Well-posedness of the Two and Half Dimensional Hall MHD

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INTRODUCTION

The incompressible and resistive MHD are

$$
\begin{aligned}
\text{Momentum Equation:} \quad & u_t + u \cdot \nabla u - J \times B + \nabla p - \mu \Delta u = 0, \\
\text{Incompressibility:} \quad & \operatorname{div} u = 0, \\
\text{Faraday's Law:} \quad & \operatorname{curl} E = -B_t, \\
\text{Ohm's Law:} \quad & E + u \times B = \nu J, \\
\text{Incompressibility:} \quad & \operatorname{div} B = 0.
\end{aligned}
$$

- u is the velocity field, p is the pressure, and B is the magnetic field
- μ and ν are the viscosity and the resistivity constants, respectively
- $J = \operatorname{curl} B$
- $J \times B$: Lorentz force

In terms of (u, p, B), MHD (with $\mu = \nu = 1$) is

$$
\begin{aligned}
&u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla p - \Delta u = 0, \\
&B_t + u \cdot \nabla B - B \cdot \nabla u - \Delta B = 0, \\
&\operatorname{div} u = 0, \\
&\operatorname{div} B = 0,
\end{aligned}
$$

where we use

$$
J \times B = B \cdot \nabla B - \frac{1}{2} \nabla |B|^2, \quad p \mapsto p + \frac{1}{2} |B|^2.
$$

- MHD model provides a macroscopic description of a plasma.
- MHD is deficient in many respect: it does not explain magnetic reconnection.
- Some of these deficiencies are accounted for by extending Ohm's Law.

- Generalized Ohm's law:

$$E + u \times B = J + (J \times B - \nabla p_e), \quad J = \operatorname{curl} B$$

$\Longrightarrow$ Hall MHD

$$u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla p - \Delta u = 0,$$

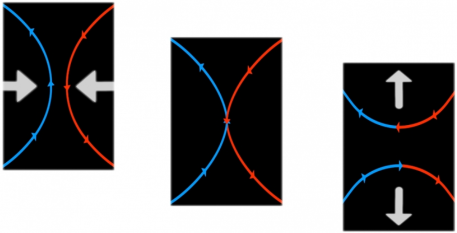
$$B_t + u \cdot \nabla B - B \cdot \nabla u + \underbrace{\operatorname{curl}((\operatorname{curl} B) \times B)}_{\text{Hall Term}} - \Delta B = 0$$

- Chae-Degond-Liu, Chae-Schonbek, Chae-Lee
- Chae-Weng, Dai, Danchin-Tan
- $\cdots$

MAGNETIC RECONNECTION

- The Hall term plays a critical role in magnetic reconnection processes.

- When magnetic fields lines on the sun come together, they can realign into a new configuration.



The process is called magnetic reconnection.

$2\frac{1}{2}$ DIMENSIONAL HALL MHD

- Magnetic reconnection: 2D feature
- $2\frac{1}{2}$ dimensional Hall MHD with u and B (Litvinenko–McMahon (2015))

$$u(t,x,y) = \left(\nabla^{\perp}\phi(t,x,y), W(t,x,y)\right) = (\phi_y(t,x,y), -\phi_x(t,x,y), W(t,x,y))\,,$$

$$B(t,x,y) = \left(\nabla^{\perp}\psi(t,x,y), Z(t,x,y)\right) = (\psi_y(t,x,y), -\psi_x(t,x,y), Z(t,x,y))\,.$$

$\Longrightarrow$

$$\begin{aligned} \psi_t - \Delta\psi &= [\psi, Z] - [\psi, \phi], \\ Z_t - \Delta Z &= [\Delta\psi, \psi] - [Z, \phi] + [W, \psi], \\ W_t - \Delta W &= -[W, \phi] - [\psi, Z], \\ \Delta\phi_t - \Delta\Delta\phi &= -[\Delta\phi, \phi] + [\Delta\psi, \psi], \end{aligned} \tag{1}$$

where $[f,g] = \nabla^{\perp} f \cdot \nabla g = f_x g_y - f_y g_x$

$2\frac{1}{2}$ DIMENSIONAL HALL EQUATIONS

When $u = p = 0$ (because the Hall term is dominant)

$$\begin{aligned} \psi_t - \Delta\psi &= [\psi, Z], \\ Z_t - \Delta Z &= [\Delta\psi, \psi] \end{aligned} \tag{2}$$

- **B.-Kang (2022)**
  - ▷ LWP and Blow-up criterion
  - ▷ GWP and decay rates when initial data is sufficiently small
  - ▷ Asymptotic profiles of (ψ, Z)
  - ▷ Perturbations near harmonic functions

Let $(\nabla\psi_0, Z_0) \in H^2$ and

$$M(t) = \|\nabla\psi(t)\|_{H^2}^2 + \|Z(t)\|_{H^2}^2, \quad M(0) = \|\nabla\psi_0\|_{H^2}^2 + \|Z_0\|^2,$$
$$N(t) = \left\|\nabla^2\psi(t)\right\|_{H^2}^2 + \|\nabla Z(t)\|_{H^2}^2, \quad \mathcal{E}(t) = M(t) + \int_0^t N(s)ds.$$

Theorem 1. (i) There exists $T^* = T(M(0))$ such that there exists a unique solution with $\mathcal{E}(T^*) < \infty$.

(ii) The maximal existence time $T^* < \infty$ if and only if

$$\lim_{T \nearrow T^*} \int_0^T \|\nabla Z(t)\|_{L^p}^q \, dt = \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \le q < \infty.$$

REMARKS

- Compared to Chae-Degond-Liu, regularity of initial data is the borderline case:

$$B_0 = \left(\nabla^{\perp}\psi_0, Z_0\right) \in H^2, \quad 2 = \frac{d}{2} + 1 = \frac{2}{2} + 1.$$

- The blow-up criterion is stated only in terms of the third component of B.

A similar blow-up criterion can be derived in terms of $\Delta\psi$:

$$\lim_{T \nearrow T^*} \int_0^T \|\Delta\psi\|_{L^p}^q \, dt = \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty.$$

- Blow-up criteria are related to the scaling invariant property :

$$\psi(t,x,y) \mapsto \psi_\lambda(t,x,y) = \lambda^{-1}\psi\left(\lambda^2 t, \lambda x, \lambda y\right)$$
$$Z(t,x,y) \mapsto Z_\lambda(t,x,y) = Z\left(\lambda^2 t, \lambda x, \lambda y\right)$$

- The scaling invariant property also used to establish weak-strong uniqueness.

Theorem 2. Let $B_1 = \left(\nabla^\perp \psi_1, Z_1\right)$ and $B_2 = \left(\nabla^\perp \psi_2, Z_2\right)$ be weak solutions with the same initial data $(\nabla \psi_0, Z_0) \in L^2$. Then, $B_1 = B_2$ on $[0, T]$ if B_2 satisfies

$$(\Delta \psi_2, \nabla Z_2) \in L^p\left([0, T] : L^q\right), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty.$$

- Chae-Degond-Liu: weak-strong uniqueness with $B_2 \in L^2\left([0, T] : W^{1,\infty}(\mathbb{R}^3)\right)$

PROOF OF THEOREM 1: A PRIORI ESTIMATES

- Some properties of the commutator $[f, g] = \nabla f \cdot \nabla^{\perp} g = f_x g_y - f_y g_x$:

$$[f, f] = 0, \quad [f, g] = -[g, f]$$
$$\Delta[f, g] = [\Delta f, g] + [f, \Delta g] + 2[f_x, g_x] + 2[f_y, g_y],$$
$$\int f[f, g] = 0,$$
$$\int f[g, h] = \int g[h, f].$$

- L^2 bound:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right) + \|\Delta\psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 = -\int \Delta\psi[\psi, Z] + \int Z[\Delta\psi, \psi] = 0$$

- $\dot{H}^1$ bound:

$$\frac{1}{2}\frac{d}{dt}\left(\|\Delta\psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2\right) + \|\nabla\Delta\psi\|_{L^2}^2 + \|\Delta Z\|_{L^2}^2 = \int \Delta^2\psi[\psi, Z] - \int \Delta Z[\Delta\psi, \psi]$$

$$\int \Delta^2\psi[\psi, Z] - \int \Delta Z[\Delta\psi, \psi] \simeq C\|\nabla\psi\|_{L^\infty} \|\nabla\Delta\psi\|_{L^2} \|\Delta Z\|_{L^2} + \cdots$$

▷ Need smallness condition? No

- Cancellation of highest order terms:

$$\int \Delta^2\psi[\psi, Z] - \int \Delta Z[\Delta\psi, \psi] = \int \Delta\psi\Delta[\psi, Z] - \int \Delta Z[\Delta\psi, \psi]$$

$$= \int \Delta\psi[\psi, \Delta Z] - \int \Delta Z[\Delta\psi, \psi] + 2\int \Delta\psi\left([\psi_x, Z_x] + [\psi_y, Z_y]\right)$$

$$= 2\int \Delta\psi\left([\psi_x, Z_x] + [\psi_y, Z_y]\right) \leq C\|\Delta Z\|_{L^2} \|\Delta\psi\|_{L^4}^2$$

$\Longrightarrow$

$$\frac{d}{dt}\left(\|\nabla\psi\|_{H^1}^2+\|Z\|_{H^1}^2\right)+\|\Delta\psi\|_{H^1}^2+\|\nabla Z\|_{H^1}^2\le C\,\|\Delta\psi\|_{L^2}^2\,\|\nabla\Delta\psi\|_{L^2}^2$$
$$\le C\left(\|\nabla\psi\|_{H^1}^2+\|Z\|_{H^1}^2\right)\left(\|\Delta\psi\|_{H^1}^2+\|\nabla Z\|_{H^1}^2\right)$$

- Similarly, we can obtain $\dot{H}^2$ bound.

- Blow-up criterion:

$$\int \Delta\psi\left([\psi_x,Z_x]+[\psi_y,Z_y]\right)\le C\,\|\nabla Z\|_{L^p}\left\|\nabla^2\psi\right\|_{L^q}\left\|\nabla^3\psi\right\|_{L^2},\quad \frac{1}{p}+\frac{1}{q}=\frac{1}{2}$$
$$\le C\le C\,\|\nabla Z\|_{L^p}\left\|\nabla^2\psi\right\|_{L^2}^{\frac{2}{q}}\|\nabla\Delta\psi\|_{L^2}^{2-\frac{2}{q}}$$
$$\le C\,\|\nabla Z\|_{L^p}^q\,\|\Delta\psi\|_{L^2}^2+\|\nabla\Delta\psi\|_{L^2}^2$$

Theorem 3. (i) If $\|\Delta\psi_0\|_{L^2}^2 + \|\nabla Z_0\|_{L^2}^2 \ll 1$, $T^* = \infty$ in **Theorem 1**.

(ii) If $(\nabla\psi_0, Z_0) \in L^1$ in addition, $(\Delta\psi, \nabla Z)$ decays in time as follows

$$\|\Delta\psi(t)\|_{L^2} + \|\nabla Z(t)\|_{L^2} \le \frac{C}{1+t}, \quad \|\nabla\Delta\psi(t)\|_{L^2} + \|\Delta Z(t)\|_{L^2} \le \frac{C}{(1+t)^{3/2}}.$$

- From the linear part, we expect the decay rates of the form

$$\|\Delta\psi(t)\|_{L^2} + \|\nabla Z(t)\|_{L^2} \le \frac{C}{\sqrt{1+t}}, \quad \|\nabla\Delta\psi(t)\|_{L^2} + \|\Delta Z(t)\|_{L^2} \le \frac{C}{1+t}.$$

▷ Improved by combining with the decay rate of weak solutions

- Smallness condition: $\dot{B}^{3/2}_{2,1}$ (Chae-Lee) $\mapsto \dot{H}^1$

DECAY RATE OF WEAK SOLUTION

- Hall equation:

$$\frac{1}{2}\frac{d}{dt}\|B\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = -\int \mathrm{curl}(J \times B)\cdot B = -\int (J \times B)\cdot J = 0$$

▷ Chae-Schonbek: decay rate with $B_0 \in L^2 \cap L^1$

$$\|B(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}}$$

- $2\frac{1}{2}$ dimensional Hall equation:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\psi\|_{L^2}^2 + \|Z\|_{L^2}^2\right) + \|\Delta\psi\|_{L^2}^2 + \|\nabla Z\|_{L^2}^2 = 0$$

▷ If $(\nabla\psi_0, Z_0) \in L^2 \cap L^1$,

$$\|\nabla\psi(t)\|_{L^2} + \|Z(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}}$$

- Further improvement of the decay rate of ψ:

$$\psi_t + \nabla^\perp Z \cdot \nabla\psi - \Delta\psi = 0$$

Let $(\nabla\psi_0, Z_0) \in H^2$. If $\psi_0 \in L^1 \cap L^2$ in addition,

$$\|\nabla\psi(t)\|_{L^2} \le \frac{C}{1+t}, \quad \|\Delta\psi(t)\|_{L^2} \le \frac{C}{(1+t)^{3/2}}$$

- How to derive these? For example,

$$\frac{1}{2}\frac{d}{dt}\Big(\frac{\|\psi\|_{L^2}^2}{t} + \|\nabla\psi\|_{L^2}^2\Big) + \frac{t}{C}\Big(\frac{\|\psi\|_{L^2}^2}{t} + \|\nabla\psi\|_{L^2}^2\Big)^2 \le 0.$$

$$\Longrightarrow \frac{\|\psi(t)\|_{L^2}^2}{t} + \|\nabla\psi(t)\|_{L^2}^2 \le \frac{C}{(1+t)^2}$$

Asymptotic behaviors

- Upper bounds of decay rates:

$$\|\nabla\psi(t)\|_{L^2} \le \frac{C}{1+t}, \quad \|\nabla Z(t)\|_{L^2} \le \frac{C}{1+t}.$$

Although there is no embedding relationship between $\dot{H}^1$ and L^∞, we can expect similar decay results in L^∞ if we establish the asymptotic behavior of (ψ, Z).

- Motivation from the asymptotic behavior of the vorticity of the incompressible Navier-Stokes equations in 2D (Carpio, Gallay, Giga-Giga).

$$\lim_{t\to\infty} t\,\|\omega(t) - \gamma\Gamma(t)\|_{L^\infty} = 0, \quad \int_{\mathbb{R}^2} \omega_0(x)dx = \gamma$$

- Assumptions (in addition to H^2)

$$\psi_0 \in L^1, \quad Z_0 \in L^1, \tag{3a}$$

$$\langle x \rangle \psi_0 \in L^1, \quad \langle x \rangle Z_0 \in L^1, \tag{3b}$$

$$\int_{\mathbb{R}^2} \psi_0(x) dx = \gamma, \quad \int_{\mathbb{R}^2} Z_0(x) dx = \eta, \tag{3c}$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$. Let

$$\Gamma(t, x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}$$

be the two dimensional heat kernel.

Theorem 4. (i) Suppose $(\nabla\psi_0, Z_0) \in H^2$ and we assume (3a). Then,

$$\psi(t,x) = \Gamma(t) * \psi_0 + O(t^{-3/2}), \quad Z(t,x) = \Gamma(t) * Z_0 + O(t^{-2})$$

(ii) If we assume (3b) and (3c),

$$\psi(t,x) = \gamma\Gamma(t,x) + O(t^{-3/2}), \quad Z(t,x) = \eta\Gamma(t,x) + O(t^{-3/2}).$$

- We observe that constant multiples of Γ are solutions of

$$\begin{aligned} \psi_t - \Delta\psi &= [\psi, Z], \\ Z_t - \Delta Z &= [\Delta\psi, \psi] \end{aligned}$$

because Γ and Γ_t are radial functions and so

$$\nabla^\perp\Gamma \cdot \nabla\Gamma = 0, \quad \nabla^\perp\Gamma \cdot \nabla\Delta\Gamma = \nabla^\perp\Gamma \cdot \nabla\Gamma_t = 0.$$

- Let $\psi = \widetilde{\psi} + \gamma\Gamma$ and $Z = \widetilde{Z} + \eta\Gamma$. Then,

$$\widetilde{\psi}(t) = \Gamma(t) * (\psi_0 - \gamma\delta_0) - \int_0^t \Gamma(t-s) * (\nabla^\perp Z \cdot \nabla\psi)(s)ds,$$
$$\widetilde{Z}(t) = \Gamma(t) * (Z_0 - \eta\delta_0) - \int_0^t \Gamma(t-s) * (\nabla^\perp \psi \cdot \nabla\Delta\psi)(s)ds.$$

- By combining the decay rate of Γ and decay rate of (ψ, Z)

$$\left\|\widetilde{\psi}(t)\right\|_{L^\infty} = O(t^{-\frac{3}{2}})$$

$\Longrightarrow$

$$\psi(t,x) = \gamma\Gamma(t,x) + \widetilde{\psi}(t,x) = \Gamma(t) * \psi_0 + O(t^{-3/2})$$
$$\psi(t,x) = \gamma\Gamma(t,x) + \widetilde{\psi}(t,x) = \gamma\Gamma(t,x) + O(t^{-3/2})$$

PERTURBATION AROUND HARMONIC FUNCTIONS

- Perturbation around zero solution $\longmapsto$ perturbation around harmonic functions.

- Case 1: Let $\overline{\psi}$ be a harmonic function such that $\|\nabla^2\overline{\psi}\|_{L^\infty} < \infty$.

For example, $\overline{\psi}(x,y) = x^2 - y^2$ or $\overline{\psi}(x,y) = xy$. Let $\psi = \rho + \overline{\psi}$

$$\rho_t - \Delta\rho = [\rho, Z] + [\overline{\psi}, Z], \quad Z_t - \Delta Z = [\Delta\rho, \rho] + [\Delta\rho, \overline{\psi}].$$

- Case 2: Let $\overline{Z}$ be a harmonic function such that $\|\nabla\overline{Z}\|_{L^\infty} < \infty$.

For example, $\overline{Z}(x,y) = ax + by$. Let $Z = \omega + \overline{Z}$.

$$\psi_t - \Delta\psi = [\psi, \omega] + [\psi, \overline{Z}], \quad \omega_t - \Delta\omega = [\Delta\psi, \psi].$$

- Global-in-time solutions with small initial data

- Harmonic functions are not necessarily small

FULL $2\frac{1}{2}$ DIMENSIONAL HALL MHD

The $2\frac{1}{2}$ dimensional Hall MHD with u and B of the form

$$u(t,x,y) = \left(\nabla^{\perp}\phi(t,x,y), W(t,x,y)\right) = \left(\phi_y(t,x,y), -\phi_x(t,x,y), W(t,x,y)\right),$$
$$B(t,x,y) = \left(\nabla^{\perp}\psi(t,x,y), Z(t,x,y)\right) = \left(\psi_y(t,x,y), -\psi_x(t,x,y), Z(t,x,y)\right).$$

$\Longrightarrow$

$$\begin{aligned}
&\psi_t - \Delta\psi = [\psi, Z] - [\psi, \phi],\\
&Z_t - \Delta Z = [\Delta\psi, \psi] - [Z, \phi] + [W, \psi],\\
&W_t - \Delta W = -[W, \phi] - [\psi, Z],\\
&\Delta\phi_t - \Delta\Delta\phi = -[\Delta\phi, \phi] + [\Delta\psi, \psi],
\end{aligned}$$

where $[f,g] = \nabla^{\perp} f \cdot \nabla g = f_x g_y - f_y g_x$.

- Let

$$P(t) = \|\nabla\psi(t)\|_{H^2}^2 + \|Z(t)\|_{H^2}^2 + \|\nabla\phi(t)\|_{H^2}^2 + \|W(t)\|_{H^2}^2 ,$$
$$Q(t) = \|\Delta\psi(t)\|_{H^2}^2 + \|\nabla Z(t)\|_{H^2}^2 + \|\Delta\phi(t)\|_{H^2}^2 + \|\nabla W(t)\|_{H^2}^2 ,$$
$$\mathcal{E}(t) = P(t) + \int_0^t Q(s)ds.$$

LWP and Blow-up criterion. Let $(\nabla\psi_0, Z_0, \nabla\phi_0, W_0) \in H^2$. There exists $T^* = T(\mathcal{E}_0) > 0$ such that there exists a unique solution with $\mathcal{E}(T^*) < \infty$.

Moreover, the maximal existence time $T^* < \infty$ if and only if

$$\lim_{T \nearrow T^*} \int_0^T \|\nabla Z(t)\|_{L^p}^q \, dt = \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}, \quad 2 \leq q < \infty.$$

- Blow-up criterion is specified only in terms of Z even when the fluid part enters.

This is because the Hall term is dominant.

GENERALIZED OHM'S LAW

Momentum Equation: $u_t + u \cdot \nabla u - J \times B + \nabla p - \Delta u = 0,$

Ohm's Law: $E + u \times B = J + \underbrace{(J \times B - \nabla p_e)}_{\text{Hall term}} + \underbrace{(J_t + \epsilon\, \mathrm{div}(uJ + Ju))}_{\text{Electron Inertial term}},$

- When $\epsilon = 0$,

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|J\|_{L^2}^2 + \|B\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = 0.$$

- When $\epsilon = 1$, No energy conservation:

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|J\|_{L^2}^2 + \|B\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = -\int u \cdot (J \cdot \nabla J).$$

- Lüst (1959)

$$
\begin{aligned}
&u_t + u\cdot\nabla u - B\cdot\nabla B + \nabla p - \Delta u = -J\cdot\nabla J,\\
&E + u\times B = J + (J\times B - \nabla p_e) + (J_t + \operatorname{div}(uJ + Ju)) - J\cdot\nabla J
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
u_t + u\cdot\nabla u - B\cdot\nabla B + \nabla p - \Delta u &= -J\cdot\nabla J\\
B_t - \Delta B_t - \Delta B + u\cdot\nabla B - B\cdot\nabla u + \operatorname{curl}(J\times B) &= -\operatorname{curl}(\operatorname{div}(uJ + Ju))\\
&\quad + \operatorname{curl}(J\cdot\nabla J)
\end{aligned}
$$

- Energy conservation

$$
\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|J\|_{L^2}^2 + \|B\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = 0.
$$

▷ Not enough to construct a weak solution

Theorem (B. -Shin) (i) Let $(u_0, B_0, J_0) \in H^3$. There exists T only depending on (u_0, B_0) such that there exists a unique solution (u, B, J) of Lüst model on $[0, T)$.

(ii) Moreover, $T^* > 0$ is the maximal existence time of solutions if and only if

$$\limsup_{t \nearrow T^*} \int_0^{T^*} \left(\|\operatorname{curl} u\|_{\mathrm{BMO}} + \|\nabla J\|_{\mathrm{BMO}} \right) dt = \infty.$$

- Commutator estimate (Kato-Ponce)

$$\sum_{|\beta| \le m} \left\| D^\beta (fg) - f D^\beta g \right\|_{L^2} \le C \left(\|\nabla f\|_{L^\infty} \left\| D^{m-1} g \right\|_{L^2} + \|D^m f\|_{L^2} \|g\|_{L^\infty} \right)$$

- Cancellation property: for smooth vector fields F, G, H with $\operatorname{div} F = 0$, we have

$$\int (F \cdot \nabla G) \cdot H + \int (F \cdot \nabla H) \cdot G = \int F \cdot \nabla (G \cdot H)) = 0.$$

- A priori estimates

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^3}^2 + \|B\|_{H^3}^2 + \|J\|_{H^3}^2\right) + \|\nabla u\|_{H^3}^2 + \|\nabla B\|_{H^3}^2$$
$$\leq C\left(\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty} + \|\nabla J\|_{L^\infty}\right)\left(\|u\|_{H^3}^2 + \|B\|_{H^3}^2 + \|J\|_{H^3}^2\right).$$

- Beale-Kato-Majda type inequality in BMO (Kozono-Taniuchi):

$$\|f\|_{L^\infty} \leq C\left(1 + \|f\|_{\mathrm{BMO}}\right)\log\left(1 + \|f\|_{H^s}\right), \qquad s > \frac{d}{2}$$

- Blow-up criterion: Let $\mathcal{E}(t) = \|u\|_{H^3}^2 + \|B\|_{H^3}^2 + \|J\|_{H^3}^2$

$$\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty} + \|\nabla J\|_{L^\infty}$$
$$\leq C\left(1 + \|\nabla u\|_{\mathrm{BMO}} + \|\nabla B\|_{\mathrm{BMO}} + \|\nabla J\|_{\mathrm{BMO}}\right)\log\left(1 + \mathcal{E}(t)\right)$$
$$\leq C\left(1 + \|\mathrm{curl}\, B\|_{\mathrm{BMO}} + \|J\|_{L^2} + \|\nabla J\|_{\mathrm{BMO}}\right)\log\left(1 + \mathcal{E}(t)\right)$$

$2\frac{1}{2}$ dimensional Lüst model?

Let

$$u(t,x,y) = \left(\nabla^{\perp}\phi(t,x,y), W(t,x,y)\right), \quad B(t,x,y) = \left(\nabla^{\perp}\psi(t,x,y), Z(t,x,y)\right)$$

$\Longrightarrow$

$$\begin{aligned}
&\psi_t - \Delta\psi_t - \Delta\psi = [Z,\Delta\psi] + [\phi,\Delta\psi] + [W,Z] + [\psi,Z] + [\phi,\psi],\\
&Z_t - \Delta Z_t - \Delta Z = [Z,\Delta Z] + [\phi,\Delta Z] + [Z,\Delta\phi] + [\Delta\psi,\psi] + [\phi,Z] + [W,\psi]\\
&W_t - \Delta W = [Z,\Delta\psi] + [\phi,W] + [Z,\psi],\\
&\Delta\phi_t - \Delta^2\phi = -[Z,\Delta Z] + [\phi,\Delta\phi] + [\Delta\psi,\psi],
\end{aligned}$$

- Start without the fluid effects: let $\mathcal{E}_0 = \|\nabla\psi_0\|_{H^3} + \|Z_0\|_{H^3}$ and

$$\mathcal{E}(t) = \|\nabla\psi(t)\|_{H^3}^2 + \|\Delta\psi(t)\|_{H^3}^2 + \|Z(t)\|_{H^2}^3 + \|\nabla Z(t)\|_{H^3}^2 .$$

Theorem (B. -Shin). There exists T depending on $\mathcal{E}_0$ such that there exists a unique solution $(\nabla^\perp\psi, Z)$ on $[0,T)$.

Moreover, $T^* > 0$ is the maximal existence time of solutions if and only if

$$\limsup_{t\nearrow T^*} \int_0^{T^*} \left\|\nabla^2 Z(t)\right\|_{L^\infty} dt = \infty.$$

- Blow-up criterion is specified only in terms of Z:

$$\|\nabla J\|_{\mathrm{BMO}} \longmapsto \left\|\nabla^2 Z(t)\right\|_{L^\infty}$$

- Full $2\frac{1}{2}$ dimensional Lüst model: In preparation

- A priori estimates:

$$\frac{d}{dt}\mathcal{E}(t) + \|\Delta\psi\|_{H^3}^2 + \|\nabla Z\|_{H^3}^2 \le C\left(\left\|\nabla^2 Z\right\|_{L^\infty} + \left\|\nabla^2\psi\right\|_{L^\infty} + \left\|\nabla^3\psi\right\|_{L^\infty}\right)\mathcal{E}(t).$$

- Blow-up criterion by Beale-Kato-Majda type inequality in BMO:

$$\begin{aligned}&\left\|\nabla^2 Z\right\|_{L^\infty} + \left\|\nabla^2\psi\right\|_{L^\infty} + \left\|\nabla^3\psi\right\|_{L^\infty}\\ &\le C\left(1 + \left\|\nabla^2 Z\right\|_{\mathrm{BMO}} + \|\Delta\psi\|_{L^2} + \left\|\nabla^3\psi\right\|_{\mathrm{BMO}}\right)\log\left(1+\mathcal{E}(t)\right).\end{aligned}$$

- Embedding of BMO in 2D: $\left\|\nabla^3\psi\right\|_{\mathrm{BMO}} \le C\|\nabla^3\psi\|_{H^1}$
- H^3 bound:

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla\psi\|_{H^2}^2 + \|\Delta\psi\|_{H^2}^2 + \|Z\|_{H^2}^2 + \|\nabla Z\|_{H^2}^2\right) + \|\nabla\Delta Z\|_{L^2}^2\\ &\le C\left\|\nabla^2 Z\right\|_{L^\infty}\left(\|\nabla\psi\|_{H^2}^2 + \|\Delta\psi\|_{H^2}^2 + \|Z\|_{H^2}^2 + \|\nabla Z\|_{H^2}^2\right)\end{aligned}$$

Thank you for your attention!!