On well-posedness at critical regularity for a family of active scalar equations arising in hydrodynamics

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Generalized dissipative SQG equation

Introduced by Chae, Constantin, Cordoba, Gancedo, Wu (2012)

Let $\Omega = \mathbb{R}^2$, $\kappa \in (0,2)$, $\beta \in [0,2)$, and $\gamma > 0$. Then gSQG given by

$$\partial_t \theta + \gamma \Lambda^{\kappa} \theta = -u \cdot \nabla \theta, \quad u = -\nabla^{\perp} \Lambda^{\beta - 2} \theta, \qquad (gSQG)$$

where $\Lambda = \sqrt{-\Delta}$.

- $u \sim \Lambda^{\beta-1}\theta, \quad R^{\perp} \sim \nabla^{\perp}\Lambda^{-1}$.
- $\kappa > \beta$ subcritical
- $\kappa = \beta$ critical
- $\kappa < \beta$ supercritical
- $\beta = 0$ dissipative Euler equation
- $\beta = 1$ dissipative SQG; analogy to 3D NSE (Constantin, Majda, Tabak 1994)

$$\partial_t v + \gamma \Lambda v + (R^{\perp}\theta \cdot \nabla) v = -(R^{\perp} v) \cdot v, \quad v = \nabla \theta.$$

- $\beta = 2$ implies $u \cdot \nabla \theta = -\nabla^{\perp}\theta \cdot \nabla \theta = 0$
- Define $u := (\log(I - \Delta))^{\mu}\theta$, $\mu > 0$ to replace $\beta = 2$ case.

General structure

Let $A = -\Delta$ and $\kappa \in (0, 2], \alpha \in [-1, 1)$.

Equation (up to homogeneity) has the following structure:

$$\underbrace{\partial_t v + A^{\kappa/2} v}_{\text{linear evolution}} = \underbrace{F(A^{\alpha/2} v, A^{1/2} v)}_{\text{polynomial}}$$

Structural Criticality.

- Subcritical. $\kappa > 1$;
- Critical. $\kappa = 1$;
- Supercritical. $\kappa < 1$.

Examples.

- *NSE*: subcritical
- *mSQG* ($\kappa = \beta$): critical ($\beta = 1$), super. ($\beta < 1$), sub. ($\beta > 1$)
- *gSQG* ($\kappa < 1$): supercritical

General structure

Let $A = -\Delta$ and consider $\kappa \in (0,2], \alpha \in [-1,1)$.

$$\partial_t v + A^{\kappa/2} v = F(A^{\alpha/2} v, A^{1/2} v).$$

Questions.

1. *Well-posedness? "Best" regularity class?*
2. *Assumptions on F?*

Expectations.

1. *F real analytic in its arguments*
2. *Propagation of regularity from the linear evolution*

Classical results

Let $A = -\Delta$ and $\Omega = \mathbb{T}^d$, $d = 2, 3$. Consider the Cauchy problem:

$$\partial_t u + \nu A u = -(u\cdot\nabla)u - \nabla p + f, \quad \nabla\cdot u = 0, \quad u(x,0) = u_0(x).$$

Theorem (Foias, Temam 1989)

Let $u_0 \in \dot{H}^1_{sol.}$, f real analytic with radius $\lambda > 0$. Then $\exists T = T(\|u_0\|_{H^1}, \|e^{\lambda A^{1/2}} f\|_{L^2})$ such that

$$\exists! \text{ solution } u(t) \in D(A^{1/2} e^{tA^{1/2}}), \quad \forall t \in (0, T)$$

Key Idea. Develop bilinear estimates in Gevrey classes.

Literature.

1960s: Masuda (1967), Kahane (1968)

1990s: Doering, Titi (1995), Grujic, Kukavica (1998), Kukavica (1998), Ferrari, Titi (1998), Lemarié-Rieusset (1999)

2000s: Giga, Sawada (2002), Mattingly, Shirikyan (2002), Germain (2007), Pavlovic, Staffilani (2007), Biswas-Swanson (2007)

2010s: Herbst, Skibsted (2011), Biswas, Jolly, M, Titi (2014), Bae, Biswas (2014), Bae (2015), M, Zhao (2017)

General results

Let $\Omega = \mathbb{T}^d$, $d \geq 2$. Consider $v : \mathbb{T}^d \to \mathbb{R}^m$, $m \geq 1$ and

$$v_t + Av = F(v), \quad v(x,0) = v_0(x),$$

where F is real analytic in each of its arguments.

Theorem (Ferrari, Titi 1998)

Let $v_0 \in H^s$, $s > d/2$, such that $\|v_0\|_{H^s} \leq M$. Then $\exists T = T(M, F)$ such that

$$\exists! \text{ solution } u(t) \in D(A^{s/2} e^{tA^{1/2}}), \quad \forall t \in (0, T).$$

Key Idea: Develop product estimates in a Banach algebra.

General results (ctd.)

Let $\Omega = \mathbb{T}^d$, $d \geq 2$. Consider

$$v_t + Av = F(v, A^{1/2}v), \quad v(x,0) = v_0(x).$$

Theorem (Ferrari, Titi 1998)

Let $v_0 \in H^s$, $s > d/2$, such that $\|v_0\|_{H^s} \leq M$.

1. *If $s > d/2 + 1$, then $\exists T = T(M, F)$ such that*

   $$\exists! \text{ solution } u(t) \in D(A^{s/2} e^{tA^{1/2}}), \quad \forall t \in (0, T).$$

2. *If $s \in (d/2, d/2 + 1)$ and*

   $$F(u, A^{1/2}u) = g(u)A^{1/2}u + h(u), \quad g, h \quad \text{real analytic},$$

   then $\exists T = T(M, g, h)$ such that

   $$\exists! \text{ solution } u(t) \in D(A^{s/2} e^{tA^{1/2}}), \quad \forall t \in (0, T).$$

General results (ctd.)

Let $\Omega = \mathbb{T}^d$, $d \geq 2$. Consider

$$v_t + A^{\gamma/2} v = G(v), \quad v(x,0) = v_0(x),$$

where $G(v) = A^{\gamma_0/2} G(A^{\gamma_1/2} v, \ldots, A^{\gamma_n/2} v)$ analytic and

$$\gamma_0 + \bar{\gamma} < \gamma, \quad \bar{\gamma} := \max_{j \geq 1} \gamma_j.$$

Theorem (Bae, Biswas 2014)

Let $(I + A^{1/2})^s u_0 \in L^p$, $s > d/p$, with $p \in (1, \infty)$, such that $\|(I + A^{1/2})^s u_0\|_{L^p} \leq M$.

1. *If $s > d/p + \bar{\gamma}$, then $\exists T = T(M, G)$ such that $\exists!$ solution $u(t) \in D(A^{s/2} e^{tA^{1/2}})$, $\forall t \in (0, T)$;*
2. *If $s \in (d/p, d/p + \bar{\gamma})$ and* ***under certain restrictions*** *on G, then $\exists T = T(M, G)$ such that $\exists!$ solution $u(t) \in D(A^{s/2} e^{tA^{\gamma/2}})$, $\forall t \in (0, T)$;*

Note: (Ferrari-Titi), (Bae-Biswas) only treat struct. subcrit. systems

Scaling-Critical Sobolev spaces

Scaling symmetry.

$$\theta_\lambda = \lambda^{\kappa-\beta}\theta(\lambda^\kappa t, \lambda x) \text{ is a solution of } gSQG$$

whenever θ is a solution of *gSQG*.

Critical norm.

$$\|\theta_\lambda\|_{\dot{H}^{\beta+1-\kappa}} = \|\theta\|_{\dot{H}^{\beta+1-\kappa}}.$$

In particular

$$\dot{H}^{\beta+1-\kappa} \quad \text{is a } \textit{scaling-critical} \text{ space for } gSQG$$

Some history

Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0. \tag{NSE}$$

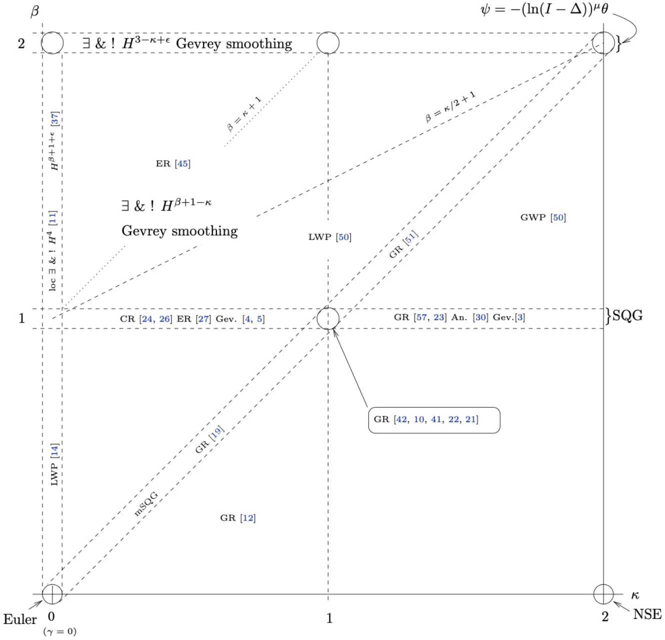
Critical Spaces.

$$\dot{H}^{d/2-1} \hookrightarrow L^d \hookrightarrow \dot{B}^{-1+d/p}_{p,\infty} \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty} \subset \mathcal{S}'$$

Classical.
Fujita, Kato (1964): $L^d(\mathbb{R}^d)$ and $\dot{H}^{d/2-1}(\mathbb{R}^d)$
Cannone (1997), Planchon (1998): $\dot{B}^{-1+d/p}_{p,\infty}(\mathbb{R}^d)$
Koch, Tataru (2001): $BMO^{-1}(\mathbb{R}^d)$
Germain (2008), Bourgain, Pavlovic (2008): ill-posedness in $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d)$
Friedlander, Rusin (2013): second iterate of active scalars

Smoothing.
Grujic, Kukavica (1998): analyticity in $L^d(\mathbb{R}^d)$
Germain, Pavlovic, Staffilani (2007): analyticity in $BMO^{-1}(\mathbb{R}^d)$
Bae, Biswas, Tadmor (2012): analyticity (Gevrey) and decay in $\dot{B}^{-1+d/p}_{p,\infty}(\mathbb{R}^d)$

Map of results for gSQG



Sobolev and Gevrey norms

(Homogeneous) Sobolev space.
Let $\sigma \in \mathbb{R}$. Then

$$\|f\|^2_{\dot{H}^\sigma} := \Sigma_{j\in\mathbb{Z}} 2^{2j\sigma} \|\triangle_j f\|^2_{L^2} < \infty,$$

where

$$\mathrm{supp}(\widehat{\triangle_j f}) \subset Ann(2^{j-1}, 2^{j+1}).$$

Gevrey norm.
Let $\alpha > 0$ and $\lambda \geq 0$. Define

$$G^\lambda_\alpha := \exp(\lambda \Lambda^\alpha)$$

Then

$$\|f\|_{\dot{G}^\lambda_{\alpha,\sigma}} := \|G^\lambda_\alpha f\|_{\dot{H}^\sigma}.$$

Sobolev and Gevrey norms

Immediate consequences.
For $k \geq 0$

$$\|D^k f\|_{\dot{H}^\sigma} \leq C^k \frac{k^{k/\alpha}}{(\alpha\lambda)^{k/\alpha}} \|G_\alpha^\lambda f\|_{\dot{H}^\sigma}$$

In particular

$$\alpha = 1 \rightsquigarrow \text{analytic Gevrey class}$$

and

$$\lambda = \lambda(t) \rightsquigarrow \text{temporal decay of derivatives.}$$

History.
Foias, Temam (1989), Doering, Titi (1995), Ferrari, Titi (1998), Biswas, Swanson (2007), Kukavica, Vicol (2008), Herbst, Skibsted (2011), Biswas, Jolly, M, Titi (2014), Bae, Biswas, Tadmor (2014), Bae (2015), Camilyurt, Kukavica, Vicol (2018), Friendlander, Suen (2019), Chemin, Gallagher, Zhang (2020), Ambrose, Lopes-Filho, Nussenzveig-Lopes (2022), many, many others...

Exist. & Uniq. for supercritical SQG ($\beta = 1$)

Let $\kappa \in (0,1)$ and $\gamma > 0$. Recall

$$\partial_t \theta + \gamma \Lambda^\kappa \theta = -u \cdot \nabla \theta, \quad u = -R^\perp \theta. \tag{SQG}$$

Theorem (Miura, 2006)

Let $\theta_0 \in H^{\sigma_c}(\mathbb{R}^2)$, where $\sigma_c := 2 - \kappa$.

1. *There exists $T^* = T^*(\theta_0)$ such that the unique solution θ satisfies*

$$\sup_{0<t<T^*} t^{\rho/\kappa} \|\theta(\,\cdot\,, t)\|_{\dot{H}^{\sigma_c+\rho}} < \infty,$$

   for some $\rho > 0$, $\lambda = \lambda(t)$ such that $\lambda(0) = 0$.

2. *$\exists \epsilon > 0$ such that $\|\theta_0\|_{\dot{H}^{\sigma_c}} \leq \epsilon$ implies $T_* = \infty$.*

Main insight.
Fujita-Kato approach insufficient.
Propose iteration scheme that preserves commutator structure:

$$\partial_t \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} = -\gamma \Lambda^\kappa \theta^{n+1}.$$

Gevrey regularity for supercritical SQG ($\beta = 1$)

Let $\kappa \in (0, 1)$ and $\gamma > 0$. Recall

$$\partial_t \theta + \gamma \Lambda^\kappa \theta = -u \cdot \nabla \theta, \quad u = -R^\perp \theta. \tag{SQG}$$

Theorem (Biswas, 2013)

Let $\theta_0 \in H^{\sigma_c}(\mathbb{R}^2)$, where $\sigma_c := 1 + \beta - \kappa$. Then there exists $T^* = T^*(\theta_0)$ such that the unique solution θ satisfies

$$\sup_{0<t<T^*} t^{\rho/\kappa} \| G_\alpha^{\lambda(t)} \theta(\,\cdot\,, t) \|_{\dot{H}^{\sigma_c+\rho}} < \infty,$$

for some $0 < \rho < \kappa$, $\lambda = \lambda(t)$ such that $\lambda(0) = 0$.

Main insight.

Iteration scheme of Miura can accommodate Gevrey operator G_α^λ.

Extend product and commutator estimates to Gevrey norms.

Exist. & Uniq. for supercritical generalized SQG

Let $\kappa \in (0,1)$ and $\beta \in (0,2)$. Recall

$$\partial_t\theta + \gamma\Lambda^\kappa\theta = -u\cdot\nabla\theta, \quad u = -R^\perp\Lambda^{\beta-1}\theta. \qquad (gSQG)$$

Main issues.

- Commutator structure more nuanced; exploit skew self-adjointness of constitutive law (Chae, Constantin, Cordoba, Gancedo, Wu, 2012), (Hu, Kukavica, Ziane, 2014).
- Approximation scheme of Miura is insufficient.
- Naive modifications of Miura's scheme also insufficient, e.g.,

$$\partial_t\theta^{n+1} + u^n\cdot\nabla\theta^{n+1} = -\gamma\Lambda^\kappa\theta^{n+1}$$

$$\partial_t\theta^{n+1} + u^{n+1}\cdot\nabla\theta^n = -\gamma\Lambda^\kappa\theta^{n+1}$$

$$\partial_t\theta^{n+1} + \frac{1}{2}\left(u^n\cdot\nabla\theta^{n+1} + u^{n+1}\cdot\nabla\theta^n\right) = -\gamma\Lambda^\kappa\theta^{n+1}$$

- Apparent structural obstructions

$$\partial_t\theta = -\gamma\Lambda^\kappa\theta + F(\Lambda^{\beta-1}\theta, \Lambda\theta).$$

Thus "strongly" quasilinear when $\kappa \le \beta - 1$.

Exist./Uniq./Gev. for supercritical generalized SQG

Let $\kappa \in (0,1)$ and $\beta \in (0,2)$. Recall

$$\partial_t \theta + \gamma \Lambda^\kappa \theta = -u \cdot \nabla \theta, \quad u = -R^\perp \Lambda^{\beta-1} \theta. \qquad (gSQG)$$

Theorem (Jolly, Kumar, M 2021)

Let $\theta_0 \in H^{\sigma_c}(\mathbb{R}^2)$, where $\sigma_c := 1 + \beta - \kappa$. Then $\exists!$ solution θ s.t.

1. *For some $T^* = T^*(\theta_0)$, θ satisfies*

$$\sup_{0<t<T^*} t^{\rho/\kappa} \| G_\alpha^{\lambda(t)} \theta(\,\cdot\,, t) \|_{\dot{H}^{\sigma_c + \rho}} < \infty,$$

   for some $0 < \rho < \kappa$ and $\lambda = \lambda(t)$ such that $\lambda(0) = 0$.

2. *$\exists \epsilon > 0$ such that $\|\theta_0\|_{\dot{H}^{\sigma_c}} \leq \epsilon$ implies $T_* = \infty$.*

Exist./Uniq./Gev. for supercritical gSQG (endpoint)

Let $\kappa \in (0,1)$, $\mu > 0$, "$\beta = 2$". Recall

$$\partial_t \theta + u \cdot \nabla \theta = -\gamma \Lambda^\kappa \theta, \quad u = (\log(I - \Delta))^\mu \nabla^\perp \theta.$$

Theorem (Jolly, Kumar, M 2021)

Let $\theta_0 \in H^3(\mathbb{R}^2)$. Then $\exists!$ solution θ such that for some $T^* = T^*(\|\theta_0\|_{H^3})$, θ satisfies

$$\sup_{0 \le t < T^*} \|G_\alpha^{\lambda(t)} \theta(\,\cdot\,, t)\|_{\dot{H}^3} < \infty,$$

for some $\lambda = \lambda(t)$ such that $\lambda(0) = 0$.

Note. Highest order term given by velocity; added structural complication

Proof sketch

Critical space phenomenon

Let $\theta_j := \triangle_j \theta$, where $\triangle_j$ is the j–th Littlewood-Paley block.

Consider the critical regularity parameter

$$\sigma_c = \beta + 1 - \kappa.$$

Assuming proper bounds on the nonlinearity, one always obtains

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{\dot{H}^{\sigma_c}} + \gamma \|\Lambda^{\kappa/2}\theta\|^2_{\dot{H}^{\sigma_c}} \leq C \|\Lambda^{\kappa/2}\theta\|^2_{\dot{H}^{\sigma_c}} \|\theta\|_{\dot{H}^{\sigma_c}}.$$

Therefore, to construct solutions at critical regularity, one can (must) appeal to heat kernel.

The heat kernel can be effectively exploited at level of dyadic blocks.

A priori estimates

Let $\theta_j := \triangle_j \theta$, where $\triangle_j$ is the j–th Littlewood-Paley block.

With appropriate commutator estimates

$$\frac{d}{dt}\|\theta_j\|_{\dot{H}^{\sigma_c}} + c_\kappa \gamma 2^{\kappa j}\|\theta_j\|_{\dot{H}^{\sigma_c}} \leq C 2^{(\kappa/2)j} c_j \|\theta\|_{\dot{H}^{\sigma_c+\kappa/2}}\|\theta\|_{\dot{H}^{\sigma_c}}$$

$$\frac{d}{dt}\|\theta_j\|_{\dot{H}^{\sigma_c+\kappa/2}} + c_\kappa \gamma 2^{\kappa j}\|\theta_j\|_{\dot{H}^{\sigma_c+\kappa/2}} \leq C 2^{(\kappa-\delta)j} c_j \|\theta\|_{\dot{H}^{\sigma_c+\kappa/2}}\|\theta\|_{\dot{H}^{\sigma_c+\delta}}$$

$$\frac{d}{dt}\|\theta_j\|_{\dot{H}^{\sigma_c+\delta}} + c_\kappa \gamma 2^{\kappa j}\|\theta_j\|_{\dot{H}^{\sigma_c+\delta}} \leq C 2^{(\kappa-\delta)j} c_j \|\theta\|^2_{\dot{H}^{\sigma_c+\delta}},$$

where $\{c_j\} \in \ell^2$ and $\delta \in (0,\kappa)$. Then Gronwall implies

$$\|\theta_j(t)\|_{\dot{H}^{\sigma_c+\delta}}$$

$$\leq e^{-c_\kappa \gamma 2^{\kappa j} t}\|\theta_j(0)\|_{\dot{H}^{\sigma_c+\delta}} + C c_j \int_0^t \underbrace{e^{-c_\kappa \gamma 2^{\kappa j}(t-s)} 2^{(\kappa-\delta)j}}_{\sim (t-s)^{-1+\delta/\kappa}} \|\theta(s)\|^2_{\dot{H}^{\sigma_c+\delta}}\, ds.$$

Hardy-Littlewood-Sobolev yields $L^{\kappa/\delta}$ integrability-in-time.

Need suitable approximation scheme to “linearize” resulting bounds.

Approximation scheme

For $n = 0$

$$\begin{cases} \partial_t \theta^0 + \gamma \Lambda^\kappa \theta^0 = 0, \quad \theta^0(x,0) = \theta_0(x). \end{cases}$$

For $n > 0$

$$\begin{cases} \partial_t \theta^{n+1} + \operatorname{div} F_{-\theta^n}(\theta^{n+1}) = -\gamma \Lambda^\kappa \theta^{n+1}, \quad \theta^{n+1}(x,0) = \theta_0(x), \end{cases}$$

where

$$F_{-\theta^n}(\theta^{n+1}) = \begin{cases} -(\nabla^\perp \Lambda^{\beta-2}\theta^n)\theta^{n+1} & \text{if } \kappa > \beta - 1, \\ -(\nabla^\perp \Lambda^{\beta-2}\theta^n)\theta^{n+1} - \Lambda^{\beta-2}((\nabla^\perp \theta^{n+1})\theta^n) & \text{if } \kappa \le \beta - 1. \end{cases}$$

Observe that

$$\operatorname{div} \cdot \Lambda^{\beta-2}((\nabla^\perp \theta)\theta) = 0.$$

Local energy balance...

Let $\lambda(t) := \lambda t^{\alpha/\kappa}$ and $\tilde{\theta}_j := G_\alpha^{\lambda(t)} \triangle_j \theta$.

When $j \in \mathbb{Z}$, evolution of $\tilde{\theta}_j^{n+1}$ given by

$$\partial_t \tilde{\theta}_j^{n+1} + \gamma \Lambda^\kappa \tilde{\theta}_j^{n+1} + G_\alpha^\lambda \triangle_j \operatorname{div} F_{-\theta^n}(\theta^{n+1}) = \frac{\alpha}{\kappa} \frac{\lambda(t)}{t} \Lambda^\alpha \tilde{\theta}_j.$$

Then local energy balance given by

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}_j^{n+1}\|_{L^2}^2 + \gamma \|\Lambda^{\kappa/2}\widetilde{\theta}_j^{n+1}\|_{L^2}^2 \\ &\quad = \frac{\alpha}{\kappa}\frac{\lambda(t)}{t}\|\Lambda^{\sigma_c+\alpha/2}\widetilde{\theta}_j^{n+1}\|_{L^2}^2 - \underbrace{\langle G_\alpha^\lambda \triangle_j \operatorname{div} F_{-\theta^n}(\theta^{n+1}), \tilde{\theta}_j^{n+1}\rangle}_{J}. \end{aligned}$$

On the other hand, by skew-self adjointness of $\Lambda^{\beta-2}\nabla\cdot$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta^{n+1}\|_{L^2}^2 + \gamma\|\Lambda^{\kappa/2}\theta^{n+1}\|_{L^2}^2 &= \langle \Lambda^{\beta-2}\nabla\cdot((\nabla^\perp\theta^{n+1})\theta^n), \theta^{n+1}\rangle \\ &= -\frac{1}{2}\langle [\nabla\Lambda^{\beta-2}, \nabla^\perp\theta^n]\theta^{n+1}, \theta^{n+1}\rangle. \end{aligned}$$

...and commutators

Let $q = -\theta^n$ and drop $n+1$ superscript.

Strongly quasilinear regime: $\kappa \leq \beta - 1$

$$J = \underbrace{\left\langle G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} (\nabla^\perp \Lambda^{\beta-2} q \cdot \nabla\theta), \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle}_{J^a} - \underbrace{\left\langle G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} \Lambda^{\beta-2} (\nabla^\perp q \cdot \nabla\theta), \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle}_{J^b}.$$

Then

$$J^a = \left(J^a - \underbrace{\left\langle (G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} \nabla^\perp \Lambda^{\beta-2} q \cdot \nabla)\theta, \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle}_{J_1^a} - \underbrace{\left\langle (\nabla^\perp \Lambda^{\beta-2} q \cdot \nabla G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} \theta), \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle}_{J_2^a} \right)$$
$$+ J_1^a + J_2^a.$$

Similarly

$$J^b = \left(J^b - \underbrace{\left\langle \nabla^\perp \Lambda^{\beta-2} \cdot ((G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} q) \nabla\theta), \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle}_{J_1^b} - \underbrace{\left\langle \Lambda^{\frac{\beta-2}{2}} (\nabla^\perp q \cdot \nabla G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} \Lambda^{\frac{\beta-2}{2}} \theta), \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle}_{J_2^b} \right)$$
$$+ J_1^b + J_2^b.$$

Observe that $J_2^a = J_2^b = 0$.

...and commutators

Let $q = -\theta^n$ and drop $n+1$ superscript.

Strongly quasilinear regime: $\kappa \leq \beta - 1 \implies \sigma_c \geq 2$

$$J^a = \left(J^a - J_1^a - J_2^a\right) + J_1^a.$$

Observe

$$\Lambda_j^{\sigma c+\delta} = -\Lambda_j^{\sigma c+\delta-2}\partial_l^2.$$

Then for $A_\ell = \Lambda^{\beta-2}\nabla_\ell^\perp$, where $\ell = 1, 2$

$$\begin{aligned} J^a - J_1^a - J_2^a = &- \left\langle G_\alpha^{\lambda(t)}\Lambda_j^{\sigma c+\delta-2}\partial_l(\partial_l A_\ell q\, \partial_\ell \theta), \Lambda^{\sigma c+\delta}\widetilde{\theta}_j\right\rangle + \left\langle (G_\alpha^{\lambda(t)}\Lambda_j^{\sigma c+\delta-2}\partial_l(\partial_l A_\ell q))\partial_\ell \theta, \Lambda^{\sigma c+\delta}\widetilde{\theta}_j\right\rangle \\ &- \left\langle G_\alpha^{\lambda(t)}\Lambda_j^{\sigma c+\delta-2}\partial_l(A_\ell q\, \partial_l \partial_\ell \theta), \Lambda^{\sigma c+\delta}\widetilde{\theta}_j\right\rangle + \left\langle A_\ell q(G_\alpha^{\lambda(t)}\Lambda_j^{\sigma c+\delta-2}\partial_l(\partial_l \partial_\ell \theta)), \Lambda^{\sigma c+\delta}\widetilde{\theta}_j\right\rangle \\ = &- \left\langle [G_\alpha^{\lambda(t)}\Lambda_j^{\sigma c+\delta-2}\partial_l, \partial_\ell \theta]\partial_l A_\ell q, \Lambda^{\sigma c+\delta}\widetilde{\theta}_j\right\rangle - \left\langle [G_\alpha^{\lambda(t)}\Lambda_j^{\sigma c+\delta-2}\partial_l, A_\ell q]\partial_l \partial_\ell \theta, \Lambda^{\sigma c+\delta}\widetilde{\theta}_j\right\rangle \\ = &\widetilde{J}_1^a + \widetilde{J}_2^a, \end{aligned}$$

(Observation of Hu, Kukavica, Ziane (2014))

...and commutators

Let $q = -\theta^n$ and drop $n+1$ superscript.

Strongly quasilinear regime: $\kappa \leq \beta - 1 \implies \sigma_c \geq 2$

$$J^b = \left(J^b - J_1^b - J_2^b\right) + J_1^b.$$

Similarly

$$\begin{aligned}
&J^b - J_1^b - J_2^b \\
&= \left\langle G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta-2} \partial_l(\nabla^\perp(\partial_l q)\cdot \nabla\theta), \Lambda^{\sigma c+\delta+\beta-2}\widetilde{\theta}_j \right\rangle - \left\langle (G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta-2} \nabla^\perp \Delta q)\cdot \nabla\theta, \Lambda^{\sigma c+\delta+\beta-2}\widetilde{\theta}_j \right\rangle \\
&\quad + \left\langle G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} \Lambda^{\beta/2-3} \partial_l((\nabla^\perp q\cdot \nabla(\partial_l\theta)), \Lambda^{\sigma c+\delta+\beta/2-1}\widetilde{\theta}_j \right\rangle \\
&\quad - \left\langle (\nabla^\perp q\cdot \nabla) G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta+\beta/2-3} \Delta\theta, \Lambda^{\sigma c+\delta+\beta/2-1}\widetilde{\theta}_j \right\rangle \\
&= \left\langle [G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta-2}\partial_l, \partial_\ell\theta]\partial_\ell^\perp \partial_l q, \Lambda^{\sigma c+\delta+\beta-2}\widetilde{\theta}_j \right\rangle + \left\langle [G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta+\beta/2-3}\partial_l, \partial_\ell^\perp q]\partial_\ell \partial_l\theta, \Lambda^{\sigma c+\delta+\beta/2-1}\widetilde{\theta}_j \right\rangle \\
&= \widetilde{J}_1^b + \widetilde{J}_2^b.
\end{aligned}$$

(Variation of Hu, Kukavica, Ziane (2014))

...and commutators

Let $q = -\theta^n$ and drop $n+1$ superscript.

Strongly quasilinear regime: $\kappa \leq \beta - 1 \implies \sigma_c \geq 2$

$$J_1^a = \left\langle (G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} \nabla^\perp \Lambda^{\beta-2} q \cdot \nabla)\theta, \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle$$
$$J_1^b = \left\langle \nabla^\perp \Lambda^{\beta-2} \cdot ((G_\alpha^{\lambda(t)} \Lambda_j^{\sigma c+\delta} q)\nabla\theta), \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle$$

Observe that

$$\begin{aligned} J_1^a - J_1^b &= -\left\langle [\Lambda^{\beta-2} \partial_\ell^\perp, \partial_\ell \theta] G_\alpha^{\lambda(t)} \Lambda^{\sigma c+\delta} q_j, \Lambda^{\sigma c+\delta} \widetilde{\theta}_j \right\rangle \\ &= \widetilde{J}^c \end{aligned}$$

Therefore

$$J = \widetilde{J}_1^a + \widetilde{J}_2^a + \widetilde{J}_1^b + \widetilde{J}_2^b + \widetilde{J}^c,$$

where $\widetilde{J}_k^i$ are all commutators.

Commutator estimates (Type A)

Let $[A, B] := AB - BA$ denote commutator bracket and define

$$D = \Lambda \text{ or } \partial_\ell, \quad \ell = 1, 2, \quad \text{and} \quad \mathcal{F}(E_\alpha^\lambda f)(\xi) := \left(\int_0^1 e^{\lambda \tau^\alpha |\xi|^\alpha} d\tau \right) \hat{f}(\xi).$$

Lemma (Jolly, Kumar, M 2021)

Let $\lambda \geq 0$, $\sigma \in [0, 1)$, $\alpha \in (0, 1]$, $\zeta \in [0, 1)$, $\nu \in (0, 1)$, and $\rho \in \mathbb{R}$. Suppose $f, g, h \in L^2$ such that $\operatorname{supp} \hat{h} \subset Ann(2^{j-1}, 2^{j+1})$. Then there exists a sequence $\{c_j\} \in \ell^2(\mathbb{Z})$ such that $\|\{c_j\}\|_{\ell^2} \leq 1$ and

$$\begin{aligned} &|\langle [G_\alpha^\lambda \Lambda^{\sigma+\rho} D\triangle_j, g] f, h \rangle| \\ &\leq C c_j 2^{\nu j} \min \left\{ \|f\|_{\dot{G}^\lambda_{\alpha,1-\nu}} \|g\|_{\dot{G}^\lambda_{\alpha,\sigma+1}}, \|g\|_{\dot{G}^\lambda_{\alpha,2-\nu}} \|f\|_{\dot{G}^\lambda_{\alpha,\sigma}} \right\} \|\Lambda^\rho h\|_{L^2} \\ &+ C\lambda 2^{(\sigma+1+\alpha-\zeta)j} \|E_\alpha^\lambda S_{j-3} g\|_{\dot{H}^{1+\zeta}} \|G_\alpha^\lambda \triangle_j f\|_{L^2} \|\Lambda^\rho h\|_{L^2}, \end{aligned}$$

for some constant $C > 0$, depending only on $\sigma, \alpha, \zeta, \nu, \rho$.

Commutator estimates (Type B)

Lemma (Jolly, Kumar, M 2021)

Let $\mu, \rho > 0$, $\epsilon \in (0,1)$, *and* $\delta \in (0, 2\mu)$. *Suppose* $f, g, h \in L^2$. *Then there exists a constant* $C > 0$, *depending only on* μ, ϵ, δ, *such that*

$$
\begin{aligned}
&|\langle [(\log(I - \Delta))^{\mu} \partial_\ell, g] f, h \rangle| \\
&\le C \|g\|_{\dot{H}^{2-\epsilon+\rho}}^{\frac{1}{1+\rho}} \|g\|_{\dot{H}^{1-\epsilon}}^{\frac{\rho}{1+\rho}} \left(\|f\|_{\dot{H}^{\epsilon+\delta}} \|h\|_{L^2} + \|f\|_{L^2} \|h\|_{\dot{H}^{\epsilon+\delta}} \right),
\end{aligned}
$$

for $\ell = 1, 2$.

Proof of Commutator Type A

Suppose $\hat{h} \subset Ann(2^{j-1}, 2^{j+1})$.

Observe

$$\langle [G^\lambda_\alpha \Lambda^{\sigma+\rho} D\triangle_j, g]f, h\rangle = \iint\limits_{\substack{\xi\sim 2^j\\ \eta\in\mathbb{R}^2}} m^\lambda_{\alpha,\sigma,j,\ell}(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\overline{\hat{h}(\xi)}d\eta d\xi,$$

where

$$m(\xi,\eta) = e^{\lambda|\xi|^\alpha}\phi_j(\xi)|\xi|^\sigma\xi_\ell - e^{\lambda|\xi-\eta|^\alpha}\phi_j(\xi-\eta)|\xi-\eta|^\sigma(\xi-\eta)_\ell.$$

Triangle inequality yields

$$\begin{aligned}|m(\xi,\eta)| \le\ & ||\xi|^\sigma\xi_\ell - |\xi-\eta|^\sigma(\xi-\eta)_\ell|e^{\lambda|\xi|^\alpha}|\xi-\eta|^\rho\phi_j(\xi-\eta)\\ &+ ||\xi|^\rho\phi_j(\xi) - |\xi-\eta|^\rho\phi_j(\xi-\eta)|e^{\lambda|\xi|^\alpha}|\xi|^{\sigma+1}\\ &+ |e^{\lambda|\xi|^\alpha} - e^{\lambda|\xi-\eta|^\alpha}||\xi-\eta|^{\sigma+\rho+1}\phi_j(\xi-\eta)\\ =\ & m_1(\xi,\eta)+m_2(\xi,\eta)+m_3(\xi,\eta).\end{aligned}$$

Observe that

$$\begin{aligned}|m_1(\xi,\eta)| &\le C2^{\rho j}e^{\lambda|\xi-\eta|^\alpha}e^{\lambda|\eta|^\alpha}\phi_j(\xi-\eta)|\eta|\left(|\xi-\eta|^\sigma+|\eta|^\sigma\right)\\ |m_2(\xi,\eta)| &\le C2^{\rho j}e^{\lambda|\xi-\eta|^\alpha}e^{\lambda|\eta|^\alpha}|\eta|\left(|\xi-\eta|^\sigma+|\eta|^\sigma\right).\end{aligned}$$

Proof of Commutator Type A (ctd.)

Suppose $\hat{h} \subset Ann(2^{j-1}, 2^{j+1})$. Then

$$\langle [G_\alpha^\lambda \Lambda^{\sigma+\rho} D\triangle_j, g]f, h\rangle = \iint_{\substack{\xi \sim 2^j \\ \eta \in \mathbb{R}^2}} m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\overline{\hat{h}(\xi)}d\eta d\xi.$$

Also

$$\begin{aligned} m_3(\xi,\eta) &= |e^{\lambda|\xi|^\alpha} - e^{\lambda|\xi-\eta|^\alpha}||\xi-\eta|^{\sigma+\rho+1}\phi_j(\xi-\eta) \\ &= \underbrace{m_3(\xi,\eta)\mathbb{1}_{\mathcal{A}_j}(\xi)\mathbb{1}_{\mathcal{A}_j}(\xi-\eta)\mathbb{1}_{\mathcal{B}_{j-3}}(\eta)}_{m_3^a(\xi,\eta)} + \underbrace{m_3(\xi,\eta)\mathbb{1}_{\mathcal{A}_j}(\xi)\mathbb{1}_{\mathcal{A}_j}(\xi-\eta)\mathbb{1}_{\mathcal{A}_{j-3,j+2}}(\eta)}_{m_3^b(\xi,\eta)}. \end{aligned}$$

Then by mean value theorem

$$m_3^a(\xi,\eta) \leq C\lambda|\eta|2^{(\alpha-1+\rho)j}e^{\lambda|\xi-\eta|^\alpha}\left(\int_0^1 e^{\lambda\tau^\alpha|\eta|^\alpha}d\tau\right)|\xi-\eta|^{\sigma+1}\phi_j(\xi-\eta)\mathbb{1}_{\mathcal{A}_j}(\xi)\mathbb{1}_{\mathcal{B}_{j-3}}(\eta)$$

$$m_3^b(\xi,\eta) \leq C2^{\rho j}e^{\lambda|\xi-\eta|^\alpha}e^{\lambda|\eta|^\alpha}|\eta||\xi-\eta|^\sigma\phi_j(\xi-\eta)\mathbb{1}_{\mathcal{A}_j}(\xi)\mathbb{1}_{\mathcal{A}_{j-3,j+2}}(\eta).$$

Remarks

1. Distinction between strongly quasilinear regime NOT detected in formal analysis!
2. L^p-based Besov: $p < \infty$, $\kappa \in (0, 1)$, $\beta = 1$

   Established by Chen, Miao, Zhang (2005) and Biswas, M, Silva (2015)
3. Endpoint Besov: $p = \infty$, $\kappa = 1$, $\beta = 1$

   Resolved by T. Iwabuchi (2020)
4. “Optimal Gevrey": $\alpha = \kappa$, $\beta = 1 \rightsquigarrow \lambda(t) = \lambda t^{\alpha/\kappa} = \lambda t$

   Resolved by D. Li (2021)

Other results

Inviscid case ($\gamma = 0$)

Consider following *modification* to gSQG:

$$\partial_t \theta + m(D)\theta = -u \cdot \nabla \theta, \quad u = -\nabla^{\perp} p(D)\theta,$$

where $\mathcal{F}(m(D)f) = m(\xi)\mathcal{F}f(\xi)$. For example

$$m(D) = \Lambda^{\kappa}, \quad p(D) = \Lambda^{\beta-2} \rightsquigarrow \text{dissipative gSQG equation}$$

Ill-posedness in critical spaces for inviscid equation ($m(D) \equiv 0$)

- Bourgain, Li (2014), $\beta = 0$ (Euler), H^1
- Elgindi, Masmoudi (2019), $\beta = 1$ (SQG) and several others, C^1
- Kwon (2020), $\beta = 0$ (logarithmically regularized Euler), H^1
- Cordoba, Zoroa-Martinez & Jeong, Kim (2021), $0 < \beta \leq 1$ (gSQG), $H^{\beta+1}$

Recover well-posedness with mild (inviscid) regularizations

- Chae, Wu (2012) $0 < \beta \leq 1$ (gSQG), $p(D) = \log(e - \Delta)^{\mu}$, $H^{\beta+1}$

Open: $1 < \beta < 2$ and dissipative reg. weaker than $m(D) = \Lambda^{\kappa}$

Local well-posed. mild inviscid regularization of gSQG

Consider

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = -\nabla^{\perp} p(D)\theta, \quad m(D) \equiv 0.$$

where $p(D) = \log(e - \Delta)^{\mu}$

Theorem (Jolly, Kumar, M 2021)

Let $\theta_0 \in H^{\sigma_c}(\mathbb{R}^2)$, where $\sigma_c := 1 + \beta$. For $\mu < -1/2$, there exists $T^* = T^*(\theta_0)$ such that a unique solution θ satisfies

$$\sup_{0<t<T^*} \|\theta(\,\cdot\,, t)\|_{\dot{H}^{\sigma_c}} < \infty.$$

Moreover, if $\gamma \geq 2\mu + 1$, then

$$\sup_{0<t<T^*} \|\theta(t)\|_{\dot{H}^{\beta+1+\lambda t}} < \infty,$$

for some $\lambda > 0$.

Local well-posed. mild dissip. regularization of gSQG

Consider

$$\partial_t \theta + u \cdot \nabla \theta = -m(D)\theta, \quad u = -\nabla^{\perp} p(D)\theta,$$

where $m(D) = \log(1 - \Delta)^{\gamma}$, $p(D) = \log(1 - \Delta)^{\mu \geq 0}$ and $p(D) = \log(e - \Delta)^{\mu < 0}$.

Theorem (Jolly, Kumar, M 2022)

Let $\theta_0 \in H^{\sigma_c}(\mathbb{R}^2)$, where $\sigma_c := 1 + \beta$. For $\gamma > \mu + 1/2$ and $\mu \geq -1/2$, there exists $T^* = T^*(\theta_0)$ such that a unique solution θ satisfies

$$\sup_{0<t<T^*} \|\theta(\,\cdot\,, t)\|_{\dot{H}^{\sigma_c}} < \infty.$$

Moreover, if $\gamma \geq 2\mu + 1$, then

$$\sup_{0<t<T^*} \|\theta(t)\|_{\dot{H}^{\beta+1+\lambda t}} < \infty,$$

for some $\lambda > 0$. Lastly, if $\gamma = 1$ and $\beta = \mu = 0$ (Euler with mild diss. reg.), then the solution is global in time.

Current and Future Directions

- extend to L^p

  as in Biswas, M, Silva (2015)

- complete existing critical-space program of NSE for gSQG

  non-endpoint well-posedness of NSE resolved by Farwig, Giga, Hsu (2019)

- refined general theorem in spirit of Ferrari & Titi and Bae & Biswas for parabolic systems?

Thank you!