Regularity structure of conservative solutions to the Hunter-Saxton equation

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Joint work with Hao Liu and Tak Kwong Wong

Outline

Background

The Hunter-Saxton equation for $(x,t) \in \mathbb{R} \times \mathbb{R}$ (J. K. Hunter and R. Saxton, SIAM J. Appl. Math., 51(6), 1991):

$$u_t + uu_x = \frac{1}{2} \int_{-\infty}^{x} u_y^2(y,t)\,\mathrm{d}y,$$

or

$$u_t + uu_x = \frac{1}{4} \left(\int_{-\infty}^{x} u_y^2(y,t)\,\mathrm{d}y - \int_{x}^{\infty} u_y^2(y,t)\,\mathrm{d}y \right).$$

The HS equation describes the propagation of waves in a massive director field of a nematic liquid crystal.

Formal calculations

Take derivative of x:

$$u_{xt} + uu_{xx} = -\frac{1}{2}u_x^2.$$

(Flow map, Riccati type equation, finite time blow up)

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Energy conservation:

$$(u_x^2)_t + (uu_x^2)_x = 0.$$

($\|u_x(\cdot,t)\|_{L^2}$ is conserved?)

Formal calculations

Take second order derivative of x:

$$u_{xxt} + 2u_x u_{xx} + u u_{xxx} = 0.$$

Set $m := u_{xx}$. Then

$$m_t + 2m u_x + m_x u = 0, \quad m = u_{xx}.$$

It resembles the Camassa-Holm equation:

$$m_t + 2m u_x + m_x u = 0, \quad m = u - u_{xx}.$$

(Integrable systems)

Outline

Flow map

Let initial datum $\bar{u}$ be smooth and $\bar{u}_x \in L^2(\mathbb{R})$. Consider flow map:

$$\frac{\partial}{\partial t} X(\xi, t) = u(X(\xi, t), t), \quad X(\xi, 0) = \xi.$$

Then

$$\frac{\partial^2}{\partial t^2} X(\xi, t) = \frac{1}{2} \int_{-\infty}^{X(\xi,t)} u_x^2(y, t)\, \mathrm{d}y = \frac{1}{2} \int_{-\infty}^{\xi} \bar{u}_x^2(y)\, \mathrm{d}y.$$

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$\Rightarrow$ Global characteristics:

$$X(\xi, t) = \xi + \bar{u}(\xi) t + \frac{t^2}{4} \int_{-\infty}^{\xi} \bar{u}_x^2(y)\,\mathrm{d}y.$$

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$\Rightarrow$

$$X_\xi(\xi,t) = \left[1 + \frac{t}{2}\bar{u}_x(\xi)\right]^2 \geq 0.$$

Finite time blow up

Recall: $u_{xt} + uu_{xx} = -\frac{1}{2}u_x^2$, which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} u_x(X(\xi,t),t) = -\frac{1}{2}u_x^2(X(\xi,t),t).$$

$\Rightarrow$

$$u_x(X(\xi,t),t) = \frac{2\bar{u}_x(\xi)}{2 + t\bar{u}_x(\xi)}.$$

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Hence, when $\inf_{x\in\mathbb{R}} \bar{u}_x(x) < 0$,

$$\inf_x u_x(x,t) \to -\infty \quad \text{as} \quad t \to T_0 := -2/\inf_{x\in\mathbb{R}} \bar{u}_x(x)$$

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Do we have conservation of $\|u_x(\cdot,t)\|_{L^2}$?

An example

Consider

$$\bar{u}(x) = \begin{cases} 0, & x \le 0, \\ -x, & 0 < x < 1, \\ -1, & x \ge 1, \end{cases} \qquad \bar{u}_x(x) = \begin{cases} 0, & x \le 0, \\ -1, & 0 < x < 1, \\ 0, & x \ge 1. \end{cases}$$

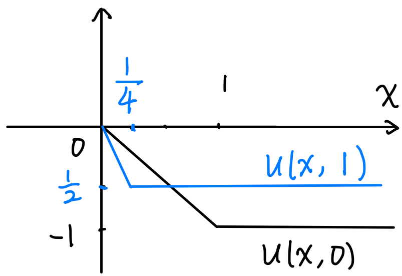
For $t \in [0, 2)$, we have

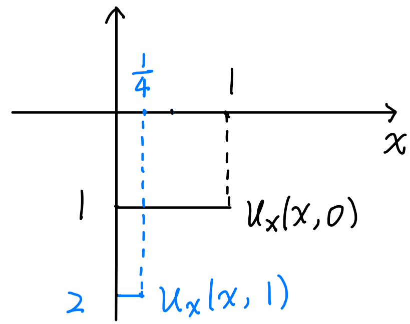
$$u(x,t) = \begin{cases} 0, & x \le 0, \\ -\dfrac{x}{1 - t/2}, & 0 < x < (1 - t/2)^2, \\ -(1 - t/2), & x > (1 - t/2)^2, \end{cases}$$

$$u_x(x,t) = \begin{cases} -\dfrac{1}{1 - t/2}, & 0 < x < (1 - t/2)^2, \\ 0, & \text{other } x, \end{cases} \qquad \|u_x(\cdot, t)\|_{L^2} = 1.$$

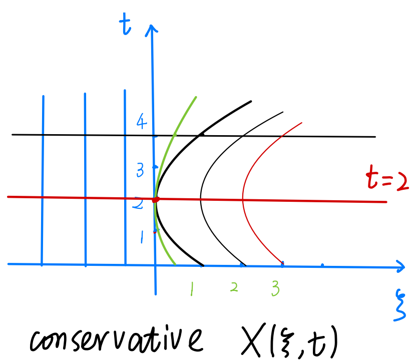
Conservative or dissipative after $t = 2$?

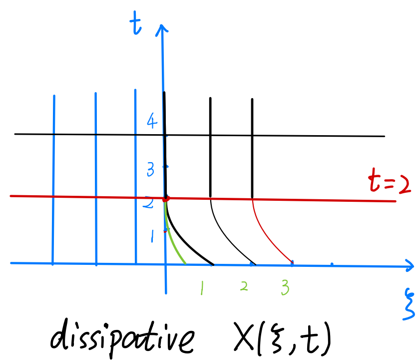
Energy “disappears”





$$u(x,t) \xrightarrow{C_b(\mathbb{R})} 0, \quad u_x^2(\cdot,t)\,\mathrm{d}x \overset{*}{\rightharpoonup} \delta \quad \text{as} \quad t \to 2.$$





Conservative and dissipative solutions

Dissipative and conservative:
Hunter-Zheng (1995, $\bar{u}_x \in BV(\mathbb{R}^+)$, regularized method of characteristics; 1995, Zero-viscosity, dispersion limit (numerical), spatial initial datum as in the above example)
Zhang-Zheng (1998, $0 \le \bar{u}_x \in L^p(\mathbb{R}^+)$, $p > 2$; 1999, $0 \le \bar{u}_x \in L^2(\mathbb{R}^+)$; 2000, $\bar{u}_x \in L^2(\mathbb{R}^+)$, methods of Young measures)

Dissipative:
Bressan-Constantin (2005, existence, uniqueness)
Dafermos (2011, 2012, uniqueness), Cieślak-Jamróz (2016, uniqueness)

Conservative:
Bressan-Zhang-Zheng (2007, existence and uniqueness, general models)
Bressan-Holden-Raynaud (2010, Lipschitz stability)
Carrillo-Grunert-Holden (2019, Lipschitz stability)
Grunert-Holden (2021, ArXiv, uniqueness)

λ-dissipative ($\lambda \in [0, 1]$):
Grunert-Tandy (2021, ArXiv, Lipschitz stability)

Outline

Energy measure and generalized framework

Space for conservative solutions:

Definition

Let $\mathcal{D}$ be the set of pairs (u, μ) satisfying

(i) $u \in C_b(\mathbb{R})$, $u_x \in L^2(\mathbb{R})$;

(ii) $\mu \in \mathcal{M}_+(\mathbb{R})$;

(iii) $\mathrm{d}\mu_{ac} = u_x^2 \,\mathrm{d}x$, where μ_{ac} is the absolutely continuous part of measure μ with respect to the Lebesgue measure $\mathcal{L}$.

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Generalized Framework:

$$\begin{cases} u_t + uu_x = \dfrac{1}{2} \displaystyle\int_{-\infty}^{x} \mathrm{d}\mu(t), \\ \mu_t + (u\mu)_x = 0, \\ \mathrm{d}\mu_{ac}(t) = u_x^2(\cdot, t) \, \mathrm{d}x. \end{cases}$$

Outline

flow map $X(\xi, t)$

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ be an initial datum. When $\bar{u}_x^2 \,\mathrm{d}x = \mathrm{d}\bar{\mu}$, the global flow map is

$$X(\xi, t) = \xi + \bar{u}(\xi)t + \frac{t^2}{4} \int_{-\infty}^{\xi} \bar{u}_x^2(y) \,\mathrm{d}y. \tag{1}$$

Consider an initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. The flow map $X(\xi, t)$ is no longer suitable when $\mathrm{d}\bar{\mu} \neq \mathrm{d}\bar{\mu}_{ac}$.

Problem: how to understand the cumulative energy distribution:

$$\int_{-\infty}^{x} \mathrm{d}\bar{\mu}?$$

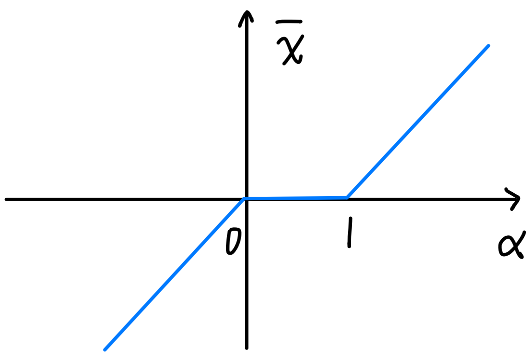
An example

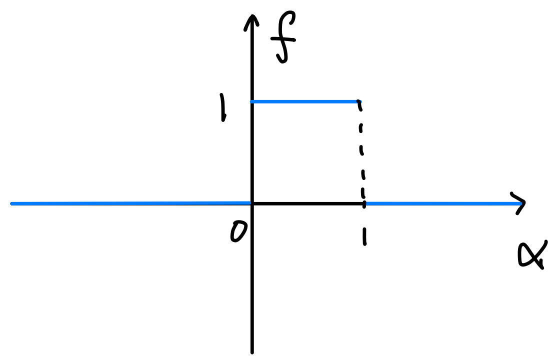
Consider $\bar{\mu} = \delta$, and define

$$\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \le \alpha \le \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

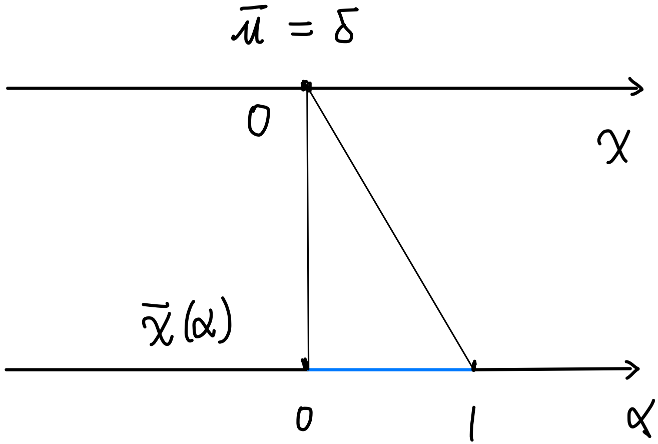
Then, we have $\bar{\mu} = \delta = \bar{x}\#f$ and

$$\bar{x}(\alpha) = \begin{cases} \alpha, & \alpha < 0, \\ 0, & 0 \le \alpha \le 1, \\ \alpha - 1, & \alpha > 1, \end{cases} \qquad f(\alpha) = 1 - \bar{x}'(\alpha) = \begin{cases} 0, & \alpha < 0, \\ 1, & 0 \le \alpha \le 1, \\ 0, & \alpha > 1. \end{cases}$$



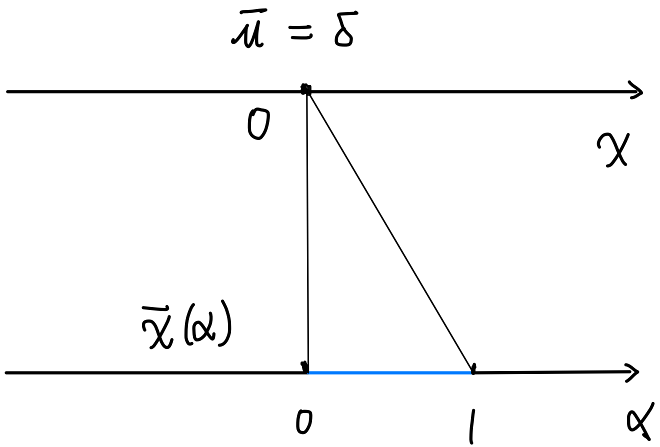


Energy density



$$\begin{cases} f(\alpha) = 1 - \bar{x}'(\alpha) \\ \bar{u} = \bar{x} \# f \end{cases}$$

Energy density



$$\begin{cases} f(\alpha) = 1 - \bar{x}'(\alpha) \\ \bar{\mu} = \bar{x} \# f \end{cases}$$

Function f: the energy density in the α-variable

$$\bar{\mu}((-\infty, \bar{x}(\alpha))) \leq \int_{(-\infty,\alpha)} f(\theta)\,\mathrm{d}\theta = \alpha - \bar{x}(\alpha) \leq \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

The energy changes continuously from 0 to 1, which helps to understand $\int_{-\infty}^{x} \mathrm{d}\bar{\mu}$ in the α-variable!

α-variable system

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. "Flatten" the singular part of $\bar{\mu}$ by

$$\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \leq \alpha \leq \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

When $\bar{\mu}$ is absolutely continuous:

$$\bar{x}(\alpha) + \int_{-\infty}^{\bar{x}(\alpha)} \mathrm{d}\bar{\mu} = \bar{x}(\alpha) + \int_{-\infty}^{\bar{x}(\alpha)} \bar{u}_x^2(y)\,\mathrm{d}y = \alpha.$$

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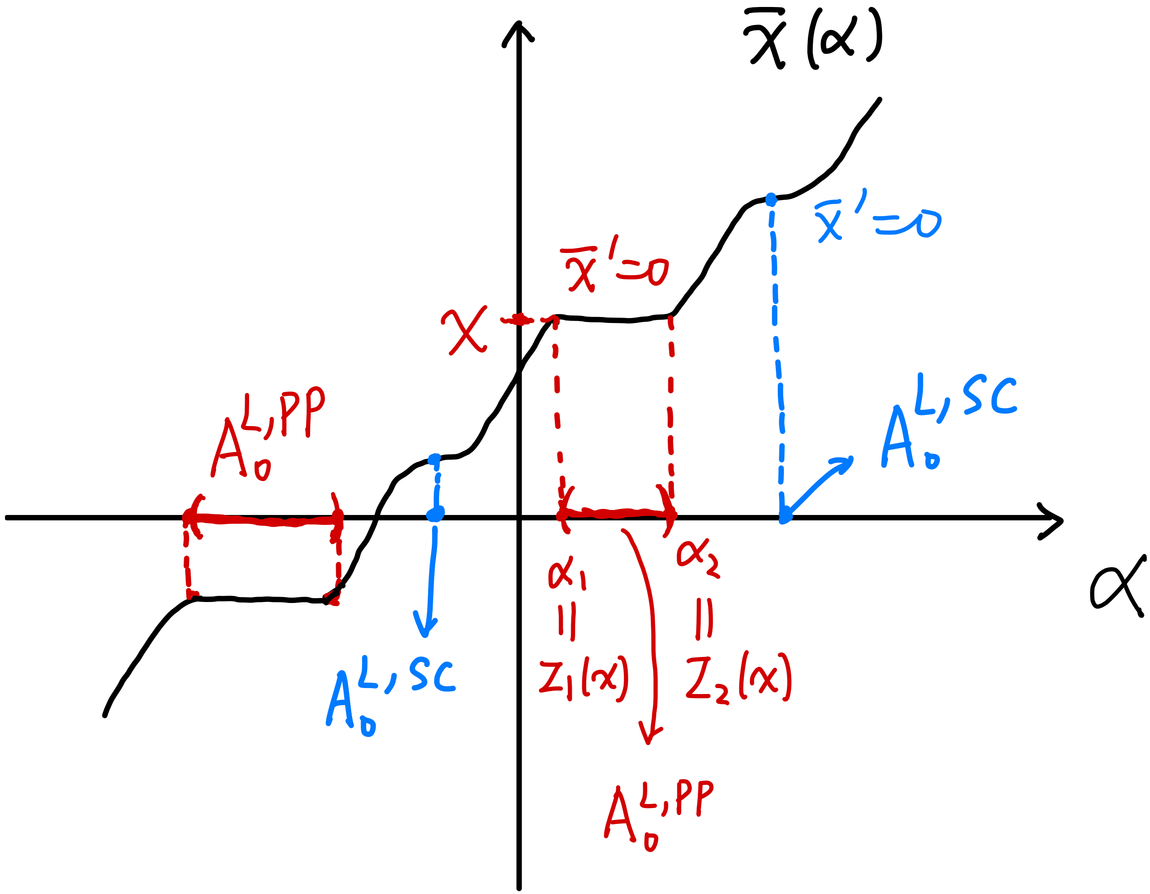
Two pseudo-inverses:

$$z_1(x) = \inf\{\alpha : \ \bar{x}(\alpha) = x\}, \ z_2(x) = \sup\{\alpha : \ \bar{x}(\alpha) = x\}.$$

Three sets:
$B_0^L = \{\alpha : \ \ \bar{x}'(\alpha) > 0\}, \ A_0^{L,pp} = \{\alpha : \ \bar{x}'(\alpha) = 0, \ z_1(\bar{x}(\alpha)) < z_2(\bar{x}(\alpha))\},$

$$A_0^{L,sc} = \{\alpha : \ \bar{x}'(\alpha) = 0, \ z_1(\bar{x}(\alpha)) = z_2(\bar{x}(\alpha))\}.$$

Definitions related to $\bar{x}$



Decomposition of $\bar{\mu}$ by $\bar{x}$:

Assume $\bar{\mu} = \bar{\mu}_{ac} + \bar{\mu}_{pp} + \bar{\mu}_{sc}$. Here, $\bar{\mu}_{ac}$: absolutely continuous part, $\bar{\mu}_{pp}$: pure point part, $\bar{\mu}_{sc}$: singular continuous part.

Proposition

(i) Lipschitz continuity: $|\bar{x}(\alpha) - \bar{x}(\beta)| \leq |\alpha - \beta|, \forall \alpha, \beta \in \mathbb{R}$.
(ii) Define $f(\alpha) := 1 - \bar{x}'(\alpha)$. Then

$$\bar{x}\#(f\,\mathrm{d}\alpha) = \bar{\mu}, \quad \|f\|_{L^1} = \bar{\mu}(\mathbb{R}).$$

(iii) Decomposition of $\bar{\mu}$:

$$\bar{\mu}_{pp} = \bar{x}\#(f|_{A_0^{L,pp}}\,\mathrm{d}\alpha),\ \bar{\mu}_{sc} = \bar{x}\#(f|_{A_0^{L,sc}}\,\mathrm{d}\alpha),\ \bar{\mu}_{ac} = \bar{x}\#(f|_{B_0^L}\,\mathrm{d}\alpha).$$

$$\bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha) = f(\alpha), \quad \alpha \in B_0^L.$$

A useful lemma for push forward measures

Lemma (Structure for push forward measures)

$$\left.\begin{array}{l} X:\mathbb{R}\to\mathbb{R} \;\; \textit{continuous, increasing, surjective} \\ 0\le g\in L^1(\mathbb{R}) \end{array}\right\} \Rightarrow \mu := X\#(g\,\mathrm{d}\xi)$$

Two pseudo-inverses:

$$Z_1(x)=\inf\{\xi:\ X(\xi)=x\},\ Z_2(x)=\sup\{\xi:\ X(\xi)=x\}.$$

Three sets:

$$\begin{aligned} A^{pp} &= \{\xi:\ X_\xi(\xi)=0,\ Z_1(X(\xi))<Z_2(X(\xi))\},\\ A^{sc} &= \{\xi:\ X_\xi(\xi)=0,\ Z_1(X(\xi))=Z_2(X(\xi))\},\\ B &= \{\xi:\ X_\xi(\xi)>0\}. \end{aligned}$$

and $g_1=g\cdot 1_B$, $g_2=g\cdot 1_{A^{pp}}$, $g_3=g\cdot 1_{A^{sc}}$. *Then:*

(i) $\mathcal{L}(X(A^{pp}\cup A^{sc}))=0$, $X(A^{pp})$ *is a countable set;*

(ii) $\mathrm{d}\mu_{ac}=X\#(g_1\,\mathrm{d}\xi)$, $\mathrm{d}\mu_{pp}=X\#(g_2\,\mathrm{d}\xi)$, $\mathrm{d}\mu_{sc}=X\#(g_3\,\mathrm{d}\xi)$.

Ideas for global characteristics in α-variable

Formal calculations for a smooth solution u:

For any $\beta \in \mathbb{R}$, we define $x(\beta, t)$ by

$$x(\beta, t) + \int_{(-\infty, x(\beta,t))} u_x^2(y, t)\, \mathrm{d}y = \beta, \quad t \in \mathbb{R}.$$

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$\Rightarrow$

$$\partial_t x(\beta,t) = \frac{(uu_x^2)(x(\beta,t),t)}{1 + u_x^2(x(\beta,t),t)}, \qquad \partial_\beta x(\beta,t) = \frac{1}{1 + u_x^2(x(\beta,t),t)},$$

Define functions:

$$\begin{cases} U(\beta,t) = u(x(\beta,t),t) \\ H(\beta,t) = \beta - x(\beta,t). \end{cases}$$

Ideas for global characteristics in α-variable

We formally obtain the system (Grunert-Holden, 2021, ArXiv, uniqueness):

$$\begin{cases} x_t(\beta,t) + Ux_\beta(\beta,t) = U(\beta,t), \\ H_t(\beta,t) + UH_\beta(\beta,t) = 0, \\ U_t(\beta,t) + UU_\beta(\beta,t) = \dfrac{1}{2}H(\beta,t). \end{cases}$$

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Drawbacks:

no explicit formula, non-uniqueness (unique in equivalent class sense)

$x_t(\beta,t) \neq u(x(\beta,t),t) \quad \Rightarrow \quad \int_{(-\infty,x(\beta,t))} u_x^2(y,t)\,\mathrm{d}y$ not conserved.

Global characteristics in α-variable

Introduce a reformulation function $\beta(t)$ with $\beta(0) = \alpha$ such that

$$x(\beta(t), t) + \int_{-\infty}^{x(\beta(t),t)} u_x^2(y,t)\,\mathrm{d}y = \beta(t), \quad t \in \mathbb{R},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} x(\beta(t), t) = u(x(\beta(t), t), t).$$

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and

$$\frac{\mathrm{d}}{\mathrm{d}t} x(\beta(t), t) = u(x(\beta(t), t), t).$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\beta(t) - x(\beta(t), t)] = 0,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\beta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(x(\beta(t), t), t) = \int_{-\infty}^{x(\beta(t),t)} u_x^2(y, t)\,\mathrm{d}y = \beta(t) - x(\beta(t), t).$$

Global characteristics in α-variable

Hence

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}x(\beta(t),t) = \frac{\mathrm{d}^3}{\mathrm{d}t^3}\beta(t) = 0$$

with initial data:

$$x(\beta(0),0) = \bar{x}(\alpha), \quad \frac{\mathrm{d}}{\mathrm{d}t}x(\beta(t),t)\Big|_{t=0} = \bar{u}(\bar{x}(\alpha)),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(\beta(t),t)\Big|_{t=0} = \int_{-\infty}^{\bar{x}(\alpha)} \bar{u}_x^2(y)\,\mathrm{d}y = \alpha - \bar{x}(\alpha).$$

$\Rightarrow$ Global characteristics:

$$y(\alpha,t) := x(\beta(t),t) = \bar{x}(\alpha) + \bar{u}(\bar{x}(\alpha))t + \frac{t^2}{4}(\alpha - \bar{x}(\alpha)). \tag{2}$$

Advantages:

1. only need information of initial datum;
2. can be generalized to any $(\bar{u},\bar{\mu}) \in \mathcal{D}$.

Recovery of solution $(u(t), \mu(t))$

From global $y(\alpha, t)$ to global $(u(t), \mu(t))$:

$$\begin{cases} u(x,t) = \dfrac{\partial}{\partial t} y(\alpha,t) = \bar{u}(\bar{x}(\alpha)) + \dfrac{t}{2}(\alpha - \bar{x}(\alpha)) \ \text{ for } \ x = y(\alpha,t), \\ \mu(t) = y(\cdot,t)\#(f\,\mathrm{d}\alpha), \quad f(\alpha) = 1 - \bar{x}'(\alpha). \end{cases}$$

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Two pseudo-inverses:

$$z_1(x,t) = \inf\{\alpha : \ y(\alpha,t) = x\}, \ z_2(x,t) = \sup\{\alpha : \ y(\alpha,t) = x\}.$$

Three sets:

$$\begin{aligned} A_t^{L,pp} &= \{\alpha : \ y_\alpha(\alpha,t) = 0, \ z_1(y(\alpha,t),t) < z_2(y(\alpha,t),t)\}, \\ A_t^{L,sc} &= \{\alpha : \ y_\alpha(\alpha,t) = 0, \ z_1(y(\alpha,t),t) = z_2(y(\alpha,t),t)\}, \\ B_t^L &= \{\alpha : \ y_\alpha(\alpha,t) > 0\}. \end{aligned}$$

Outline

Properties of $\mu(t)$

Theorem

(i) Energy conservation: $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$ *and* $\mu(t)(\mathbb{R}) = \bar{\mu}(\mathbb{R})$, $t \in \mathbb{R}$.
(ii) Decomposition: $\mu_{pp}(t) = y(\cdot, t)\#(f|_{A_t^{L,pp}}\, \mathrm{d}\alpha)$, $\mu_{sc}(t) = y(\cdot, t)\#(f|_{A_t^{L,sc}}\, \mathrm{d}\alpha)$,

$$\mu_{ac}(t) = y(\cdot, t)\#(f|_{B_t^L}\, \mathrm{d}\alpha).$$

Coordinates cause singular sets:

$$A_t^{L,pp} \cup A_t^{L,sc} = \left\{ \alpha \in B_0^L : \quad \bar{u}_x(\bar{x}(\alpha)) = -\frac{2}{t} \right\} := A_t^L, \quad t \in \mathbb{R}.$$

Support of singular parts:

$$\mathrm{supp}\Big(\mu_{pp}(t) + \mu_{sc}(t)\Big) \subset \left\{ x + \bar{u}(x)t + \frac{t^2}{4}\bar{\mu}((-\infty, x)) : \; x \in A_t^E = \bar{x}(A_t^L) \right\}.$$

(iii) Countably many time $t \in \mathbb{R}$ *for singular measures:* $T_p := \{t : \; \mu_{pp}(t) \neq 0\}$ *and* $T_s := \{t : \; \mu_{sc}(t) \neq 0\}$ *are countable.*

Key ingredients for proof

1. Energy density in the α-variable:

$$f(\alpha) = 1 - \bar{x}'(\alpha) = \begin{cases} \bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha), & \alpha \in B_0^L, \\ 1, \quad \alpha \in A_0^L = A_0^{L,pp} \cup A_0^{L,sc}, \end{cases}$$

2. Derivative of $y(\alpha, t)$:

$$y_\alpha(\alpha, t) = \begin{cases} \bar{x}'(\alpha)\left[1 + \frac{t}{2}\bar{u}_x(\bar{x}(\alpha))\right]^2, & \alpha \in B_0^L, \\ \frac{t^2}{4}, \quad \alpha \in A_0^L. \end{cases}$$

3. The real line $\mathbb{R}$ cannot be written as the union of uncountably many disjoint subsets with positive measures.

Mass for singular measures

Recall: $\bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha) = f(\alpha)$ for $\alpha \in B_0^L$.

Remark

- Mass of pure point parts: $\bar{x}'(\alpha) = \frac{t^2}{t^2+4}$ and $f(\alpha) = \frac{4}{t^2+4}$ on A_t^L. We consider a point $x_0 \in y(A_t^{L,pp}, t)$ and $\alpha_1 := z_1(x_0,t) < z_2(x_0,t) =: \alpha_2$. Then, $\bar{u}_x(x) = -\frac{2}{t}$ for all $x \in [x_1, x_2] := [\bar{x}(\alpha_1), \bar{x}(\alpha_2)]$. The mass concentrated at x is calculated by

$$\mu(t)(\{x_0\}) = \int_{[\alpha_1,\alpha_2]} f(\alpha)\,\mathrm{d}\alpha = \frac{4}{t^2+4}(\alpha_2 - \alpha_1) = \frac{4}{t^2}(x_2 - x_1).$$

- Mass of singular continuous part: define $A_t^{E,sc} = \bar{x}(A_t^{L,sc})$ and then $\bar{u}_x(x) = -\frac{2}{t}$ for $x \in A_t^{E,sc}$. We have

$$\mu_{sc}(t)(\mathbb{R}) = \frac{4}{t^2}\mathcal{L}(A_t^{E,sc}).$$

Properties of $u(t)$

Theorem

(iv) For all time $t \in \mathbb{R}$, the function $u(\cdot, t)$ is globally absolutely continuous and

$$\mathrm{d}\mu_{ac}(t) = u_x^2(x,t)\,\mathrm{d}x.$$

Moreover,

$$u \in C(\mathbb{R}; C_b(\mathbb{R})) \cap C^{1/2}_{loc}(\mathbb{R}\times\mathbb{R}), \quad u_x \in L^\infty(\mathbb{R}; L^2(\mathbb{R})), \quad u_t \in L^2_{loc}(\mathbb{R}\times\mathbb{R}).$$

(v) If $\bar{u}(-\infty) := \lim_{x\to-\infty} \bar{u}(x)$ exists, then we have

$$\lim_{x\to-\infty} u(x,t) = \bar{u}(-\infty).$$

On the other hand, if $\bar{u}(+\infty) := \lim_{x\to+\infty} \bar{u}(x)$ exists, then we also have

$$\lim_{x\to+\infty} u(x,t) = \bar{u}(+\infty) + \frac{1}{2}\bar{\mu}(\mathbb{R})t.$$

Key ingredients for proof

From the formulas for $u(y(\alpha,t),t)$ and $y(\alpha,t)$, it is easy to show

$$u_x^2(y(\alpha,t),t)y_\alpha(\alpha,t) = f(\alpha), \quad \alpha \in B_t^L.$$

Remark

Usually $u_x \notin C(\mathbb{R}; L^2(\mathbb{R}))$, since

$$\int_{\mathbb{R}} u_x^2(x,t)\,\mathrm{d}x = \mu_{ac}(t)(\mathbb{R}) < \mu(t)(\mathbb{R}) = \bar{\mu}(\mathbb{R}), \quad \text{for all } t \in T_s \cup T_p.$$

Relation between $y(\alpha, t)$ and $X(\xi, t)$

Theorem

(vi) Consider a time $s \in \mathbb{R}$ such that $\mu(s)$ is absolutely continuous with respect to the Lebesgue measure. Let $\tilde{u}(x) = u(x, s)$, and $X(\xi, t)$ be defined by $\tilde{u}$ via

$$X(\xi, t) = \xi + \tilde{u}(\xi)t + \frac{t^2}{4} \int_{-\infty}^{\xi} \tilde{u}_x^2(y)\, \mathrm{d}y.$$

Then, we have

$$\tilde{u} \in C_b(\mathbb{R}), \quad \tilde{u}_x \in L^2(\mathbb{R}), \quad \|\tilde{u}_x^2\|_{L^1} = \bar{\mu}(\mathbb{R}).$$

For any $t \in \mathbb{R}$, we also have

$$y(\cdot, t) = X(\cdot, t - s) \circ y(\cdot, s), \quad \mu(t) = X(\cdot, t - s)\#(\tilde{u}_x^2\, \mathrm{d}x),$$

and

$$u(x, t) = \frac{\partial}{\partial t} X(\xi, t - s) = \tilde{u}(\xi) + \frac{(t - s)}{2} \tilde{F}(\xi), \quad \text{for } x = X(\xi, t - s).$$

Outline

Definition

Definition (Conservative solutions)

For initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, $(u(t), \mu(t))$ is said to be a conservative solution if

(i) $u \in C(\mathbb{R}; C_b(\mathbb{R})) \cap C^{1/2}_{loc}(\mathbb{R} \times \mathbb{R})$, $u_t \in L^2_{loc}(\mathbb{R} \times \mathbb{R})$, $u_x(\cdot, t) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$, and $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$;

(ii) $(u(\cdot, 0), \mu(0)) = (\bar{u}, \bar{\mu})$, and $\mathrm{d}\mu(t) = u_x^2(\cdot, t)\,\mathrm{d}x$ for a.e. $t \in \mathbb{R}$;

(iii) the equations

$$\int_{\mathbb{R}} \int_{\mathbb{R}} u\phi_t - \phi\left(uu_x - \frac{1}{2}F\right)\mathrm{d}x\,\mathrm{d}t = 0, \quad \int_{\mathbb{R}} \int_{\mathbb{R}} (\phi_t + u\phi_x)\,\mathrm{d}\mu(t)\,\mathrm{d}t = 0$$

hold for all $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$ and $F(x, t) := \int_{-\infty}^{x} \mathrm{d}\mu(t)$;

(iv) $\mathrm{d}\mu_{ac}(t) = u_x^2(\cdot, t)\,\mathrm{d}x$ for all $t \in \mathbb{R}$.

Existence

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. Define

$$\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \leq \alpha \leq \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

Then define

$$y(\alpha, t) = \bar{x}(\alpha) + \bar{u}(\bar{x}(\alpha))t + \frac{t^2}{4}(\alpha - \bar{x}(\alpha)),$$

and

$$\begin{cases} u(x,t) = \frac{\partial}{\partial t} y(\alpha, t) = \bar{u}(\bar{x}(\alpha)) + \frac{t}{2}(\alpha - \bar{x}(\alpha)) \ \text{ for } \ x = y(\alpha, t), \\ \mu(t) = y(\cdot, t)\#(f \, \mathrm{d}\alpha), \quad f(\alpha) = 1 - \bar{x}'(\alpha). \end{cases}$$

Theorem (Existence)

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ be an initial datum. Let (u, μ) be defined as above. Then, $(u(t), \mu(t))$ is a global-in-time conservative solution to the generalized framework of HS equation with initial datum $(\bar{u}, \bar{\mu})$.

Outline

Uniqueness via characteristics

Theorem (Uniqueness of characteristics and conservative solutions)

Let (v,ν) be a conservative solution to the HS equation with initial datum $(\bar{u},\bar{\mu})\in\mathcal{D}$. Then, there exists a unique characteristic $y_1(\alpha,t)$ satisfying

$$\frac{\partial}{\partial t}y_1(\alpha,t)=v(y_1(\alpha,t),t),\quad y_1(\alpha,0)=\bar{x}(\alpha),$$

and

$$\nu(t)((-\infty,y_1(\alpha,t)))\le\alpha-\bar{x}(\alpha)\le\nu(t)((-\infty,y_1(\alpha,t)]),$$

for any $\alpha\in\mathbb{R}$ and a.e $t\in\mathbb{R}$. The uniqueness of characteristics and conservative solutions follows, i.e., $(v,\nu)=(u,\mu)$, where (u,μ) is constructed in the existence theorem.

Difficulty: $v\in C_b(\mathbb{R})$, $v_x\in L^2(\mathbb{R})$, function v is not Lipschitz.

(Idea: Bressan-Chen-Zhang, 2014, uniqueness CH eqn.)

Key lemma

Lemma

Let (v, ν) be a conservative solution to the HS equation. Consider the time t and τ such that ν is absolutely continuous. Then, for any fixed $y \in \mathbb{R}$ and $\varepsilon_0 > 0$,

$$\int_{(-\infty, y+a_-(t-\tau))} v_x^2(x,t)\,\mathrm{d}x \le \int_{(-\infty,y)} v_x^2(x,\tau)\,\mathrm{d}x \le \int_{(-\infty, y+a_+(t-\tau))} v_x^2(x,t)\,\mathrm{d}x,$$

provided that $t - \tau > 0$ is small enough (depending on v, y and ε_0), where $a_\pm := v(y,\tau) \pm \varepsilon_0$.

Key lemma

Lemma

Let (v, ν) be a conservative solution to the HS equation. Consider the time t and τ such that ν is absolutely continuous. Then, for any fixed $y \in \mathbb{R}$ and $\varepsilon_0 > 0$,

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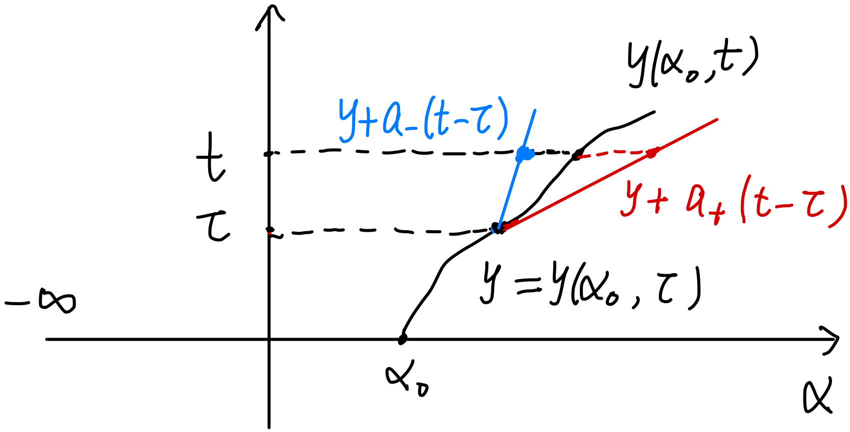
provided that $t - \tau > 0$ is small enough (depending on v, y and ε_0), where $a_\pm := v(y,\tau) \pm \varepsilon_0$. Moreover, for any $T > 0$ and any $-T \le \tau < t \le T$,

$$\int_{(-\infty, y-C_T(t-\tau))} v_x^2(x,t)\,\mathrm{d}x \le \int_{(-\infty,y)} v_x^2(x,\tau)\,\mathrm{d}x \le \int_{(-\infty, y+C_T(t-\tau))} v_x^2(x,t)\,\mathrm{d}x,$$

for all C_T satisfying $\|v\|_{C_b(\mathbb{R}\times[-T,T])} \le C_T$.

Ideas for the proof of the lemma:



$$a_\pm = v(y, \tau) \pm \varepsilon_0$$

Thank you!