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RC-positivity, rational connectedness and Yau's conjecture

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In this paper, we introduce a concept of RC-positivity for Hermitian holomorphic vector bundles and prove that, if E is an RC-positive vector bundle over a compact complex manifold X, then for any vector bundle A, there exists a positive integer $c_A = c(A, E)$ such that

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0$$

for $\ell \geq c_A(k+1)$ and $k \geq 0$. Moreover, we obtain that, on a compact Kähler manifold X, if $\Lambda^p T_X$ is RC-positive for every $1 \leq p \leq \dim X$, then X is projective and rationally connected. As applications, we show that if a compact Kähler manifold (X, ω) has positive holomorphic sectional curvature, then $\Lambda^p T_X$ is RC-positive and $H^{p,0}_{\bar\partial}(X) = 0$ for every $1 \leq p \leq \dim X$, and in particular, we establish that X is a projective and rationally connected manifold, which confirms a conjecture of Yau ([57, Problem 47]).

Keywords and phrases: RC-positivity, vanishing theorem, holomorphic sectional curvature, rationally connected.

1. Introduction

In this paper, we give a geometric interpretation of Mumford's conjecture on rational connectedness by using curvature conditions and propose a differential geometric approach to attack this conjecture. As an application of this approach, we confirm a conjecture of Yau ([57, Problem 47]) that a compact Kähler manifold with positive holomorphic sectional curvature is a projective and rationally connected manifold. This project is motivated by a number of well-known conjectures proposed by Yau, Mumford, Demailly, Campana, Peternell and etc., and we refer to [55, 57, 10, 33, 32, 16, 17, 38, 27, 7, 12, 14, 28, 6, 49, 11, 15] and the references therein.

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A projective manifold X is called *rationally connected* if any two points of X can be connected by some rational curve. It is easy to show that on a rationally connected projective manifold, one has

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \text{for every} \quad m\geq 1.$$

A well-known conjecture of Mumford says that the converse is also true.

Conjecture 1.1 (Mumford)**.** *Let X be a projective manifold. If*

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \textit{for every} \quad m\geq 1,$$

then X is rationally connected.

This conjecture holds when $\dim X\leq 3$ ([33]) and not much has been known in higher dimensions, and we refer to [35], [12] and [32] for more historical discussions. In [6], Brunebarbe and Campana also proposed a stronger conjecture that X is rationally connected if and only if

$$H^0(X,\mathrm{Sym}^{\otimes\ell}\Omega_X^p)=0 \quad \text{for every } \ell>0 \text{ and } 1\leq p\leq \dim X. \tag{1.1}$$

In order to give geometric interpretations on rational connectedness, we introduce the following concept for Hermitian vector bundles:

Definition 1.2. Let (E,h) be a Hermitian holomorphic vector bundle over a complex manifold X and $R^{(E,h)}\in\Gamma(X,\Lambda^{1,1}T_X^*\otimes\mathrm{End}(E))$ be its Chern curvature tensor. E is called *RC-positive* (resp. *RC-negative*) if for any nonzero local section $a\in\Gamma(X,E)$, there exists some local section $v\in\Gamma(X,T_X)$ such that

$$R^{(E,h)}(v,\overline{v},a,\overline{a})>0. \quad (\quad \text{resp.} \quad <0)$$

For a line bundle (L,h), it is RC-positive if and only if its Ricci curvature has at least one positive eigenvalue at each point of X. This terminology has many nice properties. For instances, quotient bundles of RC-positive bundles are also RC-positive; subbundles of RC-negative bundles are still RC-negative (see Theorem 3.5); the holomorphic tangent bundles of Fano manifolds can admit RC-positive metrics (see Corollary 3.8). The first main result of our paper is

Theorem 1.3. *Let X be a* compact complex manifold. *If E is an RC-positive vector bundle, then for any vector bundle A, there exists a positive integer $c_A=c(A,E)$ such that*

$$H^0(X,\mathrm{Sym}^{\otimes\ell}E^*\otimes A^{\otimes k})=0$$

for $\ell \geq c_A(k+1)$ and $k \geq 0$. Moreover, if X is a projective manifold, then any invertible subsheaf $\mathcal{F}$ *of $\mathcal{O}(E^*)$ is not pseudo-effective.*

As a straightforward application of Theorem 1.3 and Campana-Demailly-Peternell's criterion for rational connectedness ([12, Theorem 1.1]) (see also [22, Corollary 1.7], [38], [35, Proposition 2.1] and [11, Proposition 1.3]), we obtain the second main result of our paper

Theorem 1.4. *Let X be a compact Kähler manifold of complex dimension n. Suppose that for every $1 \leq p \leq n$, there exists a smooth Hermitian metric h_p on the vector bundle $\Lambda^p T_X$ such that $(\Lambda^p T_X, h_p)$ is RC-positive, then X is projective and rationally connected.*

Theorem 1.3 and Theorem 1.4 also hold if we replace the RC-positivity by a weaker condition defined in Definition 3.3. In the following, we shall verify that several classical curvature conditions in differential geometry can imply the RC-positivity in Theorem 1.4.

Corollary 1.5. *Let X be a compact Kähler manifold. If there exist a Hermitian metric ω on X and a (possibly different) Hermitian metric h on T_X such that*

$$\mathrm{tr}_\omega R^{(T_X,h)} \in \Gamma(X, \mathrm{End}(T_X)) \tag{1.2}$$

is positive definite, then X is projective and rationally connected.

We need to point out that, when X is projective, Corollary 1.5 can also be implied by the "Generalized holonomy principle" for *positive curvature* and the main theorem in [12], although the precise result is not stated there. This special case is a refinement under the RC positive curvature condition. In particular, by the celebrated Calabi-Yau theorem ([56]), we obtain the classical result of Campana ([10]) and Kollár-Miyaoka-Mori ([33]) that Fano manifolds are rationally connected.

Let's describe another application of Theorem 1.4. In his "Problem section", S.-T. Yau proposed the following well-known conjecture ([57, Problem 47]):

Conjecture 1.6 (Yau)**.** *Let X be a compact Kähler manifold. If X has a Kähler metric with positive holomorphic sectional curvature, then X is a projective and rationally connected manifold.*

As applications of Theorem 1.4, we confirm Yau's Conjecture 1.6. More generally, we obtain

Theorem 1.7. *Let (X,ω) be a compact Kähler manifold with positive holomorphic sectional curvature. Then for every $1 \leq p \leq \dim X$, $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive and $H^{p,0}_{\bar\partial}(X) = 0$. In particular, X is a projective and rationally connected manifold.*

Remark 1.8. We need to point out that, recently, Heier-Wong also confirmed Yau's conjecture in the special case when X is projective ([28]). *One should see clearly the significant difference in the proofs.* Our method crucially relies on the geometric properties of RC-positivity (Theorem 1.3) which *we prove by using techniques in non-Kähler geometry*, and also a minimum principle for Kähler metrics with positive holomorphic sectional curvature (Lemma 6.1 and Theorem 6.3), while their method builds on an average argument and certain integration by parts on projective manifolds.

Remark 1.9. For the negative holomorphic sectional curvature case, in the recent breakthrough paper [46] of Wu and Yau, they proved that any projective Kähler manifold with negative holomorphic sectional curvature must have ample canonical line bundle. This result was obtained by Heier et. al. earlier under the additional assumption of the Abundance Conjecture ([25]). In [41], Tosatti and the author proved that any compact Kähler manifold with nonpositive holomorphic sectional curvature must have nef canonical line bundle, and with that in hand, we were able to drop the projectivity assumption in the aforementioned Wu-Yau Theorem. More recently, Diverio and Trapani [20] further generalized the result by assuming that the holomorphic sectional curvature is only *quasi-negative,* namely, nonpositive everywhere and negative somewhere in the manifold. In [47], Wu and Yau give a direct proof of the statement that any compact Kähler manifold with quasi-negative holomorphic sectional curvature must have ample canonical line bundle. We refer to [43, 25, 44, 26, 46, 47, 41, 20, 54, 42] for more details. On the other hand, it is well-known that the anti-canonical bundle of a compact Kähler manifold with positive holomorphic sectional curvature is not necessarily nef (e.g. [49]).

It is clear that, the RC-positivity is defined for Hermitian metrics. We also obtain

Corollary 1.10. *Let X be a compact Kähler surface. If there exists a* Hermitian metric ω *with positive holomorphic sectional curvature, then X is a projective and rationally connected manifold. Moreover, the Euler characteristic $\chi(X) > 0$.*

As motivated by Theorem 1.7 and Corollary 1.10, we propose

Conjecture 1.11. *Let X be a compact complex manifold of complex dimension $n > 2$. Suppose X has a Hermitian metric with positive holomorphic sectional curvature, then $\Lambda^p T_X$ admits a smooth RC-positive metric for every $1 < p < n$. In particular, if in addition X is Kähler, then X is a projective and rationally connected manifold.*

For some related topics on positive holomorphic sectional curvature, we refer to [27, 28, 2, 49, 3, 48, 1] and the references therein.

Organizations. In Section 2, we recall some background materials on Hermitian manifolds. In Section 3, we introduce the concept of RC-positivity for vector bundles and investigate its geometric properties. In Section 4, we obtain vanishing theorems for RC-positive vector bundles and prove Theorem 1.3. In Section 5, we establish the relation between RC-positive bundles and rational connectedness, and prove Theorem 1.4 and Corollary 1.5. In Section 6, we study positive holomorphic sectional curvature and rational connectedness, and prove Theorem 1.7, and Corollary 1.10. In Section 7, we study the relations between several open conjectures and propose some further questions.

2. Background materials

Let (E, h) be a Hermitian holomorphic vector bundle over a complex manifold X with Chern connection ∇. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^{(E,h)} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \mathrm{End}(E))$ has components

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}. \tag{2.1}$$

(Here and henceforth we sometimes adopt the Einstein convention for summation.) We have the trace $\mathrm{tr} R^{(E,h)} \in \Gamma(X, \Lambda^{1,1}T_X^*)$ which has components

$$R_{i\bar{j}\alpha}^{\alpha} = h^{\alpha\bar{\beta}} R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 \log \det(h_{\alpha\bar{\beta}})}{\partial z^i \partial \bar{z}^j}.$$

For any Hermitian metric ω_g on X, we can also define $\mathrm{tr}_{\omega_g} R^{(E,h)} \in \Gamma(X, \mathrm{End}(E))$ which has components

$$g^{i\bar{j}} R_{i\bar{j}\alpha}^{\beta}.$$

In particular, if (X,ω_g) is a Hermitian manifold, then the Hermitian vector bundle (T_X, g) has Chern curvature components

$$R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 g_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{p\overline{\ell}}}{\partial \overline{z}^j}. \tag{2.2}$$

The (first) Chern-Ricci form

$$\mathrm{Ric}(\omega_g) = \mathrm{tr}_{\omega_g} R^{(T_X,g)} \in \Gamma(X, \Lambda^{1,1} T_X^*)$$

of (X,ω_g) has components

$$R_{i\overline{j}} = g^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 \log\det(g)}{\partial z^i \partial \overline{z}^j}$$

and it is well-known that the Chern-Ricci form represents the first Chern class of the complex manifold X (up to a factor 2π). The second Chern-Ricci tensor

$$\mathrm{Ric}^{(2)}(\omega_g) = \mathrm{tr}_{\omega_g} R^{(T_X,g)} \in \Gamma(X, \mathrm{End}(T_X))$$

of (X,ω_g) has (lowered down) components

$$R^{(2)}_{k\overline{\ell}} = g^{i\overline{j}} R_{i\overline{j}k\overline{\ell}}.$$

If ω_g is not Kähler (i.e. $d\omega_g \neq 0$), $\mathrm{Ric}(\omega_g)$ and $\mathrm{Ric}^{(2)}(\omega_g)$ are not the same. The (Chern) scalar curvature s_g of ω_g is defined as

$$s_g = g^{i\overline{j}} R_{i\overline{j}}.$$

Definition 2.1. A holomorphic vector bundle (E,h) is called *Griffiths positive* if

$$R_{i\overline{j}\alpha\overline{\beta}} v^i \overline{v}^j a^\alpha \overline{a}^\beta > 0$$

for any nonzero vectors $v = (v^i)$ and $a = (a^\alpha)$.

A Hermitian (or Kähler) manifold (X,ω) has positive (resp. semi-positive) holomorphic sectional curvature, if for any nonzero vector $\xi = (\xi^1, \cdots, \xi^n)$,

$$R_{i\overline{j}k\overline{\ell}} \xi^i \overline{\xi}^j \xi^k \overline{\xi}^\ell > 0 \quad (\text{resp. } \geq 0),$$

at each point of X.

3. RC-positive vector bundles on compact complex manifolds

In the section we introduce the concept of RC-positivity for Hermitian vector bundles and investigate its geometric properties.

Definition 3.1. Let L be a holomorphic line bundle over a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. L is called *q-positive*, if there exists a smooth Hermitian metric h on L such that the Chern curvature $R^{(L,h)} = -\sqrt{-1}\partial\overline{\partial}\log h$ has at least $(n-q)$ positive eigenvalues at every point on X.

When $q = n-1$, the concept of $(n-1)$ positivity has very nice geometric interpretations. We established in [52, Theorem 1.5] the following result:

Theorem 3.2. *Let L be a line bundle over a compact complex manifold X with $\dim_{\mathbb{C}} X = n$. The following statements are equivalent:*

(1) L is $(n-1)$-positive;
(2) The dual line bundle L^* is not pseudo-effective.

In this paper, we extend the concept of $(n-1)$-positivity to vector bundles.

Definition 3.3. A Hermitian holomorphic vector bundle (E,h) over a complex manifold X is called *RC-positive* (resp. *RC-negative*) at point $q \in X$ if for any nonzero $a = (a^1, \cdots, a^r) \in \mathbb{C}^r$, there exists a vector $v = (v^1, \cdots, v^n) \in \mathbb{C}^n$ such that

$$\sum R_{i\overline{j}\alpha\overline{\beta}} v^i \overline{v}^j a^\alpha \overline{a}^\beta > 0 \quad (\text{resp.} < 0) \tag{3.1}$$

at point q. (E,h) is called RC-positive if it is RC-positive at every point of X. E is called *weakly RC-positive* if there exists a smooth Hermitian metric h on the tautological line bundle $\mathcal{O}_E(1)$ over $\mathbb{P}(E^*)$ such that $(\mathcal{O}_E(1), h)$ is $(\dim X - 1)$-positive.

We can also define RC-semi-positivity (resp. RC-semi-negativity) in the same way.

Remark 3.4. From the definition, it is easy to see that,

(1) if (E,h) is RC-positive, then for any nonzero $u \in \Gamma(X,E)$, as a Hermitian $(1,1)$-form on X, $R^{(E,h)}(u,u) \in \Gamma(X, \Lambda^{1,1}T^*_X)$ has at least one positive eigenvalue, i.e. $R^{(E,h)}(u,u)$ is $(n-1)$-positive;
(2) if a vector bundle (E,h) is Griffiths positive, then (E,h) is RC-positive;

(3) if E is a line bundle, then E is RC-positive if and only if E is $(n-1)$-positive;
(4) if $\dim X = 1$, (E,h) is RC-positive if and only if (E,h) is Griffiths positive.

Here, we can also define RC-positivity along k linearly independent directions, i.e. for any given nonzero local section $a \in \Gamma(X,E)$, $R^{(E,h)}(\bullet,\bullet,a,\overline{a}) \in \Gamma(X,\Lambda^{1,1}T_X^*)$ is $(n-k)$-positive as a Hermitian $(1,1)$-form on X. It is also a generalization of the Griffiths positivity. This terminology will be systematically investigated in the forthcoming paper [53].

By using the monotonicity formula and Theorem 3.2, we obtain the following properties, which also hold for RC-positivity along k linearly independent directions.

Theorem 3.5. *Let (E,h) be a Hermitian vector bundle over a compact complex manifold X.*

(1) (E,h) is RC-positive if and only if (E^*,h^*) is RC-negative;
(2) If (E,h) is RC-negative, every subbundle S of E is RC-negative;
(3) If (E,h) is RC-positive, every quotient bundle Q of E is RC-positive;
(4) If (E,h) is RC-positive, every line subbundle L of E^* is not pseudo-effective.
(5) If (E,h) is an RC-positive line bundle, then for any pseudo-effective line bundle L, $E\otimes L$ is RC-positive.

Proof. (1) is obvious. (2) follows from a standard monotonicity formula. Let r be the rank of E and s the rank of S. Without loss of generality, we can assume, at a fixed point $p\in X$, there exists a local holomorphic frame $\{e_1,\cdots,e_r\}$ of E centered at point p such that $\{e_1,\cdots,e_s\}$ is a local holomorphic frame of S. Moreover, we can assume that $h(e_\alpha,e_\beta)(p) = \delta_{\alpha\beta}$, for $1\le\alpha,\beta\le r$. Hence, the curvature tensor of S at point p is

$$R^S_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i\partial\overline{z}^j} + \sum_{\gamma=1}^{s}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j} \tag{3.2}$$

where $1\le\alpha,\beta\le s$. The curvature tensor of E at point p is

$$R^E_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i\partial\overline{z}^j} + \sum_{\gamma=1}^{r}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial\overline{z}^j} \tag{3.3}$$

where $1 \leq \alpha, \beta \leq r$. It is easy to see that

$$R^E|_S - R^S = \sqrt{-1}\sum_{i,j}\sum_{\alpha,\beta=1}^{s}\left(\sum_{\gamma=s+1}^{r}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}\right)dz^i\wedge d\overline{z}^j\otimes e^\alpha\otimes e^\beta. \tag{3.4}$$

Since R^E is RC-negative, for any nonzero vector $a_E = (a^1, \cdots, a^s, 0, \cdots, 0) \in \mathbb{C}^r$, there exists a vector $v = (v^1, \cdots, v^n)$ such that

$$R^E(v, \overline{v}, a_E, \overline{a}_E) < 0.$$

Let $a = (a^1, \cdots, a^s)$. Then

$$R^S(v,\overline{v},a,\overline{a}) = R^E(v,\overline{v},a_E,\overline{a}_E) - \sum_{i,j}\sum_{\alpha,\beta=1}^{s}\left(\sum_{\gamma=s+1}^{r}\frac{\partial h_{\alpha\overline{\gamma}}}{\partial z^i}\frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}\right)v^i\overline{v}^j a^\alpha\overline{a}^\beta < 0. \tag{3.5}$$

The proof of part (3) is similar.

(4). Let L be a line subbundle of E^* where (E, h) is RC-positive. Then by (1) and (2), we know L is RC-negative. Hence L^* is RC-positive and L^* is $(n-1)$-positive. By Theorem 3.2, $L = (L^*)^*$ is not pseudo-effective.

(5). Suppose $E \otimes L$ is not RC-positive, then by Theorem 3.2, $E^* \otimes L^*$ is pseudo-effective and so is $E^* = (E^* \otimes L^*) \otimes L$. This is a contradiction since E is RC-positive. □

Let's recall some basic linear algebra. Let V be a complex vector space and $\dim_{\mathbb{C}} V = r$. Let $A \in \mathrm{End}(V)$. For any $1 \leq p \leq r$, we define $\Lambda^p A \in \mathrm{End}(\Lambda^p V)$ as

$$(\Lambda^p A)(v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} v_1 \wedge \cdots \wedge Av_i \wedge \cdots \wedge v_p. \tag{3.6}$$

Similarly, for $k \geq 1$ we define $A^{\otimes k} \in \mathrm{End}(V^{\otimes k})$ by

$$\left(A^{\otimes k}\right)(v_1 \otimes \cdots \otimes v_p) = \sum_{i=1}^{p} v_1 \otimes \cdots \otimes Av_i \otimes \cdots \otimes v_p. \tag{3.7}$$

By choosing a basis, it is easy to see that if A is positive definite, then both $\Lambda^p A$ and $A^{\otimes k}$ are positive definite. We have the following important observation:

Theorem 3.6. *Let (X,ω) be a Hermitian manifold and (E,h) be a Hermitian holomorphic vector bundle. If*

$$\mathrm{tr}_\omega R^{(E,h)} \in \Gamma(X, \mathrm{End}(E)) \tag{3.8}$$

is positive definite, then

(1) $(\Lambda^p E, \Lambda^p h)$ *is RC-positive for every* $1 \leq p \leq \mathrm{rank}(E)$*;*
(2) $(E^{\otimes k}, h^{\otimes k})$ *is RC-positive for every* $k \geq 1$*.*

Proof. From the expression of the induced curvature tensor of $(\Lambda^p E, \Lambda^p h)$, it is easy to see that

$$R^{(\Lambda^p E, \Lambda^p h)} = \Lambda^p R^{(E,h)} \in \Gamma(X, \Lambda^{1,1} T_X^* \otimes \mathrm{End}(\Lambda^p E)),$$

and

$$\mathrm{tr}_\omega R^{(\Lambda^p E, \Lambda^p h)} = \mathrm{tr}_\omega \left(\Lambda^p R^{(E,h)} \right) = \Lambda^p \left(\mathrm{tr}_\omega R^{(E,h)} \right) \in \Gamma(X, \mathrm{End}(\Lambda^p E)),$$

is positive definite. Suppose $(\Lambda^p E, \Lambda^p h)$ is not RC-positive, then there exists a point $q \in X$ and a nonzero local section $a \in \Gamma(X, \Lambda^p E)$, such that for any local section $v \in \Gamma(X, T_X)$,

$$R^{(\Lambda^p E, \Lambda^p h)}(v, \overline{v}, a, \overline{a}) \leq 0$$

at point $q \in X$. In particular, we have

$$\left(\mathrm{tr}_\omega R^{(\Lambda^p E, \Lambda^p h)} \right)(a, \overline{a}) \leq 0$$

at point q. This is a contradiction. The proof for part (2) is similar. □

Corollary 3.7. *Let X be a compact complex manifold. If there exist a* Hermitian metric ω *on X and a (possibly different) Hermitian metric h on T_X such that*

$$\mathrm{tr}_\omega R^{(T_X,h)} > 0 \tag{3.9}$$

then

(1) $(T_X^{\otimes k}, h^{\otimes k})$ *is RC-positive for every* $k \geq 1$*;*
(2) $(\Lambda^p T_X, \Lambda^p h)$ *is RC-positive for every* $1 \leq p \leq \dim X$*.*

Corollary 3.8. *Let X be a compact Kähler manifold. If there exists a* Kähler metric ω *such that it has positive Ricci curvature, i.e.*

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log(\omega^n) > 0,$$

then

(1) $(T_X^{\otimes k}, \omega^{\otimes k})$ *is RC-positive for every* $k \geq 1$*;*
(2) $(\Lambda^p T_X, \Lambda^p \omega)$ *is RC-positive for every* $1 \leq p \leq \dim X$*.*

4. Vanishing theorems for RC-positive vector bundles

In this section, we derive vanishing theorems for RC-positive vector bundles over compact complex manifolds and prove Theorem 1.3.

Proposition 4.1. *Let* $\mathcal{O}_E(1) \to \mathbb{P}(E^*)$ *be the tautological line bundle of* $E \to X$*. Suppose* (E, h^E) *is RC-positive, then* E *is weakly RC-positive over* $\mathbb{P}(E^*)$*.*

Proof. The proof follows from a standard curvature formula for $\mathcal{O}_E(1)$ induced from (E, h). Suppose $\dim_{\mathbb{C}} X = n$. Let π be the projection $\mathbb{P}(E^*) \to X$ and $L = \mathcal{O}_E(1)$. Let $(e_1, \cdots, e_r)$ be the local holomorphic frame with respect to a given trivialization on E and the dual frame on E^* is denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on E^* are denoted by $(W_1, \cdots, W_r)$. There is a local section e_{L^*} of L^* defined by

$$e_{L^*} = \sum_{\alpha=1}^{r} W_\alpha e^\alpha \tag{4.1}$$

Its dual section is denoted by e_L. Let h^L the induced quotient metric on L by the morphism $(\pi^* E, \pi^* h^E) \to L$. If $\left(h_{\alpha\overline{\beta}}\right)$ is the matrix representation of h with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then h^L can be written as

$$h^L = \frac{1}{h^{L^*}(e_{L^*}, e_{L^*})} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta} \tag{4.2}$$

The curvature of (L, h^L) is

$$R^{h^L} = -\sqrt{-1}\partial\overline{\partial}\log h^L = \sqrt{-1}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) \tag{4.3}$$

where ∂ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(E^*)$. We fix a point $p \in \mathbb{P}(E^*)$, then there exist local holomorphic coordinates $(z^1, \cdots, z^n)$ centered at point $s = \pi(p)$ and local holomorphic basis $\{e_1, \cdots, e_r\}$ of E around s such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3) \tag{4.4}$$

Without loss of generality, we assume p is the point $(0, \cdots, 0, [a_1, \cdots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \cdots, r-1$. By formula (4.3) and (4.4)

$$R^{h^L}(p) = \sqrt{-1}\left(\sum R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \sum_{A,B=1}^{r-1} \left(1 - \frac{a_B \overline{a}_A}{|a|^2}\right) dw^A \wedge d\overline{w}^B\right) \tag{4.5}$$

where $|a|^2 = \sum_{\alpha=1}^{r} |a_\alpha|^2$. If R^E is RC-positive, then the $(n \times n)$ Hermitian matrix

$$\left[\sum R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2}\right]_{i,j=1}^{n} \tag{4.6}$$

has at least one positive eigenvalues. In particular, the curvature R^{h^L} of (L, h^L) has at least r positive eigenvalues over the projective bundle $\mathbb{P}(E^*)$. Since $\dim \mathbb{P}(E^*) = n + r - 1$, we know L is $(n-1)$-positive. $\square$

We also have the following observation for RC-positive vector bundles.

Proposition 4.2. *Let X be a compact complex manifold. If E is RC-positive, then*

$$H^0(X, E^*) = 0. \tag{4.7}$$

Proof. Let (E, h) be RC-positive and $\sigma \in H^0(X, E^*)$. We have

$$\partial\overline{\partial}|s|_g^2 = \langle \nabla s, \nabla s\rangle_g - R^{E^*}(\bullet, \bullet, s, \overline{s}). \tag{4.8}$$

Suppose $|s|_g^2$ attains its maximum at some point p and $|s|_g^2(p) > 0$. Since (E, h) is RC-positive, the induced bundle (E^*, g) is RC negative, i.e., at point q, there exists a nonzero vector v such that

$$R^{E^*}(v, \overline{v}, s, \overline{s}) < 0.$$

By applying maximum principle to (4.8), we get a contradiction. Hence, we deduce $s = 0$ and $H^0(X, E^*) = 0$. □

Corollary 4.3. *Let X be a compact Kähler manifold. Suppose $\Lambda^2 T_X$ is RC-positive, then X is projective.*

Proof. By Proposition 4.2, we have $H^{2,0}_{\overline{\partial}}(X) = H^{0,2}_{\overline{\partial}}(X) = 0$. Hence, by the Kodaira theorem ([31, Theorem 1], see also [30, Proposition 3.3.2 and Corollary 5.3.3]), the Kähler manifold X is projective. □

Lemma 4.4. *Let X be a compact complex manifold of complex dimension n. If E is weakly RC-positive, then for any vector bundle A over $\mathbb{P}(E^*)$, there exists a positive integer $c_A = c(A, E)$ such that*

$$H^{p,q}(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes A^{\otimes k}) = 0 \tag{4.9}$$

when $\ell \geq c_A(k+1)$, $k \geq 0$, $p \geq 0$ and $q > n-1$.

Proof. The proof follows from an Andreotti-Grauert type vanishing theorem. Let $F = \Omega^p_{\mathbb{P}(E^*)} \otimes K^{-1}_{\mathbb{P}(E^*)}$, then we have

$$H^{p,q}(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes A^{\otimes k}) \cong H^{n+r-1,q}(\mathbb{P}(E^*), \mathcal{O}_E(\ell) \otimes A^{\otimes k} \otimes F). \tag{4.10}$$

Suppose E is weakly RC-positive, i.e., there exists a smooth Hermitian metric on $\mathcal{O}_E(1)$ such that its curvature has at least r-positive eigenvalues at each point where $r = \operatorname{rank}(E)$. As shown in [52, Proposition 2.2], by a conformal change of the background metric, there exists a smooth Hermitian metric ω on $\mathbb{P}(E^*)$ such that $(\mathcal{O}_E(1), h)$ is uniformly $(n-1)$-positive. That is, in local holomorphic coordinates of $\mathbb{P}(E^*)$, at some point $p \in X$,

$$\omega = \sqrt{-1} \sum_i dz^i \wedge d\overline{z}^i, \qquad R^{(\mathcal{O}_E(1), h)} = \sqrt{-1} \sum_i \lambda_i dz^i \wedge d\overline{z}^i$$

where $\lambda_1 \geq \cdots \geq \lambda_{n+r-1}$, $\lambda_r > 0$ and

$$\lambda_r + \cdots + \lambda_{n+r-1} > 0. \tag{4.11}$$

Let $\mathcal{E} = \mathcal{O}_E(\ell) \otimes A^{\otimes k} \otimes F$ and $h^{\mathcal{E}}$ be the induced metric on $\mathcal{E}$ where we fix an arbitrary smooth metric h^A on A. Without loss of generality, we assume A is a line bundle. By standard Bochner formulas on compact complex manifolds ([18, Theorem 1.4 in Chapter VII]), one has

$$\Delta_{\overline{\partial}_{\mathcal{E}}} = \widehat{\Delta}_{\partial_{\mathcal{E}}} + \left[R^{\mathcal{E}}, \Lambda_\omega\right] + T_\omega \tag{4.12}$$

where $\tau = [\Lambda_\omega, \partial\omega]$, $\nabla^{\mathcal{E}} = \partial_{\mathcal{E}} + \overline{\partial}_{\mathcal{E}}$ is the decomposition of the Chern connection, $T_\omega = [\Lambda_\omega, [\Lambda_\omega, \sqrt{-1}\partial\overline{\partial}\omega]] - [\partial\omega, (\partial\omega)^*]$ and

$$\widehat{\Delta}_{\partial_{\mathcal{E}}} = (\partial_{\mathcal{E}} + \tau)(\partial_{\mathcal{E}}^* + \tau^*) + (\partial_{\mathcal{E}}^* + \tau^*)(\partial_{\mathcal{E}} + \tau).$$

Hence, for any $s \in H^{n+r-1,q}(\mathbb{P}(E^*), \mathcal{E})$, we have

$$\left\langle [R^{\mathcal{E}}, \Lambda_\omega] s, s \right\rangle = \left\langle \ell \left[R^{(\mathcal{O}_E(1),h)}, \Lambda_\omega \right] s + k \left[R^{(A,h^A)}, \Lambda_\omega \right] s + \left[R^{(F,h^F)}, \Lambda_\omega \right] s, s \right\rangle.$$

It is obvious that there exist uniform constants $c_0 \in \mathbb{R}$ and $c_1 \in \mathbb{R}$ such that

$$\left\langle \left[R^{(A,h^A)}, \Lambda_\omega \right] s, s \right\rangle \geq c_0 |s|^2 \quad \text{and} \quad \langle T_\omega s, s \rangle + \left\langle \left[R^{(F,h^F)}, \Lambda_\omega \right] s, s \right\rangle \geq c_1 |s|^2$$

hold on $\mathbb{P}(E^*)$. On the other hand, a straightforward computation shows that when $q > n - 1$

$$\ell \left\langle \left[R^{(\mathcal{O}_E(1),h)}, \Lambda_\omega \right] s, s \right\rangle \geq \ell \left(\lambda_r + \cdots + \lambda_{n+r-1} \right) |s|^2.$$

We set

$$\gamma = \inf_{p \in \mathbb{P}(E^*)} \left(\lambda_r + \cdots + \lambda_{n+r-1} \right) > 0$$

and pick a positive number $c_A = c(A, E)$ such that

$$c_A \cdot \gamma + c_0 > 0, \quad c_A \cdot \gamma + c_1 > 1.$$

Hence, for any $s \in H^{n+r-1,q}(\mathbb{P}(E^*), \mathcal{E})$, $q > n - 1$ and $\ell \geq c_A(k+1)$, we have

$$\begin{aligned} 0 = \left\langle \Delta_{\overline{\partial}_{\mathcal{E}}} s, s \right\rangle &= \left\langle \widehat{\Delta}_{\partial_{\mathcal{E}}} s, s \right\rangle + \left\langle \left[R^{\mathcal{E}}, \Lambda_\omega \right] s, s \right\rangle + \langle T_\omega s, s \rangle \\ &\geq \left\langle \widehat{\Delta}_{\partial_{\mathcal{E}}} s, s \right\rangle + k(c_A \cdot \gamma + c_0)|s|^2 + (c_A \cdot \gamma + c_1)|s|^2 \\ &\geq \left\langle \widehat{\Delta}_{\partial_{\mathcal{E}}} s, s \right\rangle + |s|^2. \end{aligned}$$

An integration over $\mathbb{P}(E^*)$ shows $s = 0$. □

Theorem 4.5. *Let E be a weakly RC-positive vector bundle over a compact complex manifold X. Then for every vector bundle A, there exists a positive integer $c_A = c(A, E)$ such that*

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0 \tag{4.13}$$

when $\ell \geq c_A(k+1)$ and $k \geq 0$. In particular, (4.13) holds if E is RC-positive.

Proof. By the classical Le Potier isomorphism over compact complex manifolds (e.g. [39, Theorem 5.16]), we have

$$H^q\left(\mathbb{P}(E^*), \mathcal{O}_E(\ell)\otimes\Omega^p_{\mathbb{P}(E^*)}\otimes(\pi^*A^*)^{\otimes k}\right) = H^q\left(X, \mathrm{Sym}^{\otimes\ell}E\otimes\Omega^p_X\otimes(A^*)^{\otimes k}\right)$$

where $\pi : \mathbb{P}(E^*) \to X$ is the projection. By Lemma 4.4, if we take $p = q = n$,

$$H^{n,n}\left(\mathbb{P}(E^*), \mathcal{O}_E(\ell)\otimes(\pi^*A^*)^{\otimes k}\right) = H^{n,n}\left(X, \mathrm{Sym}^{\otimes\ell}E\otimes(A^*)^{\otimes k}\right) = 0$$

when $\ell \geq c_A(k+1)$ and $k \geq 0$. By the Serre duality, we obtain (4.13). □

The proof of Theorem 1.3. The first part of Theorem 1.3 is contained in Theorem 4.5. For the second part, suppose-to the contrary-that there exists a pseudo-effective invertible sheaf $\mathcal{F} \subset \mathcal{O}(E^*)$. It is well-known that there exists a very ample line bundle A on the projective manifold such that

$$H^0(X, \mathcal{F}^{\otimes\ell}\otimes A) \neq 0$$

for all $\ell \geq 0$, and so

$$H^0(X, \mathrm{Sym}^{\otimes\ell}E^*\otimes A) \neq 0.$$

This is a contradiction. □

5. RC positivity and rational connectedness

In this section, we prove Theorem 1.4 and Corollary 1.5. The proofs rely on Theorem 1.3, Corollary 4.3 and a classical criterion for rational connectedness proved in [12, Theorem 1.1] (see also [38], [22, Corollary 1.7], [35, Proposition 2.1] and [11, Proposition 1.3].)

The proof of Theorem 1.4. By Corollary 4.3, we know X is a projective manifold. The rational connectedness of X follows from Theorem 1.3 and [12, Theorem 1.1] (or [11, Proposition 1.3], [38]). For readers' convenience, we give a proof here following [12, Theorem 1.1]. Indeed, if $(\Lambda^p T_X, h_p)$ is RC-positive for every $1 \leq p \leq \dim X$, then by Theorem 1.3, any invertible subsheaf $\mathcal{F}$ of Ω^p_X can not be pseudo-effective. In particular, when $p = n$, K_X is not pseudo-effective. Thanks to [7], X is uniruled. Let $\pi : X \dashrightarrow Z$ be the associated MRC fibration of X. After possibly resolving the singularities of π and Z, we may assume that π is a proper morphism and Z is smooth.

By a result of Graber, Harris and Starr [22, Corollary 1.4], it follows that the target Z of the MRC fibration is either a point or a positive dimensional variety which is not unruled. Suppose X is not rationally connected, then $\dim Z \geq 1$. Hence Z is not uniruled, by [7] again, K_Z is pseudo-effective. Since $K_Z = \Omega_Z^{\dim Z} \subset \Omega_X^{\dim Z}$ is pseudo-effective, we get a contradiction. Hence X is rationally connected. $\square$

The proof of Corollary 1.5. It follows Theorem 3.6 and Theorem 1.4. If the trace $\mathrm{tr}_\omega R^{(T_X,h)}$ is positive definite, by Theorem 3.6, $(\Lambda^p T_X, \Lambda^p h)$ is RC-positive for every $1 \leq p \leq \dim X$. Hence, by Theorem 1.4, X is projective and rationally connected. $\square$

6. A proof of Yau's conjecture on positive holomorphic sectional curvature

In this section, we describe more applications of Theorem 1.4 on RC-positive vector bundles and prove Theorem 1.7 and Corollary 1.10. More precisely, we confirm Yau's conjecture that if a compact Kähler manifold has positive holomorphic sectional curvature, then it is projective and rationally connected.

6.1. Compact Kähler manifolds with positive holomorphic sectional curvature. We begin with an algebraic curvature relation on a compact Kähler manifold (X, ω). By the Kähler symmetry, we have

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}} = R_{k\bar{j}i\bar{\ell}}.$$

At a given point $q \in X$, the minimum holomorphic sectional curvature is defined as

$$\min_{W \in T_qX, |W|=1} H(W),$$

where $H(W) := R(W, \overline{W}, W, \overline{W})$. Since X is of finite dimension, the minimum can be attained. The following result is essentially obtained in [50, Lemma 4.1] (see also some variants in [24, p. 312], [8, p. 136] and [9, Lemma 1.4]). For the sake of completeness, we include a proof here.

Lemma 6.1. *Let (X, ω) be a compact Kähler manifold and q be an arbitrary point on X. Let $e_1 \in T_qX$ be a unit vector which minimizes the holomorphic sectional curvature H of ω at point q, then*

$$2R(e_1, \bar{e}_1, W, \overline{W}) \geq \left(1 + |\langle W, e_1\rangle|^2\right) R(e_1, \bar{e}_1, e_1, \bar{e}_1) \tag{6.1}$$

for every unit vector $W \in T_qX$.

Proof. Let $e_2 \in T_qX$ be any unit vector orthogonal to e_1. Let

$$f_1(\theta) = H(\cos(\theta)e_1 + \sin(\theta)e_2), \qquad \theta \in \mathbb{R}.$$

Then we have the expansion

$$\begin{aligned} f_1(\theta) &= \cos^4(\theta)R_{1\bar{1}1\bar{1}} + \sin^4(\theta)R_{2\bar{2}2\bar{2}} \\ &\quad +2\sin(\theta)\cos^3(\theta)\,[R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}] + 2\cos(\theta)\sin^3(\theta)\,[R_{1\bar{2}2\bar{2}} + R_{2\bar{1}2\bar{2}}] \\ &\quad + \sin^2(\theta)\cos^2(\theta)[4R_{1\bar{1}2\bar{2}} + R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}}]. \end{aligned}$$

Since $f_1(\theta) \geq R_{1\bar{1}1\bar{1}}$ for all $\theta \in \mathbb{R}$ and $f_1(0) = R_{1\bar{1}1\bar{1}}$, we have

$$f_1'(0) = 0 \quad \text{and} \quad f_1''(0) \geq 0.$$

By a straightforward computation, we obtain

$$\begin{aligned} f_1'(0) &= 2(R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}) = 0, \quad f_1''(0) \\ &= 2\,(4R_{1\bar{1}2\bar{2}} + R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}}) - 4R_{1\bar{1}1\bar{1}} \geq 0. \end{aligned} \tag{6.2}$$

Similarly, if we set $f_2(\theta) = H(\cos(\theta)e_1 + \sqrt{-1}\sin(\theta)e_2)$, then

$$\begin{aligned} f_2(\theta) = \cos^4(\theta)&R_{1\bar{1}1\bar{1}} + \sin^4(\theta)R_{2\bar{2}2\bar{2}} \\ &+ 2\sqrt{-1}\sin(\theta)\cos^3(\theta)\,[-R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}] \\ &+ 2\sqrt{-1}\cos(\theta)\sin^3(\theta)\,[-R_{1\bar{2}2\bar{2}} + R_{2\bar{1}2\bar{2}}] \\ &+ \sin^2(\theta)\cos^2(\theta)[4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}]. \end{aligned}$$

From $f_2'(0) = 0$ and $f_2''(0) \geq 0$, one can see

$$-R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}} = 0, \qquad 2\,(4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) - 4R_{1\bar{1}1\bar{1}} \geq 0. \tag{6.3}$$

Hence, from (6.2) and (6.3), we obtain

$$R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = 0, \quad \text{and} \quad 2R_{1\bar{1}2\bar{2}} \geq R_{1\bar{1}1\bar{1}}. \tag{6.4}$$

For an arbitrary unit vector $W \in T_qX$, if W is parallel to e_1, i.e. $W = \lambda e_1$ with $|\lambda| = 1$,

$$2R(e_1, \bar{e}_1, W, \overline{W}) = 2R(e_1, \bar{e}_1, e_1, \bar{e}_1).$$

Suppose W is not parallel to e_1. Let e_2 be the unit vector

$$e_2 = \frac{W - \langle W, e_1\rangle e_1}{|W - \langle W, e_1\rangle e_1|}.$$

Then e_2 is a unit vector orthogonal to e_1 and

$$W = ae_1 + be_2$$

where $a = \langle W, e_1 \rangle$, $b = |W - \langle W, e_1\rangle e_1|$ and $|a|^2 + |b|^2 = 1$. Hence

$$2R(e_1, \overline{e}_1, W, \overline{W}) = 2|a|^2 R_{1\overline{1}1\overline{1}} + 2|b|^2 R_{1\overline{1}2\overline{2}},$$

since we have $R_{1\overline{1}1\overline{2}} = R_{1\overline{1}2\overline{1}} = 0$ by formula (6.4). By formula (6.4) again,

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq (2|a|^2 + |b|^2) R_{1\overline{1}1\overline{1}} = (1 + |a|^2) R_{1\overline{1}1\overline{1}}$$

which completes the proof of Lemma 6.1. □

Remark 6.2. The Kähler condition is substantially used in the proof of Lemma 6.1.

Now we are ready to prove Theorem 1.7, that is

Theorem 6.3. *Let (X, ω) be a compact Kähler manifold with positive holomorphic sectional curvature. Then for every $1 \leq p \leq \dim X$, $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive and $H^{p,0}_{\overline{\partial}}(X) = 0$. In particular, X is a projective and rationally connected manifold.*

Proof. Suppose $(\Lambda^p T_X, \Lambda^p \omega)$ is not RC-positive. By Definition 3.3, there exist a point $q \in X$ and a nonzero vector $a \in \Gamma(X, E)$ where $E = \Lambda^p T_X$ such that the Hermitian $(1,1)$-form

$$R^E(\bullet, \bullet, a, \overline{a}) \in \Gamma(X, \Lambda^{1,1} T^*_X) \tag{6.5}$$

is semi-negative at point q. We choose $e_1 \in \Gamma(X, T_X)$ at point q such that

$$R(e_1, \overline{e}_1, e_1, \overline{e}_1) = H(e_1) = \min_{W \in T_q X, |W|=1} H(W) > 0.$$

Hence, by Lemma 6.1,

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq \left(1 + |\langle W, e_1 \rangle|^2\right) R(e_1, \overline{e}_1, e_1, \overline{e}_1) > 0 \tag{6.6}$$

for every unit vector $W \in T_q X$. In particular,

$$R(e_1, \overline{e}_1, \bullet, \bullet) \in \Gamma(X, \mathrm{End}(T_X))$$

is positive definite at point q. Hence,

$$R^E(e_1, \overline{e}_1, \bullet, \bullet) \in \Gamma(X, \mathrm{End}(\Lambda^p T_X)) \tag{6.7}$$

is also positive definite at point q. Therefore,

$$R^E(e_1, \overline{e}_1, a, \overline{a}) > 0$$

which is a contradiction to (6.5). Hence, we deduce that $(\Lambda^p T_X, \Lambda^p \omega)$ is RC-positive. By Theorem 1.4, X is projective and rationally connected. $\square$

6.2. Compact complex manifolds with non-negative holomorphic sectional curvature. In this subsection, we investigate Hermitian metrics with non-negative holomorphic sectional curvature.

Proposition 6.4. *Let (X, ω) be a compact Hermitian manifold with semi-positive holomorphic sectional curvature. If the holomorphic sectional curvature is not identically zero, then*

(1) there exists a Gauduchon metric ω_G on X such that

$$\int_X \mathrm{Ric}(\omega_G) \wedge \omega_G^{n-1} > 0;$$

(2) K_X is not pseudo-effective;
(3) there exists a Hermitian metric h on K_X^{-1} such that (K_X^{-1}, h) is RC-positive.

Moreover, if in addition, X is projective, then X is uniruled.

Remark 6.5. Proposition 6.4 is a straightforward application of Theorem [49, Theorem 1.2], [52, Theorem 4.1] and the classical result of [7]. To demonstrate the essential difficulty in proving Conjecture 1.11 for higher dimensional compact complex manifolds and also the significant difference from the Kähler case, we include a detailed proof for Proposition 6.4.

Proof. By [52, Theorem 4.1], we know (1), (2) and (3) are mutually equivalent. Hence, we only need to prove one of them, for instance (1). We follow the steps in [49, Theorem 4.1] for readers' convenience. At a given point $p \in X$, the maximum holomorphic sectional curvature is defined to be

$$H_p := \max_{W \in T_p X, |W|=1} H(W),$$

where $H(W) := R(W, \overline{W}, W, \overline{W})$. Suppose the holomorphic sectional curvature is not identically zero, i.e. $H_p > 0$ for some $p \in X$. For any $q \in X$. We assume $g_{i\overline{j}}(q) = \delta_{ij}$. If $\dim_{\mathbb{C}} X = n$ and $[\xi^1, \cdots, \xi^n]$ are the homogeneous coordinates on $\mathbb{P}^{n-1}$, and ω_{FS} is the Fubini-Study metric of $\mathbb{P}^{n-1}$. At point q, we have the following well-known identity:

$$\int_{\mathbb{P}^{n-1}} R_{i\bar{j}k\bar{\ell}} \frac{\xi^i \overline{\xi}^j \xi^k \overline{\xi}^\ell}{|\xi|^4} \omega_{FS}^{n-1} = R_{i\bar{j}k\bar{\ell}} \cdot \frac{\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}}{n(n+1)} = \frac{s+\widehat{s}}{n(n+1)}. \tag{6.8}$$

where s is the Chern scalar curvature of ω and $\widehat{s}$ is defined as

$$\widehat{s} = g^{i\bar{\ell}} g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}. \tag{6.9}$$

Hence if (X,ω) has semi-positive holomorphic sectional curvature, then $s+\widehat{s}$ is a non-negative function on X. On the other hand, at point $p \in X$, $s+\widehat{s}$ is strictly positive since $H_p > 0$. By (6.8), the integrand is quasi-positive over $\mathbb{P}^{n-1}$, and so $s+\widehat{s}$ is strictly positive at $p \in X$. By [37, Section 4], we have the relation

$$s = \widehat{s} + \langle \partial\overline{\partial}^* \omega, \omega\rangle. \tag{6.10}$$

Therefore, we have

$$\int_X \widehat{s}\omega^n = \int_X s\omega^n - \int_X |\overline{\partial}^* \omega|^2 \omega^n. \tag{6.11}$$

Let $\omega_G = f_0^{\frac{1}{n-1}} \omega$ be a Gauduchon metric (i.e. $\partial\overline{\partial}\omega_G^{n-1} = 0$) in the conformal class of ω for some positive weight function $f_0 \in C^\infty(X)$. Let $s_G, \widehat{s}_G$ be the corresponding scalar curvatures with respect to the Gauduchon metric ω_G. Then we have

$$\begin{aligned} \int_X s_G \omega_G^n &= -n \int_X \sqrt{-1}\partial\overline{\partial} \log\det(\omega_G) \wedge \omega_G^{n-1} \\ &= -n \int_X f_0 \sqrt{-1}\partial\overline{\partial} \log\det(\omega) \wedge \omega^{n-1} \\ &= \int_X f_0 s\omega^n. \end{aligned} \tag{6.12}$$

By using a similar equation as (6.11) for $s_G, \widehat{s}_G$ and ω_G, we obtain

$$\int_X \widehat{s}_G \omega_G^n = \int_X f_0 \widehat{s}\omega^n. \tag{6.13}$$

Therefore, if $s+\widehat{s}$ is quasi-positive, we obtain

$$\begin{aligned}\int_X s_G\omega_G^n &= \frac{\int_X (s_G+\widehat{s}_G)\omega_G^n}{2}+\frac{\int_X (s_G-\widehat{s}_G)\omega_G^n}{2}\\ &= \frac{\int_X (s_G+\widehat{s}_G)\omega_G^n}{2}+\frac{\|\overline{\partial}_G^*\omega_G\|^2}{2}=\frac{\int_X f_0(s+\widehat{s})\omega^n}{2}+\frac{\|\overline{\partial}_G^*\omega_G\|^2}{2}>0 \end{aligned} \tag{6.14}$$

where the third equation follows from (6.12) and (6.13). □

Corollary 6.6. *Let X be a compact complex manifold. Suppose X has a Hermitian metric with positive holomorphic sectional curvature, then*

(1) T_X is RC-positive;
(2) $K_X^{-1}=\det T_X$ is RC-positive.

In particular, if in addition, X is projective, then X is uniruled.

Proof. From the definition of positive holomorphic holomorphic sectional curvature and RC-positivity, it is easy to see that if a Hermitian metric ω has positive holomorphic sectional curvature, then (T_X,ω) is RC-positive. On the other hand, by Proposition 6.4, if ω has positive holomorphic sectional curvature, then there exists a (possibly different) Hermitian metric $\widetilde{h}$ on K_X^{-1} such that $(K_X^{-1},\widetilde{h})$ is RC-positive. □

The proof of Corollary 1.10. We first show X is indeed projective. Suppose $\sigma\in H^0(X,K_X)$ is not zero, then K_X is $\mathbb{Q}$-effective. However, by Proposition 6.4, K_X is not pseudo-effective. This is a contradiction. Hence we deduce $H^{2,0}_{\overline{\partial}}(X)=H^{0,2}_{\overline{\partial}}(X)=H^0(X,K_X)=0$. We know X is projective. Now Corollary 1.10 follows from Theorem 1.4 and Corollary 6.6. □

Let $f:X\to Y$ be a smooth submersion between projective manifolds. It is well-known that if X is rationally connected, then Y is rationally connected. As analogous to this result, we have:

Proposition 6.7. *Let $f:X\to Y$ be a smooth submersion between compact complex manifolds. Suppose X admits a Hermitian metric h such that (Λ^pT_X,h) is RC-positive for some $1\le p\le\dim Y$, then Λ^pT_Y admits an RC-positive Hermitian metric.*

Proof. It follows from part (3) of Theorem 3.5. □

Similarly, we have

Corollary 6.8. *Let $f : X \to Y$ be a smooth submersion between compact complex manifolds. Suppose X admits a Hermitian metric h with positive holomorphic sectional curvature, then Y has a Hermitian metric with positive holomorphic sectional curvature. In particular, if in addition, Y is projective, then Y is uniruled.*

Proof. It follows from part (3) of Theorem 3.5, formula (3.5) and Theorem 6.6. ☐

7. RC positivity and Mumford's conjecture

In this section, we gather several conjectures in complex algebraic geometry and give their differential geometric interpretations.

7.1. Mumford's conjecture and uniruledness conjecture In [52, Theorem 4.1] and [52, Corollary 1.6], we proved that

Theorem 7.1. *Let L be a line bundle over a compact complex manifold X. The following are equivalent:*

(1) the dual line bundle L^* is not pseudo-effective;
(2) L is RC-positive;
(3) there exist a smooth Hermitian metric h on L and a smooth Hermitian metric ω on X such that the scalar curvature $\mathrm{tr}_\omega(-\sqrt{-1}\partial\overline{\partial}\log h) > 0$.

Moreover, if X is projective, then they are also equivalent to

(4) for any ample line bundle A, there exists a positive integer c_A such that
$$H^0(X, (L^*)^{\otimes \ell} \otimes A^{\otimes k}) = 0$$
for $\ell \geq c_A(k+1)$ and $k \geq 0$.

The classical result of [7] says that a projective manifold is uniruled if and only if the canonical bundle K_X is not pseudo-effective. Hence, one can formulate the uniruledness conjecture as

Conjecture 7.2. *Let X is a projective manifold. Then $\kappa(X) = -\infty$ is equivalent to one (and hence all) of the following*

(1) X is uniruled.
(2) K_X is not pseudo-effective;
(3) K_X^{-1} is RC-positive;

(4) there exists a Hermitian metric ω on X with positive (Chern) scalar curvature;

(5) for any ample line bundle A, there exists a positive integer c_A such that, for $\ell \geq c_A(k+1)$ and $k \geq 0$

$$H^0(X, K_X^{\otimes \ell} \otimes A^{\otimes k}) = 0.$$

Conjecture 7.3 (Mumford)**.** *Let X be a projective manifold. If*

$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \textit{for every} \quad m \geq 1,$$

then X is rationally connected.

It is well-known that the uniruledness conjecture can imply Conjecture 7.3 (e.g. [22, Corollary 1.7], see also Proposition 7.5).

Conjecture 7.4. *Let X be a projective manifold. If*

$$H^0(X, (T_X^*)^{\otimes m}) = 0, \quad \textit{for every} \quad m \geq 1,$$

then one (and hence all) of the following holds

(1) X is uniruled;

(2) K_X is not pseudo-effective;

(3) K_X^{-1} is RC-positive;

(4) there exists a Hermitian metric ω on X with positive (Chern) scalar curvature;

(5) for any ample line bundle A, there exists a positive integer c_A such that, for $\ell \geq c_A(k+1)$ and $k \geq 0$

$$H^0(X, K_X^{\otimes \ell} \otimes A^{\otimes k}) = 0.$$

Proposition 7.5. *We have the following relations*

$$\text{Conjecture } 7.2 \implies \text{Conjecture } 7.3 \iff \text{Conjecture } 7.4. \tag{7.1}$$

Proof. Conjecture 7.2 $\Longrightarrow$ Conjecture 7.4. Since $K_X = \det T_X^*$, it is well-known that for any positive integer ℓ, $K_X^{\otimes \ell}$ is a subbundle of $(T_X^*)^{\otimes m}$ for some large m. Hence $H^0(X, (T_X^*)^{\otimes m}) = 0$ for every $m \geq 1$ can imply $H^0(X, K_X^{\otimes \ell}) = 0$, i.e. $\kappa(X) = -\infty$. By assuming Conjecture 7.2, we obtain Conjecture 7.4.

Conjecture 7.3 $\Longrightarrow$ Conjecture 7.4. It follows from Theorem 7.1 and the fact that rationally connected manifolds are uniruled.

Conjecture 7.4 $\Longrightarrow$ Conjecture 7.3. The proof follows from Theorem 7.1 and a well-known argument in algebraic geometry (e.g. [12, Theorem 1.1], [22, Corollary 1.7], [35, Proposition 2.1]), which is also very similar to that of Theorem 1.4. Suppose

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \text{for every} \quad m\geq 1.$$

By assuming Conjecture 7.4, we know K_X is not pseudo-effective. Hence X is uniruled, thanks to the classical result of [7]. Let $\pi : X \dashrightarrow Z$ be the associated MRC fibration of X. After possibly resolving the singularities of π and Z, we may assume that π is a proper morphism and Z is smooth. By [22, Corollary 1.4], it follows that the target of the MRC fibration is either a point or a positive dimensional variety which is not unruled. Suppose X is not rationally connected, then $\dim Z\geq 1$. Hence Z is not uniruled, and by [7] again, K_Z is pseudo-effective. By assuming Conjecture 7.4, we obtain

$$H^0(Z,(T_Z^*)^{\otimes m_0})\neq 0$$

for some positive integer m_0. We obtain $H^0(X,(T_X^*)^{\otimes m_0})\neq 0$ since $(T_Z^*)^{\otimes m_0}\subset (T_X^*)^{\otimes m_0}$. This is a contradiction. $\square$

If X is rationally connected, K_X is not pseudo-effective. Hence there exists a Hermitian metric h on $K_X^{-1}=\Lambda^{\dim X}T_X$ such that (K_X^{-1},h) is RC-positive. We propose a generalization of this fact:

Question 7.6. Let X be a rationally connected projective manifold. Do there exist smooth Hermitian metrics h_p on vector bundles Λ^pT_X ($1\leq p\leq \dim X$) such that (Λ^pT_X,h_p) are all RC-positive? Do there exist smooth Hermitian metrics g_p on vector bundles $T_X^{\otimes p}$ ($p\geq 1$) such that $(T_X^{\otimes p},g_p)$ are all RC-positive?

A natural generalization of Conjecture 7.4 is

Question 7.7. Let X be a projective manifold. Suppose

$$H^0(X,(T_X^*)^{\otimes m})=0, \quad \text{for every} \quad m\geq 1.$$

We can ask the same question as in Question 7.6.

7.2. A partial converse to the Andreotti-Grauert theorem: the vector bundle version. We propose a question on vector bundles converse to Theorem 1.3:

Question 7.8. Let X be a projective manifold and E be a vector bundle. Suppose for every vector bundle A, there exists a positive integer $c_A = c(A,E)$ such that

$$H^0(X, \mathrm{Sym}^{\otimes \ell} E^* \otimes A^{\otimes k}) = 0 \tag{7.2}$$

for $\ell \geq c_A(k+1)$ and $k \geq 0$. Do there exist smooth Hermitian metrics h_p on vector bundles $\Lambda^p E$ $(1 \leq p \leq \mathrm{rank}(E))$ such that $(\Lambda^p E, h_p)$ are all RC-positive? Do there exist smooth Hermitian metrics g_p on vector bundles $E^{\otimes p}$ (resp. $\mathrm{Sym}^{\otimes p} E$) $(p \geq 1)$ such that $(E^{\otimes p}, h_p)$ (resp. $(\mathrm{Sym}^{\otimes p} E, h_p)$) are all RC-positive?

7.3. Existence of RC-positive metrics on vector bundles In this subsection, we propose several questions on the existence of RC-positive metrics. The celebrated Kodaira embedding theorem establishes that a line bundle is ample if and only if carries a smooth metric with positive curvature. The analogous correspondence for vector bundles is proposed by P. Griffiths:

Conjecture 7.9 ([23]). *If E is an ample vector bundle over a compact complex manifold X, then E admits a Griffiths positive Hermitian metric.*

When $\dim X = 1$, this conjecture is proved in [13]. The following conjecture can be implied by Griffiths' Conjecture 7.9:

Conjecture 7.10. *If E is an ample vector bundle over a projective manifold X, then there exists a smooth Hermitian metric h on E such that*

(1) $(E^{\otimes k}, h^{\otimes k})$ is RC-positive for every $k \geq 1$;
(2) $(\Lambda^p E, \Lambda^p h)$ is RC-positive for every $1 \leq p \leq \mathrm{rank}(E)$.

As a converse to Proposition 4.1, we also propose the following

Question 7.11. Let E be a vector bundle over a compact complex manifold X. If E is weakly RC-positive, is E necessarily RC-positive?

When $\dim X = 1$, Question 7.11 has an affirmative answer, thanks to [13].

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