# RC-positivity and the generalized energy density I: Rigidity

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Abstract. In this paper, we introduce a new energy density function $\mathscr{Y}$ on the projective bundle $\mathbb{P}(T_M) \to M$ for a smooth map $f : (M, h) \to (N, g)$ between Riemannian manifolds

$$\mathscr{Y} = g_{ij} f^i_\alpha f^j_\beta \frac{W^\alpha W^\beta}{\sum h_{\gamma\delta} W^\gamma W^\delta}.$$

We get new Hessian estimates to this energy density and obtain various new Liouville type theorems for holomorphic maps, harmonic maps and pluri-harmonic maps. For instance, we show that there is no non-constant holomorphic map from a compact *Hermitian manifold* with positive (resp. non-negative) holomorphic sectional curvature to a *Hermitian manifold* with non-positive (resp. negative) holomorphic sectional curvature.

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## 1. Introduction

Let $f : (M, h) \to (N, g)$ be a smooth map between two Riemannian manifolds. In local coordinates $\{y^\alpha\}$ and $\{x^i\}$ on $M$ and $N$ respectively, there is an energy density function $e$ on $M$

$$e = |df|^2 = g_{ij} h^{\alpha\beta} f^i_\alpha f^j_\beta.$$

Many milestone works are achieved in the last century by using various techniques in differential geometry and function theory in analysis, and thousands of mathematicians contributed significantly in this rich field. There is a huge literature on the subject, and we refer to the classical works [Boc55, ES64, Yau75, Yau78, Siu80, Yau82, JY93, MSY93, EL78, EL83, EL88, Xin96] and the references therein.

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In this paper, we introduce a new energy density function $\mathscr{Y}$ on the projective bundle $\mathbb{P}(T_M) \to M$ for the smooth map $f : (M, h) \to (N, g)$

$$\mathscr{Y} = g_{ij} f^i_\alpha f^j_\beta \frac{W^\alpha W^\beta}{\sum h_{\gamma\delta} W^\gamma W^\delta}, \tag{1.1}$$

which is motivated by the Leray-Grothendieck spectral sequence for abstract vector bundles used in our previous paper [Yang18a]. We obtain several new Hessian estimates on this energy density (e.g. formulas (1.3), (1.7), (1.9), (1.11)). The key new ingredient is that these new Hessian estimates can work for manifolds with partially positive curvature tensors (e.g. the holomorphic sectional curvature, or more generally, the RC-positivity for abstract vector bundles introduced in [Yang18]). In this paper, we only deal with applications when $M$ is compact.

**Part I. The generalized energy density and rigidity of holomorphic maps.** Let $f : (M, h) \to (N, g)$ be a holomorphic map between two Hermitian manifolds. Let $\{z^\alpha\}_{\alpha=1}^m$ and $\{\eta^i\}_{i=1}^n$ be the local holomorphic coordinates around $p \in M$ and $q = f(p) \in N$ respectively. We consider the generalized energy density

$$\mathscr{Y} = g_{i\bar{j}} f^i_\alpha \overline{f}^j_\beta \frac{W^\alpha \overline{W}^\beta}{\sum h_{\gamma\bar{\delta}} W^\gamma \overline{W}^\delta} \tag{1.2}$$

over the projective bundle $\mathbb{P}(T_M) \to M$ where $\{W^1, \cdots, W^m\}$ are the holomorphic coordinates on the fiber $T_pM$ with respect to the given trivialization. It is easy to see that $\mathscr{Y}$ is a well-defined function on $\mathbb{P}(T_M)$. For simplicity, we set $\mathscr{H} = \sum h_{\gamma\bar{\delta}} W^\gamma \overline{W}^\delta$. It is well-known that $\mathscr{H}^{-1}$ is a Hermitian metric on the tautological line bundle $\mathscr{O}_{T^*_M}(1)$ of the projective bundle $\mathbb{P}(T_M) \to M$ ([Gri65, Lemma 9.1]). The complex Hessian of the new energy density has the following estimate.

**Theorem 1.1.** *Let $f : (M, h) \to (N, g)$ be a holomorphic map between two Hermitian manifolds. We have the following inequality on the projective bundle $\mathbb{P}(T_M) \to M$,*

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y} \geq \left(\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}\right) \cdot \mathscr{Y} - \frac{\sqrt{-1}R_{i\bar{j}k\bar{\ell}} f^i_\alpha \overline{f^j_\beta} f^k_\mu \overline{f^\ell_\nu} W^\mu \overline{W}^\nu dz^\alpha \wedge d\bar{z}^\beta}{\mathscr{H}}. \tag{1.3}$$

*In particular, if $f : M \to \mathbb{C}$ is a holomorphic function, then*

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y} \geq \left(\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}\right) \cdot \mathscr{Y}. \tag{1.4}$$

As applications of Theorem 1.1, we obtain several new rigidity theorems.

**Theorem 1.2.** *Let $f : (M, h) \to (N, g)$ be a holomorphic map between two Hermitian manifolds. Suppose $M$ is compact. If*

(1) *$(M, h)$ has positive (resp. non-negative) holomorphic sectional curvature;*

(2) *$(N, g)$ has non-positive (resp. negative) holomorphic sectional curvature,*

*then $f$ is a constant map.*

Let's recall some classical results on rigidity of holomorphic maps. The classical Schwarz-Pick lemma states that any holomorphic map from the unit disc in the complex plane into itself decreases the Poincaré metric. This was extended by Ahlfors ([Ahl38]) to maps from the disc into a hyperbolic Riemann surface, and by Chern [Che68] and Lu [Lu68] to higher-dimensional manifolds. A major advance was Yau's Schwarz Lemma [Yau78], which says that a holomorphic map from a complete Kähler manifold with *Ricci curvature* bounded below into a Hermitian manifold with holomorphic bisectional curvature bounded above by a negative constant, is distance decreasing up to a constant depending only on these bounds. In particular, there is no nontrivial holomorphic map from compact Kähler manifolds with positive Ricci curvature to Hermitian manifolds with non-positive holomorphic bisectional curvature. By using the well-known "Royden's trick" for Kähler metrics, Royden was able to improve Yau's result and show that there is no nontrivial holomorphic map from compact Kähler manifolds with positive Ricci curvature to *Kähler manifolds* with non-positive holomorphic sectional curvature ([Roy80]). Later generalizations were mainly in two directions: relaxing the curvature condition or the Kähler assumption. A general philosophy is that holomorphic maps from "positively curved" complex manifolds to "non-positively curved" complex manifolds should be constant.

We confirmed in [Yang18, Theorem 1.7] a well-known problem of Yau ([Yau82, Problem 47]) that if a compact Kähler manifold $M$ has positive holomorphic sectional curvature, then it is projective and rationally connected, i.e. any two points in $M$ can be connected by a rational curve. On the other hand, if $N$ is Brody hyperbolic, it has no rational curves. Hence, there is no non-constant holomorphic maps from compact Kähler manifolds $M$ with positive holomorphic sectional curvature to Brody hyperbolic manifolds (e.g. Hermitian manifold $N$ with non-positive holomorphic sectional curvature). Recently, Lei Ni also obtained a rigidity theorem in [Ni18] when $M$ and $N$ are both complete *Kähler manifolds* and one of the key ingredients is the Royden's trick ([Roy80]) for Kähler metrics. As it is shown in Lemma 1.1 and Theorem 1.2, the new method in this paper has the key advantage that they can work for Hermitian metrics on both $M$ and $N$, and as it is well-known, the Royden's trick does not work on such manifolds.

Theorem 1.2 can *not* be proved by using algebraic methods developed in [Yang18a]. Indeed, when $(N, g)$ has non-positive holomorphic sectional curvature, the pullback bundle $f^*(T_N^*)$ has no desired positivity as an abstract bundle. On the other hand, if $(N, g)$ has negative holomorphic sectional curvature, then $(T_N, g)$ is RC-negative. However, it is obvious that the RC-negativity is not preserved under the pull-back of $f$, unless $f$ is a holomorphic submersion. As it is shown in [Yang18, Corollary 3.8], Kähler manifolds with negative first Chern classes have RC-negative tangent bundles and some of them can contain rational curves, for instance, the quintic surface in $\mathbb{P}^3$.

As another application of the Hessian estimate in Theorem 1.1, we obtain:

**Corollary 1.3.** *Let $f : M \to N$ be a holomorphic map from a compact complex manifold $M$ to a complex manifold $N$. If $\mathscr{O}_{T^*_M}(-1)$ is RC-positive, and $N$ has non-positive holomorphic bisectional curvature, then $f$ is a constant map.*

Let's explain the curvature condition on $M$ briefly. A holomorphic line bundle $\mathscr{L}$ over a complex manifold $X$ is called *RC-positive* if it admits a smooth Hermitian metric $h^{\mathscr{L}}$ such that its Chern curvature tensor $-\sqrt{-1}\partial\overline{\partial}\log h^{\mathscr{L}}$ has at least one positive eigenvalue everywhere. The RC-positivity of $\mathscr{O}_{T^*_M}(-1)$ is a very weak curvature condition. Indeed, we proved in [Yang17, Theorem 1.4] that $\mathscr{O}_{T^*_M}(-1)$ is RC-positive if and only if $\mathscr{O}_{T^*_M}(1)$ is not pseudo-effective. When restricted to the fibers of $\mathbb{P}(T_M)$, one can deduce $\mathscr{O}_{T^*_M}(-1)|_F \cong \mathscr{O}_{\mathbb{P}^{m-1}}(-1)$ is negative. *Roughly speaking*, the RC-positivity of $\mathscr{O}_{T^*_M}(-1)$ over the projective bundle $\mathbb{P}(T_M)$ means that $\mathscr{O}_{T^*_M}(-1)$ has at least one "positive direction" along the base $M$ directions. Moreover, complex manifolds $M$ with RC-positive tangent bundles have RC-positive $\mathscr{O}_{T^*_M}(-1)$ ([Yang18a]). Recall that, a Hermitian holomorphic vector bundle $(\mathscr{E}, h^{\mathscr{E}})$ over a complex manifold $X$ is called *RC-positive*, if for any $q \in X$ and any nonzero vector $v \in \mathscr{E}_q$, there exists **some** nonzero vector $u \in T_qX$ such that

$$R^{\mathscr{E}}(u, \overline{u}, v, \overline{v}) > 0. \tag{1.5}$$

There are many Kähler and non-Kähler complex manifolds with RC-positive tangent bundles and we list some of them for readers' convenience and for more details, we refer to [Yang18, Yang18a, Yang18b] and the references therein.

- complex manifolds with positive holomorphic sectional curvature;
- Fano manifolds [Yang18, Corollary 3.8];
- manifolds with positive second Chern-Ricci curvature [Yang18, Corollary 3.7];
- Hopf manifolds $\mathbb{S}^1 \times \mathbb{S}^{2n+1}$ ([LY14, formula (6.4)]);
- products of complex manifolds with RC-positive tangent bundles.

We need to point out that Corollary 1.3 can also be deduced from [Yang18a] (see also Corollary 1.6).

We can define some other energy densities for $f : (M, h) \to (N, g)$. For instance, on the projective bundle $\mathbb{P}(f^*T^*_N) \xrightarrow{\pi_1} M$, we have the energy density

$$\mathscr{Y}_1 = h^{\alpha\overline{\beta}} f^i_\alpha \overline{f}^j_\beta \frac{X_i\overline{X}_j}{\sum g^{k\overline{\ell}} X_k \overline{X}_\ell} \tag{1.6}$$

where $\{X_i\}_{i=1}^n$ are the holomorphic coordinates on the fiber $f^*T^*_qN$. For simplicity, we set $\mathscr{H}_1 = \sum g^{k\overline{\ell}} X_k \overline{X}_\ell$.

**Proposition 1.4.** *Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. Then we have the following estimate over $\mathbb{P}(f^*T_N^*)\stackrel{\pi_1}{\to}M$*

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y}_1\geq\left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_1^{-1}\right)\cdot\mathscr{Y}_1+\frac{\sqrt{-1}R_{\alpha\overline{\beta}\gamma\overline{\delta}}h^{\gamma\overline{\nu}}h^{\mu\overline{\delta}}f_\mu^k\overline{f_\nu^\ell}X_k\overline{X}_\ell dz^\alpha\wedge d\overline{z}^\beta}{\mathscr{H}_1}. \tag{1.7}$$

Similarly, we can define

$$\mathscr{Y}_2=f_\alpha^i\overline{f}_\beta^j\frac{X_i\overline{X}_j}{\sum g^{k\overline{\ell}}X_k\overline{X}_\ell}\cdot\frac{W^\alpha\overline{W}^\beta}{\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta} \tag{1.8}$$

over the projective bundle $\mathbb{P}(\pi^*f^*T_N^*)\to\mathbb{P}(T_M)$ where $\pi:\mathbb{P}(T_M)\to M$ is the natural projection. Recall that $\mathscr{H}=\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta$ and $\mathscr{H}_1=\sum g^{k\overline{\ell}}X_k\overline{X}_\ell$.

**Proposition 1.5.** *Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. Then we have the following inequality over $\mathbb{P}(\pi^*f^*T_N^*)\to\mathbb{P}(T_M)$*

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y}_2\geq\left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}^{-1}+\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_1^{-1}\right)\cdot\mathscr{Y}_2. \tag{1.9}$$

As applications of *generalized versions* of Theorem 1.1, Proposition 1.4 or Proposition 1.5, and some deep approximations established by Demailly-Peternell-Schneider ([DPS94]), we conclude:

**Corollary 1.6.** *Let $f:M\to N$ be a holomorphic map between compact complex manifolds. If $\mathscr{O}_{T_M^*}(-1)$ is RC-positive and $\mathscr{O}_{T_N^*}(1)$ is nef, then $f$ is a constant map.*

Note that $\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_1^{-1}$ is the curvature tensor of the line bundle $\left(\mathscr{O}_{f^*(T_N)}(-1),\mathscr{H}_1\right)$. When $\dim_{\mathbb{C}}N\geq2$, $f^*\left(\mathscr{O}_{T_N^*}(1)\right)$ and $\mathscr{O}_{f^*T_N}(-1)$ are not isomorphic. Actually, the restriction of $\mathscr{O}_{f^*(T_N)}(-1)$ on each fiber $\mathbb{P}(f^*T_q^*N)\cong\mathbb{P}^{n-1}$ is isomorphic to $\mathscr{O}_{\mathbb{P}^{n-1}}(-1)$. Hence, $\mathscr{O}_{f^*(T_N)}(-1)$ can not be nef in any case. It is easy to see that if $(N,h)$ has non-positive holomorphic bisectional curvature, then $\mathscr{O}_{f^*(T_N)}(-1)$ is RC-nonnegative along the base $M$ directions. As analogous to the proof of Corollary 1.3, it is not hard to see that we can prove Corollary 1.6 by using the generalized version of inequality (1.3) or (1.9) for *some other new energy density* since the desired asymptotic metrics constructed in [DPS94, Theorem 1.12] are on the vector bundle $\mathrm{Sym}^{\otimes k}T_N^*$. It worths to point out that, Corollary 1.6 was firstly established in [Yang18a, Theorem 1.1] by using the Leray-Grothendieck spectral sequences and isomorphisms of various cohomology groups for *abstract vector bundles*. The algebraic proof in [Yang18a] is much more effective in this setting!

In the same spirit of Theorem 1.2 and Corollary 1.6, we have:

**Corollary 1.7.** *Let $f:M\to N$ be a holomorphic map between complex manifolds. Suppose $M$ is compact. If*

(1) $\mathcal{O}_{T_M}(1)$ *is nef;*

(2) $N$ *has a Hermitian metric with negative holomorphic sectional curvature,*

*then* $f$ *is a constant map.*

Recently, there are many important works on holomorphic sectional curvature with various positivity in the Kähler setting, and we refer to [HLW10, HW12, HW12, HLW14, HW15, ACH15, Liu16, WY16, WY16a, Nom16, Yang16, YZ16, YbZ16, AHZ16, CH17, Yang18, Yang18a, Yang18b, NZ18a, NZ18b, Ni18, Mat18a, Mat18b, Gue18, Zha18] and the references therein. The results in this paper demonstrate certain similarity between Hermitian metrics and Kähler metrics with such curvature positivity. It might be an interesting problem to ask whether all compact Hermitian manifolds with positive holomorphic sectional curvature are Kähler or projective.

**Part II. Rigidity of harmonic maps and pluri-harmonic maps.** Let $f:(M,h)\to(N,g)$ be a smooth map from a Hermitian manifold to a Riemannian manifold. Let $\{z^\alpha\}_{\alpha=1}^m$ and $\{x^i\}_{i=1}^n$ be the local holomorphic coordinates and real coordinates on $M$ and $N$ respectively. We consider the generalized energy density

$$\mathscr{Y}=g_{ij}f^i_\alpha\overline{f}^j_\beta\frac{W^\alpha\overline{W}^\beta}{\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta} \tag{1.10}$$

over the projective bundle $\mathbb{P}(T_M)$, where $(W^1,\cdots,W^m)$ are the holomorphic coordinates on the fiber $T_pM$. We set $\mathscr{H}=h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta$.

It is well-known that pluri-harmonic maps are generalizations of holomorphic maps and harmonic maps. Indeed, a smooth map $f$ from a complex manifold $M$ to a Kähler (or Riemannian) manifold $N$ is *pluri-harmonic* if and only if for any holomorphic curve $i\colon C\to M$, the composition $f\circ i$ is harmonic. On the other hand, $\pm$ holomorphic maps between Kähler manifolds are pluri-harmonic. As analogous to Theorem 1.1, we obtain:

**Theorem 1.8.** *If* $f:(M,h)\to(N,g)$ *is a pluri-harmonic harmonic map, then we have the following inequality on* $\mathbb{P}(T_M)$

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y}\geq\left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}^{-1}\right)\cdot\mathscr{Y}-\frac{\sqrt{-1}R_{i\ell kj}f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}W^\gamma\overline{W}^\delta dz^\alpha\wedge d\overline{z}^\beta}{\mathscr{H}}. \tag{1.11}$$

As applications of Theorem 1.8, we study the geometry of compact Kähler manifolds with *negative/non-positive Riemannian sectional curvature* by using harmonic maps and pluri-harmonic maps into such manifolds. The motivation is the following conjecture proposed by S.-T. Yau ([Yau82, Problem 37]) :

**Conjecture 1.9.** *Let* $(X,\omega)$ *be a compact Kähler manifold with* $\dim_{\mathbb{C}}X>1$. *Suppose* $(X,\omega)$ *has negative Riemannian sectional curvature, then* $X$ *is rigid, i.e.* $X$ *has only one complex structure.*

It is a fundamental problem on the rigidity of Kähler manifolds with negative curvature. S.-T. Yau proved in [Yau77, Theorem 6] that when $X$ is covered by a 2-ball, then any complex surface oriented homotopic to $X$ must be biholomorphic to $X$. By using the terminology of "strongly negativity", Y.-T. Siu established in [Siu80, Theorem 2] that a compact Kähler manifold of the same homotopy type as a compact Kähler manifold $(X, \omega)$ with strongly negative curvature and $\dim_{\mathbb{C}} X > 1$ must be either biholomorphic or conjugate biholomorphic to $X$. It is well-known that the strongly negative curvature condition can imply the negativity of the Riemannian sectional curvature. Hence, Conjecture 1.9 holds under that stronger curvature condition. Note that when $\dim_{\mathbb{C}} X = 2$, Conjecture 1.9 has been completely solved in [Zhe95] by F.-Y. Zheng. When $\dim_{\mathbb{C}} X \geq 3$, Conjecture 1.9 is still widely open since there is no effective method to deal with the Riemannian sectional curvature on complex manifolds.

Before stating the applications of Theorem 1.8, we recall the strategy in establishing Siu's strong rigidity mentioned above. Let $f : (M, h) \to (N, g)$ be a smooth map between two compact Kähler manifolds.

(A) Suppose $(N, g)$ has strongly non-positive curvature in the sense of Siu. If $f$ is a harmonic map, then $f$ is pluri-harmonic (see Lemma 4.8);

(B) Suppose $(N, g)$ has strongly negative curvature in the sense of Siu. Then any pluri-harmonic map $f : M \to (N, g)$ is holomorphic or anti-holomorphic provided $\operatorname{rank}_{\mathbb{R}} df \geq 4$ (see Remark 4.5).

As inspired by Yau's conjecture 1.9 and Siu's strategy in steps (A) and (B), we attempt to investigate Riemannian (or Kähler) manifolds $(N, g)$ with non-positive Riemannian sectional curvature. As an application of Theorem 1.8, we obtain:

**Theorem 1.10.** *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold $M$ to a Riemannian manifold $(N, g)$ with* non-positive Riemannian sectional curvature. *If $\mathcal{O}_{T^*_M}(-1)$ is RC-positive, then $f$ is a constant map.*

Theorem 1.10 still holds when the target manifold is Kähler:

**Theorem 1.11.** *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold $M$ to a Kähler manifold $(N, g)$ with* non-positive Riemannian sectional curvature. *If $\mathcal{O}_{T^*_M}(-1)$ is RC-positive, then $f$ is a constant map.*

As we pointed out before, there are many Kähler and non-Kähler manifolds with RC-positive $\mathcal{O}_{T^*_M}(-1)$. In particular, we get:

**Corollary 1.12.** *Let $f : M \to (N, g)$ be a pluri-harmonic map from a compact complex manifold $M$ to a Kähler or Riemannian manifold $(N, g)$ with* non-positive Riemannian sectional curvature. *If $M$ has a Hermitian metric with positive holomorphic sectional curvature, then $f$ is a constant map.*

As an application of Theorem 1.10 and ideas in [Siu80, Sam85, Sam86, YZ91, JY91, JY93, LY14, WY18], we consider harmonic maps from complex manifolds into Riemannian manifolds.

**Theorem 1.13.** *Let $f:(M,h)\to(N,g)$ be a harmonic map from a compact Kähler manifold $(M,h)$ to a Riemannian manifold $(N,g)$ with* non-positive complex sectional curvature. *If $\mathcal{O}_{T_M^*}(-1)$ is RC-positive, then $f$ is a constant map.*

Theorem 1.13 still holds if $f$ is a Hermitian harmonic map from an astheno-Kähler manifold $(M,h)$ (i.e. $\partial\overline{\partial}\omega_h^{m-2}=0$) (introduced by Jost-Yau in [JY93]) to a Riemannian manifold $(N,g)$ with non-positive complex sectional curvature. In [WY18], we obtained some results by using the classical Chern-Lu type inequality (2.16) under a much stronger condition that $T_M$ is uniformly RC-positive.

**Remark 1.14.** As analogous to the classical theory of harmonic maps, there are many further applications of the generalized energy density (1.2). They are discussed briefly in Section 5. For instances,

- the RC-positivity for Riemannian curvature tensor;
- the extension of Yau's function theory on complete manifolds [Yau75, Yau78];
- the first and second variations of the generalized energy functions, and the applications in investigating the existence of rational curves on manifolds with RC-positive curvature by analytical methods ([SU81, SY80]);
- the analytical extension of methods in this paper to hyperbolic manifolds;
- the generalized energy density on Grassmannian manifolds $\mathrm{Gr}(k,T_M)$ for RC-positivity in $k$ linearly independent directions.

The Ricci-flow and Kähler-Ricci flow approaches ([Ham82, Cao85]) in this setting and the applications of parabolic estimates corresponding to formulas (1.3), (1.7), (1.9) and (1.11) are also expectable. The details of some topics listed above will appear somewhere else.

This paper is organized as follows. In Section 2, we describe the relationship between the classical energy identity and the generalized energy density. In Section 3, we prove Theorem 1.1, Theorem 1.2, Corollary 1.3 and Corollary 1.6. In Section 4, we investigate harmonic maps and pluri-harmonic from complex manifolds to Riemannian manifolds and Kähler manifolds, and establish Theorem 1.8, Theorem 1.10, Theorem 1.11 and Theorem 1.13.

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## 2. The classical energy density and the generalized energy density

2.1. **Holomorphic maps between complex manifolds.** Let $f:(M,h)\to(N,g)$ be a smooth map between two Hermitian manifolds. Let $\{z^\alpha\}_{\alpha=1}^m$ and $\{\eta^i\}_{i=1}^n$ be the local holomorphic coordinates around $p\in M$ and $q=f(p)\in N$ respectively. The classical $\partial$-energy density is defined as

$$u=|\partial f|^2=g_{i\bar j}h^{\alpha\bar\beta}f_\alpha^i\overline{f}_\beta^j. \tag{2.1}$$

Here $\partial f$ can be regarded as a section of the complex vector bundle $E=T_M^*\otimes f^*T_N$ and $u$ is the norm square of $\partial f$ with respect to the induced metric on $E$. As it is well-known, if $f$ is a holomorphic map, by using standard Bochner technique, one has the following Chern-Lu inequality ([Che68, Lu68], see also [Yang18a, Lemma 5.1]).

**Lemma 2.1.** *Let $f:(M,h)\to(N,g)$ be a holomorphic map between two Hermitian manifolds. Then*

$$\sqrt{-1}\partial\bar\partial u\geq\sqrt{-1}\left(R_{\alpha\bar\beta\gamma\bar\delta}h^{\mu\bar\delta}h^{\gamma\bar\nu}g_{i\bar j}f_\mu^i\overline{f_\nu^j}-R_{i\bar jk\bar\ell}f_\alpha^i\overline{f_\beta^j}\left(h^{\mu\bar\nu}f_\mu^k\overline{f_\nu^\ell}\right)\right)dz^\alpha\wedge d\bar z^\beta, \tag{2.2}$$

*and*

$$\mathrm{tr}_{\omega_h}\left(\sqrt{-1}\partial\bar\partial u\right)\geq\left(h^{\alpha\bar\beta}R_{\alpha\bar\beta\gamma\bar\delta}\right)h^{\mu\bar\delta}h^{\gamma\bar\nu}\left(g_{i\bar j}f_\mu^i\overline{f_\nu^j}\right)-R_{i\bar jk\bar\ell}\left(h^{\alpha\bar\beta}f_\alpha^i\overline{f_\beta^j}\right)\left(h^{\mu\bar\nu}f_\mu^k\overline{f_\nu^\ell}\right). \tag{2.3}$$

Formulas (2.2), (2.3) and their parabolic analogs have many fantastic applications in differential geometry and the theory of Ricci flows, and we refer to [Che68, Lu68, Yau78, Roy80, EL83, EL88, Tos07, Che09, YZ16, Ni18] and the references therein.

**Remark 2.2.** In the formula 2.2, if we choose $h_{\alpha\bar\beta}=\delta_{\alpha\bar\beta}$ at some point, then the curvature term on the domain manifold is $\sqrt{-1}R_{\alpha\bar\beta\gamma\bar\delta}g_{i\bar j}f_\gamma^i\overline{f_\delta^j}dz^\alpha\wedge d\bar z^\beta$. In the approach by using maximum principle, it is hard to get the desired positivity from this term.

Next, we introduce the generalized energy density on the projective bundle $\mathbb{P}(T_M)$. The points of the projective bundle $\mathbb{P}(T_M)$ of $T_M^*\to M$ can be identified with the hyperplanes of $T_M^*$. Note that every hyperplane $V$ in $T_p^*M$ corresponds bijectively to the line of linear forms in $T_pM$ which vanish on $V$. Let $\pi:\mathbb{P}(T_M)\to M$ be the natural projection. Suppose $(W^1,\cdots,W^m)$ are the holomorphic coordinates on the fiber of $TM$. The generalized $\partial$-energy density on the projective bundle $\mathbb{P}(T_M)\to M$ is defined as

$$\mathscr{Y}=g_{i\bar j}f_\alpha^i\overline{f}_\beta^j\frac{W^\alpha\overline{W}^\beta}{\sum h_{\gamma\bar\delta}W^\gamma\overline{W}^\delta}. \tag{2.4}$$

It is easy to easy see that (2.4) is well-defined on $\mathbb{P}(T_M)$. The classical energy density (2.1) and the generalized energy density (2.4) are related in the following way.

**Proposition 2.3.** *We have the following relation*

$$|\partial f|^2=m\pi_*(\mathscr{Y}), \tag{2.5}$$

*where $\pi_*$ is the fiberwise integration with respect to the fiberwise Fubini-Study metric $\omega = -\sqrt{-1}\partial\overline{\partial}\log(\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta)$.*

*Proof.* Indeed,

$$\pi_*(\mathscr{Y}) = \int_{\mathbb{P}(T_pM)} g_{i\overline{j}}f^i_\alpha\overline{f}^j_\beta\frac{W^\alpha\overline{W}^\beta}{\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta}\cdot\frac{\omega^{m-1}}{(m-1)!} = g_{i\overline{j}}f^i_\alpha\overline{f}^j_\beta\cdot\frac{h^{\alpha\overline{\beta}}}{m} = \frac{|\partial f|^2}{m},$$

where we use the well-known identity for the Fubini-Study metric $\omega_{\text{FS}}$ on $\mathbb{P}^{m-1}$

$$\int_{\mathbb{P}^{m-1}}\frac{W^\alpha\overline{W}^\beta}{|W|^2}\frac{\omega_{\text{FS}}^{m-1}}{(m-1)!} = \frac{\delta^{\alpha\overline{\beta}}}{m}. \tag{2.6}$$

□

We can also define a conformal change for the generalized energy density

$$\mathscr{Y}_\varphi = e^\varphi\mathscr{Y} = e^\varphi g_{i\overline{j}}f^i_\alpha\overline{f}^j_\beta\frac{W^\alpha\overline{W}^\beta}{\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta} \tag{2.7}$$

for any $\varphi\in C^\infty(\mathbb{P}(T_M),\mathbb{R})$. As we pointed out before, $\mathscr{H} = \sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta$ is a (local) Hermitian metric on $\mathcal{O}_{T^*_M}(-1)$, and so $\mathscr{H}e^{-\varphi}$ is also a Hermitian metric on $\mathcal{O}_{T^*_M}(-1)$. Actually, any Hermitian metric on it takes such a form.

2.2. **Pluri-harmonic maps into Riemannian manifolds.** Let $(M.h)$ be a complex Hermitian manifold, $(N,g)$ be a Riemannian manifold and $f: M\to N$ be a smooth map. We denote by $\mathscr{E} = f^*(TN)$ and endow it with the induced Levi-Civita connection $\nabla^{\mathscr{E}}$ from $TN$. There is a natural decomposition $\nabla^{\mathscr{E}} = \overline{\partial}_{\mathscr{E}} + \partial_{\mathscr{E}}$ according to the complex structure on $M$. Let $\{z^\alpha\}_{\alpha=1}^m$ be the local holomorphic coordinates on $M$ and $\{x^i\}_{i=1}^n$ the local coordinates on $N$. Let $e_i = f^*(\frac{\partial}{\partial x^i})$. There are three $\mathscr{E}$-valued 1-forms, i.e.

$$\overline{\partial}f = \frac{\partial f^i}{\partial\overline{z}^\beta}d\overline{z}^\beta\otimes e_i,\quad \partial f = \frac{\partial f^i}{\partial z^\alpha}dz^\alpha\otimes e_i,\quad df = \overline{\partial}f + \partial f. \tag{2.8}$$

It is easy to see that

$$u = |\partial f|^2 = g_{ij}h^{\alpha\overline{\beta}}f^i_\alpha\overline{f}^j_\beta,\quad\text{and}\quad u = \frac{1}{2}|df|^2 = |\overline{\partial}f|^2 = |\partial f|^2. \tag{2.9}$$

$f$ is called a *harmonic map* if it is a critical point of the Euler-Lagrange equation of the total energy $E(f) = \int_M u dV_M$.

We consider the generalized energy density

$$\mathscr{Y} = g_{ij}f^i_\alpha\overline{f}^j_\beta\frac{W^\alpha\overline{W}^\beta}{\sum h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta} \tag{2.10}$$

over the projective bundle $\mathbb{P}(T_M)$, where $(W^1, \cdots, W^m)$ are the holomorphic coordinates on the fiber $T_pM$. We set $\mathscr{H} = h_{\gamma\bar{\delta}} W^\gamma \overline{W}^\delta$. Similarly, we have

$$|df|^2 = 2m\pi_*(\mathscr{Y}). \tag{2.11}$$

**Definition 2.4.** A smooth map $f : M \to (N, g)$ from a complex manifold to a Riemannian manifold is called *pluri-harmonic* if it satisfies $\partial_{\mathscr{E}} \overline{\partial} f = 0$, i.e.

$$\left( \frac{\partial^2 f^i}{\partial z^\alpha \partial \bar{z}^\beta} + \Gamma^i_{jk} \frac{\partial f^j}{\partial \bar{z}^\beta} \frac{\partial f^k}{\partial z^\alpha} \right) dz^\alpha \wedge d\bar{z}^\beta \otimes e_i = 0 \tag{2.12}$$

where $\Gamma^i_{jk}$ is the Christoffel symbol of the Levi-Civita connection on $(N, g)$.

It is easy to see that $\pm$ holomorphic maps from complex manifolds to Kähler manifolds are pluri-harmonic.

Let $(N, g)$ be a Riemannian manifold with the Levi-Civita connection $\nabla$. Its curvature tensor is defined as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any $X, Y, Z \in \Gamma(N, TN)$. In the local coordinates $\{x^i\}$ of $N$, we adopt the convention

$$R(X, Y, Z, W) = g(R(X, Y)Z, W) = R_{ijk\ell} X^i Y^j Z^k W^\ell.$$

It is easy to see that

$$R^\ell_{ijk} = \frac{\partial \Gamma^\ell_{kj}}{\partial x^i} - \frac{\partial \Gamma^\ell_{ki}}{\partial x^j} + \Gamma^p_{kj} \Gamma^\ell_{pi} - \Gamma^p_{ki} \Gamma^\ell_{pj}, \quad \text{and} \quad R_{ijk\ell} = g_{s\ell} R^s_{ijk}. \tag{2.13}$$

The following constraint equation for pluri-harmonic maps is well-known (e.g. [OU91, Lemma 1.3] and [WY18, Lemma 2.4]), and it follows by taking first order derivatives on the pluri-harmonic equation (2.12).

**Lemma 2.5.** *If $f : M \to (N, h)$ is a pluri-harmonic map, then*

$$R_{ikj\ell} f^i_\alpha f^j_{\bar{\beta}} f^k_\gamma = 0 \tag{2.14}$$

*for any $\alpha, \beta, \gamma$ and $\ell$ where $f^i_\alpha = \frac{\partial f^i}{\partial z^\alpha}$ and $f^j_{\bar{\beta}} = \frac{\partial f^j}{\partial \bar{z}^\beta}$.*

**Remark 2.6.** By using the constraint equation (2.14), we have

$$\widehat{C} = R_{ik\ell j} \left( h^{\alpha\bar{\beta}} f^i_\alpha f^j_{\bar{\beta}} \right) \left( h^{\gamma\bar{\delta}} f^k_\gamma f^\ell_{\bar{\delta}} \right) = 0. \tag{2.15}$$

If $(N, g)$ has positive or negative constant Riemannian sectional curvature, one can deduce from (2.15) that $\operatorname{rank}_{\mathbb{R}} df \leq 2$ ([Sam85, Sam86]).

As analogous to the Chern-Lu inequalities for holomorphic maps in Lemma 2.1, Wang and the author obtained in [WY18, Proposition 3.2] the following inequalities for pluri-harmonic maps.

**Lemma 2.7.** *Let $f:(M,h)\to(N,g)$ be a pluri-harmonic map from a Hermitian manifold $M$ to a Riemannian manifold $(N,g)$. Then we have*

$$\sqrt{-1}\partial\overline{\partial}u \geq \sqrt{-1}\left(R_{\alpha\overline{\beta}\gamma\overline{\delta}}h^{\gamma\overline{\nu}}h^{\mu\overline{\delta}}g_{ij}f^i_{\mu}f^j_{\overline{\nu}} - R_{i\ell kj}f^i_{\alpha}f^j_{\overline{\beta}}\left(h^{\gamma\overline{\delta}}f^k_{\gamma}f^{\ell}_{\overline{\delta}}\right)\right)dz^{\alpha}\wedge d\overline{z}^{\beta}. \tag{2.16}$$

*and*

$$\mathrm{tr}_{\omega_h}\left(\sqrt{-1}\partial\overline{\partial}u\right) \geq \left(h^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)h^{\mu\overline{\delta}}h^{\gamma\overline{\nu}}\left(g_{ij}f^i_{\mu}f^j_{\overline{\nu}}\right) - R_{i\ell kj}\left(h^{\alpha\overline{\beta}}f^i_{\alpha}f^j_{\overline{\beta}}\right)\left(h^{\mu\overline{\nu}}f^k_{\mu}f^{\ell}_{\overline{\nu}}\right). \tag{2.17}$$

**Remark 2.8.** The formulations of curvature terms in Lemma 2.7 and Lemma 2.1 on the target manifolds are different.

For more detailed discussions on various harmonic maps and pluri-harmonic maps, we refer to [EL78, Siu82, EL83, Ohn87, EL88, Uda88, CT89, OU90, OV90, YZ91, MSY93, Uda94, JZ97, Lou99, Ni99, DEP03, Tos07, Zha12, YHD13, Dong13, LY14, Sch15, YZ16, Yang18a, Yang18b, Ni18, WY18] and the references therein.

## 3. The Hessian estimates and rigidity of holomorphic maps

In this section, we prove Theorem 1.1, Theorem 1.2, Corollary 1.3 and Corollary 1.6. The proofs of Proposition 1.4 and Proposition 1.5 are similar to that in Theorem 1.1. To avoid identifications in algebraic geometry, we shall use straightforward local computations for readers' convenience.

*The proof of Theorem 1.1.* For simplicity, we write $\mathscr{F}=\mathscr{Y}\cdot\mathscr{H}$. A straightforward calculation on $\mathbb{P}(T_M)$ yields

$$\partial\overline{\partial}\mathscr{Y} = \left(\partial\overline{\partial}\log\mathscr{H}^{-1}\right)\mathscr{Y} + \frac{\partial\overline{\partial}\mathscr{F}}{\mathscr{H}} + \frac{\mathscr{F}\partial\mathscr{H}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^3} - \frac{\partial\mathscr{H}\wedge\overline{\partial}\mathscr{F}+\partial\mathscr{F}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^2}. \tag{3.1}$$

Moreover, we set $F^i=f^i_{\alpha}W^{\alpha}$ and so $\mathscr{F}=g_{i\overline{j}}F^i\overline{F}^j$. Since $f$ is holomorphic, we have

$$\partial\mathscr{F} = \partial g_{i\overline{j}}\cdot F^i\overline{F^j} + g_{i\overline{j}}\cdot\partial F^i\cdot\overline{F}^j, \qquad \overline{\partial}\mathscr{F} = \overline{\partial}g_{i\overline{j}}\cdot F^i\overline{F^j} + g_{i\overline{j}}F^i\overline{\partial F^j}. \tag{3.2}$$

Therefore, we deduce

$$\begin{aligned}
&\frac{\partial\overline{\partial}\mathscr{F}}{\mathscr{H}} + \frac{\mathscr{F}\partial\mathscr{H}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^3} - \frac{\partial\mathscr{H}\wedge\overline{\partial}\mathscr{F}+\partial\mathscr{F}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^2}\\
=\;& \frac{\partial\overline{\partial}g_{i\overline{j}}\cdot F^i\overline{F^j} + \partial g_{i\overline{j}}\cdot F^i\cdot\overline{\partial F^j} - \overline{\partial}g_{i\overline{j}}\cdot\partial F^i\cdot\overline{F^j} + g_{i\overline{j}}\partial F^i\overline{\partial F^j}}{\mathscr{H}}\\
&+\frac{\mathscr{F}\partial\log\mathscr{H}^{-1}\wedge\overline{\partial}\log\mathscr{H}^{-1}}{\mathscr{H}}\\
&-\frac{\partial\log\mathscr{H}\cdot\left(\overline{\partial}g_{i\overline{j}}\cdot F^i\overline{F^j} + g_{i\overline{j}}F^i\overline{\partial F^j}\right) + \left(\partial g_{i\overline{j}}\cdot F^i\overline{F^j} + g_{i\overline{j}}\cdot\partial F^i\cdot F^j\right)\cdot\overline{\partial}\log\mathscr{H}}{\mathscr{H}}.
\end{aligned} \tag{3.3}$$

We define a $(1,1)$-form on $\mathbb{P}(T_M)$:

$$\mathscr{W}=g_{i\bar{j}}\left(\partial F^i+F^i\partial\log\mathscr{H}^{-1}+T^i\right)\wedge\overline{\left(\partial F^j+F^j\partial\log\mathscr{H}^{-1}+T^j\right)} \tag{3.4}$$

where $T^i=g^{i\bar{\ell}}\frac{\partial g_{k\bar{\ell}}}{\partial z^p}F^k\partial f^p$. It is easy to see that

$$g_{i\bar{j}}T^i=\frac{\partial g_{k\bar{j}}}{\partial z^p}F^k\partial f^p=(\partial g_{k\bar{j}})F^k. \tag{3.5}$$

By using (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned}&\frac{\partial\bar{\partial}\mathscr{F}}{\mathscr{H}}+\frac{\mathscr{F}\partial\mathscr{H}\wedge\bar{\partial}\mathscr{H}}{\mathscr{H}^3}-\frac{\partial\mathscr{H}\wedge\bar{\partial}\mathscr{F}+\partial\mathscr{F}\wedge\bar{\partial}\mathscr{H}}{\mathscr{H}^2}-\frac{\mathscr{W}}{\mathscr{H}}\\
=&\ \frac{\left(\partial\bar{\partial}g_{i\bar{j}}\right)F^i\overline{F^j}}{\mathscr{H}}-\frac{g_{i\bar{j}}T^i\wedge\overline{T^j}}{\mathscr{H}}\\
=&\ -\frac{R_{i\bar{j}k\bar{\ell}}f^i_\alpha\overline{f^j_\beta}f^k_\mu\overline{f^\ell_\nu}W^\mu\overline{W}^\nu dz^\alpha\wedge d\bar{z}^\beta}{\mathscr{H}}\end{aligned} \tag{3.6}$$

where the last identity follows from the facts that

$$\partial\bar{\partial}g_{i\bar{j}}=\frac{\partial^2 g_{i\bar{j}}}{\partial z^k\bar{z}^\ell}\partial f^k\wedge\overline{\partial f^\ell},\quad g_{i\bar{j}}T^i\overline{T^j}=g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial\bar{z}^\ell}F^i\overline{F^j}\partial f^k\wedge\overline{\partial f^\ell} \tag{3.7}$$

and the curvature formula of the Chern connection on $(N,g)$

$$R_{k\bar{\ell}i\bar{j}}=-\frac{\partial^2 g_{i\bar{j}}}{\partial z^k\partial\bar{z}^\ell}+g^{p\bar{q}}\frac{\partial g_{i\bar{q}}}{\partial z^k}\frac{\partial g_{p\bar{j}}}{\partial\bar{z}^\ell}. \tag{3.8}$$

Hence, by (3.1) and (3.6), we get

$$\sqrt{-1}\partial\bar{\partial}\mathscr{Y}=\left(\sqrt{-1}\partial\bar{\partial}\log\mathscr{H}^{-1}\right)\cdot\mathscr{Y}+\frac{\sqrt{-1}\mathscr{W}}{\mathscr{H}}-\frac{\sqrt{-1}R_{i\bar{j}k\bar{\ell}}f^i_\alpha\overline{f^j_\beta}f^k_\mu\overline{f^\ell_\nu}W^\mu\overline{W}^\nu dz^\alpha\wedge d\bar{z}^\beta}{\mathscr{H}}.$$

On the other hand, by the definition equation (3.4) of $\mathscr{W}$, it is easy to see that $\sqrt{-1}\mathscr{W}$ is a semi-positive $(1,1)$-form on $\mathbb{P}(T_M)$. Hence, we obtain Theorem 1.1. □

For the conformal energy density (2.7), we also have a similar inequality as in (1.3).

**Theorem 3.1.** *Let $f:(M,h)\to(N,g)$ be a holomorphic map. Then for any $\varphi\in C^\infty(\mathbb{P}(T_M),\mathbb{R})$, we have*

$$\sqrt{-1}\partial\bar{\partial}\mathscr{Y}_\varphi\geq\left(\sqrt{-1}\partial\bar{\partial}\log\mathscr{H}_\varphi^{-1}\right)\cdot\mathscr{Y}_\varphi-\frac{\sqrt{-1}R_{i\bar{j}k\bar{\ell}}f^i_\alpha\overline{f^j_\beta}f^k_\mu\overline{f^\ell_\nu}W^\mu\overline{W}^\nu dz^\alpha\wedge d\bar{z}^\beta}{\mathscr{H}_\varphi}, \tag{3.9}$$

*where $\mathscr{Y}_\varphi=e^\varphi\mathscr{Y}$ and $\mathscr{H}_\varphi=\mathscr{H}e^{-\varphi}$.*

*Proof.* We use the same notions as in the proof of Theorem 1.1. It is easy to see that $\mathscr{F}=\mathscr{Y}\cdot\mathscr{H}=\mathscr{Y}_\varphi\cdot\mathscr{H}_\varphi$. By defining a new quantity as in the equation (3.4)

$$\mathscr{W}_\varphi=g_{i\bar{j}}\left(\partial F^i+F^i\partial\log\mathscr{H}_\varphi^{-1}+T^i\right)\wedge\overline{\left(\partial F^j+F^j\partial\log\mathscr{H}_\varphi^{-1}+T^j\right)}, \tag{3.10}$$

one can use similar computations as in the proof of Theorem 1.1 to deduce (3.9). □

*The proof of Theorem 1.2.* We compute the curvature form $\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}$ over the projective bundle $X := \mathbb{P}(T_M)$ where $\pi : X \to M$ the natural projection, following the calculations in [Gri65, Proposition 9.2]. Indeed, $\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}$ is the curvature of the tautological line bundle $\left(\mathcal{O}_{T^*_M}(-1), \mathscr{H}\right)$. On the *Hermitian manifold* $M$, we can choose "normal coordinates" $\{z^\alpha\}_{\alpha=1}^m$ on a small open set $U \subset M$ centered at point $p \in M$ such that

$$h_{\alpha\overline{\beta}}(p) = \delta_{\alpha\overline{\beta}}, \qquad \frac{\partial h_{\alpha\overline{\beta}}}{\partial z^\gamma}(p) = -\frac{\partial h_{\gamma\overline{\beta}}}{\partial z^\alpha}(p). \tag{3.11}$$

By using this local trivialization chart $U$ on $M$, we know $T_M|_U \cong U \times \mathbb{C}^m$ and $\pi^{-1}(U) \cong U \times \mathbb{P}^{m-1}$. Let $q \in X$ such that $\pi(q) = p$. Locally, we write $p = (z_o^1, \cdots, z_o^m)$ and $q = (z_o^1, \cdots, z_o^m, [W_o^1, \cdots, W_o^m])$ where $[W^1, \cdots, W^m]$ are the homogeneous coordinates on the fiber $\mathbb{P}(T_pM) \cong \mathbb{P}^{m-1}$. Let $u = (W_o^1, \cdots, W_o^m, \underbrace{0, \cdots, 0}_{m-1}) \in T_qX$. We claim that at point $q \in X$, $u$ is a positive direction of $\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}$, i.e.

$$\left(\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}\right)(u, \overline{u}) > 0. \tag{3.12}$$

Indeed, if we set $H^{\gamma\overline{\delta}} = W^\gamma \overline{W}^\delta$, then $\mathscr{H} = h_{\alpha\overline{\beta}} H^{\alpha\overline{\beta}}$ and

$$\begin{aligned}
\partial\overline{\partial}\log \mathscr{H}^{-1} &= \frac{\partial\mathscr{H} \wedge \overline{\partial}\mathscr{H}}{\mathscr{H}^2} - \frac{\partial\overline{\partial}\mathscr{H}}{\mathscr{H}} \\
&= \frac{\left(H^{\alpha\overline{\delta}}\partial h_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\partial H^{\alpha\overline{\beta}}\right) \wedge \left(H^{\gamma\overline{\delta}}\overline{\partial} h_{\gamma\overline{\delta}} + h_{\gamma\overline{\delta}}\partial H^{\gamma\overline{\delta}}\right)}{\mathscr{H}^2} \\
&\quad - \frac{\partial\overline{\partial} h_{\gamma\overline{\delta}} \cdot H^{\gamma\overline{\delta}} + \partial h_{\gamma\overline{\delta}} \wedge \overline{\partial} H^{\gamma\overline{\delta}} + \partial H^{\gamma\overline{\delta}} \wedge \overline{\partial} h_{\gamma\overline{\delta}} + h_{\gamma\overline{\delta}}\partial\overline{\partial} H^{\gamma\overline{\delta}}}{\mathscr{H}}.
\end{aligned}$$

Note that $\partial h_{\alpha\overline{\beta}} = \frac{\partial h_{\alpha\overline{\beta}}}{\partial z^\mu} dz^\mu$ is along the base directions and $\partial H^{\gamma\overline{\delta}} = \partial\left(W^\gamma \overline{W}^\delta\right)$ is along the fiber directions. We evaluate the $(1,1)$ form $\sqrt{-1}\partial\overline{\partial}\log \mathscr{H}^{-1}$ at point $q = (z_o^1, \cdots, z_o^m, [W_o^1, \cdots, W_o^m])$ with $u = (W_o^1, \cdots, W_o^m, 0, \cdots, 0) \in T_qX$. Since

$$\left(\partial h_{\alpha\overline{\beta}} \wedge \overline{\partial} H^{\gamma\overline{\delta}}\right)(u, \overline{u}) = 0, \quad (\partial\overline{\partial} H^{\alpha\overline{\beta}})(u, \overline{u}) = 0, \quad \left(\partial H^{\alpha\overline{\beta}} \wedge \overline{\partial} H^{\gamma\overline{\delta}}\right)(u, \overline{u}) = 0,$$

we obtain

$$\begin{aligned}
&\left(\partial\overline{\partial}\log \mathscr{H}^{-1}\right)(u, \overline{u}) \\
&= \left(-\frac{H_o^{\gamma\overline{\delta}}\partial\overline{\partial} h_{\gamma\overline{\delta}}}{\mathscr{H}_o} + \frac{H_o^{\alpha\overline{\beta}} H_o^{\gamma\overline{\delta}} \partial h_{\alpha\overline{\beta}} \wedge \overline{\partial} h_{\gamma\overline{\delta}}}{\mathscr{H}_o^2}\right)(u, \overline{u}) \\
&= \left(-\frac{\partial^2 h_{\gamma\overline{\delta}}}{\partial z^\mu \partial \overline{z}^\nu} + h^{\alpha\overline{\beta}} \frac{\partial h_{\gamma\overline{\beta}}}{\partial z^\mu} \frac{\partial h_{\alpha\overline{\delta}}}{\partial \overline{z}^\nu}\right) \frac{W_o^\mu \overline{W}_o^\nu W_o^\gamma \overline{W}_o^\delta}{|W_o|^2} \qquad (3.13) \\
&= \frac{R_{\mu\overline{\nu}\gamma\overline{\delta}} W_o^\mu \overline{W}_o^\nu W_o^\gamma \overline{W}_o^\delta}{|W_o|^2}
\end{aligned}$$

where in the second identity we used the anti-symmetric property of the first order derivatives given in the normal coordinates (3.11).

Suppose $\mathscr{Y}$ is not identically zero on $X$. Let $q \in X$ be a maximum value point of $\mathscr{Y}$ and $p = \pi(q) \in M$. Hence $\mathscr{Y}(q) > 0$ and $\sqrt{-1}\partial\bar{\partial}\mathscr{Y}(q) \leq 0$. By using the previous setting, we can choose "normal coordinates" $\{z^\alpha\}_{\alpha=1}^m$ centered at point $p$ (e.g. 3.11) and we write $p = (z_o^1, \cdots, z_o^m)$ and $q = (z_o^1, \cdots, z_o^m, [W_o^1, \cdots, W_o^m])$ where $[W_o^1, \cdots, W_o^m]$ is the homogeneous coordinate on the fiber $\mathbb{P}(T_pM)$. Evaluating the inequality (1.3) on the vector $u = (W_o^1, \cdots, W_o^m, 0, \cdots, 0) \in T_qX$, we have

$$0 \geq \left(\partial\bar{\partial}\mathscr{Y}\right)(u, \bar{u}) \geq \left(\partial\bar{\partial}\log\mathscr{H}^{-1}\right)(u, \bar{u}) \cdot \mathscr{Y} - \frac{R_{i\bar{j}k\bar{\ell}} f_\alpha^i \overline{f_\beta^j} f_\mu^k \overline{f_\nu^\ell} W_o^\alpha \overline{W}_o^\beta W_o^\mu \overline{W}_o^\nu}{\mathscr{H}}. \tag{3.14}$$

If we set $F^i = f_\alpha^i W_o^\alpha$, then by (3.14) and (3.13), we obtain

$$0 \geq \frac{R_{\mu\bar{\nu}\gamma\bar{\delta}} W_o^\mu \overline{W}_o^\nu W_o^\gamma \overline{W}_o^\delta}{|W_o|^2} \cdot \mathscr{Y}(q) - \frac{R_{i\bar{j}k\bar{\ell}} F^i \overline{F}^j F^k \overline{F}^\ell}{\mathscr{H}}. \tag{3.15}$$

Note that $\mathscr{Y}(q) = \frac{g_{i\bar{j}} F^i \overline{F}^j}{\mathscr{H}} > 0$ and so $(F^i) \neq 0$. If $(M, h)$ has positive (resp. nonnegative) holomorphic sectional curvature and $(N, g)$ has non-positive (resp. negative) holomorphic sectional curvature, then the right hand side of (3.15) is positive, which is absurd. Therefore, $\mathscr{Y}$ must be identically zero on $X$ and so $f$ is a constant map. □

*The proof of Corollary 1.3.* Fix an arbitrary smooth metric $h$ on $M$ and let $\mathscr{H} = h_{\gamma\bar{\delta}} W^\gamma \overline{W}^\delta$. Suppose $\mathcal{O}_{T_M^*}(-1)$ is RC-positive, then there exists a Hermitian metric $\widetilde{\mathscr{H}}$ on $\mathcal{O}_{T_M^*}(-1)$ such that its curvature $-\sqrt{-1}\partial\bar{\partial}\log\widetilde{\mathscr{H}}$ has at least one positive eigenvalue at each point of $\mathbb{P}(T_M)$. Since $\mathscr{H}$ is also a smooth Hermitian metric on $\mathcal{O}_{T_M^*}(-1)$, there exists $\varphi \in C^\infty(\mathbb{P}(T_M), \mathbb{R})$ such that $\widetilde{\mathscr{H}} = \mathscr{H} e^{-\varphi}$ and

$$-\sqrt{-1}\partial\bar{\partial}\log\widetilde{\mathscr{H}} = -\sqrt{-1}\partial\bar{\partial}\log\mathscr{H} + \sqrt{-1}\partial\bar{\partial}\varphi. \tag{3.16}$$

We consider $\widetilde{\mathscr{Y}} = \mathscr{Y} e^\varphi$ on $\mathbb{P}(T_M)$. By Theorem 3.1, we obtain

$$\sqrt{-1}\partial\bar{\partial}\widetilde{\mathscr{Y}} \geq \left(\sqrt{-1}\partial\bar{\partial}\log\widetilde{\mathscr{H}}^{-1}\right) \cdot \widetilde{\mathscr{Y}} - \frac{\sqrt{-1} R_{i\bar{j}k\bar{\ell}} f_\alpha^i \overline{f_\beta^j} f_\mu^k \overline{f_\nu^\ell} W^\mu \overline{W}^\nu dz^\alpha \wedge d\bar{z}^\beta}{\widetilde{\mathscr{H}}}. \tag{3.17}$$

Suppose $\widetilde{\mathscr{Y}}$ is not identically zero on $\mathbb{P}(T_M)$. Let $p \in \mathbb{P}(T_M)$ be a maximum value point of $\widetilde{\mathscr{Y}}$. Hence, $\widetilde{\mathscr{Y}}(p) > 0$ and

$$\sqrt{-1}\partial\bar{\partial}\widetilde{\mathscr{Y}}(p) \leq 0. \tag{3.18}$$

On the other hand, since $(N, g)$ has non-positive holomorphic bisectional curvature,

$$\sqrt{-1}\partial\bar{\partial}\widetilde{\mathscr{Y}} \geq \left(\sqrt{-1}\partial\bar{\partial}\log\widetilde{\mathscr{H}}^{-1}\right) \cdot \widetilde{\mathscr{Y}}. \tag{3.19}$$

This contradicts to (3.18) since $\sqrt{-1}\partial\overline{\partial}\log\widetilde{\mathscr{H}}^{-1}$ has at least one positive eigenvalue at point $p$ and $\widetilde{\mathscr{Y}}(p)>0$. □

*The sketched proof of Corollary 1.6.* Fix an arbitrary smooth metric $h$ on $M$ and set $\mathscr{H}=h_{\gamma\overline{\delta}}W^{\gamma}\overline{W}^{\delta}$. Suppose $\mathcal{O}_{T_M^*}(-1)$ is RC-positive, then there exists $\varphi\in C^{\infty}(\mathbb{P}(T_M),\mathbb{R})$ such that $\mathscr{H}e^{-\varphi}$ is a smooth Hermitian metric on $\mathcal{O}_{T_M^*}(-1)$ and

$$-\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}+\sqrt{-1}\partial\overline{\partial}\varphi \tag{3.20}$$

has at least one positive eigenvalue at each point in $\mathbb{P}(T_M)$.

Since $\mathcal{O}_{T^*N}(1)$ is nef, by [DPS94, Theorem 1.12], for any given Hermitian metric $\omega_N$ on $N$, there exist Hermitian metrics $g_k$ on $\mathrm{Sym}^{\otimes k}T_N$ and $\varepsilon_k>0$ such that

$$R_{i\overline{j}K\overline{L}}u^i\overline{u}^jV^K\overline{V}^{\ell}\leq k\varepsilon_k|u|^2_{\omega_N}|V|^2_{g_k} \tag{3.21}$$

where $\varepsilon_k\to 0$ as $k\to\infty$. Let $\{W^A\}$ be the symmetric polynomials in $\{W^{\alpha}\}_{\alpha=1}^m$ of degree $k$. We can define the higher order energy density on $\mathbb{P}(T_M)$ as following:

$$\mathscr{Y}_k=g_{I\overline{J}}F^I_A\overline{F}^J_B\frac{W^A\overline{W}^B}{\left(\sum h_{\gamma\overline{\delta}}W^{\gamma}\overline{W}^{\delta}\right)^k}, \tag{3.22}$$

where the matrix $\left(F^I_A\right)$ is the $k$-th symmetric power product $\mathrm{Sym}^{\otimes k}\left(f^i_{\alpha}\right)$ and $\left(g_{I\overline{J}}\right)$ is the metric matrix of $g_k$ on $\mathrm{Sym}^{\otimes k}T_N$ with respect to the given trivialization on $N$. We use the conformal change (2.7) of the generalized energy density $\mathscr{Y}_k$, i.e.

$$\mathscr{Y}_{k,\varphi}=e^{k\varphi}\mathscr{Y}_k. \tag{3.23}$$

As in Theorem 1.1 and Theorem 3.1, it is not hard to deduce the inequality

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y}_{k,\varphi}\geq\left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_{\varphi}^{-k}\right)\cdot\mathscr{Y}_{k,\varphi}-\frac{\sqrt{-1}R_{i\overline{j}K\overline{L}}f^i_{\alpha}\overline{f^j_{\beta}}F^K_C\overline{F^L_D}W^C\overline{W}^Ddz^{\alpha}\wedge d\overline{z}^{\beta}}{\mathscr{H}^k_{\varphi}}, \tag{3.24}$$

where $R_{i\overline{j}K\overline{L}}$ is the curvature tensor component of the Hermitian vector bundle $\left(\mathrm{Sym}^{\otimes k}T_N,g_k\right)$. Since $M$ is compact and $\left(\mathcal{O}_{T_M^*}(-1),\mathscr{H}e^{-\varphi}\right)$ is RC-positive, we deduce

$$\min_{P\in\mathbb{P}(T_M)}\ \sup_{u\in T_P\mathbb{P}(T_M),u\neq 0}\frac{\left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_{\varphi}^{-1}\right)(u,\overline{u})}{|u|^2_{\omega}}\geq 4C \tag{3.25}$$

where $\omega$ is a fixed metric on $\mathbb{P}(T_M)$ and $C=C(\omega,\mathscr{H}_{\varphi})$ is a positive constant.

Suppose $\mathscr{Y}_{k,\varphi}$ is not identically zero on $\mathbb{P}(T_M)$. Let $P\in\mathbb{P}(T_M)$ be a maximum value point of $\mathscr{Y}_{k,\varphi}$. Hence, $\sqrt{-1}\partial\overline{\partial}\mathscr{Y}_{k,\varphi}(P)\leq 0$. Let $u\in T_P(\mathbb{P}(T_M))$ be a positive direction of $\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_{\varphi}^{-1}$ such that

$$\left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}_{\varphi}^{-1}\right)(u,\overline{u})\geq 2C|u|^2_{\omega}. \tag{3.26}$$

By (3.24), (3.21) and (3.26), we deduce

$$\left(\sqrt{-1}\partial\overline{\partial}\mathscr{Y}_{k,\varphi}\right)(u,\overline{u}) \geq 2kC|u|_\omega^2\mathscr{Y}_{k,\varphi} - k\varepsilon_k \cdot |\partial f|^2_{\omega\otimes f^*\omega_N} \cdot |u|^2_\omega \cdot \mathscr{Y}_{k,\varphi}. \tag{3.27}$$

Since $\varepsilon_k \to 0$ as $k \to \infty$, we deduce that when $k$ is large enough

$$2C - \varepsilon_k \cdot |\partial f|^2_{\omega\otimes f^*\omega_N} > C. \tag{3.28}$$

Hence, we obtain

$$\left(\sqrt{-1}\partial\overline{\partial}\mathscr{Y}_{k,\varphi}\right)(u,\overline{u}) \geq kC|u|^2_\omega\mathscr{Y}_{k,\varphi} > 0$$

which is absurd. We conclude $\mathscr{Y}_{k,\varphi}$ is identically zero and so $f$ is a constant map. □

Corollary 1.7 can be proved by using similar strategies as in the proof of Corollary 1.6, and we need to use the generalized version of the energy density (1.6) on $\mathbb{P}(f^*T_N^*)$. The negative holomorphic sectional curvature is used in a similar way as in the proof Theorem 1.2.

By using the proof in Corollary 1.6, we get the following generalization.

**Corollary 3.2.** *Let $f : M \to N$ be a holomorphic map between compact complex manifolds. If $\mathcal{O}_{\mathrm{Sym}^{\otimes k}T_M^*}(-1)$ is RC-positive for some $k \geq 1$ and $\mathcal{O}_{T_N^*}(1)$ is nef, then $f$ is a constant map.*

## 4. The Hessian estimates and rigidity of harmonic maps and pluri-harmonic maps

In this section, we use the Hessian estimate of the generalized energy density to investigate harmonic maps and pluri-harmonic from complex manifolds to Riemannian manifolds and Kähler manifolds, and establish Theorem 1.8, Theorem 1.10, Theorem 1.11 and Theorem 1.13.

4.1. **The Hessian estimate and rigidity of pluri-harmonic maps into Riemannian manifolds.** Now we are ready to prove Theorem 1.8, i.e. a projective bundle version of Lemma 2.7.

*The proof of Theorem 1.8.* We shall use similar strategies as in the proof of Theorem 1.1. We choose an arbitrary point $p \in M$ and $q = f(p) \in N$. Let $\{z^\alpha\}$ and $\{x^i\}$ be coordinates centered at point $p \in M$ and $q \in N$ respectively such that

$$h_{\alpha\overline{\beta}}(p) = \delta_{\alpha\overline{\beta}}, \quad g_{ij}(q) = \delta_{ij}, \quad \frac{\partial g_{ij}}{\partial x^\ell}(q) = 0.$$

Since $f$ is pluri-harmonic, by formula (2.12), at point $p$, we have

$$f^i_{\alpha\overline{\beta}}(p) = \frac{\partial^2 f^i}{\partial z^\alpha \partial \overline{z}^\beta}(p) = 0. \tag{4.1}$$

We write $\mathscr{F} = \mathscr{Y} \cdot \mathscr{H}$. A straightforward calculation on $X := \mathbb{P}(T_M)$ yields

$$\partial\overline{\partial}\mathscr{Y} = \left(\partial\overline{\partial}\log\mathscr{H}^{-1}\right)\mathscr{Y} + \frac{\partial\overline{\partial}\mathscr{F}}{\mathscr{H}} + \frac{\mathscr{F}\partial\mathscr{H}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^3} - \frac{\partial\mathscr{H}\wedge\overline{\partial}\mathscr{F} + \partial\mathscr{F}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^2}. \tag{4.2}$$

We write $F^i = f^i_\alpha W^\alpha$ and $\mathscr{F} = g_{ij}F^i\overline{F}^j$. Hence,

$$\partial\mathscr{F} = \partial g_{ij}\cdot F^i\overline{F^j} + g_{ij}\cdot\partial F^i\cdot F^j + g_{ij}F^i\partial f^{i}_{\overline{\beta}}\cdot\overline{W}^\beta,$$

and

$$\overline{\partial}\mathscr{F} = \overline{\partial}g_{ij}\cdot F^i\overline{F^j} + g_{ij}F^i\overline{\partial F^j} + g_{ij}\overline{F}^j\overline{\partial}f^i_\alpha\cdot W^\alpha.$$

By using (4.1), at point $P \in X$ with $\pi(P) = p$, we have

$$\partial\mathscr{F} = \partial g_{ij}\cdot F^i\overline{F^j} + g_{ij}\cdot\partial F^i\cdot F^j, \quad \overline{\partial}\mathscr{F} = \overline{\partial}g_{ij}\cdot F^i\overline{F^j} + g_{ij}F^i\overline{\partial F^j}.$$

By a similar computation as in (3.3), we deduce

$$\begin{aligned}
&\frac{\partial\overline{\partial}\mathscr{F}}{\mathscr{H}} + \frac{\mathscr{F}\partial\mathscr{H}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^3} - \frac{\partial\mathscr{H}\wedge\overline{\partial}\mathscr{F} + \partial\mathscr{F}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^2}\\
= &\frac{\partial\overline{\partial}g_{ij}\cdot F^i\overline{F^j} + \partial g_{ij}\cdot F^i\cdot\overline{\partial F^j} - \overline{\partial}g_{ij}\cdot\partial F^i\cdot\overline{F^j} + g_{ij}\partial F^i\overline{\partial F^j}}{\mathscr{H}}\\
&+\frac{g_{ij}\cdot\partial\overline{\partial}F^i\cdot\overline{F}^j + g_{ij}\cdot F^i\cdot\partial\overline{\partial}\overline{F}^j}{\mathscr{H}^2}\\
&+\frac{\mathscr{F}\partial\log\mathscr{H}^{-1}\wedge\overline{\partial}\log\mathscr{H}^{-1}}{\mathscr{H}}\\
&-\frac{\partial\log\mathscr{H}\cdot\left(\overline{\partial}g_{ij}\cdot F^i\overline{F^j} + g_{ij}F^i\overline{\partial F^j}\right) + \left(\partial g_{ij}\cdot F^i\overline{F^j} + g_{ij}\cdot\partial F^i\cdot F^j\right)\cdot\overline{\partial}\log\mathscr{H}}{\mathscr{H}}.
\end{aligned}$$

The key ingredient is to define the $(1,1)$-form $\mathscr{W}$ on $X$ as in (3.4):

$$\mathscr{W} := g_{ij}\left(\partial F^i + F^i\partial\log\mathscr{H}^{-1} + T^i\right)\wedge\overline{\left(\partial F^j + F^j\partial\log\mathscr{H}^{-1} + T^j\right)} \tag{4.3}$$

where $T^i = \Gamma^i_{pk}F^k\partial f^p$. By using a similar computation as in (3.6), we obtain

$$\begin{aligned}
&\frac{\partial\overline{\partial}\mathscr{F}}{\mathscr{H}} + \frac{\mathscr{F}\partial\mathscr{H}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^3} - \frac{\partial\mathscr{H}\wedge\overline{\partial}\mathscr{F} + \partial\mathscr{F}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^2} - \frac{\mathscr{W}}{\mathscr{H}}\\
= &\frac{\left(\partial\overline{\partial}g_{ij}\right)F^i\overline{F^j}}{\mathscr{H}} - \frac{g_{ij}T^i\wedge\overline{T^j}}{\mathscr{H}} + \frac{g_{ij}\cdot\partial\overline{\partial}F^i\cdot\overline{F}^j + g_{ij}\cdot F^i\cdot\partial\overline{\partial}\overline{F}^j}{\mathscr{H}^2}\\
= &\frac{\left(\partial\overline{\partial}g_{ij}\right)F^i\overline{F^j}}{\mathscr{H}} - \frac{g_{ij}T^i\wedge\overline{T^j}}{\mathscr{H}} + \frac{g_{ij}\cdot\left(\partial\overline{\partial}f^i_\gamma\right)W^\gamma\cdot\overline{F}^j + g_{ij}\cdot F^i\cdot\left(\partial\overline{\partial}f^j_{\overline{\delta}}\right)\overline{W}^\delta}{\mathscr{H}^2}\\
= &\frac{\left(\frac{\partial^2 g_{ij}}{\partial x^k\partial x^\ell}f^k_\alpha f^\ell_{\overline{\beta}}f^i_\gamma f^j_{\overline{\delta}} + f^i_{\alpha\overline{\beta}\gamma}f^i_{\overline{\delta}} + f^i_\gamma f^i_{\alpha\overline{\beta}\overline{\delta}}\right)W^\gamma\overline{W}^\delta dz^\alpha\wedge d\overline{z}^\beta}{\mathscr{H}}\\
= &-\frac{\left(R_{i\ell kj} + R_{ik\ell j}\right)f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}W^\gamma\overline{W}^\delta dz^\alpha\wedge d\overline{z}^\beta}{\mathscr{H}},
\end{aligned}$$

where the last identity follows from a standard computation by using the pluriharmonic equation (2.12). Indeed, by (2.12), we obtain

$$f^i_{\alpha\overline{\beta}\gamma}f^i_{\overline{\delta}} = -\frac{\partial\Gamma^\ell_{ij}}{\partial x^k}f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}, \quad f^i_{\alpha\overline{\beta}\overline{\delta}}f^i_\gamma = -\frac{\partial\Gamma^k_{ij}}{\partial x^\ell}f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}},$$

and

$$\frac{\partial^2 g_{ij}}{\partial x^k\partial x^\ell}f^k_\alpha f^l_{\overline{\beta}}f^i_\gamma f^j_{\overline{\delta}} + f^i_{\alpha\overline{\beta}\gamma}f^i_{\overline{\delta}} + f^i_\gamma f^i_{\alpha\overline{\beta}\overline{\delta}} = \left(\frac{\partial^2 g_{k\ell}}{\partial x^i\partial x^j} - \frac{\partial\Gamma^\ell_{ij}}{\partial x^k} - \frac{\partial\Gamma^k_{ij}}{\partial x^\ell}\right)f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}.$$

By using the Riemannian curvature tensor (2.13), it is easy to show that

$$\frac{\partial^2 g_{k\ell}}{\partial x^i\partial x^j} - \frac{\partial\Gamma^\ell_{ij}}{\partial x^k} - \frac{\partial\Gamma^k_{ij}}{\partial x^\ell} = -\left(R_{i\ell kj} + R_{ik\ell j}\right). \tag{4.4}$$

By using the constraint equation (2.14), we get

$$\begin{aligned} &\frac{\partial\overline{\partial}\mathscr{F}}{\mathscr{H}} + \frac{\mathscr{F}\partial\mathscr{H}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^3} - \frac{\partial\mathscr{H}\wedge\overline{\partial}\mathscr{F} + \partial\mathscr{F}\wedge\overline{\partial}\mathscr{H}}{\mathscr{H}^2} - \frac{\mathscr{W}}{\mathscr{H}} \\ = &-\frac{R_{i\ell kj}f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}W^\gamma\overline{W}^\delta dz^\alpha\wedge d\overline{z}^\beta}{\mathscr{H}}. \end{aligned}$$

Finally, we obtain

$$\sqrt{-1}\partial\overline{\partial}\mathscr{Y} = \left(\sqrt{-1}\partial\overline{\partial}\log\mathscr{H}^{-1}\right)\cdot\mathscr{Y} + \frac{\sqrt{-1}\mathscr{W}}{\mathscr{H}} - \frac{\sqrt{-1}R_{i\ell kj}f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}W^\gamma\overline{W}^\delta dz^\alpha\wedge d\overline{z}^\beta}{\mathscr{H}}$$

and the proof of Theorem 1.8 is completed. □

*The proof of Theorem 1.10.* The proof is similar to that in Corollary 1.3. Fix an arbitrary smooth metric $h$ on $M$ and let $\mathscr{H} = h_{\gamma\overline{\delta}}W^\gamma\overline{W}^\delta$. Since $\mathcal{O}_{T^*_M}(-1)$ is RC-positive, there exist a Hermitian metric $\widetilde{\mathscr{H}}$ on $\mathcal{O}_{T^*_M}(-1)$ and $\varphi\in C^\infty(\mathbb{P}(T_M),\mathbb{R})$ such that $\widetilde{\mathscr{H}} = \mathscr{H}e^{-\varphi}$ and

$$-\sqrt{-1}\partial\overline{\partial}\log\widetilde{\mathscr{H}} = -\sqrt{-1}\partial\overline{\partial}\log\mathscr{H} + \sqrt{-1}\partial\overline{\partial}\varphi \tag{4.5}$$

has at least one positive eigenvalue at each point in $\mathbb{P}(T_M)$. We consider $\widetilde{\mathscr{Y}} = \mathscr{Y}e^\varphi$ on $\mathbb{P}(T_M)$. By using similar computations as in the proof of Theorem 1.8 and Theorem 3.1, we obtain

$$\sqrt{-1}\partial\overline{\partial}\widetilde{\mathscr{Y}} \geq \left(\sqrt{-1}\partial\overline{\partial}\log\widetilde{\mathscr{H}}^{-1}\right)\cdot\widetilde{\mathscr{Y}} - \frac{\sqrt{-1}R_{i\ell kj}f^i_\alpha f^j_{\overline{\beta}}f^k_\gamma f^\ell_{\overline{\delta}}W^\gamma\overline{W}^\delta dz^\alpha\wedge d\overline{z}^\beta}{\widetilde{\mathscr{H}}}. \tag{4.6}$$

Suppose $\widetilde{\mathscr{Y}}$ is not identically zero on $\mathbb{P}(T_M)$. Let $p\in\mathbb{P}(T_M)$ be a maximum value point of $\widetilde{\mathscr{Y}}$. Hence, $\widetilde{\mathscr{Y}}(p) > 0$ and

$$\sqrt{-1}\partial\overline{\partial}\widetilde{\mathscr{Y}}(p) \leq 0. \tag{4.7}$$

Since $\sqrt{-1}\partial\overline{\partial}\log\widetilde{\mathscr{H}}^{-1}$ has at least one positive eigenvalue at point $p$, there exists a non-zero vector $u=(a_1,\cdots,a_m,v_1,\cdots,v_{m-1})$ such that

$$\sqrt{-1}\partial\overline{\partial}\log\widetilde{\mathscr{H}}^{-1}(u,\overline{u})>0. \tag{4.8}$$

Let $H^{\gamma\overline{\delta}}=W^\gamma\overline{W}^\delta$ and

$$C_{\alpha\overline{\beta}}=R_{i\ell kj}f^i_\alpha f^j_{\overline{\beta}}\left(H^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}\right). \tag{4.9}$$

By using the constraint equation (2.14) for pluri-harmonic maps, we have

$$C_{\alpha\overline{\beta}}=R_{i\ell kj}f^i_\alpha f^j_{\overline{\beta}}\left(H^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}+H^{\gamma\overline{\delta}}f^\ell_\gamma f^k_{\overline{\delta}}\right) \tag{4.10}$$

On the other hand, since $\sqrt{-1}C_{\alpha\overline{\beta}}dz^\alpha\wedge d\overline{z}^\beta$ is a real $(1,1)$ form, we obtain

$$C_{\alpha\overline{\beta}}=\overline{C_{\beta\overline{\alpha}}}=R_{i\ell kj}f^j_\alpha f^i_{\overline{\beta}}\left(H^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}+H^{\gamma\overline{\delta}}f^\ell_\gamma f^k_{\overline{\delta}}\right). \tag{4.11}$$

Therefore,

$$C_{\alpha\overline{\beta}}=\frac{1}{2}R_{i\ell kj}\left(f^i_\alpha f^j_{\overline{\beta}}+f^j_\alpha f^i_{\overline{\beta}}\right)\left(H^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}+H^{\gamma\overline{\delta}}f^\ell_\gamma f^k_{\overline{\delta}}\right). \tag{4.12}$$

If we set $u^i=\sum_\alpha f^i_\alpha a_\alpha$ and $F^k=\sum_\gamma f^k_\gamma W^\gamma$, then at point $p$,

$$\sum_{\alpha,\beta}C_{\alpha\overline{\beta}}a_\alpha\overline{a}_\beta=\frac{1}{2}\sum R_{i\ell kj}\left(u^i\overline{u}^j+u^j\overline{u}^i\right)\left(F^k\overline{F}^\ell+F^\ell\overline{F}^k\right). \tag{4.13}$$

Let $u^i=c^i+\sqrt{-1}b^i$ and $F^i=A^i+\sqrt{-1}B^i$ where $c^i,b^i,A^i,B^i$ are real numbers. Therefore we have

$$\sum_{\alpha,\beta}C_{\alpha\overline{\beta}}a_\alpha\overline{a}_\beta=2\sum R_{i\ell kj}\left(c^ic^j+b^ib^j\right)\left(A^kA^\ell+B^kB^\ell\right). \tag{4.14}$$

Since $(N,h)$ has non-positive Riemannian sectional curvature, we deduce

$$\sum_{\alpha,\beta}C_{\alpha\overline{\beta}}a_\alpha\overline{a}_\beta=2\sum R_{i\ell kj}\left(c^ic^j+b^ib^j\right)\left(A^kA^\ell+B^kB^\ell\right)\leq 0. \tag{4.15}$$

By formulas (4.6), (4.7), (4.8) and (4.15), we get a contradiction. Hence $\widetilde{\mathscr{Y}}$ is a constant and $f$ must be a constant map. □

By using formula (2.17) and the conformal change technique, we also obtain:

**Theorem 4.1.** *Let $f:M\to(N,g)$ be a pluri-harmonic map from a compact complex manifold to a Riemannian manifold $(N,g)$ with non-positive Riemannian sectional curvature. If there exist a Hermitian metric $\omega$ on $M$ and a Hermitian metric $h$ on $T^{1,0}M$ such that*

$$\mathrm{tr}_\omega R^{(T^{1,0}M,h)}\in\Gamma(M,\mathrm{End}(T^{1,0}M))$$

*is quasi-positive, then $f$ is a constant map.*

*Proof.* Let $\omega_G=e^f\omega$ be a smooth Gauduchon metric in the conformal class of $\omega$, i.e. $\partial\overline{\partial}\omega_G^{m-1}=0$. Let $\omega_G=\sqrt{-1}G_{\alpha\overline{\beta}}dz^\alpha\wedge d\overline{z}^\beta$. By taking trace of (2.16), we obtain

$$\mathrm{tr}_{\omega_G}\sqrt{-1}\partial\overline{\partial}u\geq\left(G^{\alpha\overline{\beta}}R^h_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)h^{\gamma\overline{\nu}}h^{\mu\overline{\delta}}g_{ij}f^i_\mu f^j_{\overline{\nu}}-R_{i\ell kj}\left(G^{\alpha\overline{\beta}}f^i_\alpha f^j_{\overline{\beta}}\right)\left(h^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}\right).$$

By using a similar argument as above, we have

$$R_{i\ell kj}\left(G^{\alpha\overline{\beta}}f^i_\alpha f^j_{\overline{\beta}}\right)\left(h^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}\right)=\frac{1}{2}R_{i\ell kj}\left(G^{\alpha\overline{\beta}}f^i_\alpha f^j_{\overline{\beta}}+G^{\alpha\overline{\beta}}f^j_\alpha f^i_{\overline{\beta}}\right)\left(h^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}+h^{\gamma\overline{\delta}}f^\ell_\gamma f^k_{\overline{\delta}}\right).$$

At a point $p\in M$, we can assume $h_{\alpha\overline{\beta}}(p)=\delta_{\alpha\beta}$ and $G^{\alpha\overline{\beta}}(p)=\lambda_\alpha\delta_{\alpha\beta}$ where $\lambda_\alpha>0$. Hence,

$$R_{i\ell kj}\left(G^{\alpha\overline{\beta}}f^i_\alpha f^j_{\overline{\beta}}\right)\left(h^{\gamma\overline{\delta}}f^k_\gamma f^\ell_{\overline{\delta}}\right)=2\sum_{\alpha,\gamma}R_{i\ell kj}\lambda_\alpha(A^i_\alpha A^j_\alpha+B^i_\alpha B^j_\alpha)(A^k_\gamma A^\ell_\gamma+B^k_\gamma B^\ell_\gamma)\leq 0$$

where $f^i_\alpha=A^i_\alpha+\sqrt{-1}B^i_\alpha$ and $A^i_\alpha,B^i_\alpha$ are real numbers.

On the other hand, since $\mathrm{tr}_\omega R^{(T^{1,0}M,h)}$ is quasi-positive, we know

$$\mathrm{tr}_{\omega_G}R^{(T^{1,0}M,h)}=e^{-f}\mathrm{tr}_\omega R^{(T^{1,0}M,h)}$$

is also quasi-positive. Therefore,

$$\left(G^{\alpha\overline{\beta}}R^h_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)h^{\gamma\overline{\nu}}h^{\mu\overline{\delta}}g_{ij}f^i_\mu f^j_{\overline{\nu}}\geq 0.$$

Since $\omega_G$ is Gauduchon, by (4.16) we deduce

$$\int_M\left(G^{\alpha\overline{\beta}}R^h_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)h^{\gamma\overline{\nu}}h^{\mu\overline{\delta}}g_{ij}f^i_\mu f^j_{\overline{\nu}}\cdot\omega_G^m=0$$

and $\partial f$ must be identically zero on the open set where the curvature $\left(G^{\alpha\overline{\beta}}R^h_{\alpha\overline{\beta}\gamma\overline{\delta}}\right)$ is strictly positive. Since pluri-harmonic maps are also Hermitian harmonic, by [JY93, Theorem 6], $f$ must be a constant map. □

4.2. **Rigidity of pluri-harmonic maps into Kähler manifolds.** In this subsection, we shall prove Theorem 1.11. Let $M$ be a complex manifold, $(N,g)$ be a Hermitian manifold and $f:M\to N$ be a smooth map. We denote by $\mathcal{E}=f^*(T^{1,0}N)$ and endow it with the induced **Chern connection** $\nabla^{\mathcal{E}}$ from $T^{1,0}N$. There is a natural decomposition $\nabla^{\mathcal{E}}=\overline{\partial}_{\mathcal{E}}+\partial_{\mathcal{E}}$. Let $\{z^\alpha\}_{\alpha=1}^m$ be the local holomorphic coordinates on $M$ and $\{w^i\}_{i=1}^n$ be the local holomorphic coordinates on $N$. Let $e_i=f^*(\frac{\partial}{\partial w^i})$. There are three $\mathscr{E}$-valued 1-forms, i.e.,

$$\overline{\partial}f=\frac{\partial f^i}{\partial\overline{z}^\beta}d\overline{z}^\beta\otimes e_i,\quad\partial f=\frac{\partial f^i}{\partial z^\alpha}dz^\alpha\otimes e_i,\quad df=\overline{\partial}f+\partial f. \tag{4.16}$$

**Definition 4.2.** A smooth map $f:M\to(N,g)$ from a complex manifold $M$ to a Hermitian manifold $(N,g)$ is called *pluri-harmonic* if it satisfies $\partial_{\mathcal{E}}\overline{\partial}f=0$, i.e.

$$\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial\overline{z}^\beta}+\Gamma^i_{jk}\frac{\partial f^j}{\partial z^\alpha}\frac{\partial f^k}{\partial\overline{z}^\beta}\right)dz^\alpha\wedge d\overline{z}^\beta\otimes e_i=0, \tag{4.17}$$

where $\Gamma^i_{jk}=g^{i\overline{\ell}}\frac{\partial g_{k\overline{\ell}}}{\partial z^j}$ is the Christoffel symbol of the Chern connection on $(T^{1,0}N,g)$.

**Remark 4.3.** If $(N,h)$ is not Kähler, in general, the indices $j$ and $k$ in the formula (4.17) are not symmetric. In particular, we do not have a constraint equation as in (2.14) since neither $\partial_{\mathcal{E}}\partial f=0$ nor $\overline{\partial}_{\mathcal{E}}\overline{\partial} f=0$ holds. Note that the pluri-harmonic maps in (2.12) and (4.17) are different since the metric connections on the target manifolds are different.

**Lemma 4.4.** *Let $f:M\to(N,g)$ be a pluri-harmonic map from a complex manifold $M$ to a Kähler manifold $(N,g)$. Suppose $g_{\mathbb{R}}$ is the background Riemannian metric of the Kähler metric $g$ on $N$. Then $f:M\to(N,g_{\mathbb{R}})$ is a pluri-harmonic map in the sense of (2.12).*

*Proof.* It follows from the standard complexification since the Chern connection is the same as the Levi-Civita connection when the manifold $(N,g)$ is Kähler. □

**Remark 4.5.** If $f:M\to(N,g)$ is a pluri-harmonic map from a complex manifold $M$ to a Kähler manifold $(N,g)$, we can get a similar constraint equation as in (2.14). Indeed, it is exactly the complexification of (2.14). Moreover, the complexification of the constraint term (2.15) is

$$\sum_{\alpha,\gamma} R_{i\bar{j}k\bar{\ell}}\left(\frac{\partial f^i}{\partial z^\alpha}\frac{\partial \overline{f}^j}{\partial z^\gamma}-\frac{\partial f^i}{\partial z^\gamma}\frac{\partial \overline{f}^j}{\partial z^\alpha}\right)\overline{\left(\frac{\partial f^\ell}{\partial z^\alpha}\frac{\partial \overline{f}^k}{\partial z^\gamma}-\frac{\partial f^\ell}{\partial z^\gamma}\frac{\partial \overline{f}^k}{\partial z^\alpha}\right)}=0, \tag{4.18}$$

which is the notion introduced by Siu ([Siu80]). If $(N,g)$ has strongly negative curvature in the sense of Siu and $\mathrm{rank}_{\mathbb{R}} df\geq 4$, then the pluri-harmonic map $f$ is holomorphic or anti-holomorphic.

*The proof of Theorem 1.11.* It follows from Lemma 4.4 and Theorem 1.10. □

*The proof of Corollary 1.12.* By using the formula (3.13), we deduce that if a Hermitian manifold $(M,h)$ has positive holomorphic sectional curvature, then $(\mathcal{O}_{T^*M}(-1),\mathscr{H})$ is RC-positive. Hence, Corollary 1.12 follows from Theorem 1.10. □

4.3. **Rigidity of harmonic maps into Riemannian manifolds.** In this subsection, we shall prove Theorem 1.13. Let $(M,h)$ be a compact Hermitian manifold, $(N,g)$ a Riemannian manifold and $\mathscr{E}=f^*(TN)$ with the induced Levi-Civita connection. $f$ is called *Hermitian harmonic* if it satisfies $\mathrm{tr}_{\omega_h}\partial_{\mathscr{E}}\overline{\partial} f=0$, i.e.

$$h^{\alpha\overline{\beta}}\left(\frac{\partial^2 f^i}{\partial z^\alpha\partial\overline{z}^\beta}+\Gamma^i_{jk}\frac{\partial f^j}{\partial\overline{z}^\beta}\frac{\partial f^k}{\partial z^\alpha}\right)\otimes e_i=0. \tag{4.19}$$

It is easy to see that

**Corollary 4.6.** *Let $f:(M,h)\to(N,g)$ be a smooth map from a compact Kähler manifold $(M,h)$ to a Riemannian manifold $(N,g)$. Then Hermitian harmonic maps and harmonic maps coincide.*

As analogous to Siu's strong negativity [Siu80], Sampson proposed in [Sam85] the following definition (see also [JY91, Definition 4.2]):

**Definition 4.7.** Let $(N,g)$ be a Riemannian manifold. The (complexified) curvature tensor $R$ of $(N,g)$ is said to have *non-positive complex sectional curvature* if

$$R(Z,\overline{W},W,\overline{Z})\leq 0 \tag{4.20}$$

for any $Z,W\in T_{\mathbb{C}}N$.

If $(N,g)$ has non-positive complex sectional curvature, then it has non-positive Riemannian sectional curvature. Moreover, a Kähler manifold $(N,g)$ has strongly non-positive curvature in the sense of Siu if and only if its background Riemannian metric has non-positive complex sectional curvature (e.g. [LSYY17, Theorem 4.4]).

By using Siu's $\partial\overline{\partial}$-trick and ideas in [Siu80, Sam85, Sam86, JY91, JY93], one has the following result (see [LY14, Theorem 6.11] and [WY18, Theorem 4.2]).

**Lemma 4.8.** *Let $(M,h)$ be a compact astheno-Kähler manifold (i.e. $\partial\overline{\partial}\omega_h^{m-2}=0$) and $(N,g)$ a Riemannian manifold. Let $f:(M,h)\to(N,g)$ be a Hermitian harmonic map. If $(N,g)$ has non-positive complex sectional curvature, then $f$ is pluri-harmonic.*

*Proof.* If $f$ is Hermitian harmonic, i.e., $\mathrm{tr}_{\omega_h}\partial_{\mathscr{E}}\overline{\partial}f=0$, it is easy to see that

$$\partial\overline{\partial}\{\overline{\partial}f,\overline{\partial}f\}\frac{\omega_h^{m-2}}{(m-2)!}=4|\partial_{\mathscr{E}}\overline{\partial}f|^2\frac{\omega_h^m}{m!}-4\widehat{C}\cdot\frac{\omega_h^m}{m!} \tag{4.21}$$

where

$$\widehat{C}=R_{ik\ell j}\left(h^{\alpha\overline{\beta}}f_\alpha^i f_{\overline{\beta}}^j\right)\left(h^{\gamma\overline{\delta}}f_\gamma^k f_{\overline{\delta}}^\ell\right)$$

is defined in (2.15). From integration by parts, one obtains

$$4\int_M|\partial_{\mathscr{E}}\overline{\partial}f|^2\frac{\omega_h^m}{m!}-\int_M\widehat{C}\cdot\frac{\omega_h^m}{m!}=0.$$

If $(N,g)$ has non-positive complex sectional curvature, then $\widehat{C}\leq 0$. Hence, we have $\widehat{C}\equiv 0$ and $\partial_{\mathscr{E}}\overline{\partial}f=0$, i.e. $f$ is pluri-harmonic. □

*The proof of Theorem 1.13.* Let $(M,h)$ be a compact astheno-Kähler manifold and $(N,g)$ be a Riemannian manifold with non-positive complex sectional curvature. By Lemma 4.8, every Hermitian harmonic map $f:(M,h)\to(N,g)$ is pluri-harmonic. If $\mathcal{O}_{T_M^*}(-1)$ is RC-positive, then by Theorem 1.10, the pluri-harmonic map $f:M\to(N,g)$ is constant. □

## 5. Further applications of the generalized energy density

In this section, we discuss briefly some further applications of the generalized energy, which are analogous to the classical theory of harmonic maps. For more related topics, we refer to the survey papers and books [EL78, EL83, EL88, Xin96] and the references therein.

5.1. **RC-positivity for Riemannian curvature tensor.** As analogous to the RC-positivity for abstract vector bundles, one can define it for Riemannian manifolds.

**Definition 5.1.** Let $(N, g)$ be a Riemannian manifold. The curvature tensor $R$ of $(N, g)$ is said to be *RC-positive* if at each point $p \in N$ and for any nonzero vector $Z \in T_pN$, there exists a vector $W \in T_pN$ such that

$$R(Z, W, W, Z) > 0. \tag{5.1}$$

This terminology is a generalization of positive Riemannian sectional curvature. For instances, Riemannian manifolds with positive Ricci curvature must be RC-positive. Similarly, one can define the uniform RC-positivity and other similar notions.

**Definition 5.2.** Let $(N, g)$ be a Riemannian manifold. The curvature tensor $R$ of $(N, g)$ is called *uniformly RC-positive* if at each point $p \in N$ there exists a vector $W \in T_pN$ such that for any nonzero vector $Z \in T_pN$,

$$R(Z, W, W, Z) > 0. \tag{5.2}$$

We can define the energy density function $\mathscr{Y}$ on the projective bundle $\mathbb{P}(T_M) \to M$

$$\mathscr{Y} = g_{ij} f^i_\alpha f^j_\beta \frac{W^\alpha W^\beta}{\sum h_{\gamma\delta} W^\gamma W^\delta}$$

for a smooth map $f : (M, h) \to (N, g)$ between Riemannian manifolds. By using this setting, we can obtain similar Hessian estimates as in Theorem 1.1, Proposition 1.4 and Proposition 1.5 for totally geodesic maps (or some other harmonic maps), and rigidity of such maps follow in a similar way.

5.2. **The extension of Yau's function theory on complete manifolds.** It is easy to see that by using (1.3) and (1.4), their traces or integration by parts, we can extend Yau's function theory (e.g. [Yau75, Yau78]) on complete manifolds by various generalized maximum principles. One of the key steps is established in [Yang17, Corollary 2.3].

5.3. **The first and second variations of the generalized energy function.** Let $f_t : (M, g) \to (N, h)$ be a family of smooth maps parameterized by $t \in \Delta$, the corresponding generalized energy density is denoted by $\mathscr{Y}_t$. The first and second variations of $\mathscr{Y}_t$ are powerful in analyzing the stability and related properties of harmonic maps as shown in the classical works [SU81, SY80] and also a recent work [FLW17]. The energy density (1.6) would be crucial in this context.

5.4. **The analytical extension of this method to hyperbolic manifolds.** By using the theory of RC-positivity and results in algebraic geometry (in particular, seminal works of Graber-Harris-Starr [GHS03], Boucksom-Demailly-Paun-Peternell [BDPP13] and Campana-Demailly-Peternell [CDP14]), we obtain the following rigidity theorem in [Yang18b, Corollary 1.5]

**Theorem 5.3.** *Let $(M, h)$ be a compact Kähler manifold with positive holomorphic sectional curvature (or more generally, with uniformly RC-positive tangent bundle). Then there is no non-constant homomorphic map from $M$ to a Brody hyperbolic complex manifold $N$.*

It is a natural task to explore a purely differential geometric proof of this result. More generally, we propose the following conjecture (see also similar versions in [Yang18b, Conjecture 1.9]).

**Conjecture 5.4.** *Let $M$ and $N$ be two compact complex manifolds. If $\mathcal{O}_{T^*M}(-1)$ is RC-positive and $N$ is Kobayashi hyperbolic, then there is no non-constant holomorphic map from $M$ to $N$.*

The key difficulty is that there is no Hermitian metric with desire curvature positivity on hyperbolic manifolds. It is a reasonable way to attack Conjecture 5.4 by using Proposition 1.5 and the Demailly-Semple tower method (e.g. [Dem18, BD18]).

5.5. **The generalized energy density on Grassmannian manifolds $\mathrm{Gr}(k, T_M)$.** Let's called that the curvature matrix is call RC-positive if it has at least one positive eigenvalue. In this case, we considered the projection of this positive direction on the projective bundle $\mathbb{P}(T_M)$. If the curvature matrix has $k$-positive directions ([Yang18, Yang18b]), we can consider the associated Grassmannian manifold $\mathrm{Gr}(k, T_M)$. Many results of this paper still work in this general setting.

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