + |\nabla \mathrm{d}|^2_{H^s}) \\ &+C(|\lambda_1|-\lambda_2)|\nabla \mathrm{u}|_{H^s}|\nabla \mathrm{d}|_{\dot{H}^s}(|\nabla \mathrm{d}|_{H^s} + |\nabla \mathrm{d}|^2_{H^s} + |\nabla \mathrm{d}|^3_{H^s})\,. \end{aligned} \tag{0.20}$$

Taking a positive constant $\eta = \frac{1}{2}\min\left\{1, \frac{1}{\rho_1}, \frac{|\lambda_1|}{\rho_1}\right\} \in (0, \frac{1}{2}]$, we multiply by η in the inequality (0.20) and then add it to the energy estimate obtained in the local existence.

Proof of Theorem (III)–Global solution

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\Big(|\mathrm{u}|_{H^s}^2 + \rho_1(1-\eta)|\dot{\mathrm{d}}|_{H^s}^2 + (1-\eta\rho_1)|\nabla \mathrm{d}|_{H^s}^2 \\
&\qquad + \eta\rho_1|\dot{\mathrm{d}} + \mathrm{d}|_{\dot{H}^2}^2 + \eta\rho_1|\nabla^{s+1}\mathrm{d}|_{L^2}^2 + \eta\rho_1|\lambda_1||\mathrm{d}|_{\dot{H}^s}^2\Big) \\
&+\frac{1}{2}\alpha|\nabla \mathrm{u}|_{H^s}^2 + (|\lambda_1| - \eta\rho_1)|\dot{\mathrm{d}}|_{H^s}^2 + \eta|\nabla \mathrm{d}|_{\dot{H}^s}^2 \\
&\le C'(1+\mu_1+|\lambda_1|-\lambda_2+\mu_6-\rho_1\lambda_2+\rho_1+\frac{1}{\sqrt{\rho_1}})\Big(|\mathrm{u}|_{H^s} + |\dot{\mathrm{d}}|_{H^s} + \sum_{i=1}^{4}|\nabla \mathrm{d}|_{H^s}^i\Big) \\
&\qquad \times\Big(|\nabla \mathrm{u}|_{H^s} + |\dot{\mathrm{d}}|_{H^s} + |\nabla \mathrm{d}|_{\dot{H}^s}\Big)|\nabla \mathrm{u}|_{H^s},
\end{aligned}
$$

where $\alpha = \mu_4 - 4\mu_6 - \frac{(|\lambda_1|-7\lambda_2)^2}{\eta} - \frac{2(7|\lambda_1|-2\lambda_2)^2}{|\lambda_1|} > 0.$

Proof of Theorem (III)–Global solution

We denote by

$$\mathcal{E}(t) \equiv |\mathrm{u}|_{H^s}^2 + \rho_1(1-\eta)|\dot{\mathrm{d}}|_{H^s}^2 + (1-\eta\rho_1)|\nabla \mathrm{d}|_{H^s}^2 + \eta\rho_1|\dot{\mathrm{d}} + \mathrm{d}|_{\dot{H}^s}^2$$
$$+ \eta\rho_1|\nabla^{s+1}\mathrm{d}|_{L^2}^2 + \eta\rho_1|\lambda_1||\mathrm{d}|_{\dot{H}^s}^2 \tag{0.21}$$

and

$$\mathcal{D}(t) \equiv |\nabla \mathrm{u}|_{H^s}^2 + |\dot{\mathrm{d}}|_{H^s}^2 + |\nabla \mathrm{d}|_{\dot{H}^s}^2 \,,$$

We have a new energy estimate

$$\frac{\partial}{\partial t}\mathcal{E}(t) + \theta\mathcal{D}(t) \le C_3 \sum_{q=1}^{4} [\mathcal{E}(t)]^{\frac{q}{2}} \mathcal{D}(t)\,, \tag{0.22}$$

where $\theta = \min\left\{\alpha, \eta, \frac{1}{2}|\lambda_1|\right\} > 0$. The rest is continuity argument similar as before.

$\lambda_1 = 0$ case: no damping

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \Delta u - \mathrm{div}(\nabla d \odot \nabla d) + \mathrm{div}\sigma, \\ \mathrm{div} u = 0, \\ \ddot{d} - \Delta d = (-|\dot{d}|^2 + |\nabla d|^2) d. \end{cases} \tag{0.23}$$

on $\mathbb{R}^3 \times \mathbb{R}^+$ with the constraint $|d| = 1$, where $u \in \mathbb{R}^3$ is the velocity field, $d \in \mathbb{R}^3$ is the orientation field and

$$\sigma_{ji} = \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 (d_j d_k A_{ki} + d_i d_k A_{kj}),$$

where $A_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$.
Cai-Wang first treated the case $\mu_1 = \mu_2 = 0$. (JFA 2020)

Notations

Define the perturbed angular momentum operators by

$$\tilde{\Omega}_i u = \Omega_i u + A_i u, \ \tilde{\Omega}_i d = \Omega_i d,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the rotation vector-field $\Omega = x \wedge \nabla$ and A_i is defined by

$$A_1 = e_2 \otimes e_3 - e_3 \otimes e_2, \ \ A_2 = e_3 \otimes e_1 - e_1 \otimes e_3,$$
$$A_3 = e_1 \otimes e_2 - e_2 \otimes e_1.$$

We define the scaling vector-field S by

$$S = t\partial_t + x_i \partial_{x_i}.$$

Let

$$\Gamma \in \{\partial_t, \partial_1, \partial_2, \partial_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3\}$$

and $Z^a = S^{a_1}\Gamma^{a'}$, where $a = (a_1, a') := (a_1, a_2, \cdots, a_8) \in \mathbb{Z}_+^8$, $\Gamma^{a'} = \Gamma^{a_2}\Gamma^{a_3} \cdots \Gamma^{a_8}$, we define

$$u^{(a)} := Z^a u, \ \ d^{(a)} := Z^a d.$$

Main theorem (Huang-J-Luo-Zhao, 2018)

Theorem

Assume that $N_0 := 60, N_1 := 6, h := 6$, μ_1, μ_2 *satisfy*

$$\mu_1 > -4(\mu + \frac{\mu_2}{2}), \ \ \mu + \frac{\mu_2}{2} > 0, \tag{0.24}$$

and (u_0, d_0, d_1) *are initial data near equilibrium* $(\vec{0}, \vec{i}, \vec{0})$ *satisfying the smallness assumptions*

$$\sup_{|a| \le N_1} \{ \|u_0^{(a)}\|_{H^{N(a)}} + \|\nabla d_0^{(a)}\|_{H^{N(a)}} + \|d_1^{(a)}\|_{H^{N(a)}} \} \le \epsilon_0, \tag{0.25}$$

where $N(a) = N_0 - |a|h$ *for* $0 \le |a| \le N_1$. *Then there exists a unique global solution* (u, d) *of the system (0.23) with initial data*

$$u(0) = u_0, \ \ d(0) = d_0, \ \partial_t d(0) = d_1,$$

satisfies the energy bounds

Main theorem (Huang-J-Luo-Zhao, 2018)

Theorem

$$\sup_{|a|\le N_1}\{\|u^{(a)}\|_{H^{N(a)}}+\|\nabla u^{(a)}\|^2_{L^2([0,t]:H^{N(a)})}\}+\|\nabla d\|_{H^{N(0)}}+\|\partial_t d\|_{H^{N(0)}}\lesssim \epsilon_0,$$

$$\sup_{1\le|a|\le N_1}\{\|\partial_t d^{(a)}\|_{H^{N(a)}}+\|\nabla d^{(a)}\|_{H^{N(a)}}\}\lesssim \epsilon_0(1+t)^{\bar\delta}.$$

for any $t\in[0,\infty)$, *where* $\bar\delta<10^{-7}$ *depends on* ϵ_0, N_0 *and* N_1.

Remark: 2-D case will be much harder!

Part II, Zero inertia density limit: from hyperbolic to parabolic E-L ($\lambda_1 < 0$)

Zero inertia density limit: from hyperbolic to parabolic E-L ($\lambda_1 < 0$)

Formally, the inertia constant $\rho_1 \to 0$, the hyperbolic E-L system converges to the parabolic E-L. Our goal is to *rigorously justify* this limiting process. Main difficulties: 1. changing type of equations; 2. coupled with NS.

- Special case 1: the velocity flow u is *given*. (J-Luo-Tang, 2017)
- Special case 2: completely forget u, redues to the scaling limit from wave map to heat flow into $\mathbb{S}^2$ (J-Luo-Tang-Zarnescu, 2017)
- General case: global-in-time, small initial data, from hyperbolic to parabolic E-L. (J-Luo, 2019)

Wave map

We consider a hyperbolic system for functions $\mathrm{d} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{S}^2$:

$$\partial_t \mathrm{d} = -\Box \mathrm{d} + (|\nabla \mathrm{d}|^2 - |\partial_t \mathrm{d}|^2)\mathrm{d}\,, \tag{0.26}$$

subject to initial data: for $x \in \mathbb{R}^3$,

$$\mathrm{d}|_{t=0} = \mathrm{d}^0(x) \in \mathbb{S}^2\,,\ \partial_t \mathrm{d}|_{t=0} = \tilde{\mathrm{d}}^0(x) \in \mathbb{R}^3\,, \mathrm{d}^0(x) \cdot \tilde{\mathrm{d}}^0(x) = 0\,, \tag{0.27}$$

The system (0.26) is a wave map from $\mathbb{R}^3$ to the unit sphere $\mathbb{S}^2$, with a damping term $\partial_t \mathrm{d}$. One way of interpreting this system is as follows: setting the righthand side of (0.26) equal to 0, we obtain $\Box \mathrm{d} = (|\nabla \mathrm{d}|^2 - |\partial_t \mathrm{d}|^2)\mathrm{d}$. This is the well-known *wave map*, which can be characterized variationally as a critical point of the functional

$$\mathcal{A}(\mathrm{d}) = \frac{1}{2} \iint (|\nabla \mathrm{d}|^2 - |\partial_t \mathrm{d}|^2)\, \mathrm{d}x\mathrm{d}t\,, \tag{0.28}$$

among maps d satisfying the target constraint, $\mathrm{d} : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{S}^2$. Thus the full system (0.26) can be viewed as a "*gradient flow*" of the functional (0.28).

Heat flow and wave map

The *heat flow* from $\mathbb{R}^3$ to $\mathbb{S}^2$:

$$\partial_t \mathrm{d} = \Delta \mathrm{d} + |\nabla \mathrm{d}|^2 \mathrm{d}\,. \tag{0.29}$$

The corresponding harmonic map $\Delta \mathrm{d} + |\nabla \mathrm{d}|^2 \mathrm{d} = 0$, is a critical point of the energy functional $E(\mathrm{d}) = \frac{1}{2} \int |\nabla \mathrm{d}|^2 \, dx$. Relation between (0.26) and (0.29): let $\mathrm{d}^\epsilon(t,x) := \mathrm{d}(\frac{t}{\epsilon}, \frac{x}{\sqrt{\epsilon}})$, d^ϵ satisfies the scaled wave map:

$$\partial_t \mathrm{d}^\epsilon = -(\epsilon \partial_{tt} - \Delta)\mathrm{d}^\epsilon + (|\nabla \mathrm{d}^\epsilon|^2 - \epsilon |\partial_t \mathrm{d}^\epsilon|^2)\mathrm{d}^\epsilon\,, \tag{0.30}$$

$$\mathrm{d}^\epsilon\big|_{t=0} = \mathrm{d}^{in}(x) \in \mathbb{S}^2\,,\ \partial_t \mathrm{d}^\epsilon\big|_{t=0} = \tilde{\mathrm{d}}^{in}(x) \in \mathbb{R}^3\,. \tag{0.31}$$

It is easy to see that letting $\epsilon = 0$ in (0.30) will formally give the heat flow (0.29). Question: Justify it.

Initial layer

The wave map is a system of hyperbolic equations with two initial conditions, while the heat flow is a parabolic system with only one initial condition. Usually the solution of the heat flow does not satisfy the second initial condition in (0.31). This disparity between the initial conditions of the wave map (0.30) and of the heat flow indicates that in one should expect an "initial layer" in time, appearing in the limiting process $\epsilon \to 0$. A formal derivation indicates that this should be of the form:

$$\mathrm{d}_0^I\Big(\frac{t}{\epsilon}, x\Big) = -\epsilon D(x) \exp(-\frac{t}{\epsilon}),$$

where

$$D(x) \equiv \tilde{\mathrm{d}}^{in}(x) - \Delta \mathrm{d}^{in}(x) - |\nabla \mathrm{d}^{in}(x)|^2 \mathrm{d}^{in}(x).$$

Theorem

$\nabla \mathrm{d}^{in} \in H^6$, $\tilde{\mathrm{d}}^{in} \in H^5$, and let $T > 0$ be the time interval of existence of the solution of the heat flow with initial condition d^{in}. Then, there exists an $\epsilon_0 \equiv \epsilon_0(|\nabla \mathrm{d}^{in}|_{H^6}, |\tilde{\mathrm{d}}^{in}|_{H^5}, T) \in (0, \frac{1}{2})$ s.t. for all $\epsilon \in (0, \epsilon_0)$ we have that on the interval $[0, T]$ the wave map equation (0.30) *with the initial conditions* (0.31) *admits a unique solution with the form:*

$$\mathrm{d}^{\epsilon}(t,x) = \mathrm{d}_0(t,x) + \mathrm{d}_0^I\Big(\frac{t}{\epsilon}, x\Big) + \sqrt{\epsilon}\,\mathrm{d}_R^{\epsilon}(t,x)\,,$$

where d_0 is the solution of the heat flow with initial condition d^{in} and $\mathrm{d}_0^I(\frac{t}{\epsilon}, x)$ is the initial layer. Moreover, there exists $C_0 = C_0(\mathrm{d}^{in}, \tilde{\mathrm{d}}^{in}, T) > 0$, s.t. the remainder term d_R^{ϵ} satisfies the bound

$$|\partial_t \mathrm{d}_R^{\epsilon}|^2_{L^\infty(0,T;H^2)} + \frac{1}{\epsilon}|\mathrm{d}_R^{\epsilon}|^2_{L^\infty(0,T;H^3)} \le C_0 \qquad (0.32)$$

for all $\epsilon \in (0, \epsilon_0)$.

$$
\begin{cases}
\partial_t \mathrm{u}^\epsilon + \mathrm{u}^\epsilon \cdot \nabla \mathrm{u}^\epsilon - \frac{1}{2}\mu_4 \Delta \mathrm{u}^\epsilon + \nabla p^\epsilon = -\mathrm{div}(\nabla \mathrm{d}^\epsilon \odot \nabla \mathrm{d}^\epsilon) + \mathrm{div}\sigma^\epsilon \,, \\
\mathrm{div}\mathrm{u}^\epsilon = 0 \,, \\
\epsilon \mathrm{D}^2_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon = \Delta \mathrm{d}^\epsilon + \gamma^\epsilon \mathrm{d}^\epsilon + \lambda_1 (\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon + \mathrm{B}^\epsilon \mathrm{d}^\epsilon) + \lambda_2 \mathrm{A}^\epsilon \mathrm{d}^\epsilon \,, \\
|\mathrm{d}^\epsilon| = 1 \,,
\end{cases}
\tag{0.33}
$$

where $\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon = \partial_t \mathrm{d}^\epsilon + \mathrm{u}^\epsilon \cdot \nabla \mathrm{d}^\epsilon$, and Lagrangian multiplier and stress

$$
\gamma^\epsilon = -\epsilon |\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon|^2 + |\nabla \mathrm{d}^\epsilon|^2 - \lambda_2 \mathrm{A}^\epsilon : \mathrm{d}^\epsilon \otimes \mathrm{d}^\epsilon \,,
$$

$$
\begin{aligned}
\sigma^\epsilon =& \mu_1 (\mathrm{A}^\epsilon : \mathrm{d}^\epsilon \otimes \mathrm{d}^\epsilon) \mathrm{d}^\epsilon \otimes \mathrm{d}^\epsilon + \mu_2 (\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon + \mathrm{B}^\epsilon \mathrm{d}^\epsilon) \otimes \mathrm{d}^\epsilon \\
&+ \mu_3 \mathrm{d}^\epsilon \otimes (\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon + \mathrm{B}^\epsilon \mathrm{d}^\epsilon) + \mu_5 (\mathrm{A}^\epsilon \mathrm{d}^\epsilon) \otimes \mathrm{d}^\epsilon + \mu_6 \mathrm{d}^\epsilon \otimes (\mathrm{A}^\epsilon \mathrm{d}^\epsilon) \,.
\end{aligned}
$$

the initial data

$$
\mathrm{u}^\epsilon|_{t=0} = \mathrm{u}^{in} \,,\ \mathrm{d}^\epsilon|_{t=0} = \mathrm{d}^{in} \,, (\mathrm{D}_{\mathrm{u}^\epsilon} \mathrm{d}^\epsilon)|_{t=0} = \tilde{\mathrm{d}}^{in}
$$

with the compatibilities

$$
\mathrm{div}\mathrm{u}^{in} = 0 \,,\ \mathrm{d}^{in} \cdot \tilde{\mathrm{d}}^{in} = 0 \,.
$$

General parabolic E-L

Formally $\mathrm{u}^\epsilon \to \mathrm{u}$ and $\mathrm{d}^\epsilon \to \mathrm{d}$ as $\epsilon \to 0$, the hyperbolic liquid crystal system converges to the parabolic liquid crystal model

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u} \cdot \nabla \mathrm{u} - \frac{1}{2}\mu_4 \Delta \mathrm{u} + \nabla p = -\mathrm{div}(\nabla \mathrm{d} \odot \nabla \mathrm{d}) + \mathrm{div}\sigma \,, \\ \mathrm{div}\mathrm{u} = 0 \,, \\ -\lambda_1(\mathrm{D}_\mathrm{u}\mathrm{d}_0 + \mathrm{B}\mathrm{d}) = \Delta \mathrm{d} + \gamma \mathrm{d} + \lambda_2 \mathrm{A}\mathrm{d} \,, \\ |\mathrm{d}| = 1 \,, \end{cases}$$

where the Lagrangian multiplier γ is

$$\gamma = |\nabla \mathrm{d}|^2 - \lambda_2 \mathrm{A} : \mathrm{d} \otimes \mathrm{d} \,,$$

with initial data $\mathrm{u}|_{t=0} = \mathrm{u}^{in} \,,\ \mathrm{d}|_{t=0} = \mathrm{d}^{in}$. Again, because of the disparity between the initial conditions, an initial layer is needed. The ansatz

$$\begin{aligned} \mathrm{u}^\epsilon(t,x) &= \mathrm{u}_0(t,x) + \sqrt{\epsilon}\mathrm{u}_R^\epsilon(t,x) \,, \\ \mathrm{d}^\epsilon(t,x) &= \mathrm{d}_0(t,x) + \epsilon^\beta \mathrm{d}_I(\tfrac{t}{\epsilon^\beta}, x) + \sqrt{\epsilon}\mathrm{d}_R^\epsilon(t,x) \end{aligned} \tag{0.34}$$

for a fixed $\beta > 0$ to be determined.

Define $\tau = \frac{t}{\epsilon^\beta}$, and the disparity

$$\mathrm{D}^{in} = \tilde{\mathrm{d}}^{in} - \mathrm{D}_{\mathrm{u}_0}\mathrm{d}_0|_{t=0} = \tilde{\mathrm{d}}^{in} + \mathrm{B}^{in}\mathrm{d}^{in} + \frac{1}{\lambda_1}(\Delta\mathrm{d}^{in} + \gamma_0^{in}\mathrm{d}^{in} + \lambda_2\mathrm{A}^{in}\mathrm{d}^{in}).$$

$$\begin{cases} \partial^2_{\tau\tau}\mathrm{d}_I + \frac{-\lambda_1}{\epsilon^{1-\beta}}\partial_\tau\mathrm{d}_I = \epsilon^{2\beta-1}\Delta\mathrm{d}_I\,, \\ \mathrm{d}_I(\infty, x) = \lim\limits_{\tau\to\infty}\mathrm{d}_I(\tau, x) = 0\,, \\ \partial_\tau\mathrm{d}_I(0, x) = \mathrm{D}^{in}(x)\,. \end{cases} \tag{0.35}$$

If $\beta > \frac{1}{2}$, $\epsilon^{2\beta-1}\Delta\mathrm{d}_I$ is a higher order term. Solve (0.35),

$$\begin{aligned} &\epsilon^\beta\mathrm{d}_I(\tfrac{t}{\epsilon^\beta}, x) = \epsilon\mathrm{D}_I^\epsilon(t, x) \\ &= 2\epsilon\Big(\lambda_1 - \sqrt{\lambda_1^2 + 4\epsilon\Delta}\Big)^{-1}\exp\Big(\frac{\lambda_1 - \sqrt{\lambda_1^2+4\epsilon\Delta}}{2\epsilon}t\Big)\mathrm{D}^{in}(x)\,. \end{aligned} \tag{0.36}$$

So in the ansatz, indeed, $\beta = 1$.

Main Theorem (J-Luo, 2019)

Theorem

$\mathrm{u}^{in}, \tilde{\mathrm{d}}^{in}, \nabla\mathrm{d}^{in} \in H^{2S_N}$, $\mu_4 > 0$, $\lambda_1 < 0$, $\mu_1 \geq 0$, $\mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0$, *and there exist* $\epsilon_0, \xi_0 \in (0, 1]$, *s.t.*

$$E^{in} \triangleq \|\mathrm{u}^{in}\|^2_{H^{2S_N}} + \|\tilde{\mathrm{d}}^{in}\|^2_{H^{2S_N}} + \|\nabla\mathrm{d}^{in}\|^2_{H^{2S_N}} \leq \xi_0 \tag{0.37}$$

for all $\epsilon \in (0, \epsilon_0]$, *then system* (0.33) *admits a unique solution*

$$\mathrm{u}^\epsilon, \nabla\mathrm{d}^\epsilon, \mathrm{D}_{\mathrm{u}^\epsilon}\mathrm{d}^\epsilon \in L^\infty(\mathbb{R}^+; H^N), \ \nabla\mathrm{u}^\epsilon \in L^2(\mathbb{R}^+; H^N). \tag{0.38}$$

Moreover, the solution $(\mathrm{u}^\epsilon, \mathrm{d}^\epsilon)$ *is of the form*

$$\begin{cases} \mathrm{u}^\epsilon(t,x) = \mathrm{u}_0(t,x) + \sqrt{\epsilon}\mathrm{u}^\epsilon_R(t,x), \\ \mathrm{d}^\epsilon(t,x) = \mathrm{d}_0(t,x) + \epsilon\mathrm{D}^\epsilon_I(t,x) + \sqrt{\epsilon}\mathrm{d}^\epsilon_R(t,x), \end{cases} \tag{0.39}$$

where $(\mathrm{u}_0, \mathrm{d}_0)$ *is the solution to parabolic E-L.*

The remainder equations

the remainder $(\mathrm{u}_R^\epsilon, \mathrm{d}_R^\epsilon)$ satisfies the following system

$$\begin{cases} \partial_t \mathrm{u}_R^\epsilon - \frac{1}{2}\mu_4 \Delta \mathrm{u}_R^\epsilon + \nabla p_R^\epsilon = \mu_1 \mathrm{div}\Big[(\mathrm{A}_R^\epsilon : \mathrm{d}_0 \otimes \mathrm{d}_0)\mathrm{d}_0 \otimes \mathrm{d}_0\Big] \\ \qquad + \mathcal{K}_\mathrm{u} + \mathrm{div}(\mathcal{C}_\mathrm{u} + \mathcal{T}_\mathrm{u} + \sqrt{\epsilon}\mathcal{R}_\mathrm{u}) + \epsilon \mathrm{div} \mathcal{Q}_\mathrm{u}(\mathrm{D}_I), \\ \mathrm{div} \mathrm{u}_R^\epsilon = 0, \\ \mathrm{D}^2_{\mathrm{u}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon} \mathrm{d}_R^\epsilon + \frac{-\lambda_1}{\epsilon} \mathrm{D}_{\mathrm{u}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon} \mathrm{d}_R^\epsilon - \frac{1}{\epsilon}\Delta \mathrm{d}_R^\epsilon + \partial_t(\mathrm{u}_R^\epsilon \cdot \nabla \mathrm{d}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon \cdot \nabla \mathrm{D}_I^\epsilon) \\ \qquad = \frac{1}{\epsilon}\mathcal{C}_\mathrm{d} + \frac{1}{\epsilon}\mathcal{S}_\mathrm{d}^1 + \frac{1}{\sqrt{\epsilon}}\mathcal{S}_\mathrm{d}^2 + \mathcal{R}_\mathrm{d} + \mathcal{Q}_\mathrm{d}(\mathrm{D}_I) \end{cases}$$

with the constraint

$$2\mathrm{d}_0 \cdot (\mathrm{d}_R^\epsilon + \sqrt{\epsilon}\mathrm{D}_I^\epsilon) + \sqrt{\epsilon}|\mathrm{d}_R^\epsilon + \sqrt{\epsilon}\mathrm{D}_I^\epsilon|^2 = 0,$$

with initial data

$$\begin{cases} \mathrm{u}_R^\epsilon(0,x) = 0, \\ \mathrm{d}_R^\epsilon(0,x) = -\sqrt{\epsilon}\mathrm{D}_I^\epsilon(0,x) = -\sqrt{\epsilon}\widetilde{\mathrm{D}}_\epsilon^{in}(x), \\ (\mathrm{D}_{\mathrm{u}_0 + \sqrt{\epsilon}\mathrm{u}_R^\epsilon} \mathrm{d}_R^\epsilon)(0,x) = -\sqrt{\epsilon}(\mathrm{u}_0 \cdot \nabla \mathrm{D}_I^\epsilon)(0,x) = -\sqrt{\epsilon}(\mathrm{u}^{in} \cdot \nabla \widetilde{\mathrm{D}}_\epsilon^{in})(x). \end{cases}$$

The remainder equations

The key of this work is to prove the existence for the system of $(\mathrm{u}_R^\epsilon, \mathrm{d}_R^\epsilon)$, furthermore, they satisfy the uniform bound

$$\begin{aligned}&\Big(\tfrac{1}{\epsilon}\|\mathrm{u}_R^\epsilon\|_{H^N}^2+\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2+\tfrac{1}{\epsilon}\|\mathrm{d}_R^\epsilon\|_{H^{N+1}}^2\Big)(t)\\&+\tfrac{1}{\epsilon}\int_0^t\|\nabla\mathrm{u}_R^\epsilon\|_{H^N}^2(\tau)\mathrm{d}\tau\le C\xi_0\end{aligned}\tag{0.40}$$

for all $t \geq 0$, $\epsilon \in (0, \epsilon_0]$ and for some constant $C > 0$, independent of ϵ and t.

Energy and energy-dissipation functionals-1

We now introduce the following energy functional $\mathcal{E}_{N,\epsilon}(t)$

$$\mathcal{E}_{N,\epsilon}(t) = \frac{1}{\epsilon}\|\mathrm{u}_R^\epsilon\|_{H^N}^2 + (1-\delta)\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2 + \frac{1}{\epsilon}\|\nabla \mathrm{d}_R^\epsilon\|_{H^N}^2$$
$$+(\frac{-\delta\lambda_1}{\epsilon} - \frac{5}{4}\delta)\|\mathrm{d}_R^\epsilon\|_{H^N}^2 + \|\mathrm{u}_R^\epsilon \cdot \nabla \mathrm{d}_0 + \frac{\delta}{2}\mathrm{d}_R^\epsilon\|_{H^N}^2$$
$$+\delta\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon + \mathrm{d}_R^\epsilon\|_{H^N}^2 + 2\sum_{|m|\le N}\langle\partial^m(\mathrm{u}_R^\epsilon\cdot\nabla\mathrm{d}_0), \partial^m \mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\rangle$$

and the energy dissipative rate $\mathcal{D}_{N,\epsilon}(t)$

$$\mathcal{D}_{N,\epsilon}(t) = \frac{3\mu_4}{8\epsilon}\|\nabla\mathrm{u}_R^\epsilon\|_{H^N}^2 + \frac{\delta}{2\epsilon}\|\nabla\mathrm{d}_R^\epsilon\|_{H^N}^2 - \delta\|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2$$
$$+\frac{\mu_1}{\epsilon}\sum_{|m|\le N}\|(\partial^m\mathrm{A}_R^\epsilon):\mathrm{d}_0\otimes\mathrm{d}_0\|_{L^2}^2 + \frac{1}{\epsilon}(\mu_5+\mu_6+\frac{\lambda_2^2}{\lambda_1})\sum_{|m|\le N}\|(\partial^m\mathrm{A}_R^\epsilon)\mathrm{d}_0\|_{L^2}^2$$
$$+\frac{-\lambda_1}{\epsilon}\sum_{|m|\le N}\|\partial^m\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon + (\partial^m\mathrm{B}_R^\epsilon)\mathrm{d}_0 + \frac{\lambda_2}{\lambda_1}(\partial^m\mathrm{A}_R^\epsilon)\mathrm{d}_0\|_{L^2}^2,$$

where $\delta \in (0, \frac{1}{2}]$ is a fixed constant, depending only on λ_1, λ_2 and N.

Energy and energy-dissipation functionals-2

Lemma

There is a small $\epsilon_0 > 0$*, such that the energy* $\mathcal{E}_{N,\epsilon}(t)$ *and the energy dissipative rate* $\mathcal{D}_{N,\epsilon}(t)$ *are both nonnegative for any* $\epsilon \in (0, \epsilon_0)$*. Moreover, for all* $\epsilon \in (0, \epsilon_0)$*, we have*

$$\mathcal{E}_{N,\epsilon}(t) \sim \frac{1}{\epsilon}\|\mathrm{u}_R^\epsilon\|_{H^N}^2 + \|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2 + \frac{1}{\epsilon}\|\nabla \mathrm{d}_R^\epsilon\|_{H^N}^2 + \frac{1}{\epsilon}\|\mathrm{d}_R^\epsilon\|_{H^N}^2 \,,$$

and

$$\begin{aligned}\mathcal{D}_{N,\epsilon}(t) \sim & \frac{1}{\epsilon}\|\nabla \mathrm{u}_R^\epsilon\|_{H^N}^2 + \frac{1}{\epsilon}\|\nabla \mathrm{d}_R^\epsilon\|_{H^N}^2 + \|\mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon\|_{H^N}^2 \\ & + \frac{1}{\epsilon}\sum_{|m|\leq N}\|\partial^m \mathrm{D}_{\mathrm{u}_0+\sqrt{\epsilon}\mathrm{u}_R^\epsilon}\mathrm{d}_R^\epsilon + (\partial^m \mathrm{B}_R^\epsilon)\mathrm{d}_0 + (\partial^m \mathrm{A}_R^\epsilon)\mathrm{d}_0\|_{L^2}^2 \,.\end{aligned}$$

Here the small positive constant $\beta_{S_N,0}$ *is needed in requirement of the initial data of the existence of* $(\mathrm{u}_0, \mathrm{d}_0)$*.*

Lemma

Let $(\mathrm{u}_R^\epsilon, \mathrm{d}_R^\epsilon)$ *be a sufficiently smooth solution to the remainder system on* $[0, T]$. *Then there are constants* $C > 0$ *and* $\theta_0 \gg 1$, *depending only on the Leslie coefficients and* $\beta_{S_N,0}$, *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{E}_{N,\epsilon}(t) + \theta_0 \mathcal{E}_{S_N,0}(t)\Big] + \mathcal{D}_{N,\epsilon}(t) + \frac{\theta_0}{2}\mathcal{D}_{S_N,0}(t)$$
$$\leq C\Big[\mathcal{E}_{N,\epsilon}^2(t) + \mathcal{E}_{N,\epsilon}^{\frac{1}{2}}(t) + \mathcal{E}_{S_N,\epsilon}^{\frac{1}{2}}(t)\Big]\Big[\mathcal{D}_{N,\epsilon}(t) + \frac{\theta_0}{2}\mathcal{D}_{S_N,0}(t)\Big]$$

holds for all $t \in [0, T]$ *and* $\epsilon \in (0, \epsilon_0]$, *where the small positive constant* ϵ_0 *is mentioned in the last Lemma.*

Part III, Low Mach number limit (with Guo-Li-Luo-Tang)

Low Mach number limit: scaling

- L_*, T_*, U_*: the units for (macroscopic) length, time and bulk velocity, respectively, where $U_* = L_*/T_*$.
- ρ_*, c_* and μ_*: the units of density, sound speed and viscosity, and let $\kappa_* = K/l^2$ be the unit of Frank constant, where K is the so-called elastic constant.
- l: the averaged length of the rod-like nematic liquid crystal molecular.
- Set

$$x = L_*\hat{x}\,, \quad t = T_*\hat{t}\,, \quad \mathrm{u} = U_*\hat{\mathrm{u}}\,, \quad \rho = \rho_*\hat{\rho}\,, \quad p = \rho_* c_*^2 \hat{p}\,, \quad \mathrm{d} = l\hat{\mathrm{d}}\,,$$
$$\kappa = \kappa_*\hat{\kappa}\,, \quad \mu_4 = \mu_*\hat{\mu}_4\,, \quad \mu_1 = \frac{\mu_*}{l^4}\hat{\mu}_1\,, \quad \mu_i = \frac{\mu_*}{l^2}\hat{\mu}_i \quad \text{for} \quad i = 2, 3, 5, 6\,.$$

- Define the following dimensionless constants:

$$\mathrm{Ma} = \frac{U_*}{c_*}\,, \quad \mathrm{Re} = \frac{\rho_* L_* U_*}{\mu_*}\,, \quad \mathrm{Er} = \frac{\mu_* L_* U_*}{\kappa_* l^2}\,, \quad \chi = \frac{\rho_* U_*^2}{\kappa_*}\,.$$

Low Mach number limit: scaling

- Dimensionless form (delete all the hats):

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathrm{u}) = 0\,, \\ \partial_t(\rho \mathrm{u}) + \mathrm{div}(\rho \mathrm{u} \otimes \mathrm{u}) + \frac{1}{\mathrm{Ma}^2}\nabla \mathrm{p} = \frac{1}{\mathrm{Re}}\mathrm{div}\Sigma_1 + \frac{1}{\mathrm{ReEr}}\mathrm{div}\Sigma_2 + \frac{1}{\mathrm{Re}}\mathrm{div}\Sigma_3\,, \\ \frac{\chi}{\mathrm{Er}}\rho \ddot{\mathrm{d}} = \frac{\kappa}{\mathrm{Er}}\Delta \mathrm{d} + \Gamma \mathrm{d} + \lambda_1(\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}) + \lambda_2 \mathrm{A}\mathrm{d}\,, \end{cases}$$

  where $\Gamma = -\frac{\chi}{\mathrm{Er}}\rho|\dot{\mathrm{d}}|^2 + \frac{\kappa}{\mathrm{Er}}|\nabla \mathrm{d}|^2 - \lambda_2 \mathrm{d}^\top \mathrm{A}\mathrm{d}\,.$
- Many asymptotic behavior problem can be considered here: $\chi \to 0$ (Zero inertia limit), $Re \to \infty$ (inviscid limit), $Er \to \infty$ (large Ericksen number limit), etc. (for example: Wu-Xu-Zarnescu studied $Re \& Er \to \infty$ for parabolic Q-tensor model.)
- Here, we consider the low Mach number limit: $Ma = \epsilon \to 0$:

$$\begin{cases} \partial_t \rho^\epsilon + \mathrm{div}(\rho^\epsilon \mathrm{u}^\epsilon) = 0\,, \\ \partial_t(\rho^\epsilon \mathrm{u}^\epsilon) + \mathrm{div}(\rho^\epsilon \mathrm{u}^\epsilon \otimes \mathrm{u}^\epsilon) + \frac{1}{\epsilon^2}\nabla p(\rho^\epsilon) = \mathrm{div}(\Sigma_1^\epsilon + \Sigma_2^\epsilon + \Sigma_3^\epsilon)\,, \\ \rho^\epsilon \ddot{\mathrm{d}}^\epsilon = \kappa \Delta d^\epsilon + \Gamma^\epsilon d^\epsilon + \lambda_1(\dot{\mathrm{d}}^\epsilon + \mathrm{B}^\epsilon d^\epsilon) + \lambda_2 \mathrm{A}^\epsilon d^\epsilon\,. \end{cases}$$

Low Mach number limit

Formally, supposing that the limit $(\rho^\epsilon, \mathrm{u}^\epsilon, d^\epsilon) \rightharpoonup (1, \mathrm{u}, \mathrm{d})$ exists and initially $\rho_0^\epsilon = 1 + \epsilon\phi_0^\epsilon \to 1$, $\mathrm{u}_0^\epsilon \to \mathrm{u}_0$, $d_0^\epsilon \to \mathrm{d}_0$, $\widetilde{\mathrm{d}}_0^\epsilon \to \widetilde{\mathrm{d}}_0$ as $\epsilon \to 0$, then the limiting system will be

$$\begin{cases} \mathrm{div}\,\mathrm{u} = 0\,, \\ \partial_t \mathrm{u} + (\mathrm{u}\cdot\nabla)\mathrm{u} - \frac{1}{2}\mu_4 \Delta \mathrm{u} + \nabla\pi = -\kappa \mathrm{div}(\nabla \mathrm{d} \odot \nabla \mathrm{d}) + \mathrm{div}\tilde{\sigma}_\mu(\mathrm{u}, \mathrm{d})\,, \\ \ddot{\mathrm{d}} = \kappa\Delta\mathrm{d} + \widehat{\Gamma}\mathrm{d} + \lambda_1(\dot{\mathrm{d}} + \mathrm{B}\mathrm{d}) + \lambda_2 \mathrm{A}\mathrm{d}\,, \end{cases}$$

where $\nabla\pi$ is the "limit" of $\frac{1}{\epsilon^2}\frac{a}{\rho^\epsilon}\nabla[(\rho^\epsilon)^\gamma - 1] - \kappa\frac{1}{\rho^\epsilon}\nabla(\frac{|\nabla\partial|^2}{2})$, and $\widehat{\Gamma}$ is the formal limit of Γ^ϵ:

$$\widehat{\Gamma} = -|\dot{\mathrm{d}}|^2 + \kappa|\nabla\mathrm{d}|^2 - \lambda_2 \mathrm{d}^\top \mathrm{A}\mathrm{d}\,.$$

Moreover, the system is endowed with the initial data:

$$(\mathrm{u}, \mathrm{d}, \dot{\mathrm{d}})|_{t=0} = (\mathrm{u}_0, \mathrm{d}_0, \widetilde{\mathrm{d}}_0) \in R^n \times \mathbb{S}^{n-1} \times R^n\,, \tag{0.41}$$

and the boundary condition at infinity

$$(\mathrm{u}, \mathrm{d}) \to (0, \bar{\mathrm{d}}), \quad \text{as } |x| \to \infty\,. \tag{0.42}$$

Main Theorem 1

Let $0 < \epsilon \leq 1$ and integer $s > \frac{n}{2} + 1$ with $n = 2, 3$. Let the initial data $\left(\phi_0^\epsilon, \mathrm{u}_0, d_0, \widetilde{\mathrm{d}}_0^\epsilon\right) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n$ satisfy $d_0 \cdot \widetilde{\mathrm{d}}_0^\epsilon = 0$, and

$$\|\phi_0^\epsilon\|_{L^\infty} \leq \tfrac{1}{2}\,, \quad \phi_0^\epsilon \in H^s_{p'(\rho_0^\epsilon)}\,, \quad \mathrm{u}_0,\, \widetilde{\mathrm{d}}_0^\epsilon \in H^s_{\rho_0^\epsilon}\,, \quad \nabla d_0 \in H^s\,, \tag{0.43}$$

where $\rho_0^\epsilon = 1 + \epsilon\phi_0^\epsilon$. We further assume

$$\kappa > 0\,,\ \mu_1 \geq 0\,,\ \mu_4 > 0\,,\ \tfrac{1}{2}\mu_4 + \xi \geq 0\,,\ \lambda_1 < 0\,,\ \mu_5 + \mu_6 + \frac{\lambda_2^2}{\lambda_1} \geq 0\,.$$

Then (1): If there is a small $\delta > 0$, independent of $\epsilon \in (0, 1]$, s.t.

$$\|\phi_0^\epsilon\|^2_{H^s_{p'(\rho_0^\epsilon)}} + \|\mathrm{u}_0\|^2_{H^s_{\rho_0^\epsilon}} + \|\widetilde{\mathrm{d}}_0^\epsilon\|^2_{H^s_{\rho_0^\epsilon}} + \kappa\|\nabla d_0\|^2_{H^s} \leq \delta\,, \tag{0.44}$$

then the Cauchy problem admits a unique global solution

$$\phi^\epsilon \in L^\infty(\mathbb{R}^+; H^s_{p'(\rho^\epsilon)}),\ \mathrm{u}^\epsilon, \dot{d}^\epsilon \in L^\infty(R^+; H^s_{\rho^\epsilon}),\ \nabla\mathrm{u}^\epsilon \in L^2(R^+; H^s),$$
$$\nabla d^\epsilon \in L^\infty(R^+; H^s),$$

where $\rho^\epsilon = 1 + \epsilon\phi^\epsilon$.

Main Theorem 1

Moreover, the following uniform (in $\epsilon \in (0,1]$) energy bounds hold:

$$\|\phi^\epsilon\|^2_{L^\infty(R^+;H^s_{p'(\rho^\epsilon)})} + \|\mathrm{u}^\epsilon\|^2_{L^\infty(R^+;H^s_{\rho^\epsilon})} + \|\dot{d}^\epsilon\|^2_{L^\infty(R^+;H^s_{\rho^\epsilon})}$$
$$+\kappa\|\nabla d^\epsilon\|^2_{L^\infty(R^+;H^s)} + \frac{1}{2}\mu_4\|\nabla \mathrm{u}^\epsilon\|^2_{L^2(R^+;H^s)} \le \delta\,,$$

$$\|\rho^\epsilon\|_{L^\infty(R^+;L^\infty)} = \|1+\epsilon\phi^\epsilon\|_{L^\infty(R^+;L^\infty)} \approx 1\,,$$

and

$$\|\partial_t \dot{d}^\epsilon\|^2_{L^\infty(R^+;H^{s-1})} + \|\partial_t d^\epsilon\|^2_{L^\infty(\mathbb{R}^+;H^s)} \le C_d\,,$$

where the constant $C_d > 0$ is independent of ϵ.

Main Theorem 1

(2): If besides the smallness condition (0.44), the initial data are "well-prepared", i.e. ϕ_0^ϵ and u_0^ϵ further satisfy

$$\|\phi_0^\epsilon\|_{H^s} \leq C_\phi \epsilon \quad \text{and} \quad \|\mathrm{div}\mathrm{u}_0^\epsilon\|_{H^{s-2}} \leq C_{\mathrm{u}} \epsilon$$

for some ϵ-independent constants $C_\phi, C_{\mathrm{u}} > 0$, then, ϕ^ϵ and u^ϵ have better regularity, i.e. for any $T > 0$,

$$\|\partial_t \phi^\epsilon\|^2_{L^\infty(0,T;H^{s-2}_{p'(\rho^\epsilon)})} + \|\partial_t \mathrm{u}^\epsilon\|^2_{L^\infty(0,T;H^{s-2}_{\rho^\epsilon})} \leq C_{\phi u}(T)$$

and

$$\frac{1}{\epsilon}\|\mathrm{div}\mathrm{u}^\epsilon\|_{L^\infty(0;T;H^{s-2})} + \frac{1}{\epsilon}\|p'(\rho^\epsilon)\nabla\phi^\epsilon\|_{L^\infty(0,T;H^{s-2})} \leq C'_{\phi u}(T)\,,$$

where the constants $C_{\phi u}(T), C'_{\phi u}(T) > 0$ are all independent of $\epsilon \in (0, 1]$.

Main Theorem 2

Under the same assumptions as in Theorem 1, we further assume that there exist $\mathrm{u}_0, \widetilde{\mathrm{d}}_0 \in R^n$ and $\mathrm{d}_0 \in \mathbb{S}^{n-1}$ with $\mathrm{u}_0, \nabla \mathrm{d}_0 \in H^s$, such that $\mathrm{d}_0 \cdot \widetilde{\mathrm{d}}_0 = 0$, $\mathrm{div}\mathrm{u}_0 = 0$, $\|d_0 - \mathrm{d}_0\|_{L^2} \to 0$ and

$$\left(\mathrm{u}_0^\epsilon, \widetilde{\mathrm{d}}_0^\epsilon, \nabla d_0^\epsilon\right) \to \left(\mathrm{u}_0, \widetilde{\mathrm{d}}_0, \nabla \mathrm{d}_0\right) \text{ strongly in } H^s$$

as $\epsilon \to 0$. Let $\left(\rho^\epsilon = 1 + \epsilon\phi^\epsilon, \mathrm{u}^\epsilon, d^\epsilon\right)$ be the family of solutions constructed in Theorem 1. Then there exist $\mathrm{u} \in R^n$, $\pi \in R$ and $\mathrm{d} \in \mathbb{S}^{n-1}$ with $\mathrm{u}, \dot{\mathrm{d}}, \nabla \mathrm{d} \in L^\infty(R^+; H^s) \cap C(R^+; H^{s-1}_{loc})$, $\nabla \mathrm{u} \in L^2(R^+; H^s)$ and $\pi \in L^\infty(R^+; H^{s-1})$, such that (in the sense of subsequences)

$$\rho^\epsilon \to 1 \text{ strongly in } L^\infty(R^+; H^s),$$

$$\frac{1}{\epsilon^2} p(\rho^\epsilon) - \frac{1}{2}\kappa |\nabla d|^2 \to \pi \text{ weakly-} \star \text{ for } t > 0, \text{ weakly in } H^{s-1}$$

as $\epsilon \to 0$.

Main Theorem 2

Furthermore,

$$\left(\mathrm{u}^{\epsilon}, \nabla d^{\epsilon}, \dot{d}^{\epsilon}\right) \to \left(\mathrm{u}, \nabla \mathrm{d}, \dot{\mathrm{d}}\right)$$

weakly-$\star$ for $t > 0$, weakly in H^s and strongly in $C(R^+; H^{s-1}_{loc})$ as $\epsilon \to 0$. Here $(\mathrm{u}, \pi, \mathrm{d})$ is the solution to the Ericksen-Leslie hyperbolic liquid crystal model in incompressible flow. Moreover, the following global energy bound holds:

$$\|\mathrm{u}\|^2_{L^\infty(R^+;H^s)} + \|\dot{\mathrm{d}}\|^2_{L^\infty(R^+;H^s)} + \|\nabla \mathrm{d}\|^2_{L^\infty(R^+;H^s)} + \tfrac{1}{2}\mu_4 \|\nabla \mathrm{u}\|^2_{L^2(R^+;H^s)} + \|\pi\|^2_{L^\infty(R^+;H^{s-1})} \lesssim \delta,$$

where δ is the small constant in the assumption of Theorem 1.

Main Theorem 3

Under the assumptions of the above Theorem, we further assume that

$$\|\sqrt{\rho_0^\epsilon}\mathrm{u}_0^\epsilon - \mathrm{u}_0^\epsilon\|_{L^2}^2 + \|\sqrt{\rho_0^\epsilon}\widetilde{\mathrm{d}}_0^\epsilon - \widetilde{\mathrm{d}}_0^\epsilon\|_{L^2}^2 + \kappa\|\nabla d_0^\epsilon - \nabla \mathrm{d}^\epsilon\|_{L^2}^2 + \|d_0^\epsilon - \mathrm{d}^\epsilon\|_{L^2}^2 + \langle \Pi_0^\epsilon, 1\rangle \lesssim \epsilon^{\alpha_0}$$

for some constant $\alpha_0 > 0$, where $\Pi_0^\epsilon = \frac{1}{\epsilon^2}\frac{a}{\gamma-1}\Big((\rho_0^\epsilon)^\gamma - \gamma(\rho_0^\epsilon) - 1\Big)$ satisfying

$$|\sqrt{\rho_0^\epsilon} - 1|^2 \lesssim |\rho_0^\epsilon - 1|^2 \mathbf{1}_{|\rho_0^\epsilon - 1| \le \frac{1}{2}} + |\rho_0^\epsilon - 1|^\gamma \mathbf{1}_{|\rho_0^\epsilon - 1| \ge \frac{1}{2}} \lesssim \Pi_0^\epsilon .$$

Then, for any fixed $T > 0$, we have, for $t \in [0, T]$,

$$\|\sqrt{\rho^\epsilon}\mathrm{u}^\epsilon - \mathrm{u}\|_{L^2}^2 + \|\sqrt{\rho^\epsilon}\dot{d}^\epsilon - \dot{\mathrm{d}}\|_{L^2}^2 + \kappa\|\nabla d^\epsilon - \nabla \mathrm{d}\|_{L^2}^2 + \|d^\epsilon - \mathrm{d}\|_{L^2}^2 + \langle \Pi^\epsilon, 1\rangle \le C_T \epsilon^{\beta_0}$$

where $\Pi^\epsilon = \frac{1}{\epsilon^2}\frac{a}{\gamma-1}\Big[(\rho^\epsilon)^\gamma - \gamma(\rho^\epsilon - 1) - 1\Big]$, and $\beta_0 = \min\{2, \alpha_0, 1 + \frac{\alpha_0}{2}\} > 0$ and $C_T = C(1+T)\exp(CT) > 0$ for some positive constant C, independent of ϵ.

Dissipative solutions (with F. Cheng)

- Weak solutions in the energy space of Ericksen-Leslie system are very hard to obtain, even for *parabolic* case, it is not complete yet. The hyperbolic case is even harder.
- Dissipative solutions was introduced by P.-L. Lions (1995) in 3-D incompressible Euler equations.
  - They exist globally in time for large initial data in the energy space;
  - They coincide with the unique strong solution when the latter exists (this is called weak-strong stability);
  - They allow energy dissipation phenomena occur;
  - It can be rigorously derived from renormalized solutions of Boltzmann equation (by L. Saint-Raymond, ARMA 2003)
  - We consider Ginzburg-Landau approximation:

$$\begin{cases} \partial_t \mathrm{u} + \mathrm{u}\cdot\nabla\mathrm{u} - \mu\Delta\mathrm{u} + \nabla p = -\mathrm{div}(\nabla d \odot \nabla d), \\ \mathrm{div}\mathrm{u} = 0, \\ \partial_t \omega + \mathrm{u}\cdot\nabla\omega + \omega = \Delta d - f(d), \\ \omega = \partial_t d + \mathrm{u}\cdot\nabla d, \end{cases}$$

where $f(d) = \nabla\mathcal{F}(d)$, with $\mathcal{F}(d) = \frac{1}{4\kappa^2}(|d|^2 - 1)^2$.

Weak-strong stability inequality

(u,d): a smooth solution to E-L, test functions $(\bar{u},\bar{d}) \in C^\infty([0,\infty)\times\Omega)$ s.t. $\mathrm{div}\bar{u}=0$, $\bar{u}(0,x)=\bar{u}_0, \bar{d}(0,x)=\bar{d}_0$

$$A(\bar{u},\bar{d}) = \begin{pmatrix} A_1(\bar{u},\bar{d}) \\ A_2(\bar{u},\bar{d}) \end{pmatrix} = \begin{pmatrix} -\partial_t\bar{u} - \mathbb{P}(\bar{u}\cdot\nabla\bar{u}) + \frac{1}{2}\mu_4\Delta\bar{u} - \mathbb{P}\mathrm{div}(\nabla\bar{d}\odot\nabla\bar{d}) \\ -\partial_t\bar{\omega} - \bar{u}\cdot\nabla\bar{\omega} - \bar{\omega} + \Delta\bar{d} - f(\bar{d}) \end{pmatrix}$$

where $\mathbb{P}$ is Leray projection and $\bar{\omega} = \partial_t\bar{d} + \bar{u}\cdot\nabla\bar{d}$. Growth rate:

$$\begin{aligned}\lambda(t) =& 2C\|\nabla\bar{u}\|_{L^\infty} + \|\nabla\bar{\omega}\|_{L^\infty} + \|\Delta\bar{d}\|_{L^\infty} + \frac{6C}{\mu}\|\nabla\bar{d}\|_{L^\infty}^2 + 2C\|\bar{\omega}\|_{L^3} \\ &+ C(1 + \|\nabla u\|_{L^2}^2 + \|\nabla\bar{u}\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\nabla\bar{d}\|_{L^2}^2 + \|\nabla\bar{d}\|_{L^6}^2 + \||\bar{d}|^2\|_{L^3}),\end{aligned}$$

Then, one has the following *stability inequality*

$$\begin{aligned}&\delta\mathcal{E}(t) + \frac{1}{4}\int_0^t \delta\mathcal{D}(s)e^{\int_s^t\lambda(\sigma)d\sigma}ds \\ &\leq \delta\mathcal{E}(0)e^{\int_0^t\lambda(s)ds} + \int_0^t\left[A(\bar{u},\bar{d})\cdot\begin{pmatrix} u-\bar{u} \\ \omega-\bar{\omega}\end{pmatrix}\right](s)e^{\int_s^t\lambda(\sigma)d\sigma}ds,\end{aligned}$$

Weak-strong stability

the modulated energy $\delta\mathcal{E}(t)$ and energy dissipation $\delta\mathcal{D}(t)$:

$$\begin{aligned}\delta\mathcal{E}(t) = & \frac{1}{2}\|(u-\bar{u})(t)\|_{L^2}^2 + \frac{1}{2}\|\nabla(d-\bar{d})(t)\|_{L^2}^2 + \frac{1}{2}\|(\omega-\bar{\omega})(t)\|_{L^2}^2 \\ & + \frac{1}{4}\||d-\bar{d}|^2\|_{L^2}^2, \\ \delta\mathcal{D}(t) = & \mu\|\nabla(u-\bar{u})(t)\|_{L^2}^2 + \|(\omega-\bar{\omega})(t)\|_{L^2}^2.\end{aligned}$$

By analogy with Lions' dissipative solution to the incompressible Euler system, we define dissipative solution for E-L:

Definition

For any $T > 0$, we say $(u, \nabla d) \in L^\infty(0, T; L^2(\Omega)) \cap C(0, T; w\text{-}L^2(\Omega))$, s.t. $\mathrm{div} u = 0$ is a dissipative solution of E-L, if $u(0) = u_0, d(0) = d_0, \omega(0) = \omega_0$ satisfy $\mathrm{div} u_0 = 0$ and the stability inequality holds for any test functions $(\bar{u}, \bar{d}, \bar{\omega}) \in C_c^\infty([0,\infty) \times \Omega)$ such that $\mathrm{div}\bar{u} = 0$.

Regularized equations

The regularized system

$$\begin{cases} \partial_t \mathrm{u}^\epsilon + \mathrm{u}^\epsilon \cdot \nabla \mathrm{u}^\epsilon + \nabla p^\epsilon - \mu \Delta \mathrm{u}^\epsilon = -\mathrm{div}(\nabla d^\epsilon \odot \nabla d^\epsilon), \\ \mathrm{div} \mathrm{u}^\epsilon = 0, \\ \partial_t \omega^\epsilon + \mathrm{u}^\epsilon \cdot \nabla \omega^\epsilon + \omega^\epsilon = \Delta d^\epsilon - f(d^\epsilon), \\ \omega^\epsilon = \partial_t d^\epsilon + \mathrm{u}^\epsilon \cdot \nabla d^\epsilon - \epsilon \Delta d^\epsilon, \end{cases}$$

Formally, let $\epsilon = 0$ in the above system, it then reduces to E-L. It is supplemented with the following initial data

$$\mathrm{u}^\epsilon|_{t=0} = \mathrm{u}_0, \quad d^\epsilon|_{t=0} = d_0, \quad \omega^\epsilon + \epsilon \Delta d^\epsilon|_{t=0} = \omega_0,$$

where of course $\mathrm{div} \mathrm{u}_0 = 0$.

Future work and open problems

- Finite time singularities.
- Global in time weak solutions.
- 2-D case for $\lambda_1 = 0$.
- Inertia density limit for $\lambda_1 = 0$.
- Twist solutions (i.e. $\mathrm{u} = 0$).