Boundary layer/spike solutions of chemotaxis models

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Experiment of boundary layers in chemotactic movement

Experimental observations of boundary layers of chemotaxis in fluid [1], where the chemical (oxygen) is saturated at the boundary (interface between air and fluid).

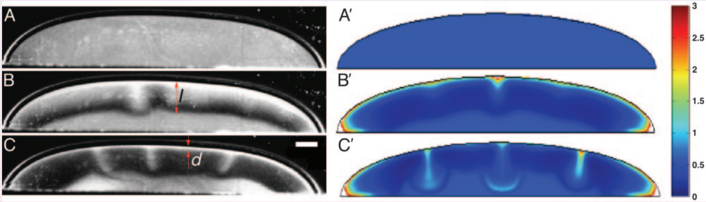

Fig. 2. Stages leading to self-concentration in a sessile drop, observed experimentally (*A–C*) and by numerical computations with the model described in text (*A′–C′*). (*A* and *A′*) Drop momentarily after placement in the chamber. (*B* and *B′*) Formation of depletion zone after 150 s, before appreciable fluid motion. (*C* and *C′*) Lateral migration of accumulation layer toward drop edges and creation of vortex (shown in Fig. 3) after 600 s. The color scheme describes the rescaled bacteria concentration ρ as defined in *Appendix*. (Scale bar, 0.5 mm.)

Two salient experimental observations

- Boundary layer (aggregation) and vortex creation;
- Boundary layer thickness is different at different part of air-water interface (boundary curvature)

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[1] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E. (2005). Bacterial swimming and oxygen transport near contact lines. *PNAS*, 102:2277-2282, 2005

Chemotaxis-fluid coupled system

u- bacterial density; w- oxygen concentration; $\mathbf{v}$- fluid velocity

$$\begin{cases} u_t + \mathbf{v}\cdot\nabla u = D_u\Delta u - \chi\nabla\cdot(u\phi(w)\nabla w), & \text{in } \Omega \\ w_t + \mathbf{v}\cdot\nabla w = D_w\Delta w - uf(w), & \text{in } \Omega \\ \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla P = \varepsilon\Delta\mathbf{v} - u\nabla\psi, & \text{in } \Omega \\ \nabla\cdot\mathbf{v} = 0 \end{cases} \tag{1}$$

where $\nabla\psi = V_b g(\rho_b - \rho)\mathbf{z}$ describes the gravitational force exerted by bacteria onto the fluid along the upward unit vector $\mathbf{z}$ proportional to the bacterial volume V_b, the gravitational constant g and the bacterial density ρ_b;

$$\phi(w) - \text{chemotactic sensitivity function}$$

$$f(w) - \text{oxygen consumption function}$$

Physical boundary conditions in *Tuval etc. PNAS 2005* are

$$D_u\partial_\nu u - \chi u\phi(w)\partial_\nu w = 0,\ w = w_b,\ \mathbf{v} = 0 \quad \text{on } \partial\Omega \tag{2}$$

Namely zero-flux boundary for bacterial cells, Dirichlet boundary condition for oxygen and no-slip boundary condition for fluid.

Results on (1)

1. Numerical results with physical BC (26):
   - Chertock-Feller-Kurganov-Lorz-Markowich. J. Flui. Mech., 694(2012):155-190. Observe the boundary layer formed in 2D chamber;
   - Lee-Kim. Eur. J. Mech. B Fluids, 52(2015), 120-130. Observe the boundary layer formed in 3D chamber;
2. Analytical results with physical BC (26):
   - A. Lorz (M3AS, 2010): local existence of weak solutions in Ω - water-drop shape (2D and 3D) for $\phi(w) = w$;

3. Other results on $\mathbb{R}^N$ or bounded domain with Neumann boundary conditions: Duan-Lord-Markowich (CPDE 2009), Chae-Kang-Lee (DCDS 2012, CPDE 2014), Zhang-Zheng (SIMA 2014), Lorz (CMS 2012, M3AS 2010), Winkler (CPDE 2012, ARMA 2014, CVPDE 2015, AIHPAN 2016), Tao-Winkler (AIHPAN 2016), Duan-Li-Xiang (JDE 2017), Liu-Lorz (AIHPAN 2011), Zhang-Li (DCDSB 2015, JDE 2015), Peng-Xiang (M3AS 2018)

Remarks and goal

Remark: No analytical results on the boundary layer (aggregation) for the chemotaxis-fluid system with physical boundary conditions proposed by Tuval *et al* 2015 (PNAS).

Goal: Exploit the boundary-layer solutions. Consider a 1-D interval or radially symmetric domain, then $\mathbf{v} = 0$ and the system

$$\begin{cases} u_t + \cancel{\mathbf{v} \cdot \nabla u} = D_u \Delta u - \chi \nabla \cdot (u\phi(w)\nabla w), \\ w_t + \cancel{\mathbf{v} \cdot \nabla w} = D_w \Delta w - uf(w), \\ \cancel{\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla P = \varepsilon \Delta \mathbf{v} - \kappa u \nabla \psi}, \\ \cancel{\nabla \cdot \mathbf{v} = 0} \end{cases}$$

reduces to the following Keller-Segel (1971 Keller & Segel, JTB) type chemotaxis system

$$\begin{cases} u_t = D_u \Delta u - \chi \nabla \cdot (u\phi(w)\nabla w), \\ w_t = D_w \Delta w - uf(w), \end{cases} \tag{3}$$

where $f(w) = w^m (m \geq 0)$ and

$$\phi(w) = 1/w \text{ (logarithmic/singular sensitivity)},$$

$$\phi(w) = 1 \text{ (linear sensitivity)}.$$

Type 1: K-S system with singular sensitivity

Goal: consider the KS system with physical boundary conditions:

$$\begin{cases} u_t = u_{xx} - \chi\left(\frac{u}{w} w_x\right)_x, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ w_t = \varepsilon w_{xx} - u w^m, \\ (u,w)(x,0) = (u_0(x), w_0(x)) \end{cases} \tag{4}$$

where $\mathbb{R}_+ = [0,\infty)$, with boundary conditions

$$\begin{cases} (u_x - \chi \frac{u}{w} w_x)(0,t) = 0, \ w(0,t) = b, \\ (u,w)(+\infty,t) = (0,0). \end{cases} \tag{5}$$

The zero-flux boundary condition on u preserves the mass which will be hence prescribed: $\lambda := \int_0^\infty u(x,t)dx$.

The steady state (U,W) of (4)-(5) satisfies

$$\begin{cases} U_{xx} - \chi\left(\frac{U}{W} W_x\right)_x = 0, \quad \varepsilon W_{xx} - U W^m = 0, \quad \int_0^\infty U(x)dx = \lambda > 0, \\ \left(U_x - \chi \frac{U}{W} W_x\right)(0) = 0, \ W(0) = b, \ (U,W)(+\infty) = (0,0). \end{cases} \tag{6}$$

Boundary spiky/layer steady states

Theorem 1 (Carrillo-Li-W., PLMS 2020): Assume $m \geq 0$ and $\chi > |1-m|$. Then
(i) The problem (6) has a unique solution (U, W) with $U'(x) < 0$, $W'(x) < 0$, and

$$U(x) = \frac{\lambda^2(\chi+1-m)^2}{2\varepsilon(\chi+m+1)b^{1-m}} \cdot \left(1 + \frac{\lambda(\chi+m-1)(\chi+1-m)}{2\varepsilon(\chi+m+1)b^{1-m}} \cdot x\right)^{\frac{-2\chi}{\chi+m-1}},$$

$$W(x) = b\left(1 + \frac{\lambda(\chi+m-1)(\chi+1-m)}{2\varepsilon(\chi+m+1)b^{1-m}} \cdot x\right)^{\frac{-2}{\chi+m-1}}.$$

(ii) As $\chi \to \infty$, U concentrates at $x = 0$ and

$U(x) \to \lambda\delta(x)$ in the sense of distribution,

$W(x) \to b$ uniformly on any bounded interval.

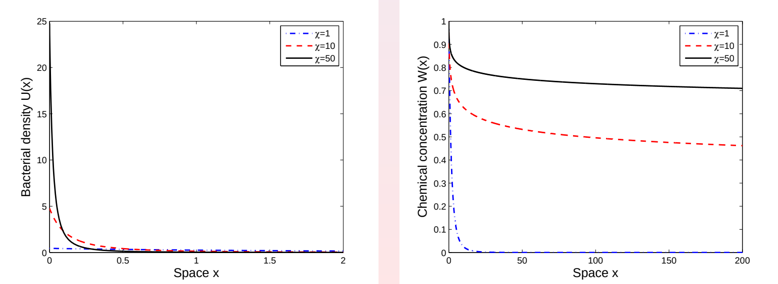


Figure: Profiles of spiky steady state $(U, W)(x)$ with respect to $\chi > 0$, where $b = \lambda = \varepsilon = 1, m = 0.5$

Boundary spik/layer profile- numerical illustration

(iii) As $\varepsilon \to 0$, U concentrates at $x = 0$ and $W(x)$ forms a (boundary) layer at $x = 0$. Namely

$$U(x) \to \lambda\delta(x) \text{ in the sense of distribution as } \varepsilon \to 0$$

and there is a constant $\theta = \theta(\varepsilon)$ satisfying $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\lim_{\varepsilon\to 0} \|W\|_{C[\theta,\infty]} = 0, \quad \liminf_{\varepsilon\to 0} \|W\|_{C[0,\theta)} > 0.$$

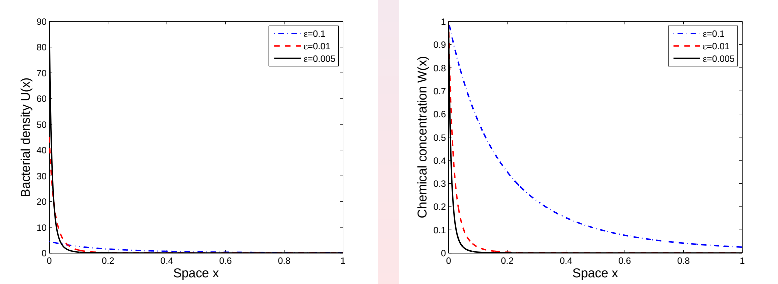


Figure: Profiles of spiky-layer steady state $(U, W)(x)$ with respect to $\varepsilon > 0$, where $b = \lambda = \chi = 1, m = 0.5$

Challenges/ideas on stability

Recall

$$\begin{cases} u_t = u_{xx} - \chi\left(\frac{u}{w} w_x\right)_x, \quad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ w_t = \varepsilon w_{xx} - u w^m, \\ (u,w)(x,0) = (u_0(x), w_0(x)) \\ (u_x - \chi \frac{u}{w} w_x)(0,t) = 0, \ w(0,t) = b, \\ (u,w)(+\infty, t) = (0,0). \end{cases} \tag{7}$$

Challenge: singularity due to $\inf\limits_{x \in \mathbb{R}_+} w(x) = 0$.

Idea: using Cole-Hopf transformation (no better way that we can have so far)

$$v := -\frac{w_x}{w}, \ \text{ i.e. } \ w(x,t) = b e^{-\int_0^x v(y,t)dy} \tag{8}$$

which now turns (7) into a nonlocal system of conservation laws

Key observations

$$
\begin{cases}
u_t = u_{xx} + \chi(uv)_x, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
v_t = \varepsilon v_{xx} - (\varepsilon v^2 - uw^{m-1})_x, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\
w(x,t) = be^{-\int_0^x v(y,t)dy}, & \\
(u,v)(x,0) = (u_0(x), v_0(x)) &
\end{cases}
\tag{9}
$$

with zero-flux boundary conditions

$$
\begin{cases}
u_x + \chi uv = \varepsilon v_x - (\varepsilon v^2 - uw^{m-1}) = 0, & x = 0 \\
(u,v) \to (0,0), & x \to +\infty
\end{cases}
$$

Observations:

- Price paid: though singularity vanishes, a new nonlocal term is present, which is unfavorable for analysis but more manageable than the singularity ⌣̈;

Key observations

$$\begin{cases} u_t = u_{xx} + \chi(uv)_x, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ v_t = \varepsilon v_{xx} - (\varepsilon v^2 - uw^{m-1})_x, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+ \\ w(x,t) = be^{-\int_0^x v(y,t)dy}, \\ (u,v)(x,0) = (u_0(x), v_0(x)) \end{cases} \tag{9}$$

with zero-flux boundary conditions

$$\begin{cases} u_x + \chi uv = \varepsilon v_x - (\varepsilon v^2 - uw^{m-1}) = 0, & x = 0 \\ (u,v) \to (0,0), & x \to +\infty \end{cases}$$

Observations:

- Price paid: though singularity vanishes, a new nonlocal term is present, which is unfavorable for analysis but more manageable than the singularity ⌣̈;
- The nonlocality can be removed when taking the antiderivative of v which is also a (necessary) technique often used for conservation laws. Hence the technique of taking antiderivative works for both “conservation law” and “removal of nonlocality” (kill two birds with one stone).

Keys in proof

Hence we decompose the solution (u, v) as

$$u = \phi_x + U, \; v = \psi_x + V, \text{ i.e. } (\phi, \psi)(x, t) = \int_0^x (u(y, t) - U(y), v(y, t) - V(y))dy.$$

Then (ϕ, ψ) satisfies

$$\begin{cases} \phi_t = & \phi_{xx} + \chi V \phi_x + \chi U \psi_x + \chi \phi_x \psi_x, \; (x, t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ \psi_t = & \varepsilon \psi_{xx} - 2\varepsilon V \psi_x - U W^{m-1}(1 - e^{-(m-1)\psi}) + W^{m-1}\phi_x \\ & -\varepsilon \psi_x^2 - W^{m-1}(1 - e^{-(m-1)\psi})\phi_x, \end{cases} \tag{10}$$

where the initial value $(\phi, \psi)(x, 0)$ is given by

$$(\phi, \psi)(x, 0) = (\phi_0, \psi_0)(x) \to (0, 0) \text{ as } x \to +\infty,$$
$$(\phi, \psi)(0, t) = (0, 0), \; (\phi, \psi)(+\infty, t) = (0, 0).$$

Technical treatment in the proof (method of energy estimates):

- Making a term $W^{m-1}\phi_x$;
- Taylor expansion $1 - e^{-(m-1)\psi} = (m - 1)\psi - \sum_{n=2}^{\infty} \frac{(1-m)^n \psi^n}{n!}$;
- Hardy inequality.

Stability of spiky/layer steady states

Theorem 2 (Carrillo-Li-W. PLMS 2020): Assume that $m \geq 0$ and that $\chi > |1-m|$. Let (U, W) be the unique steady state of system (4)-(5) obtained in Theorem 1. Assume that the initial perturbation satisfies $\phi_0(\infty) = \psi_0(\infty) = 0$ where

$$\phi_0(x) = \int_0^x (u_0(y) - U(y))dy, \ \psi_0(x) = -\ln w_0(x) + \ln W(x).$$

Define the weight functions $w_1 = 1/U, \ w_2 = W^{1-m}, \ w_3 = W^{m-1}/U$. Then the following results hold:

1. If $m \geq 1$, then there exists a constant $\delta_1 > 0$ such that if $\|\phi_0\|^2_{H^1_{w_1}} + \|\psi_0\|^2_{H^1_{w_2}} + \|\phi_{0xx}\|^2 + \|\psi_{0xx}\|^2 \leq \delta_1$, then system (4)-(5) has a unique global solution $(u, w)(x, t)$ satisfying
$$\sup_{x \in \mathbb{R}_+} |(u, w)(x, t) - (U, W)(x)| \to 0 \ \text{ as } \ t \to +\infty. \tag{11}$$
2. If $0 \leq m < 1$ and $\chi \gg 1$, then there exists a constant $\delta_2 > 0$ such that if $\|\phi_0\|^2_{H^1_{w_3}} + \|\phi_{0xx}\|^2 + \|\psi_0\|^2_2 \leq \delta_2$, then system (4)-(5) has a unique global solution $(u, w)(x, t)$ satisfying
$$\sup_{x \in \mathbb{R}_+} |(u, w)(x, t) - (U, W)(x)| \to 0 \ \text{ as } \ t \to +\infty.$$

Problem in a unit ball

Consider

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla \ln w), \\ w_t = \varepsilon \Delta w - uw, \end{cases} \tag{12}$$

with the following initial and boundary conditions

$$u(x,0) = u_0(x),\ w(x,0) = w_0(x),\ x \in B_1, \tag{13}$$

and

$$\frac{\partial u}{\partial \mathbf{n}} - \chi u \frac{\partial \ln w}{\partial \mathbf{n}} = 0,\ w = b > 0,\ x \in \partial B_1,\ t > 0, \tag{14}$$

where $\mathbf{n}$ is the outward unit normal vector field on ∂B_1.

Questions

- Existence and uniqueness of steady state;
- Nonlinear stability (local) of the unique steady state.

Spiky steady states in a unit ball

Consider the steady state problem

$$\begin{cases} \Delta U - \chi \nabla \cdot (U \nabla \ln W) = 0, & x \in B_1, \\ \Delta W - UW = 0, & x \in B_1, \\ \frac{\partial U}{\partial \mathbf{n}} - \chi U \frac{\partial \ln W}{\partial \mathbf{n}} = 0, \ W = b > 0, & x \in \partial B_1, \\ \int_{B_1} U dx = \lambda. \end{cases} \tag{15}$$

Theorem 3 (Li-W. 2023). Assume that $b > 0$ is a constant. Then

1. The system (15) has a unique smooth positive solution (U, W) that is radially symmetric in B_1, satisfying $U_r > 0$ and $W_r > 0$ with $r := |x|$.
2. As $\chi \to \infty$, U concentrates on the boundary ∂B_1 and W converges to the boundary value b. That is as $\chi \to \infty$

$$\omega_{n-1} r^{n-1} U(r) \to m\delta(r-1) \text{ in the sense of distribution,} \tag{16}$$

$$W(r) \to b \text{ in } C(\overline{B_1}), \tag{17}$$

where ω_{n-1} is the surface area of the $(n-1)$ dimensional unit sphere and $\delta(r-1)$ is the Dirac function centered at $r = 1$.

Spiky steady states in a unit ball

Theorem 4 (Stability) (Li-W. 2023). Assume that the initial datum (u_0, w_0) is radially symmetric, and that $u_0 \in H^2(B_1)$, $w_0 - b \in H_0^1(B_1) \cap H^2(B_1)$ with $u_0 > 0$ and $w_0 > 0$. Let (U, W) be the unique steady state obtained in Theorem 3 with $\int_{B_1} U(x)dx = \int_{B_1} u_0(x)dx$. Then there exists a constant $\varepsilon_0 > 0$ such that if the initial datum satisfies

$$\|(u_0 - U, w_0 - W)\|_{H^2} \leq \varepsilon_0,$$

then the system (12)-(14) admits a unique global radial solution $(u, w) \in C([0, +\infty); H^2(B_1))$, which satisfies

$$\|(u - U, w - W)(t)\|_{H^2} \leq C\|(u_0 - U, w_0 - W)\|_{H^2} e^{-\mu t}, \tag{18}$$

where C and μ are positive constants independent of t.

Remark. By the Sobolev imbedding theorem, we further have the following asymptotic convergence:

$$\|(u - U, w - W)(\cdot, t)\|_{C(\overline{B_1})} \leq Ce^{-\mu t}.$$

Type2: K-S system with linear response

Consider

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla w) & \text{in } \Omega, \\ w_t = \varepsilon^2 \Delta w - uw & \text{in } \Omega, \\ \partial_\nu u - u \partial_\nu w = 0, \ w = w_b & \text{on } \partial\Omega, \end{cases} \tag{19}$$

Note: global well-posedness results (Tao 2011, Tao-Winkler 2012, Fan-Jin 2017) for Neumann boundary conditions: $\partial_\nu u|_{\partial\Omega} = \partial_\nu w|_{\partial\Omega} = 0$.

Aim: stationary boundary-layer solutions

Integration of the first equation of (19) gives

$$\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx := \lambda$$

where $\lambda > 0$ denotes the total mass of cells. Therefore the stationary problem of (19) reads as

$$\begin{cases} \Delta u - \nabla \cdot (u \nabla w) = 0 & \text{in } \Omega, \\ \varepsilon^2 \Delta w - uw = 0 & \text{in } \Omega, \\ \partial_\nu u - u \partial_\nu w = 0, \ w = w_b & \text{on } \partial\Omega, \\ \int_\Omega u(x)dx = \lambda. \end{cases} \tag{20}$$

Type2: K-S system with linear response

The first equation of (20) gives

$$u = c_0 e^w$$

for some positive constant $c_0 > 0$. Since $\lambda = \int_\Omega u(x)dx$, we get $c_0 = \frac{\lambda}{\int_\Omega e^w dx}$. Therefore the problem (20) is equivalent to the following nonlocal semilinear elliptic Dirichlet problem

$$\begin{cases} \varepsilon^2 \Delta w = \dfrac{\lambda}{\int_\Omega e^w dx} w e^w & \text{in } \Omega, \\ w = w_b & \text{on } \partial\Omega, \end{cases} \tag{21}$$

with

$$u = \frac{\lambda}{\int_\Omega e^w dx} e^w, \tag{22}$$

To do:

- Existence/uniqueness of boundary-layer solutions with thickness
- Effect of curvature on the boundary-layer profile (steepness and thickness)

Type2: K-S system with linear response

$$\Omega_\delta = \{p \in \Omega \mid \mathrm{dist}(p, \partial\Omega) > \delta\}$$

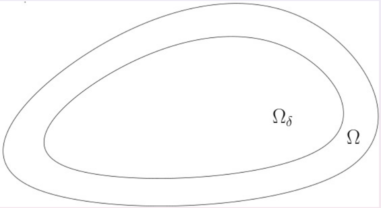


Figure: An illustration of Ω_δ in Ω.

We shall give a description of solutions in a general domain as $\varepsilon \to 0$. WLOG, we may assume $0 \in \Omega$ throughout the paper and set

$$\Omega^\varepsilon = \{y \mid \varepsilon y \in \Omega\}\,.$$

Define $W_\varepsilon(y)$ to be the unique solution of the following local problem

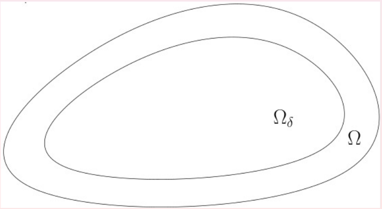
$$\begin{cases} \Delta_y W_\varepsilon = \frac{\lambda}{|\Omega|} W_\varepsilon e^{W_\varepsilon} & \text{in } \Omega^\varepsilon, \\ W_\varepsilon = w_b & \text{on } \partial\Omega^\varepsilon. \end{cases} \tag{23}$$

Main results

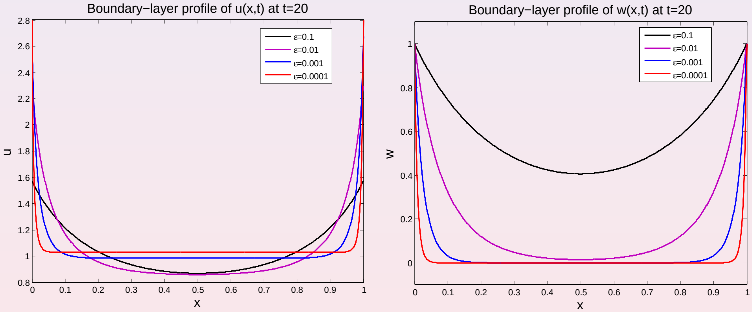
Theorem 1 (Lee-W.-Yang, Nonlinearity 2020): Let Ω be a bounded smooth domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary, and let λ and w_b be given positive constants independent of ε. Then

1. For $\varepsilon > 0$, the Dirichlet problem (21) admits a unique classical solution $w_\varepsilon \in C^1(\overline{\Omega}) \cap C^\infty(\Omega)$, and hence the elliptic system (20) admits a unique solution which is non-degenerate.
2. There exists a non-negative constant $\delta(\varepsilon) \sim O(\varepsilon)$ such that u_ε has the following property:

$$\lim_{\varepsilon \to 0} \|w_\varepsilon\|_{L^\infty(\overline{\Omega_{\delta(\varepsilon)}})} = 0, \quad \|w_\varepsilon(x) - W_\varepsilon(x/\varepsilon)\|_{L^\infty(\Omega)} = O(\varepsilon).$$



Numerical illustration of boundary layers



Effect of boundary curvature

It turns out that this is a difficult problem for general domain due to the non-local term. To be educational, we consider a simplified geometry $\Omega = B_R(0)$ with radius $R > 0$, where the curvature is $\frac{1}{R}$. We denote by ω_N the volume of $B_R(0) \subset \mathbb{R}^N$ and $\alpha(N) = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ the volume of unit ball in $\mathbb{R}^N$, where $\omega_N = \alpha(N)R^N$. For convenience, we define

$$f(s) := se^s \text{ and } F(s) := \int_0^s f(\tau)\mathrm{d}\tau = (s-1)e^s + 1 \geq 0, \text{ for } s \geq 0.$$

Then by the uniqueness and the classical moving plane method (Gidas-Ni-Nirenberg 1979), $u_\varepsilon(x) = \psi_\varepsilon(|x|) = \psi_\varepsilon(r)$ is radially symmetric in $B_R(0)$, where ψ_ε uniquely solves

$$\begin{cases} \varepsilon^2 \left(\psi_\varepsilon'' + \frac{N-1}{r}\psi_\varepsilon'\right) = \rho_\varepsilon f(\psi_\varepsilon), \ r \in (0, R), \\ \rho_\varepsilon := \rho_\varepsilon(\psi_\varepsilon) = \frac{\lambda}{N\alpha(N)} \left(\int_0^R e^{\psi_\varepsilon(s)} s^{N-1} ds\right)^{-1}, \\ \psi_\varepsilon'(0) = 0, \ \psi_\varepsilon(R) = w_b. \end{cases} \tag{24}$$

Key: Equation (24) is equivalent to an integro-ODEs

$$\frac{\varepsilon^2}{2}\psi_\varepsilon'^2(r) + \varepsilon^2 \int_{\frac{R}{2}}^r \frac{N-1}{s}\psi_\varepsilon'^2(s)\, ds = \rho_\varepsilon F(\psi_\varepsilon(r)) + \mathrm{K}_\varepsilon, \ r \in [0, R],$$

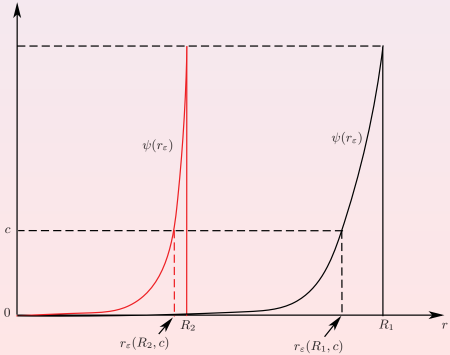
Effect of boundary curvature

For $R > 0$ and $c \in (0, w_b)$, we define

$$r_\varepsilon(R, c) := \psi_\varepsilon^{-1}(c) \text{ and } \Gamma_\varepsilon(R, c) := \{r \in [0, R] : \psi_\varepsilon(r) \in [c, w_b]\} = [r_\varepsilon(R, c), R]$$

as functions of R and c, where $\Gamma_\varepsilon(R, c)$ is a closed interval with width (boundary-layer thickness)

$$R - r_\varepsilon(R, c) = O(\varepsilon)$$



Theorem 2 (Lee-W.-Yang, Nonlinearity 2020):

(i) The slope of the boundary layer profile at the boundary has the asymptotic expansion as

$$\psi'_\varepsilon(R) = \frac{1}{\varepsilon R^{N/2}} \sqrt{\frac{2\lambda F(w_b)}{\alpha(N)}} + \frac{1}{R}\left(N \sqrt{\frac{F(w_b)}{2}} J_0 - (N-1) \int_0^{w_b} \sqrt{\frac{F(t)}{F(w_b)}}\, \mathrm{d}t \right) + o_\varepsilon(1)$$

where

$$J_0 = -\sqrt{2F(w_b)} + \int_0^{w_b} \sqrt{\frac{F(t)}{2}}\, dt.$$

Simply speaking

$$\psi'_\varepsilon(R) \sim \frac{1}{\varepsilon R^{N/2}} + \frac{1}{R} + o_\varepsilon(1)$$

Main results

Theorem 2 (Lee-W.-Yang Nonlinearity 2020):

(ii) For each $c \in (0, w_b)$, we have

$$R - r_\varepsilon(R,c) = \sqrt{\alpha(N)}R^{N/2}\Psi^{-1}(c)\varepsilon + \frac{\varepsilon^2}{2}\alpha(N)R^{N-1}\Big[-\frac{N}{\sqrt{\lambda}}\Psi^{-1}(c)J_0 + \frac{N-1}{\lambda}\int_c^{w_b}\left(\frac{1}{F(s)}\int_0^s\sqrt{\frac{F(t)}{F(s)}}dt\right)ds + o_\varepsilon(1)\Big].$$

where Ψ denotes the unique positive solution of

$$\begin{cases} -\Psi'(\xi) = \sqrt{2\lambda F(\Psi(\xi))}, & \xi > 0, \\ \Psi(0) = w_b > 0, \ \Psi(\infty) = 0. \end{cases}$$

Simply speaking $R - r_\varepsilon(R,c)$ (BL-thickness) is strictly increasing with respect to R.

Effect of curvature on boundary-layer profile

Remark. The result of Theorem 2-(i) implies that the slope of boundary layer profile near the boundary increases with respect to the boundary curvature (i.e. decrease with respect to R). The result of Theorem 2-(ii) implies that the boundary-layer thickness decreases with respect to the boundary curvature (i.e. increases with respect to R).

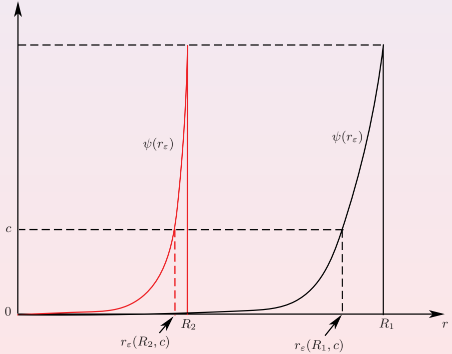


Figure: Schematic of the curvature effect on boundary-layer steepness and thickness.

Stability of boundary layer solutions

We consider the problem in a bounded interval $\mathcal{I} := (0, 1)$

$$\begin{cases} u_t = u_{xx} - (uv_x)_x & \text{in } \mathcal{I}, \\ v_t = \varepsilon v_{xx} - uv & \text{in } \mathcal{I}, \\ (u, v)(x, 0) = (u_0, v_0)(x) & \text{in } \mathcal{I} \end{cases} \tag{25}$$

subject to the following boundary conditions:

$$\begin{cases} (u_x - uv_x)|_{x=0,1} = 0, \quad v(0,t) = v(1,t) = v_* & \text{if } \varepsilon > 0 \\ (u_x - uv_x)|_{x=0,1} = 0 & \text{if } \varepsilon = 0. \end{cases} \tag{26}$$

By integrating the first equation along with the boundary condition, one immediately finds that cell mass is conserved:

$$\int_{\mathcal{I}} u(x,t)\mathrm{d}x = \int_{\mathcal{I}} u_0(x)\mathrm{d}x := M.$$

Lemma (Lee-W.-Yang, Nonlinearity 2020). For any $M \in (0, \infty)$, the problem (25)-(26) with $\varepsilon > 0$ admits a unique classical non-constant solution $(\bar{u}, \bar{v}) \in C^1(\overline{\mathcal{I}}) \cap C^\infty(\mathcal{I})$ such that

$$\bar{u} = \frac{M}{\int_{\mathcal{I}} \mathrm{e}^{\bar{v}}\, \mathrm{d}x} \mathrm{e}^{\bar{v}}, \quad \bar{u} > 0, \quad 0 < \bar{v} \leq v_* \quad \text{for any } x \in \overline{\mathcal{I}}.$$

Stability of boundary layer solutions

Theorem (Hong-W. QAM 2021). Suppose that $u_0 \in H^1$ and $v_0 \in H^2$ with $u_0 \geq 0, v_0 \geq 0$ such that $\int_{\mathcal{I}} u_0 \,\mathrm{d}x = M$. Let $(\bar{u}, \bar{v})$ be the steady state solution of (25)-(26) with $\int_{\mathcal{I}} \bar{u} \,\mathrm{d}x = M$ and define

$$\varphi_0(x) = \int_0^x (u_0(y) - \bar{u}(y))\,\mathrm{d}y.$$

Then there exists a constant $\delta_0 > 0$ such that if

$$\|\varphi_0\|_{H^1}^2 + \|v_0 - \bar{v}\|_{L^2}^2 \leq \delta_0,$$

then the initial-boundary value problem (25)-(26) admits a unique global solution (u, v) satisfying

$$u \in C\left([0,\infty); H^1\right) \cap L^2\left(0,\infty; H^2\right), \quad v \in C\left([0,\infty); H^2\right) \cap L^2\left(0,\infty; H^3\right),$$

and the following asymptotic decay:

$$\|(u - \bar{u}, v - \bar{v})(\cdot, t)\|_{L^\infty} \leq C\mathrm{e}^{-\alpha t} \text{ for any } t \geq 0,$$

where α and C are positive constants independent of t.

Stability

When $\varepsilon = 0$, (25) becomes a PDE-ODE system which has a unique constant steady state $(M, 0)$ with $M = \int_{\mathcal{I}} u_0 \, dx$ satisfying the boundary condition (26).

Theorem (Hong-W. QAM 2021). Let $(u_0, v_0) \in H^1 \times H^2$ with $u_0 \geq 0, v_0 \geq 0$ such that $\int_{\mathcal{I}} u_0 \, dx = M$ and define

$$w_0(x) = \int_0^x (u_0(y) - M) \, dy.$$

Then there exists a constant $\delta_1 > 0$ such that if $\|w_0\|_{H^1}^2 + \|v_0\|_{H^1}^2 \leq \delta_1$, then the initial-boundary value problem (25)-(26) admits a unique solution (u, v) in $\mathcal{I} \times (0, \infty)$ satisfying

$$u \in C\left([0, \infty); H^1\right) \cap L^2\left(0, \infty; H^2\right), \quad v \in C\left([0, \infty); H^2\right).$$

Furthermore, we have the following decay estimates:

$$\|(u - M, v)(\cdot, t)\|_{L^\infty} \leq C e^{-\alpha_0 t} \text{ for any } t > 0,$$

where α_0 and $C > 0$ are positive constants independent of t.

Ongoing works

Convergence of BL-solutions (Carrillo-Hong-W. 2023)

Theorem. Assume that $(u_0, v_0) \in H^6 \times H^7$ with $u_0 \geq, \not\equiv 0, v_0 \geq 0$ and $(\sqrt{v_0})_x \in L^2$ satisfying some **compatibility conditions**. Then for any $v_* > 0$, there exists constants $T_0(v_*) > 0$ and $\varepsilon_0 > 0$, where $T_0(v_*) \to \infty$ as $v_* \to 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, the problem (25)-(26) admits a unique solution $(u^\varepsilon, v^\varepsilon) \in L^\infty(0, T_0; H^1 \times H^2)$ which satisfies for any $x \in [0, 1]$

$$\begin{aligned} u^\varepsilon(x,t) &= u^{I,0}(x,t) + u^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) + u^{b,0}\Big(\frac{1-x}{\sqrt{\varepsilon}}, t\Big) + O(\varepsilon^{1/4}), \\ v^\varepsilon(x,t) &= v^{I,0}(x,t) + v^{B,0}\Big(\frac{x}{\sqrt{\varepsilon}}, t\Big) + v^{b,0}\Big(\frac{1-x}{\sqrt{\varepsilon}}, t) + O(\varepsilon^{1/2}), \end{aligned} \tag{27}$$

where $(u^{I,0}, v^{I,0})(x,t)$ denotes the unique solution of (25)-(26) with $\varepsilon = 0$ (outer-layer solution), $(u^{B,0}, v^{B,0}(z,t)$ and $(^{b,0}, v^{b,0})(\xi, t)$ with $z := \frac{x}{\sqrt{\varepsilon}}$ and $\xi := \frac{x-1}{\sqrt{\varepsilon}}$ are called the left and right leading-order boundary layer (inner layer) solutions of problems (25)-(26) with $\varepsilon > 0$.

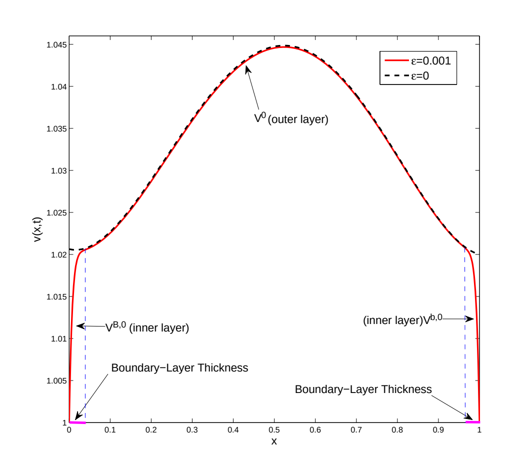
Several remarks

- The equations for outer-layer and boundary-layer profiles can be explicitly derived based on the (formal) asymptotic analysis and their existence and uniqueness can be proved (non-trivial).
- From the refined solution structure given in (27), without difficulty we can show for any $\delta = O(\varepsilon^{\alpha}) > 0$ $(0 < \alpha < 1/2)$, it holds that

$$\lim_{\varepsilon\to 0} \left\| u^{\varepsilon} - u^{I,0} \right\|_{L^{\infty}([\delta,1-\delta]\times[0,T_0])} = 0, \ \liminf_{\varepsilon\to 0} \left\| u^{\varepsilon} - u^{I,0} \right\|_{L^{\infty}([0,1]\times[0,T_0])} > 0,$$
$$\lim_{\varepsilon\to 0} \left\| v^{\varepsilon} - v^{I,0} \right\|_{L^{\infty}([\delta,1-\delta]\times[0,T_0])} = 0, \ \liminf_{\varepsilon\to 0} \left\| v^{\varepsilon} - v^{I,0} \right\|_{L^{\infty}([0,1]\times[0,T_0])} > 0,$$

  which indicates that the solution $(u^{\varepsilon}, v^{\varepsilon})$ of (25)-(26) will develop a boundary layer profile with thickness of order $\varepsilon^{1/2}$ as $\varepsilon \to 0$, which consists of out-layer profile $(u^{I,0}, v^{I,0})$ and boundary (inner) layer profiles $(u^{B,0}, v^{B,0})$ at the left boundary $x = 0$ and $(u^{b,0}, v^{b,0})$ at the right boundary $x = 1$, with an error at the order of $\varepsilon^{1/4}$ for u^{ε} and of $\varepsilon^{1/2}$ for v^{ε} as $\varepsilon \to 0$.

Illustration of boundary layer profile



Several remarks

- Though the boundary values of u^ε are elusive in the zero-flux boundary condition of u prescribed for u in (26), the expansion (27) not only indicates that $u^\varepsilon(x,t)$ has boundary layer profiles $u^{B,0}(z,t)$ near $x=0$ and $u^{b,0}(z,t)$ near $x=1$, but also gives the approximate boundary value of u for $0<\varepsilon\ll 1$

$$u^\varepsilon(0,t) = u^{I,0}(0,t)\exp\left(v_* - v^{I,0}(0,t)\right) + O(\varepsilon^{1/4}),$$
$$u^\varepsilon(1,t) = u^{I,0}(1,t)\exp\left(v_* - v^{I,0}(1,t)\right) + O(\varepsilon^{1/2}).$$

- When $v_*=0$, according to our analysis, the boundary layer profiles in (27) will vanish, which leads to $(u^\varepsilon, v^\varepsilon)\to(u^{I,0}, v^{I,0})$ in L^∞ as $\varepsilon\to 0$.
- The compatibility conditions require that $\min\limits_{x\in\bar{I}} u_0 = 0$. If $\min\limits_{x\in\bar{I}} u_0 > 0$, then initial layers will be present. We shall investigate this case in a separate work using different approaches.

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