Mathematical Properties of Kinetic Equations with Radiative Transfer

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Part I. Kinetic Equations with Radiative Transfer

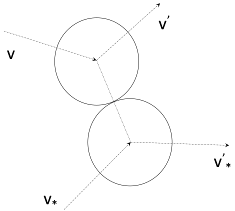
The (elastic, single-species) Boltzmann equation

- The Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \;\; x \in \Omega \subset \mathbb{R}^3, \;\; v \in \mathbb{R}^3, \;\; t \geq 0$$

- $F = F(t, x, v)$: velocity distribution function

$$Q(F, G)(t, x, v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\omega \; B(v - v_*, \theta) \times [F(t, x, v'_*) G(t, x, v') - F(t, x, v_*) G(t, x, v)].$$



- Energy-momentum conservation laws for a binary collision:

$$v + v_* = v' + v'_*, \;\; |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

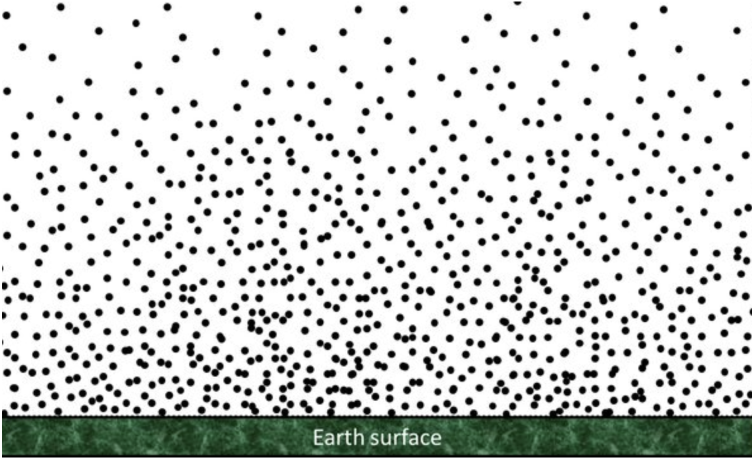
Connections to physical quantities

- mass/charge density: $\rho(t,x) = \int_{\mathbb{R}^3} F(t,x,v)dv$
- macroscopic velocity: $u(t,x) = \frac{1}{\rho}\int_{\mathbb{R}^3} vF(t,x,v)dv$
- temperature: $T(t,x) = \frac{1}{3\rho}\int_{\mathbb{R}^3} |u-v|^2 F(t,x,v)dv$
- pressure: $p(t,x) = \rho T$
- The entropy functional is defined as

$$S(t) = -H(t) \overset{\text{def}}{=} -\int_{\Omega\times\mathbb{R}^3} F(t,x,v)\log F(t,x,v)dvdx.$$

- Boltzmann H-theorem: $\frac{dS}{dt} \geq 0$.
- Maxwellian equilibrium: $M(x,v;\rho,u,T) = \rho\frac{e^{-\frac{|v-u|^2}{2k_B T}}}{(2\pi k_B T)^{3/2}}$.

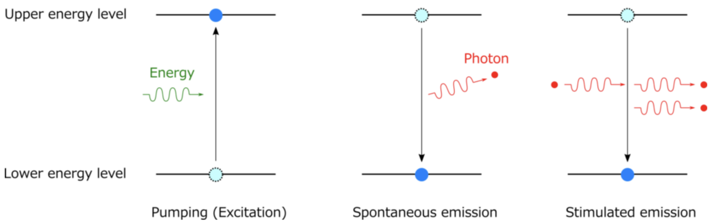
Radiation added to the system



A three-particle system and simplication assumptions

- molecules of the gas can be in two different states
- the ground state and the excited state, which we will denote as A and $\bar{A}$
- the radiation is monochromatic and it consists of collection of photons with frequency $\nu_0 > 0$.
- all the photons of the system have the same energy $\epsilon_0 = h\nu_0$ where h is the Planck constant.
- no Doppler effect: valid if non-relativistic $\left|\frac{v}{c}\right| \simeq 0$

Two-level molecules and radiation



- Elastic collisions between molecules

$$A + A \rightleftarrows A + A,$$
$$A + \bar{A} \rightleftarrows A + \bar{A},$$
$$\bar{A} + \bar{A} \rightleftarrows \bar{A} + \bar{A}$$

- Inelastic collisions

$$A + A \rightleftarrows A + \bar{A}.$$

- Collisions between a molecule and a photon

$$A + \phi \rightleftarrows \bar{A}.$$

Conservation laws for the inelastic collisions

- The photon energy $\epsilon_0 = h\nu_0$: required to form an excited
- Conservation of total energy;

$$\frac{1}{2}|\bar{v}_1|^2 + \frac{1}{2}|\bar{v}_2|^2 = \frac{1}{2}|\bar{v}_3|^2 + \frac{1}{2}|\bar{v}_4|^2 + \epsilon_0.$$

- Conservation of total momentum:

$$\bar{v}_1 + \bar{v}_2 = \bar{v}_3 + \bar{v}_4.$$

- The total energy is conserved but the total kinetic energy is not conserved here.

Radiative transfer equation for photons

- Velocity distributions for the ground (A) and the excited ($\bar{A}$) states as $F^{(1)} = F^{(1)}(t,x,v)$ and $F^{(2)} = F^{(2)}(t,x,v)$, respectively.
- Intensity of the radiation at the frequency ν as $I_\nu = I_\nu(t,x,n)$ where $n \in \mathbb{S}^2$.

$$\frac{1}{c}\frac{\partial I_{\nu_0}}{\partial t} + n \cdot \nabla_x I_{\nu_0}$$
$$= \frac{\epsilon_0}{4\pi}\int_{\mathbb{R}^3} dv \ \left[\frac{2h\nu_0^3}{c^2}F^{(2)}(v)\left(1+\frac{c^2}{2h\nu_0^3}I_{\nu_0}\right) - F^{(1)}(v)I_{\nu_0}(n)\right]$$
$$=: \epsilon_0 h_{rad}[F^{(1)}, F^{(2)}, I_{\nu_0}].$$

Kinetic equations for two-species gases coupled with radiation

For $\frac{D}{Dt} \stackrel{\text{def}}{=} \partial_t + v \cdot \nabla_x$,

$$\frac{DF^{(1)}}{Dt} = \mathcal{K}_{el}^{(1,1)}[F^{(1)}, F^{(1)}] + \mathcal{K}_{el}^{(1,2)}[F^{(1)}, F^{(2)}] + \mathcal{K}_{non.el}^{(1)}[F, F] + \int_{\mathbb{S}^2} dn\ h_{rad}[F^{(1)}, F^{(2)}, I_{\nu_0}],$$

and

$$\frac{DF^{(2)}}{Dt} = \mathcal{K}_{el}^{(2,1)}[F^{(2)}, F^{(1)}] + \mathcal{K}_{el}^{(2,2)}[F^{(2)}, F^{(2)}] + \mathcal{K}_{non.el}^{(2)}[F, F] - \int_{\mathbb{S}^2} dn\ h_{rad}[F^{(1)}, F^{(2)}, I_{\nu_0}],$$

Elastic and non-elastic Boltzmann operators

$$
\begin{aligned}
&\mathcal{K}_{el}^{(i,j)}[F,G](v_1) \\
&\stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv_2 \int_{\mathbb{S}^2} d\omega\, B_{el}^{(i,j)}(|v_1-v_2|,(v_1-v_2)\cdot\omega)(F(v_3)G(v_4)-F(v_1)G(v_2)),
\end{aligned}
$$

$$
\mathcal{K}_{non.el}^{(1)}[F,F] \stackrel{\text{def}}{=} 2\mathcal{K}_{1,1}[F,F] + \mathcal{K}_{1,2}^{(1)}[F,F],
$$

$$
\begin{aligned}
\mathcal{K}_{1,1}[F,F](\bar v_1) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3} d\bar v_2 \int_{\mathbb{S}^2} d\omega \frac{\sqrt{|\bar v_1-\bar v_2|^2-4\epsilon_0}}{2|\bar v_1-\bar v_2|} \\
&\times B_{non.el}(|\bar v_1-\bar v_2|,\omega\cdot(\bar v_1-\bar v_2))(\bar F_3^{(2)}\bar F_4^{(1)}-\bar F_1^{(1)}\bar F_2^{(1)}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_{1,2}^{(1)}[F,F](\bar v_4) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3} d\bar v_3 \int_{\mathbb{S}^2} d\omega \frac{\sqrt{|\bar v_3-\bar v_4|^2+4\epsilon_0}}{2|\bar v_3-\bar v_4|} \\
&\times B_{non.el}(|\bar v_3-\bar v_4|,\omega\cdot(\bar v_3-\bar v_4))(\bar F_1^{(1)}\bar F_2^{(1)}-\bar F_3^{(2)}\bar F_4^{(1)}),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{K}_{non.el}^{(2)}[F,F](\bar v_3) \stackrel{\text{def}}{=} &\int_{\mathbb{R}^3} d\bar v_4 \int_{\mathbb{S}^2} d\omega \frac{\sqrt{|\bar v_3-\bar v_4|^2+4\epsilon_0}}{2|\bar v_3-\bar v_4|} \\
&\times B_{non.el}(|\bar v_3-\bar v_4|,\omega\cdot(\bar v_3-\bar v_4))(\bar F_1^{(1)}\bar F_2^{(1)}-\bar F_3^{(2)}\bar F_4^{(1)}).
\end{aligned}
$$

Part II: LTE and Non-LTE: Existence and Non-existence of a stationary solution, arXiv:2109.10071

Local Thermodynamic Equilibrium (LTE)

- Saha-Boltzmann Ratio: $\frac{\rho_2}{\rho_1} = e^{-\frac{\epsilon_0}{k_B T}}$ and let $k_B = \frac{1}{2}$
- degeneracy of energy levels $= 1$
- LTE: ρ_1, ρ_2 satisfy approximately the Boltzmann ratio AND the distribution of velocities at each point can be approximated by a Maxwellian distribution

$$M(x, v; \rho_i, u, T) \stackrel{\text{def}}{=} \frac{\rho_i}{(\pi T)^{3/2}} \exp\left(-\frac{1}{T}|v-u|^2\right).$$

- $\tilde{\rho} = \rho_1 + \rho_2 = \rho_1(1 + e^{-\frac{2\epsilon_0}{T}})$ and

$$F^{(1)}(t,x,v) \cong M(x, v; \rho_1, u, T),\ \ F^{(2)}(t,x,v) \cong e^{-\frac{2\epsilon_0}{T}} F^{(1)}(v).$$

Non-LTE

- Non-LTE if the assumption of LTE fails.
- The failure of the local Maxwellian approximation is rare.
- Restrict only to situations, in which the distributions of velocities are the Maxwellians, but with different temperature T_1, T_2 for each of the species, and ρ_1, ρ_2 not satisfying the Boltzmann ratio.
- Each species can have different temperatures T_1 and T_2.

Chapman-Enskog approximations yielding LTE

- $F_{eq} = (F_{eq}^{(1)}, F_{eq}^{(2)})^\top$ and the Local Maxwellians

$$F_{eq}^{(j)} = F_{eq}^{(j)}(\rho, u, T) \stackrel{\text{def}}{=} \frac{c_0 \rho}{T^{3/2}} \exp\left(-\frac{1}{T}\left(|v-u|^2 + 2\epsilon_0 \delta_{j,2}\right)\right).$$

- Rescaled Kinetic System ($t \to \alpha t$, $x \to y = \alpha x$ and $I_\nu \to G = \frac{c^2}{2h\nu_0^3} I$):

$$\partial_t F + v \cdot \nabla_y F = \frac{1}{\alpha}\left(\mathcal{K}_{el}[F,F] + \eta \mathcal{K}_{non.el}[F,F]\right) + \mathcal{R}_p[F,G],$$
$$n \cdot \nabla_y G = \mathcal{R}_r[F,G].$$

- Chapman-Enskog expansion with $\alpha \to 0^+$:

$$F^{(j)} = F_{eq}^{(j)}(1 + f^{(j)}) = F_{eq}^{(j)}(1 + \alpha f_1^{(j)} + \alpha^2 f_2^{(j)} + \cdots).$$

$$
\mathcal{R}_p[F,G] \stackrel{\text{def}}{=} \begin{pmatrix} \int_{\mathbb{S}^2}[F^{(2)}(1+G) - F^{(1)}G]dn \\ -\int_{\mathbb{S}^2}[F^{(2)}(1+G) - F^{(1)}G]dn \end{pmatrix},
$$

$$
\mathcal{R}_r[F,G] \stackrel{\text{def}}{=} \epsilon_0 \int_{\mathbb{R}^3}[F^{(2)}(1+G) - F^{(1)}G]dv,
$$

$$
\rho \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} F^{(1)}dv,
$$

$$
u_i \stackrel{\text{def}}{=} \frac{1}{\rho}\int_{\mathbb{R}^3} v_i F^{(1)}dv, \text{ for each } i = 1,2,3,
$$

$$
T \stackrel{\text{def}}{=} \frac{2}{3\rho}\int_{\mathbb{R}^3} |v-u|^2 F^{(1)}dv.
$$

Chapman-Enskog approximations yielding LTE

- $\mathcal{K}[F_{eq}, F_{eq}] = 0$.
- $\mathcal{K}[F, F] =: \mathcal{K}[F] = \mathcal{K}[F_{eq}(1 + \alpha f_1 + \alpha^2 f_2 + \cdots)] = \alpha L[F_{eq}; f_1] + O(\alpha^2)$.
- $\langle \phi, L[F_{eq}; f_1] \rangle = 0$ for $\phi = 1, v - u, |v - u|^2 + 2\epsilon_0 \delta_{j,2}$.
- Taylor approximation:

$$\partial_t F_{eq} + v \cdot \nabla F_{eq}$$
$$\cong \frac{\partial F_{eq}}{\partial \rho}[\partial_t \rho + v \cdot \nabla \rho] + \sum_{i=1}^{3} \frac{\partial F_{eq}}{\partial u_i}[\partial_t u_i + v \cdot \nabla u_i] + \frac{\partial F_{eq}}{\partial T}[\partial_t T + v \cdot \nabla T].$$

Euler-like system coupled with radiative transfer equation (LTE)

$$
\begin{aligned}
\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot (\tilde{\rho} u) &= 0, \\
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{\nabla p}{\tilde{\rho}} &= 0, \\
\frac{\partial (\tilde{\rho} e)}{\partial t} + \nabla \cdot (\tilde{\rho} u e) + p \nabla \cdot u &= \epsilon_0 \rho \int_{\mathbb{S}^2} dn \, \left[G(1 - e^{-\frac{\epsilon_0}{k_B T}}) - e^{-\frac{\epsilon_0}{k_B T}} \right], \\
n \cdot \nabla_y G &= \epsilon_0 \rho \left[e^{-\frac{2\epsilon_0}{T}} (1 + G) - G \right].
\end{aligned}
$$

Boundary-value problem for the stationary system (LTE)

$$
\begin{aligned}
\nabla \cdot (\tilde{\rho} u) &= 0, \\
(u \cdot \nabla) u + \frac{\nabla p}{\tilde{\rho}} &= 0, \\
\nabla \cdot (\tilde{\rho} u e) + p \nabla \cdot u &= \epsilon_0 \rho \int_{\mathbb{S}^2} dn \, [G(1 - e^{-\frac{\epsilon_0}{k_B T}}) - e^{-\frac{\epsilon_0}{k_B T}}], \\
n \cdot \nabla G &= \epsilon_0 \rho \left[e^{-\frac{2\epsilon_0}{T}} (1 + G) - G \right].
\end{aligned}
$$

- Domain Ω: convex with C^1 boundary
- Specular reflection boundary conditions for F: $F(t, y, v) = F(t, y, v - 2(n_y \cdot v) n_y)$ for $y \in \partial\Omega$
- $\rightarrow$ boundary condition for macroscopic velocity $u \cdot n_y = 0$.

Different scaling limits for LTE

- Linearization around constant steady states $(\tilde{\rho}_0, 0, T_0, \frac{e^{-2\epsilon/T}}{1-e^{-2\epsilon/T}})$:

  $$\tilde{\rho} = \tilde{\rho}_0(1+\zeta), \;\; T = T_0(1+\theta) \text{ for } |\zeta| + |\theta| + |u| \ll 1,$$

  such that $\frac{2\epsilon_0}{T_0}|\theta| \ll 1$.
- A scaling limit yielding constant absorption rate (and nonlinear emission rate) with $\frac{2\epsilon_0}{T_0}|\theta| \approx 1$

Linearized system near the constant states with $\frac{2\epsilon_0}{T_0}|\theta| \ll 1$

$$
\begin{aligned}
\frac{\partial \zeta}{\partial t} + \nabla_y \cdot u &= 0, \\
\frac{\partial u}{\partial t} + \frac{T_0}{2}\nabla_y(\zeta + \theta) &= 0, \\
\lambda_0 \epsilon_0 \frac{\partial \theta}{\partial t} + \frac{p_0}{\tilde{\rho}_0}\nabla_y \cdot u &= \frac{\epsilon_0 G_0}{1 + e^{-\frac{2\epsilon_0}{T_0}}} \int_{\mathbb{S}^2} dn \ \left[\frac{h}{1+G_0} - \frac{2\epsilon_0}{T_0}\theta\right] \\
n \cdot \nabla_y h &= \frac{\epsilon_0 \tilde{\rho}_0 G_0}{1 + e^{-\frac{2\epsilon_0}{T_0}}} \left[\frac{2\epsilon_0}{T_0}\theta - \frac{h}{1+G_0}\right],
\end{aligned}
$$

where $\lambda_0 \overset{\text{def}}{=}$ and $p_0 = \tilde{\rho}_0 T_0$.

- Mass conservation: $\int_\Omega \xi \ dy = b_2$.

Theorem

The linearized stationary system with the incoming boundary condition has a unique solution (ζ, θ) with $u = 0$.

A system with nonlinear emission rate with $\frac{2\epsilon_0}{T_0}|\theta| \approx 1$

- a new scaling limit $|\zeta| + |u| + |\theta| \ll 1$ with $\frac{2\epsilon_0}{T_0}|\theta| \approx 1$, $\zeta = \frac{T_0}{2\epsilon_0}\xi$ and $\theta = \frac{T_0}{2\epsilon_0}\vartheta$
- leading-order system with an exponential dependence in the temperature:

$$
\begin{aligned}
&\tilde{\rho}_0 \nabla_y \cdot u = 0, \\
&\frac{T_0}{2} \nabla_y (\xi + \theta) = 0, \\
&\epsilon_0 e^{-\frac{2\epsilon_0}{T_0}} \int_{\mathbb{S}^2} dn \, \left[H - e^{\vartheta} \right] = 0, \\
&n \cdot \nabla_y H = \epsilon_0 \tilde{\rho}_0 \left[e^{\vartheta} - H \right], \\
&\int_\Omega \tilde{\rho} dy = \int_\Omega \tilde{\rho}_0 \left(1 + \frac{T_0}{2\epsilon_0} \xi \right) dy = M_0 \text{ given.}
\end{aligned}
$$

Boundary-value problem with the nonlinear emission rate

- Incoming boundary conditions: at any given $y_0 \in \partial\Omega$, define $\nu = \nu_{y_0}$ as the outward normal vector at y_0. For any $n \in \mathbb{S}^2$, if $n \cdot \nu_{y_0} < 0$, then
$$H(y_0, n) = f(n),$$
for a given profile f.

Theorem

For $\Omega \in \mathbb{R}^3$ convex and bounded with $\partial\Omega \in C^1$, there exists a unique solution with $u = 0$ to the system of nonlinear emission rate with the incoming boundary condition.

Strategy for the proof

- Given $y \in \Omega$ and $n \in \mathbb{S}^2$, there exist unique $y_0 = y_0(y,n) \in \partial\Omega$ and $s = s(y,n)$ such that

$$y = y_0(y,n) + s(y,n)n.$$

- $s = s(y,n)$: optical length
- Using $n \cdot \nabla H = e^{\vartheta} - H =: w - H$ with the boundary condition, we have

$$H(y,n) = f(n)e^{-s(y,n)} + \int_0^{s(y,n)} e^{-(s(y,n)-\xi)} w(y_0(y,n)+\xi n)d\xi.$$

- Then the flow $\vec{J} \overset{\text{def}}{=} \int dn\, nH$ satisfies

$$\begin{aligned} 0 &= div(\vec{J}) \\ &= div(\vec{R}) + div \int_{\mathbb{S}^2} n \left(\int_0^{s(y,n)} e^{-(s(y,n)-\xi)} w(y_0(y,n)+\xi n)d\xi \right) dn. \end{aligned}$$

Strategy for the proof

- Goal: to derive that w actually satisfies the Fredholm integral equation of the second kind
- Key idea: to raise the integral into a 5-fold one:

$$\int_{\mathbb{S}^2} n \left(\int_0^{s(y,n)} e^{-(s(y,n)-\xi)} w(y_0(y,n) + \xi n) d\xi \right) dn$$
$$= \int_{\partial\Omega} dz \int_{\mathbb{S}^2} n \left(\int_0^{s(y,n)} e^{-(s(y,n)-\xi)} w(z + \xi n) \delta(z - y_0(y,n)) d\xi \right) dn.$$

- change of variables $\xi \mapsto \hat{\xi} = s(y,n) - \xi$ and then $(\hat{\xi}, n) \mapsto \eta \overset{\text{def}}{=} y - \hat{\xi} n \in \Omega$ with the Jacobian $J(\eta, n) \overset{\text{def}}{=} \left| \frac{\partial(\hat{\xi}, n)}{\partial \eta} \right| = \frac{1}{|y-\eta|^2}$.
- Since $n = n(y-\eta) = \frac{y-\eta}{|y-\eta|}$ and $\hat{\xi} = |y-\eta|$, we have

$$\int_{\partial\Omega} dz \int_{\mathbb{S}^2} n \left(\int_0^{s(y,n)} e^{-\hat{\xi}} w(z + (s - \hat{\xi}) n) \delta(z - y_0(y,n)) d\hat{\xi} \right) dn$$
$$= \int_\Omega \frac{1}{|y-\eta|^2} \frac{y-\eta}{|y-\eta|} e^{-|y-\eta|} w(\eta) d\eta.$$

Strategy for the proof

- Using

$$div\left(\frac{1}{|y-\eta|^2}\frac{y-\eta}{|y-\eta|}e^{-|y-\eta|}\right)=div\left(\frac{e^{-r}}{r^2}\hat{r}\right)=-\frac{e^{-r}}{r^2}+4\pi\delta(r),$$

  we have

$$w(y)=\int_\Omega \frac{e^{-|y-\eta|}}{4\pi|y-\eta|^2}w(\eta)d\eta-\frac{1}{4\pi}div(\vec{R}).$$

- $w(\eta)=0$ if $\eta\notin\Omega$ and $w\in L^\infty(\Omega)$.
- $\int_\Omega \frac{e^{-|y-\eta|}}{4\pi|y-\eta|^2}d\eta<1$.
- A unique solution exists.

Non-LTE with two different temperatures

- $F^{(j)}$ in local Maxwellians, but $F^{(2)} \neq e^{-\frac{2\epsilon_0}{T}} F^{(1)}$
- Two different temperatures T_1 and T_2 for A and $\bar{A}$.
- Additional assumption: not sufficient mixing of A and $\bar{A}$ via the elastic collisions $\mathcal{K}_{el}^{(1,2)}$ and $\mathcal{K}_{el}^{(2,1)}$
- Local Maxwellian equilibria $M^{(j)}$ for each type of molecules $j = 1, 2$:

$$M^{(j)} = M^{(j)}(x, v; \rho_j, u_j, T_j) \stackrel{\text{def}}{=} \frac{c_0 \rho_j}{T_j^{\frac{3}{2}}} \exp\left(-\frac{|v - u_j|^2}{T_j}\right), \; j = 1, 2.$$

- Densities, velocities, temperatures:

$$\rho_j \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} F^{(j)} dv,$$

$$u_j \stackrel{\text{def}}{=} \frac{1}{\rho_j} \int_{\mathbb{R}^3} v F^{(j)} dv, \text{ for } i = 1, 2, 3,$$

$$T_j \stackrel{\text{def}}{=} \frac{2}{3\rho_j} \int_{\mathbb{R}^3} |v - u_j|^2 F^{(j)} dv,$$

Euler-like system for non-LTE

- Chapman-Enskog Expansion with $\sigma \overset{\text{def}}{=} \frac{\eta}{\alpha} \approx 1$ and $\alpha \to 0^+$:

$$F^{(j)} \cong M^{(j)}(1 + \alpha f_1^{(j)} + \cdots).$$

- Euler-like System for Non-LTE:

$$\begin{aligned}
&\partial_t \rho_1 + \nabla_y \cdot (\rho_1 u_1) = \sigma H^{(1)} + Q^{(1)}, \\
&\partial_t \rho_2 + \nabla_y \cdot (\rho_2 u_2) = \sigma H^{(2)} + Q^{(2)}, \\
&\partial_t (\rho_1 u_1) + \nabla_y \cdot (\rho_1 u_1 \otimes u_1) + \nabla_y \cdot S^{(1)} = \sigma J_m^{(1)} + \Sigma^{(1)}, \\
&\partial_t (\rho_2 u_2) + \nabla_y \cdot (\rho_2 u_2 \otimes u_2) + \nabla_y \cdot S^{(2)} = \sigma J_m^{(2)} + \Sigma^{(2)}, \\
&\partial_t (\rho_1 T_1) + \nabla_y \cdot (\rho_1 u_1 T_1 + J_q^{(1)}) = \sigma J_e^{(1)} + J_r^{(1)}, \\
&\partial_t \left(\rho_2 T_2 + \frac{4}{3} \epsilon_0 \rho_2 \right) + \nabla_y \cdot \left(\rho_2 u_2 T_2 + \frac{4}{3} \epsilon_0 u_2 \rho_2 + J_q^{(2)} \right) = \sigma J_e^{(2)} + J_r^{(2)}.
\end{aligned}$$

$$H^{(j)} \stackrel{\text{def}}{=} \int \mathcal{K}^{(j)}_{non.el}[M, M] dv,$$

$$Q^{(j)} \stackrel{\text{def}}{=} \int \mathcal{R}^{(j)}_{p}[M, G] dv,$$

$$J^{(j)}_{m} \stackrel{\text{def}}{=} \int v \mathcal{K}^{(j)}_{non.el}[M, M] dv,$$

$$\Sigma^{(j)} \stackrel{\text{def}}{=} \int v \mathcal{R}^{(j)}_{p}[M, G] dv,$$

$$S^{(j)} \stackrel{\text{def}}{=} \int (v - u_j) \otimes (v - u_j) M^{(j)} dv,$$

$$J^{(j)}_{q} \stackrel{\text{def}}{=} \int \frac{4}{3} \left(\frac{|v - u_j|^2}{2} + \epsilon_0 \delta_{j,2} \right) (v - u_j) M^{(j)} dv = 0,$$

$$J^{(j)}_{e} = \int \frac{4}{3} \left(\frac{|v - u_j|^2}{2} + \epsilon_0 \delta_{j,2} \right) \mathcal{K}^{(j)}_{non.el}[M, M] dv,$$

$$J^{(j)}_{r} \stackrel{\text{def}}{=} \int \frac{4}{3} \left(\frac{|v - u_j|^2}{2} + \epsilon_0 \delta_{j,2} \right) \mathcal{R}^{(j)}_{p}[M, G] dv,$$

$$\mathcal{R}_p[F, G] \stackrel{\text{def}}{=} \begin{pmatrix} \int_{\mathbb{S}^2} [F^{(2)}(1 + G) - F^{(1)} G] dn \\ -\int_{\mathbb{S}^2} [F^{(2)}(1 + G) - F^{(1)} G] dn \end{pmatrix}.$$

Stationary equations with zero velocities (Non-LTE)

$$\sigma H^{(1)} + Q^{(1)} = -\sigma H^{(2)} - Q^{(2)} = 0,$$
$$\nabla_y \cdot S^{(1)} = -\nabla_y \cdot S^{(2)} = 0,$$
$$\sigma J_e^{(1)} + J_r^{(1)} = 0,$$
$$\sigma J_e^{(2)} + J_r^{(2)} = 0.$$

$$H^{(2)} = -H^{(1)} = \rho_1^2 e^{-\frac{2\epsilon_0}{T_1}} \mathcal{P}(T_1) - \rho_1\rho_2 \mathcal{P}(T_2, T_1),$$
$$Q^{(1)} = -Q^{(2)} = \rho_2 \int_{\mathbb{S}^2} dn\ (1+G) - \rho_1 \int_{\mathbb{S}^2} dn\ G,$$
$$S^{(j)} = p^{(j)} I = \frac{1}{2}\rho_j T_j I,$$
$$J_e^{(1)} = -J_e^{(2)} = -\frac{4}{3}\left[(\rho_1^2 e^{-\frac{2\epsilon_0}{T_1}} - \rho_2\rho_1)\mathcal{A}(T_1;\epsilon_0) + \rho_1\rho_2\mathcal{B}(T_1, T_2)\right],$$
$$J_r^{(1)} = -\rho_1 T_1 \int_{\mathbb{S}^2} G dn + \rho_2 T_2 \int_{\mathbb{S}^2} (1+G) dn,$$
$$J_r^{(2)} = \rho_1 T_1 \int_{\mathbb{S}^2} G dn - \rho_2 T_2 \int_{\mathbb{S}^2} (1+G) dn + \frac{4\epsilon_0}{3} Q^{(2)}.$$

$$\mathcal{P}(T_k, T_l, u_k, u_l) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} dv \int_{\mathbb{R}^3} dv_4 \int_{\mathbb{S}^2} d\omega\ W_+(v, v_4; v_1, v_2)\mathcal{Z}(v, u_k, T_k)\mathcal{Z}(v_4, u_l, T_l).$$

$$\mathcal{A}(T_1; \epsilon_0) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \left(\frac{|v|^2}{2} + \epsilon_0\right) dv \int_{\mathbb{R}^3} dv_4 \int_{\mathbb{S}^2} d\omega\ W_+(v, v_4; v_1, v_2) \times \mathcal{Z}(v, 0, T_1)\mathcal{Z}(v_4, 0, T_1)$$

$$\mathcal{B}(T_1, T_2) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \frac{|v|^2}{2} dv \int_{\mathbb{R}^3} dv_4 \int_{\mathbb{S}^2} d\omega\ W_+(v, v_4; v_1, v_2)\mathcal{Z}(v_4, 0, T_1) \times (\mathcal{Z}(v, 0, T_1) - \mathcal{Z}(v, 0, T_2)).$$

Radiation and mass conservation

- Radiative transfer equation:

$$n\cdot\nabla_y G = \epsilon_0 \int_{\mathbb{R}^3} [M^{(2)}(1+G) - M^{(1)}G]dv = \epsilon_0(\rho_2(1+G) - \rho_1 G).$$

- Mass conservation:

$$\int_{\Omega_y} dy\ (\rho_1 + \rho_2) = m_0.$$

Non-existence theorem for non-LTE

Theorem

Let $m_0 > 0$ given and let Ω be a bounded convex domain with $\partial\Omega \in C^1$. Assume that $L(T_1, T_2) = \frac{m_0}{|\Omega|}$ defines a smooth curve in the plane $(T_1, T_2) \in \mathbb{R}^2_+$ for given m_0 and Ω. Assume the incoming boundary condition for each boundary point $y_0 \in \partial\Omega$, $G(y_0, n) = f(n)$, for some given f. Then the system of the Euler-like system coupled with radiation in the non-LTE case with the boundary condition does not have a solution unless the given boundary profile f is chosen specifically so that it satisfies

$$div\left(\int_{\mathbb{S}^2} nf(n)e^{-A_2 s(y,n)}dn\right) = 0,$$

for any $y \in \Omega$ and for some $A_2 > 0$, where for each $y \in \Omega$ and $n \in \mathbb{S}^2$, $y_0 = y_0(y, n) \in \partial\Omega$ and $s = s(y, n)$ are determined uniquely such that

$$y = y_0(y, n) + s(y, n)n.$$

Proof of non-existence

- Observation: $Q^{(2)} = 0$. Thus $H^{(j)} = Q^{(j)} = 0$, $j = 1, 2$.
- obtain the relation

$$\frac{4\sigma}{3}\left[(\rho_1^2 e^{-\frac{2\epsilon_0}{T_1}} - \rho_2\rho_1)\mathcal{A}(T_1;\epsilon_0) + \rho_1\rho_2\mathcal{B}(T_1, T_2)\right] = \frac{4\pi\rho_1\rho_2(T_2 - T_1)}{\rho_1 - \rho_2},$$

and $\nabla p^{(j)} = 0$.

- $\rho_j = \frac{C_j}{T_j}$ and

$$\mathcal{C} \stackrel{\text{def}}{=} \frac{C_2}{C_1} = \frac{T_2}{T_1} e^{-\frac{2\epsilon_0}{T_1}} \frac{\mathcal{P}(T_1)}{\mathcal{P}(T_2, T_1)} = H(T_1, T_2).$$

- mass conservation implies

$$L \stackrel{\text{def}}{=} \frac{\frac{4\pi \frac{\mathcal{C}}{T_1 T_2}(T_2 - T_1)}{\frac{1}{T_1} - \frac{\mathcal{C}}{T_2}}}{\frac{4\sigma}{3}\frac{1}{T_1}\left[\left(\frac{e^{-\frac{2\epsilon_0}{T_1}}}{T_1} - \frac{\mathcal{C}}{T_2}\right)\mathcal{A}(T_1;\epsilon_0) + \frac{\mathcal{C}}{T_2}\mathcal{B}(T_1, T_2)\right]}\left(\frac{1}{T_1} + \frac{H(T_1, T_2)}{T_2}\right) = \frac{m_0}{|\Omega|}.$$

- We can parametrize $T_j = T_j(\tau)$ for τ on some interval I_L.

Proof of non-existence

- Now use

$$n \cdot \nabla_y G = \epsilon_0(\rho_2(1+G) - \rho_1 G) = (A_1 - A_2 G)\,.$$

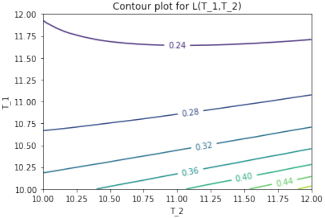
- Follow the same trick as in the LTE case with $w = e^{\vartheta}$ replaced by A_1.
- Finally we have

$$\frac{1}{4\pi} div \int_{\mathbb{S}^2} nf(n) e^{-A_2 s(y,n)} dn = A_1 A_2 \int_{\mathbb{R}^3} \frac{e^{-A_2|y-\eta|}}{4\pi|y-\eta|^2} d\eta - A_1 = 0.$$

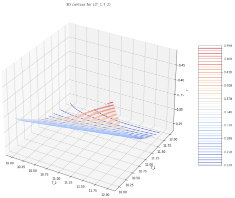
- Note that $A_2 > 0$ is a constant that depends only on τ and ϵ_0.
- $s = s(y,n)$: optical length
- contradiction for a general incoming boundary profile f.

Contour plots of $L(T_1, T_2)$ for the hard-sphere kernel

$$L(T_1, T_2) \stackrel{\text{def}}{=} \frac{\dfrac{4\pi \frac{H(T_1,T_2)}{T_1 T_2}(T_2 - T_1)}{\frac{1}{T_1} - \frac{H(T_1,T_2)}{T_2}}}{\frac{4\sigma}{3}\frac{1}{T_1}\left[\left(\frac{e^{-\frac{2\epsilon_0}{T_1}}}{T_1} - \frac{H(T_1,T_2)}{T_2}\right)\mathcal{A}(T_1;\epsilon_0) + \frac{H(T_1,T_2)}{T_2}\mathcal{B}(T_1, T_2)\right]} \times \frac{1}{T_1}\left(1 + e^{-\frac{2\epsilon_0}{T_1}}\frac{\mathcal{P}(T_1)}{\mathcal{P}(T_2, T_1)}\right).$$



Contour



3D Contour

Figure: Contour level curves for $L(T_1, T_2)$ for $(T_1, T_2) \in [10, 12]^2$

True non-LTE with three-level molecules

- Three-level molecules: A_1 (ground), A_2 (second level $= A_1 + \epsilon_1$), A_3 (third level $= A_2 + \epsilon_2$)
- No artificial no-mixing conditions
- More freedom from the energy equation $\epsilon_1 Q^{(1)} + \epsilon_2 Q^{(2)} = 0$.
- Existence of a unique stationary solution with $u_1 = u_2 = u_3 = 0$.

Further generalization

1. General black-body emission (Planck distribution)
2. no monochromatic condition
3. scattering of the radiation
4. Doppler effects and widening of the spectrum

Thank you for your attention.