Yamabe flow and its soliton on manifolds with boundary

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If we write $g = u^{\frac{4}{n-2}} g_0$, then

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The Yamabe problem is to find $0 < u \in C^\infty(M)$ with $R_g \equiv c$.

Yamabe problem

The **Yamabe constant** of (M, g_0) is

$$Y(M, g_0) = \inf\{E_{g_0}(u) : 0 < u \in C^\infty(M)\}.$$

Here

$$E_{g_0}(u) = \frac{\int_M \frac{4(n-1)}{n-2}|\nabla_{g_0} u|^2 + R_{g_0} u^2 dV_{g_0}}{\left(\int_M u^{\frac{2n}{n-2}} dV_{g_0}\right)^{\frac{n-2}{n}}} = \frac{\int_M R_g dV_g}{\left(\int_M dV_g\right)^{\frac{n-2}{n}}}$$

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is the Yamabe energy, where $g = u^{\frac{4}{n-2}} g_0$.

Fact: If u is a positive minimizer, i.e. $E_{g_0}(u) = Y(M, g_0)$, then u solves (YamabePDE).

Proof: $\frac{d}{dt} E_{g_0}(u + tv) = 0$ for all $v \in C^\infty(M)$.

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The Yamabe problem was solved by Aubin, Trudinger, and Schoen.

Yamabe flow

Hamilton introduced the **Yamabe flow**:

$$\frac{\partial}{\partial t} g(t) = -(R_{g(t)} - \overline{R}_{g(t)}) g(t), \quad g(0) = g_0, \qquad \text{(YF)}$$

where $\overline{R}_{g(t)}$ is the average of $R_{g(t)}$:

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Hamilton proved the long time existence of (YF). Also, he proved the convergence of (YF) when $Y(M, g_0) \leq 0$, i.e. $g(t) \to g_\infty$ as $t \to \infty$ for some metric g_∞ with $R_{g_\infty} \equiv c$.

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When $Y(M, g_0) > 0$, the convergence of (YF) was studied by Brendle, Chow, Schwetlick and Struwe, and Ye.

Theorem (Carlotto-Chodosh-Rubinstein)

If g_∞ is an integrable cirtical point of Yamabe energy, then (YF) converges exponentially to g_∞, i.e.

$$\|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq Ce^{-\delta t}$$

for some $\delta > 0$.

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Fact: g_∞ is an integrable critical point of Yamabe energy if $\ker\mathcal{L}_\infty=\{0\}$, where

$$\mathcal{L}_\infty v=(n-1)\Delta_{g_\infty}v+R_{g_\infty}v$$

is the linearized Yamabe operator of g_∞.

Yamabe flow

Theorem (Carlotto-Chodosh-Rubinstein)

There exists g_∞ such that (YF) does not converge exponentially to g_∞. More precisely,

$$C^{-1}(1+t)^{-\frac{1}{p-2}} \leq \|g(t)-g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq C(1+t)^{-\frac{1}{p-2}}$$

for $p=3$ and for some $p \geq 3$.

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Fact: g_∞ satisfies AS_3 if there exists $v \not\equiv 0$ such that

$$\mathcal{L}_\infty v = (n-1)\Delta_{g_\infty} v + R_{g_\infty} v = 0 \quad \text{and} \quad R_{g_\infty} \int_M v^3 dV_{g_\infty} \neq 0.$$

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Goal: Find g_∞ with $R_{g_\infty}\equiv c>0$ such that $\int_M v^3 dV_{g_\infty}\neq 0$.

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$(\mathbb{S}^n_+, g_{\mathbb{S}^n_+})$ is an example of (i), and (D^n, g_{flat}) is an example of (ii).

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$$\begin{aligned} -\frac{4(n-1)}{n-2}\Delta_{g_0} u + R_{g_0} u &= R_g u^{\frac{n+2}{n-2}} \text{ in } M, \\ \frac{2(n-1)}{n-2}\frac{\partial u}{\partial \nu_{g_0}} + H_{g_0} u &= H_g u^{\frac{n}{n-2}} \text{ on } \partial M. \end{aligned}$$

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The Yamabe probelm with boundary was later studied by Almaraz, S. Chen, Marques, Mayer and Ndiaye, etc.

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To study (I), we consider the **Yamabe flow with boundary**:

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To study (II), we consider the **conformal mean curvature flow**:

$$\begin{cases} \frac{\partial}{\partial t}g(t) = -(H_{g(t)} - \overline{H}_{g(t)})g(t) \text{ on } \partial M, \\ R_{g(t)} = 0 \text{ in } M, \quad g(0) = g_0. \end{cases} \tag{CMCF}$$

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- When the Yamabe constant is positive, the convergence of (YFB) is proved by Almaraz-Sun, and the convergence of (CMCF) is proved by Almaraz.

Yamabe flow with boundary

Theorem (H.-Shin)

If g_∞ is an integrable cirtical point of Yamabe energy, then (YFB) (respectively $(CMCF)$) converges exponentially to g_∞, i.e.

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for $p = 3$.

Sketlov eigenvalue

For a Riemannian manifold (M, g) with boundary ∂M, the **Dirichlet-to-Neumann map** $DN : C^\infty(\partial M) \to C^\infty(\partial M)$ is given by

$$DN(f) = \frac{\partial \hat{f}}{\partial \nu_g},$$

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Along the unnormalized CMCF

$$\frac{\partial}{\partial t} g(t) = -H_{g(t)} g(t) \text{ on } \partial M \quad \text{and} \quad R_{g(t)} = 0 \text{ in } M,$$

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Using CMCF, we also obtained some estimates of the first nonzero Steklov eigenvalue of (M, g).

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Suppose that M is without boundary, possibly noncompact.

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More precisely, the **Yamabe soliton** is the solution $g(t)$ of the (unnormalized) Yamabe flow:

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$$g(t) = \sigma(t)\psi_t^*(g_0)$$

where $\sigma : [0,\infty) \to (0,\infty)$ with $\sigma(0) = 1$,
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This is **Yamabe soliton from flow point of view**.

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Differentiating $g(t) = \sigma(t)\psi_t^*(g_0)$ with respect to t gives

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Evaluating it at $t = 0$, using $\psi_0 = id_M$ and $\sigma(0) = 1$, we get

$$-R_{g_0}g_0 = \dot{\sigma}(0)g_0 + L_X g_0,$$

where L_X is the Lie derivative with respect to X, and X is the vector field generated by ψ_t.

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If we write $\lambda = -\dot{\sigma}(0)$, then

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If we write $\lambda = -\dot{\sigma}(0)$, then

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If the vector field is gradient, i.e. $X = \frac{1}{2}\nabla_{g_0} f$ for some function f, then

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where $\nabla^2_{g_0} f$ is the Hessian of f. This is called **gradient Yamabe soliton**.

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Fact: The Yamabe soliton from the flow point of view is equivalent to the Yamabe soliton from the equation point of view.

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Theorem (Daskalopoulos-Sesum)

Suppose that (M, g_0) *is a gradient Yamabe soliton (from equation point of view) which is locally conformally flat and has positive sectional curvature. Then* (M, g_0) *has a warped-product structure.*

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Conformal mean curvature soliton–the flow point of view

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$$g(t) = \sigma(t)\psi_t^*(g_0)$$

where $\sigma : [0, \infty) \to (0, \infty)$ with $\sigma(0) = 1$,
and $\psi_t : M \to M$ is a family of diffeomorphisms with $\psi_0 = id_M$.

Conformal mean curvature soliton–the flow point of view

The **conformal mean curvature soliton** is the solution $g(t)$ of the (unnormalized) conformal mean curvature flow:

$$\frac{\partial}{\partial t}g(t) = -H_{g(t)}g(t) \text{ on } \partial M \quad \text{and} \quad R_{g(t)} = 0 \text{ in } M$$

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These are the Yamabe soliton with boundary and the conformal mean curvature soliton from flow point of view.

Results

From the flow point of view, we have

Theorem (Chen-H.)

Any compact Yamabe soliton with boundary must have constant scalar curvature in M (and vanishing mean curvature on ∂M).

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Any compact conformal mean curvature soliton must have constant mean curvature on ∂M (and vanishing scalar curvature in M).

Proof

Proof: Since the (unnormalized) Yamabe flow with boundary preserves conformal structure, $g(t) = u(t)^{\frac{4}{n-2}} g_0$.

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$$\frac{\partial}{\partial t} u(t) = -\frac{n-2}{4} R_{g(t)} u(t) \text{ in } M \quad \text{and} \quad \frac{\partial u(t)}{\partial \nu_{g_0}} = 0 \text{ on } \partial M.$$

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Therefore,

$$\frac{d}{dt}\left(\frac{\int_M R_{g(t)} dV_{g(t)}}{(\int_M dV_{g(t)})^{\frac{n-2}{n}}}\right) = \frac{d}{dt} E_{g_0}(u(t))$$

$$= -\frac{n-2}{2} \cdot \frac{(\int_M R_{g(t)}^2 dV_{g(t)})(\int_M dV_{g(t)}) - (\int_M R_{g(t)} dV_{g(t)})^2}{(\int_M dV_{g(t)})^{\frac{n-2}{n}+1}}$$

Proof

Cauchy-Schwarz inequality $\Rightarrow \frac{d}{dt}\left(\frac{\int_M R_{g(t)}dV_{g(t)}}{(\int_M dV_{g(t)})^{\frac{n-2}{n}}}\right) \leq 0$, and equality holds if and only if $R_{g(t)}$ is constant.

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On the other hand, if $g(t) = \sigma(t)\psi_t^*(g_0)$, we have

$$\frac{\int_M R_{g(t)}dV_{g(t)}}{\left(\int_M dV_{g(t)}\right)^{\frac{n-2}{n}}} = \frac{\int_M R_{\psi_t^*(g_0)}dV_{\psi_t^*(g_0)}}{\left(\int_M dV_{\psi_t^*(g_0)}\right)^{\frac{n-2}{n}}} = \frac{\int_M R_{g_0}dV_{g_0}}{\left(\int_M dV_{g_0}\right)^{\frac{n-2}{n}}}. \qquad \blacksquare$$

The equation point of view

Goal: Define the Yamabe soliton with boundary and the conformal mean curvature soliton from equation point of view.

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The **Yamabe soliton with boundary from equation point of view** is defined as

$$(\lambda - R_{g_0})g_0 = L_X g_0 \text{ in } M \quad \text{and} \quad H_{g_0} = 0, \langle X, \nu_{g_0} \rangle = 0 \text{ on } \partial M.$$

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The **conformal mean curvature soliton from equation point of view** is defined as

$$R_{g_0} = 0 \text{ in } M \quad \text{and} \quad (\lambda - H_{g_0})g_0 = L_X g_0, \langle X, \nu_{g_0} \rangle = 0 \text{ on } \partial M.$$

The equation point of view

Theorem (H.-Shin)

The Yamabe soliton with boundary from flow point of view is equivalent to the Yamabe soliton with boundary from equation point of view.

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Example: Consider $(\mathbb{R}^n_+, g_e)$.

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Thus, if we choose $f = f(x_1, ..., x_n) = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$, then $X = \frac{1}{2}\nabla_{g_e} f = (\frac{a_1}{2}, ..., \frac{a_{n-1}}{2}, 0)$

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Therefore, we have

$$-R_{g_e} g_e = L_X g_e \text{ in } \mathbb{R}^n_+ \quad \text{and} \quad H_{g_e} = 0, \langle X, \nu_{g_e}\rangle = 0 \text{ on } \partial\mathbb{R}^n_+$$

i.e. $(\mathbb{R}^n_+, g_e)$ is a (steady, nontrivial) gradient Yamabe soliton with boundary (from equation point of view).

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i.e. $(\mathbb{R}^n_+, g_e)$ is a (steady, nontrivial) gradient Yamabe soliton with boundary (from equation point of view).

Similarly, $(\mathbb{R}^n_+, g_e)$ is a (steady, nonotrivial) gradient conformal mean curvature soliton (from equation point of view).

The equation point of view

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Any compact gradient Yamabe soliton with boundary from equation point of view must have constant scalar curvature in M (and vanishing mean curvature on ∂M).

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Any compact gradient conformal mean curvature soliton from equation point of view must have constant mean curvature on ∂M (and vanishing scalar curvature in M).

The equation point of view

Proof: If (M, g_0, f) is a gradient Yamabe soliton with boundary, then

$$\nabla^2_{g_0} f = (\lambda - R_{g_0}) g_0 \text{ in } M \quad \text{and} \quad \frac{\partial f}{\partial \nu_{g_0}} = 0 \text{ on } \partial M.$$

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Taking trace $\Rightarrow \Delta_{g_0} f = n(\lambda - R_{g_0})$ in M.

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Taking trace $\Rightarrow \Delta_{g_0} f = n(\lambda - R_{g_0})$ in M. Also, one has

$$(n-1)\Delta_{g_0} R_{g_0} = \frac{1}{2}\langle \nabla_{g_0} R_{g_0}, \nabla_{g_0} f\rangle + R_{g_0}(\lambda - R_{g_0}) \text{ in } M, \quad \frac{\partial R_{g_0}}{\partial \nu_{g_0}} = 0 \text{ on } \partial M.$$

The equation point of view

Proof: If (M, g_0, f) is a gradient Yamabe soliton with boundary, then

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Integrating the first equation over M, we get

$$(n-1)\int_M \Delta_{g_0} R_{g_0} = \frac{1}{2}\int_M \langle \nabla_{g_0} R_{g_0}, \nabla_{g_0} f \rangle + \int_M R_{g_0}(\lambda - R_{g_0}),$$

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Proof: We get

$$0 = \left(1 - \frac{n}{2}\right) \int_M R_{g_0}(\lambda - R_{g_0}). \tag{1}$$

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Finally, integrating $\Delta_{g_0} f = n(\lambda - R_{g_0})$ over M and using $\dfrac{\partial f}{\partial \nu_{g_0}} = 0$ on ∂M, we obtain

$$0 = \int_M (\lambda - R_{g_0}) \qquad (2)$$

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Proof: We get

$$0 = \left(1 - \frac{n}{2}\right) \int_M R_{g_0}(\lambda - R_{g_0}). \tag{1}$$

Finally, integrating $\Delta_{g_0} f = n(\lambda - R_{g_0})$ over M and using $\dfrac{\partial f}{\partial \nu_{g_0}} = 0$ on ∂M, we obtain

$$0 = \int_M (\lambda - R_{g_0}) \tag{2}$$

Combining (1) and (2), we get

$$\int_M (\lambda - R_{g_0})^2 = \int_M R_{g_0}(\lambda - R_{g_0}) - \lambda \int_M (\lambda - R_{g_0}) = 0,$$

which gives $R_{g_0} \equiv \lambda$.

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Suppose that (M, g_0) is a Yamabe soliton with boundary from equation point of view. If (M, g_0) is locally conformal flat and has positive sectional curvature,

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$$(M, g_0) = (\mathbb{R}, dr^2) \times_{w(r)} (\mathbb{S}^{n-1}_+, g_{\mathbb{S}^{n-1}_+}).$$

$\dim M = 2$

Suppose M is a 2-dimensional manifold with boundary ∂M.

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The (unnormalized) **Gauss curvature flow with boundary** is

$$\frac{\partial}{\partial t}g(t) = -K_{g(t)}g(t) \text{ in } M \quad \text{and } k_{g(t)} = 0 \text{ on } M.$$

Here, $K_{g(t)}$ is the Gauss curvature, and $k_{g(t)}$ is the geodesic curvature.

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The Gauss curvature flow with boundary is a two-dimensional version of the Yamabe flow with boundary.

dim $M = 2$

Suppose M is a 2-dimensional manifold with boundary ∂M.

The (unnormalized) **geodesic curvarure flow** is

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We have defined the **Gauss curvature soliton with boundary**, which is the self-similar solution of the Gauss curvature flow with boundary.

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Theorem (H.-Shin)

(i) Any compact Gauss curvature soliton with boundary must have constant Gauss curvature in M.

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Theorem (H.-Shin)

(i) Any compact Gauss curvature soliton with boundary must have constant Gauss curvature in M.
(ii) Any compact geodesic curvature soliton must have constant geodesic curvature in M.

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Thank you very much for your attention!