Reconstruction of the values of meromorphic functions on a compact Riemann surface via Hermite-Padé polynomials

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Padé polynomials

Let f be a multivalued analytic function on Riemann sphere $\widehat{\mathbb{C}}$ outside the finite number of points $\{a_1, a_2, \dots, a_p\}$ and $\infty \notin \{a_1, \dots, a_p\}$. Notation: $f \in \mathcal{A}(\widehat{\mathbb{C}} \setminus \{a_1, \dots, a_p\})$.

Let f_∞ be a germ of f at ∞.
Let $f_\infty(z) = \sum_{k=0}^{\infty} \dfrac{c_k}{z^k}$ be the Taylor expansion of f_∞ at ∞.

Definition

Padé polynomials for f_∞ at ∞ (of order n): $\deg P_{n,j} \le n, j = 0, 1,$

$$(P_{n,0} + P_{n,1} f_\infty)(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

$-\dfrac{P_{n,0}}{P_{n,1}}$ is called Padé approximant.

Stahl's theorem, 1985

Recall that $f \in \mathcal{A}(\widehat{\mathbb{C}} \setminus \{a_1, \dots, a_p\})$, f_∞ is a germ of f at ∞,

$$(P_{n,0} + P_{n,1} f_\infty)(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

Theorem

There exists the compact set $S = S(f_\infty)$ (it is known that $\widehat{\mathbb{C}} \setminus S$ is connected and $S = \bigcup_{j=0}^{J} s_j$, where s_j are analytic arcs) such that

1) f_∞ can be extended as meromorphic function in $\widehat{\mathbb{C}} \setminus S$;

2) $\frac{1}{n}\mu(P_{n,j}) \xrightarrow{*} \frac{1}{2\pi} \mathrm{dd^c}\, g_{\widehat{\mathbb{C}}\setminus S}(\cdot, \infty)$ in $\mathrm{Meas}(\mathbb{C})$;

3) $-\frac{P_{n,0}}{P_{n,1}} \xrightarrow{\mathrm{cap}} f_\infty$ compactly in $\mathbb{C} \setminus S$.

For arbitrary polynomial P we denote $\mu(P) := \sum_{x:P(x)=0} \delta_x$,

and $g_{\widehat{\mathbb{C}}\setminus S}(\cdot, \infty)$ is the Green function of $\widehat{\mathbb{C}} \setminus S$ with the singularity at ∞.

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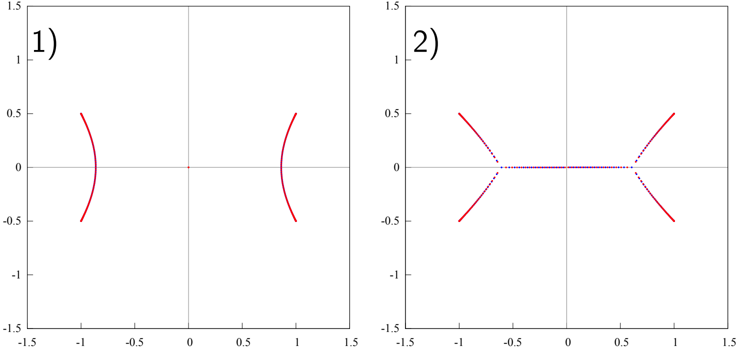
blue points are zeroes of $P_{300,0}$ for f

red points are zeroes of $P_{300,1}$ for f



$$1) f(z) = \sqrt{\left(1 - \frac{1+i/2}{z}\right)\left(1 - \frac{1-i/2}{z}\right)\left(1 - \frac{-1+i/2}{z}\right)\left(1 - \frac{-1-i/2}{z}\right)},$$

$$2) f(z) = \sqrt[4]{\left(1 - \frac{1+i/2}{z}\right)\left(1 - \frac{1-i/2}{z}\right)\left(1 - \frac{-1+i/2}{z}\right)\left(1 - \frac{-1-i/2}{z}\right)}$$

Hermite-Padé polynomials

Let us now fix an arbitrary $m \in \mathbb{N}$.

Definition

Hermite-Padé polynomials for $[1, f_\infty, f_\infty^2, \ldots, f_\infty^m]$ *at* ∞ *(of order* n*):* $\deg Q_{n,j} \leq n, j = 0, \ldots, m,$

$$\left(Q_{n,0} + Q_{n,1} f_\infty + Q_{n,2} f_\infty^2 + \cdots + Q_{n,m} f_\infty^m\right)(z) = O\left(\frac{1}{z^{m(n+1)}}\right)$$

as $z \to \infty$.

When $m = 1$, we have usual Padé polynomials

$$\left(Q_{n,0} + Q_{n,1} f_\infty\right)(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

Let now $f \in M(\mathfrak{R})$

Let $\mathfrak{R}$ be a compact Riemann surface.
Let $\pi : \mathfrak{R} \to \widehat{\mathbb{C}}$ be a (m+1)-sheeted branched covering of $\widehat{\mathbb{C}}$. (For points that lie above z we use notation $\mathbf{z}$, so $\pi : \mathbf{z} \to z$.)
By Σ denote the set of branch points of π. Let $\infty \notin \Sigma$.

Let now f be a meromorphic function on $\mathfrak{R}$.
With the help of π^{-1} we have that $f \in \mathcal{A}(\widehat{\mathbb{C}} \setminus \Sigma)$.
Choose $\boldsymbol{\infty}^{(0)} \in \pi^{-1}(\infty)$.
Let f_∞ be the germ of f at $\boldsymbol{\infty}^{(0)}$.

Now we consider Hermite-Padé polynomials for the tuple $[1, f_\infty, f_\infty^2, \dots, f_\infty^m]$.

Nuttall' 1984
Chirka, Palvelev, Suetin, K.' 2017

Nuttall partition of $\mathfrak{R}$

Let $u(\mathbf{z})$ be the harmonic function on $\mathfrak{R} \setminus \pi^{-1}(\infty)$ with the following singularities at $\pi^{-1}(\infty)$:

$$u(\mathbf{z}) = -m \log |z| + O(1), \quad \mathbf{z} \to \boldsymbol{\infty}^{(0)},$$
$$u(\mathbf{z}) = \log |z| + O(1), \quad \mathbf{z} \to \pi^{-1}(\infty) \setminus \boldsymbol{\infty}^{(0)}.$$

Let $z \in \mathbb{C} \setminus \Sigma$. Then

$$\pi^{-1}(z) = \{\mathbf{z}^{(0)}, \mathbf{z}^{(1)}, \dots, \mathbf{z}^{(j)}, \dots, \mathbf{z}^{(m)}\}$$

and we order these points with respect to non-decreasing of the values of u:

$$u(\mathbf{z}^{(0)}) \le u(\mathbf{z}^{(1)}) \le \cdots \le u(\mathbf{z}^{(j)}) \le \cdots \le u(\mathbf{z}^{(m)}).$$

$$\mathfrak{R}^{(0)} := \{\mathbf{z}^{(0)} \in \mathfrak{R} : u(\mathbf{z}^{(0)}) < u(\mathbf{z}^{(1)})\};$$
$$\mathfrak{R}^{(j)} := \{\mathbf{z}^{(j)} \in \mathfrak{R} : u(\mathbf{z}^{(j-1)}) < u(\mathbf{z}^{(j)}) < u(\mathbf{z}^{(j+1)})\}, \quad j = 1, \dots, m-1;$$
$$\mathfrak{R}^{(m)} := \{\mathbf{z}^{(m)} \in \mathfrak{R} : u(\mathbf{z}^{(m-1)}) < u(\mathbf{z}^{(m)})\}.$$

Nuttall partition of $\mathfrak{R}$

$$u(\mathbf{z}^{(0)}) \le u(\mathbf{z}^{(1)}) \le \cdots \le u(\mathbf{z}^{(j)}) \le \cdots \le u(\mathbf{z}^{(m)}).$$
$$\mathfrak{R}^{(m)} := \{\mathbf{z}^{(m)} \in \mathfrak{R} : u(\mathbf{z}^{(m-1)}) < u(\mathbf{z}^{(m)})\}.$$

Let us now define the compact set

$$F := \{z \in \widehat{\mathbb{C}} : u(\mathbf{z}^{(m-1)}) = u(\mathbf{z}^{(m)})\}.$$

Lemma. *$F = \bigcup_{j=0}^{J'} \gamma_j$, where γ_j are analytic arcs.*

For $z \in \mathbb{C}$ **define** $u_j(z) := u(\mathbf{z}^{(j)})$, $j = 0, \ldots, m$.

Lemma. *u_j is a continious function on $\mathbb{C}$, $j = 0, \ldots, m$.*
Lemma. *u_m is a subharmonic function on $\mathbb{C}$ (and harmonic on $\mathbb{C} \setminus F$).*

Nuttall partition of $\mathfrak{R}$

$$u(\mathbf{z}^{(0)}) \leq u(\mathbf{z}^{(1)}) \leq \cdots \leq u(\mathbf{z}^{(j)}) \leq \cdots \leq u(\mathbf{z}^{(m)}).$$

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Let us now define the compact set

$$F := \{z \in \widehat{\mathbb{C}} : u(\mathbf{z}^{(m-1)}) = u(\mathbf{z}^{(m)})\} = \pi\left(\partial \mathfrak{R}^{(m)}\right).$$

Lemma. *$F = \bigcup_{j=0}^{J'} \gamma_j$, where γ_j are analytic arcs.*

For $z \in \mathbb{C}$ **define** $u_j(z) := u(z^{(j)})$, $j = 0, \ldots, m$.

Lemma. *u_j is a continious function on $\mathbb{C}$, $j = 0, \ldots, m$.*
Lemma. *u_m is a subharmonic function on $\mathbb{C}$ (and harmonic on $\mathbb{C} \setminus F$).*

Some analogue of Stahl's theorem

Theorem

1) $\frac{1}{n}\mu(Q_{n,j}) \xrightarrow{*} \frac{1}{2\pi}\,\mathrm{dd^c}\,u_m + \delta_\infty$ *in* $\mathrm{Meas}(\widehat{\mathbb{C}})$.

2) $\frac{Q_{n,j}}{Q_{n,m}} \xrightarrow{\text{cap}} (-1)^{m-j}\sigma_{m-j}(f(z^{(0)}), f(z^{(1)}), \dots, f(z^{(m-1)}))$

compactly in $\mathbb{C} \setminus F$, $j = 0, \dots, m-1$.

Here σ_j is the j-th symmetric polynomial of m variables.

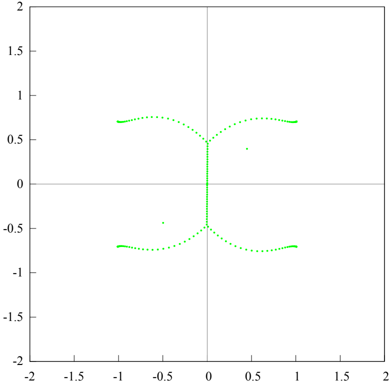
Let us consider the corresponding polynomial equation:

$$\frac{Q_{n,0}}{Q_{n,m}} + \frac{Q_{n,1}}{Q_{n,m}}w + \dots + \frac{Q_{n,m-1}}{Q_{n,m}}w^{m-1} + w^m = 0. \qquad (1)$$

From the theorem it follows that solutions of (1) *asymptotically reconstruct (as* $n \to \infty$*) the values of our function* f *on first* m *sheets of Nuttall partition of* $\mathfrak{R}$ *outside the preimage of the compact set* F.

$\mathfrak{R}: w^4 = (1 + \frac{1+0.7i}{z})(1 - \frac{1+0.7i}{z})(-1 + \frac{1+0.7i}{z})(1 + \frac{-1-0.7i}{z})$, $m = 3$

Numerical modelling of the compact set F for some initial point $\infty^{(0)} \in \mathfrak{R}$.



Green points are zeroes of $Q_{200,3}$ for some germ f_∞ at ∞ of $f := w$.

Properties of Nuttall partition

Statement. $\mathfrak{R} \setminus \overline{\mathfrak{R}^{(m)}}$ is connected.

Why is this statement interesting?

Stahl's theorem, 1985

Recall the definition of Padé polynomials:

$$(P_{n,0} + P_{n,1} f_\infty)(z) = O\left(\frac{1}{z^{n+1}}\right) \text{ as } z \to \infty.$$

Theorem

There exists the compact set S (it is known that $\widehat{\mathbb{C}} \setminus S$ is connected and $S = \bigcup_{j=0}^{J} s_j$, where s_j are analytic arcs) such that

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3) $-\frac{P_{n,0}}{P_{n,1}} \xrightarrow{\text{cap}} f_\infty$ compactly in $\mathbb{C} \setminus S$.

Nuttall partition's dependence on $\boldsymbol{\infty}^{(0)}$

Recall that $u(\mathbf{z})$ is the harmonic function on $\mathfrak{R} \setminus \pi^{-1}(\infty)$ with the following singularities at $\pi^{-1}(\infty)$:

$$u(\mathbf{z}) = -m \log |z| + O(1), \quad \mathbf{z} \to \boldsymbol{\infty}^{(0)},$$
$$u(\mathbf{z}) = \log |z| + O(1), \quad \mathbf{z} \to \pi^{-1}(\infty) \setminus \boldsymbol{\infty}^{(0)}.$$

We denote

$$\pi^{-1}(z) = \{\mathbf{z}^{(0)}, \mathbf{z}^{(1)}, \dots, \mathbf{z}^{(j)}, \dots, \mathbf{z}^{(m)}\}$$

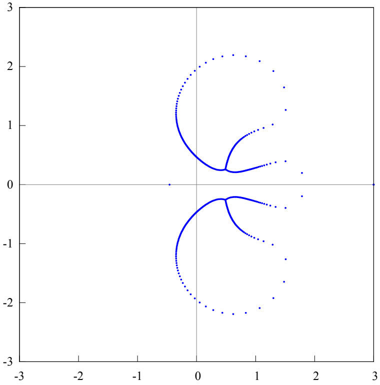
and we order these points with respect to non-decreasing of the values of u:

$$u(\mathbf{z}^{(0)}) \le u(\mathbf{z}^{(1)}) \le \cdots \le u(\mathbf{z}^{(j)}) \le \cdots \le u(\mathbf{z}^{(m)}).$$

$$F := \{z \in \widehat{\mathbb{C}} : u(\mathbf{z}^{(m-1)}) = u(\mathbf{z}^{(m)})\} = \pi\left(\partial \mathfrak{R}^{(m)}\right).$$

$\mathfrak{R}: w^3 - 3(\frac{1}{z} - 3)^2 w + 2(\frac{3}{z} - 1)^3$, $m = 2$

Numerical modelling of the compact set F *for two different* initial points $\infty^{(0)} \in \mathfrak{R}$.



Blue points are zeroes of $Q_{200,0}$ *for two different* germs f_∞ at ∞ of $f := w$.

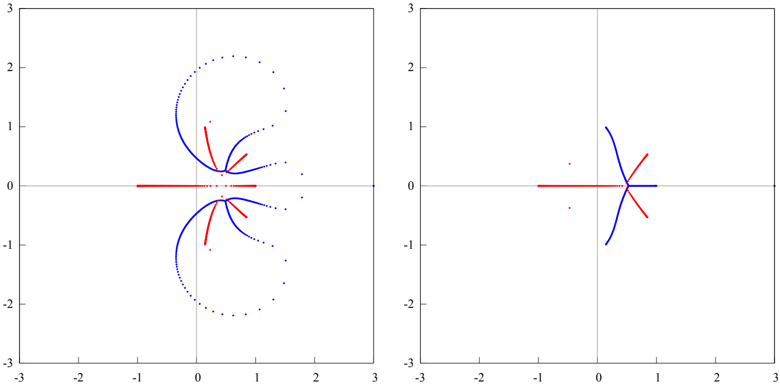
Property of Nuttall partition

Statement. $\mathfrak{R} \setminus \overline{\mathfrak{R}^{(m)}}$ *is connected.*

Conjecture. $\mathfrak{R} \setminus \overline{\bigcup_{j=k}^{m} \mathfrak{R}^{(j)}}$ *is connected for* $k = 1, \ldots, m$*.*
In particular, the sheet $\mathfrak{R}^{(0)}$ *is connected.*

$\mathfrak{R}: w^3 - 3(\frac{1}{z} - 3)^2 w + 2(\frac{3}{z} - 1)^3$, $m = 2$

Numerical modelling of the Stahl compact set S and the compact set F *for two different* initial points $\infty^{(0)} \in \mathfrak{R}$.



Red points are zeroes of $P_{300,1}$ *for two different* germs f_∞ at ∞ of $f := w$, blue points are zeroes of $Q_{200,0}$ for the same germs.