SOLITON EQUATIONS
AND THEIR HOLOMORPHIC SOLUTIONS

The 4th Sino-Russian Conference in Mathematics, Hong Kong, 15/X/2018

1. **Heat equation.**
2. **Soliton equations.**
3. **Construction of soliton equations.**
4. **Construction of their solutions.**
5. **Further properties of solutions.**

1. Heat equation

CAUCHY–KOVALEVSKAYA THEOREM. *Let P be holomorphic with respect to (x,t) at $(x_0,t_0) \in \mathbb{C}^2$ and polynomial in other variables. Then the Cauchy problem*

$$\partial_t^m u = P(x,t,\{\partial_x^k \partial_t^l u\}_{k+l\leqslant m,(k,l)\neq(0,m)});$$
$$\partial_t^j u(x,t_0) = \varphi_j(x), \quad 0 \leqslant j \leqslant m-1$$

has a unique local holomorphic solution $u(x,t)$ at (x_0,t_0) for any choice of the holomorphic germs $\varphi_0,\varphi_1,\ldots,\varphi_{m-1}$ at x_0.

In 1842 Cauchy proved this in a series of four long *Comptes Rendus* notes using reduction to first-order systems and the majorant method. In 1873 Weierstrass posed it as a thesis problem for Kovalevskaya, without the condition $k+l \leqslant m$, $(k,l) \neq (0,m)$. She has found a counterexample and eventually proved the correct result.

Kovalevskaya considered the heat equation at $(0,0) \in \mathbb{C}^2$:

$$(*) \qquad \partial_t u = \partial_x^2 u, \qquad u(x,0) = u_0(x).$$

Since $\partial_t^l u = \partial_x^{2l} u$ for all l, there is a unique *formal* power series solution

$$u(x,t) = \sum_{l=0}^{\infty} \frac{u_0^{(2l)}(x)}{l!} t^l \qquad \left(\text{rewritten as } \sum_{k,l=0}^{\infty} b_{kl} x^k t^l\right).$$

In particular, writing the initial condition in the form

$$u(x,0) = \sum_{k=0}^{\infty} c_k x^k, \qquad \text{we have} \qquad u(0,t) = \sum_{k=0}^{\infty} \frac{(2k)!}{k!} c_{2k} t^k.$$

The factor $\frac{(2k)!}{k!} \geqslant k!$ increases rapidly as $k \to \infty$. Hence the last series has zero radius of convergence and $(*)$ has no local holomorphic solution even when $u_0(x) = 1/(1-x)$ and $c_{2k} = 1$ for all k.

"...Today, a century and a quarter later, it is difficult to appreciate how surprising such examples were in 1874. The discovery of them often requires a young and fresh mind like Kovalevskaya's, to wander more freely along routes that experience had trained older minds to ignore as unpromising..."
(Roger L. Cooke, *The Cauchy–Kovalevskaya theorem*, 2001)

Theorem (S. von Kowalevsky, 1875; J. Le Roux, 1895). *The Cauchy problem* $(*)$ *is locally soluble at* $(0,0)$ $\iff$ $u_0(x) = \sum_{k=0}^{\infty} c_k x^k$, *where* $|c_k|^{1/k} \leqslant C/\sqrt{k}$ *for some* $C > 0$ $\iff$ $u_0(x)$ *is an entire functions of order at most* 2 *and finite type, i.e.* $u_0 \in \mathcal{O}(\mathbb{C})$ *and* $|u_0(x)| \leqslant Ae^{B|x|^2}$ *for some* $A, B > 0$.

Formally speaking, Kovalevskaya only proved that these bounds for $|c_k|$ are necessary for the local solubility of $(*)$, whence $u_0(x)$ must be an entire function, and showed by the example $u_0(x) = \sum_{k=0}^{\infty} x^k/(k!)^{1/3}$ that not every entire function can be taken for $u_0(x)$. This already yields a corollary on analytic extension of solutions of the heat equation.

Corollary. *Every solution* $u \in \mathcal{O}(D_1 \times D_2)$ *of the equation*

$$u_t = au_{xx}, \qquad a \in \mathbb{C} \setminus \{0\}$$

in a bidisk $D_1 \times D_2 \subset \mathbb{C}^2$ *extends analytically to* $\widetilde{u} \in \mathcal{O}(\mathbb{C} \times D_2)$ *and, generally speaking, admits no further analytic extension.*

Note that *generic solutions* (this means that $u(x, t_0)$ is an entire function of order < 2 for at least one $t_0 \in D_2$) extend to an entire function on the whole of $\mathbb{C}^2$.

Theorem and Corollary also hold for all equations

$$\partial_t u = (b\partial_x^m + c_1\partial_x^{m-1} + \cdots + c_m)u, \quad b \in \mathbb{C} \setminus \{0\},\ c_1, \ldots, c_m \in \mathbb{C}$$

with integer $m \geqslant 2$ if we replace "entire function of order 2" by "entire functions of order $m/(m-1)$" (G.S.Salekhov, 1950). Thus the solubility condition becomes more restrictive as m grows. Even more general equations of the form

$$\partial_x^m u = \sum_{k+l<m} c_{kl}\partial_x^k \partial_t^l u, \qquad c_{kl} \in \mathbb{C},$$

satisfy Corollary (C.Kiselman, 1969) and appropriately modified Theorem (Yu.F.Korobeinik, 1997). When c_{kl} are entire functions of x and holomorphic in t, we have an analogue of Corollary (M.Zerner, 1972) and part $\Rightarrow$ of Theorem (S.Ōuchi, 1983; M.Garay, 2016).

2. Soliton equations

Well-known examples of soliton equations in dimension $1+1$:

(1) $$u_t = au_{xxx} + buu_x, \qquad a,b \in \mathbb{C}\setminus\{0\},$$

(2) $$u_{tt} = au_{xxxx} + buu_{xx} + bu_x^2, \qquad a,b \in \mathbb{C}\setminus\{0\},$$

(3) $$iu_t = au_{xx} + bu|u|^2, \qquad a,b \in \mathbb{R}\setminus\{0\},$$

with $|u|^2$ in (3) understood as $u(x,t)\overline{u(\overline{x},\overline{t})}$. The inverse scattering method (Gardner, Green, Kruskal, Miura 1967) for the Korteweg–de Vries equation (1) (long waves on shallow water): when the potential $u(x,t)$ changes in time according to (1), the evolution of its scattering data (some spectral characteristsics of the operator $L = \partial_x^2 + u(x,t)$ on $L^2(\mathbb{R}_x^1)$) is linear and explicitly integrable. Explanation (Lax 1968): equation (1) with $a = 1/4$, $b = 3/2$ is the condition for solubility of the *auxiliary linear problem*

(4) $$L\psi = \lambda\psi, \qquad \psi_t = P\psi$$

where $L := \partial_x^2 + u$ and $P := \partial_x^3 + (3/2)u\partial_x + (3/4)u_x$, i.e. (1) is of the form $L_t = [P,L] \Longrightarrow$ the evolution of L is its conjugaton by a t-dependent unitary operator on $L^2(\mathbb{R}_x^1) \Longrightarrow$ the evolution of its spectral characteristics is simple.

The nonlinear Schrödinger equation (3) (propagation of wave packages in nonlinear dispersive media) was considered by V. E. Zakharov and A. B. Shabat (1971) who replaced the scalar second-order differential equation $L\psi = \lambda\psi$ in (4) by a first-order 2×2-matrix equation:

(5) $$E_x = UE, \qquad E_t = VE$$

where $U(x,t,z)$ and $V(x,t,z)$ are polynomials of degree 1 and 2 in the spectral parameter $z \in \mathbb{C}$ (related to λ in (4) by the formula $\lambda = z^2$). Then (3) is a reduction of the zero curvature equation

(6) $$U_t - V_x + [U,V] = 0$$

(first explcictly written and applied by S. P. Novikov, 1974).

The Boussinesq equation (2) (describing water waves that can move left or right) by V. E. Zakharov (1973): replace 2×2-matrices in (5) and the second-order operator L in (4) by 3×3-matrices and a third order operator.

"...To date there is no proof that the Korteweg–de Vries equation possesses the Painlevé property. The main problem lies in the lack of methods for obtaining the global analytic description of a locally defined solution in the space of several complex variables..."
(Martin D. Kruskal, *Analytic and asymptotic methods for nonlinear singularity analysis*: *a review and extension of tests for the Painlevé property*, 1997–2004)

Here is the strongest form of the Painlevé property for (1)–(3).

Theorem 1 (D., 2008–2012). *For each equation* (1)–(3), *every holomorphic solution* $u(x,t)$ *in an arbitrary bidisk* $D_1 \times D_2 \subset \mathbb{C}^2$ (*centred on* $\mathbb{R}^2$ *in case* (3)) *extends analytically to a meromorphic function* $\widetilde{u}(x,t)$ *in the strip* $S = \mathbb{C} \times D_2 \subset \mathbb{C}^2$.

Remark 1. The theorem is *non-empty and non-improvable.* Indeed, the Cauchy–Kovalevskaya theorem (with x as time variable) gives a local holomorphic solution $u(x,t)$ of (1) with $\partial_x^j u(x_0,t) = \varphi_j(t)$, $j = 0,1,2$, where $\varphi_j(t)$ are *any prescribed* holomorphic germs. One can choose them in such a way that the extension $\widetilde{u}$ cannot be further extended holomorphically to any boundary point of S.

Remark 2. To prove Theorem 1, we develop a local version of the inverse scattering method for soliton equations of parabolic type (the existsing hyperbolic version of I. M. Krichever 1981–1983 derives completely different conclusions for completely different equations), where the potentials are holomorphic germs *without any boundary conditions.* This makes a step towards another dream (how to study finite-gap and rapidly decreasing solutions in a unified manner):

"...comment marier les solutions géometriques attachées aux courbes algebriques [...] avec les diffusions qui viennent du scattering-inverse (solutions L^2 de KdV par exemple)?"
(Daniel Bennequin, *Hommage à Jean–Louis Verdier: Au jardin des systèmes intégrables*, 1993)

3. Construction of soliton equations

The main tool is the zero curvature equation (6):

$$U_t - V_x + [U, V] = 0,$$

where $U, V : \mathbb{R}^2_{xt} \times \mathbb{C}^1_z \to \mathrm{gl}(n, \mathbb{C})$ are polynomials in z (with coefficients depending on x, t) of the form $U(x, t, z) = az + q(x, t)$ and $V(x, t, z) = az^m + r_1(x, t)z^{m-1} + \cdots + r_m(x, t)$ for some constant matrix $a \in \mathrm{gl}(n, \mathbb{C})$ and smooth functions $q, r_1, \ldots, r_m : \mathbb{R}^2 \to \mathrm{gl}(n, \mathbb{C})$. The function

$$(x, t, z) \mapsto U_t - V_x + [U, V] = q_t - V_x + [U, V]$$

is a polynomial of degree $\leqslant m+1$ in z. We require it to be a polynomial of degree 0, that is, to be *independent of* z. This condition gives m differential relations (with respect to x, therefore in what follows we *omit the dependence on* t) between $q, r_1, \ldots, r_m$. If
1) $a = \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \in \mathrm{gl}(n, \mathbb{C})$ is diagonal and $\alpha_i \neq \alpha_j$ for $i \neq j$,
2) $q(x)$ is off-diagonal, that is, $q_{jj}(x) \equiv 0$ при $j = 1, \ldots, n$,
then these m relations determine $r_1, \ldots, r_m$ almost uniquely as *differential polynomials* of q in x:

$$r_j = F_j(q), \quad j = 1, \ldots, m.$$

Lemma 1. *Fix any point $x_0 \in \mathbb{C}$ and put*

$$\mathcal{O}(x_0) := \{\textit{holomorphic } \mathrm{gl}(n, \mathbb{C})\textit{-valued germs } q(x) \textit{ at } x_0\},$$
$$\mathcal{O}(x_0)^{od} := \{\textit{all off-diagonal } q \in \mathcal{O}(x_0)\}.$$

Then there is a unique sequence of differential polynomials $F_j : \mathcal{O}(x_0) \to \mathcal{O}(x_0)$, $j = 0, 1, 2, \ldots$ such that
1) *$F_0(q) \equiv a$ for all $q \in \mathcal{O}(x_0)$,*
2) *$F_j(0) \equiv 0$ for all $j \geqslant 1$,*
3) *the formal power series $F(q, z) := \sum_{j=0}^{\infty} F_j(q)z^{-j}$ satisfies*

$$\partial_x F = [az + q, F] \quad \textit{for all } q \in \mathcal{O}(x_0)^{od}.$$

For every $m = 0, 1, 2, \ldots$ the zero curvature condition (6) with

$$U = az + q \quad \text{and} \quad V = az^m + F_1(q)z^{m-1} + \cdots + F_m(q)$$

takes the form

$$(\mathrm{Eq}_m) \qquad q_t = [a, F_{m+1}(q)],$$

where $q(x,t)$ is the unknown off-diagonal $\mathrm{gl}(n,\mathbb{C})$-valued germ at the point $(x_0, t_0) \in \mathbb{C}^2$. By fixing the matrix a and a reduction (i.e. dependence of the matrix q on a scalar u) and letting $m = 1, 2, \ldots$, we obtain a *hierarchy* (an infinite sequence of commuting flows).

Examples of reductions. 1) The hierarchy of the heat equation:

$$a = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ 0 & 0 \end{pmatrix} \quad \Longrightarrow$$
$$\Longrightarrow \quad (\mathrm{Eq}_m) \quad \text{takes the form} \quad \partial_t u = \partial_x^m u.$$

2) The Korteweg–de Vries equation:

$$a = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ 1 & 0 \end{pmatrix} \quad \Longrightarrow$$
$$\Longrightarrow \quad (\mathrm{Eq}_3) \quad \text{takes the form} \quad u_t = -6uu_x + u_{xxx}.$$

3) The nonlinear Schrödinger equation:

$$a = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}, \quad q(x,t) = \begin{pmatrix} 0 & u(x,t) \\ -\overline{u(\overline{x},\overline{t})} & 0 \end{pmatrix} \quad \Longrightarrow$$
$$\Longrightarrow \quad (\mathrm{Eq}_2) \quad \text{takes the form} \quad iu_t = -u_{xx} - 2u|u|^2.$$

THEOREM $1'$ (D. 2008). *If the Cauchy problem $q(x.t_0) = q_0(x)$ for an equation Eq_m, $m \geq 2$, has a local holomorphic solution $q(x,t)$ in a neighbourhood of (x_0, t_0), then the germ $q_0(x)$ extends to a meromorphic matrix-valued function on the whole of $\mathbb{C}^1$.*

Theorem 1 (along with many other results) follows from Theorem $1'$. To prove Theorem $1'$, we construct all local holomorphic solutions of the equations Eq_m in terms of appropriate scattering data.

4. Construction of the solutions

LEMMA 2. *For every* $q \in \mathcal{O}(x_0)^{od}$ *there is a unique formal series*

$$\mu(x,z) = I + \frac{\mu_1(x)}{z} + \frac{\mu_2(x)}{z^2} + \dots \quad \textit{with } \mu_j \in \mathcal{O}(x_0) \textit{ such that}$$

$$\mu_x = (az+q)\mu - \mu az \quad \textit{and the series } \mu(x_0,z) - I \textit{ is off-diagonal.}$$

The equation for μ is equivalent to $\partial_x - (az+q) = \mu(\partial_x - az)\mu^{-1}$. Geometrically it means that μ is a gauge transformation of the connection $\partial_x - az$ (with zero potential) to the connection $\partial_x - (az + q(x))$ (with potential $q(x)$).

We define the *local scattering data* of any potential $q \in \mathcal{O}(x_0)^{od}$ as the formal power series

$$Lq(z) := \mu(x_0,z) - I = \frac{\mu_1(x_0)}{z} + \frac{\mu_2(x_0)}{z^2} + \dots.$$

Its radius of convergence is zero for almost all potentials q. To measure its divergence, we introduce the *Gevrey class* α for every $\alpha \geqslant 0$:

$$\mathrm{Gev}_\alpha := \{\text{formal power series } \varphi = \sum_{k=1}^{\infty} \frac{\varphi_k}{z^k} \text{ with off-diagonal}$$

$$\varphi_k \in \mathrm{gl}(n,\mathbb{C}) \text{ such that } \sum_{k=1}^{\infty} \frac{|\varphi_k|}{k!^\alpha} A^k < \infty \text{ for some } A > 0\}.$$

LEMMA 3. (A) *The Cauchy problem* $q(x,t_0) = q_0(x)$ *for the equation* (Eq_m) *has a local holomorphic solution* $q(x,t)$ *at the point* (x_0,t_0) *if and only if* $Lq_0 \in \mathrm{Gev}_{1/m}$.

(B) *The map* $q_0 \mapsto Lq_0$ *is a bijection of* $\mathcal{O}(x_0)^{od}$ *onto* Gev_1.

(C) *If* $q_0 \in \mathcal{O}(x_0)^{od}$ *is such that* $Lq_0 \in \mathrm{Gev}_\alpha$ *for some* $\alpha < 1$, *then the germ* $q_0(x)$ *extends analytically to an off-diagonal meromorphic function* $Q_0 \in \mathcal{M}(\mathbb{C})$.

REMARK. Theorem $1'$ follows immediately from Lemma 3(A),(C).

Proof of Lemma 3(A). Part "only if" is technical and based on a theorem of Ya. Sibuya (1991) on Gevrey solutions of singularly peturbed systems of ODE.

To prove part “if”, we define

$$\mathrm{Ent}_m := \{\text{all holomorphic } E : \mathbb{C} \to \mathrm{GL}(n,\mathbb{C}) \text{ such that } |E(z)| \leqslant Ae^{B|z|^m} \text{ for some } A, B > 0\}$$

and consider the *Riemann factorization problem*: given any $\varphi \in \mathrm{Gev}_{1/m}$ and $E \in \mathrm{Ent}_m$, find $\psi \in \mathrm{Gev}_{1/m}$ and $F \in \mathrm{Ent}_m$ such that

$$E(z)(I+\varphi(z))^{-1} = (I+\psi(z))^{-1}F(z)$$

as formal Laurent series. If we choose $E(x,t,z) = e^{a((x-x_0)z+(t-t_0)z^m)}$ for all $(x,t) \in \mathbb{C}^2$, then the Riemann problem

$$e^{a((x-x_0)z+(t-t_0)z^m)} f^{-1}(z) = \gamma_-^{-1}(x,t,z)\gamma_+(x,t,z)$$

has a solution $\gamma_\pm(x,t,z)$ for all (x,t) in some neighbourhood $\Omega = \Omega(\varphi)$ of the point (x_0,t_0) in $\mathbb{C}^2$. This is because the problem is equivalent to solving a linear equation $(I+K(x,t))u(x,t) = u_0(x,t)$ in an appropriate Banach space, where $K(x,t)$ is a *holomorphic family of bounded linear operators* on $\mathbb{C}^2_{xt}$ with $K(x_0,t_0) = 0$.

Taking the logarithmic derviatives of both parts of the equality $\gamma_+ = \gamma_- e^{a((x-x_0)z+(t-t_0)z^m)} f^{-1}$, where $(x,t) \in \Omega$, $z \in \mathbb{C}P^1$, we have $(\partial_x\gamma_+)\gamma_+^{-1} = (\partial_x\gamma_- + \gamma_- az)\gamma_-^{-1}$. Separate the positive and negative parts of the Laurent series:

$$\begin{cases} (\partial_x\gamma_+)\gamma_+^{-1} & = \{\gamma_- az\gamma_-^{-1}\}_+ \\ 0 & = (\partial_x\gamma_-)\gamma_-^{-1} + \{\gamma_- az\gamma_-^{-1}\}_-. \end{cases}$$

This can be rewritten as

$$\begin{cases} \partial_x\gamma_+ = & (az+q(x,t))\gamma_+ \\ \partial_x\gamma_- = & (az+q(x,t))\gamma_- - \gamma_- az, \end{cases} \tag{7}$$

where

$$q(x,t) = (B\varphi)(x,t) := \lim_{z\to\infty} z[\gamma_-(x,t,z),a].$$

Repeating the whole argument for t instead of x, we similarly get the second equation of the *auxiliary linear system* (5):

$$\partial_x\gamma_+ = U(q)\gamma_+, \qquad \partial_t\gamma_+ = V(q)\gamma_+,$$

where $U(q) := az + q$ and $V(q) := az^m + F_1(q)z^{m-1} + \cdots + F_m(q)$. Therefore, by cross-differentiation, $q(x,t)$ satisfies the *zero curvature condition* $U_t - V_x + [U,V] = 0$, which is equivalent to (Eq_m). If we now choose $\varphi = Lq_0$ (which lies in $\mathrm{Gev}_{1/m}$ by the hypothesis), then the initial condition $q(x,t_0) = q_0(x)$ will also hold. □

Remark. The equalities (7) reveal the geometric meaning of $\gamma_\pm$: the columns of γ_+ form a parallel frame field (a trivialization) of the flat connection $\nabla(q) := (\partial_x - U(q))\,dx + (\partial_t - V(q))\,dt$ on $\Omega \times \mathbb{C}P^1_z$, and γ_- is a formal gauge transformation of the connection $\nabla(0)$ to the connection $\nabla(q)$ on $\Omega \times \mathbb{C}P^1_z$, Here $\Omega \subset \mathbb{C}^2_{xt}$ is the neighbourhood of (x_0, t_0) where the Riemann problem is soluble.

Proof of Lemma 3(B). When $m = 1$ and $t = t_0$, the proof of Lemma 3(A) yields that the maps $L : \mathcal{O}(x_0)^{od} \to \mathrm{Gev}_1$ and

$$B : \mathrm{Gev}_1 \to \mathcal{O}(x_0)^{od}, \qquad B\varphi(x) := \lim_{z\to\infty} z[\gamma_-(x,z), a],$$

are inverse to each other: $B \circ L = \mathrm{Id}$, $L \circ B = \mathrm{Id}$. □

Proof of Lemma 3(C). If $Lq_0 \in \mathrm{Gev}_\alpha$ for some $\alpha < 1$, then the Riemann problem (with $m = 1$, $t = t_0$)

$$e^{a(x-x_0)z}(I + Lq_0(z))^{-1} = \gamma_-^{-1}(x,z)\gamma_+(x,z)$$

is equivalent to solving a linear equation $(I + K(x))u(x) = u_0(x)$ in an appropriate Banach space, where $K(x)$ is a holomorphic family of *compact* linear operators parametrized by $\mathbb{C}^1_x$ with $K(x_0) = 0$. Hence the following lemma is applicable.

Lemma 4 (meromorphic Fredholm alternative). *Let X be a complex Banach space, D a Stein manifold with $H^2(D,\mathbb{Z}) = 0$, and $K : D \to \mathcal{B}(X)$ a holomorphic family of compact operators such that $I + K(x_0)$ is invertible for some $x_0 \in D$. Then there is a holomorphic function $\tau \in \mathcal{O}(D)$ with $\tau(x_0) = 1$ such that*

$$I + K(x) \quad \text{is invertible} \quad \Longleftrightarrow \quad \tau(x) \neq 0,$$

and the map $x \mapsto \tau(x)(I + K(x))^{-1}$ extends to the zero set of τ as a holomorphic map $D \to \mathcal{B}(X)$. Thus $(I + K(x))^{-1}$ is a meromorphic operator-valued function on D.

The matrix-valued function

$$Q_0(x) := BLq_0(x) = \lim_{z\to\infty} z[\gamma_-(x, t_0, z), a]$$

is defined for all $x \in \mathbb{C}$ such that $\tau(x) \neq 0$. We have $Q_0 \in \mathcal{M}(\mathbb{C})$ by Lemma 4 and $Q_0(x) = q_0(x)$ in a neighbourhood of x_0 in $\mathbb{C}^1_x$ by what was said in the proof of Lemma 3(B). Hence it is the desired meromorphic extension of $q_0(x)$. □

5. Further properties of solutions

1) *The heat equation again.* In this case the direct ($q \mapsto Lq$) and inverse ($\varphi \mapsto B\varphi$) scattering tranforms are reduced to the classical Laplace and Borel transforms:

$$q(x) = \sum_{k=0}^{\infty} \begin{pmatrix} 0 & c_k \\ 0 & 0 \end{pmatrix} (x - x_0)^k \quad \Longrightarrow \quad Lq(z) = -\sum_{k=0}^{\infty} \begin{pmatrix} 0 & c_k \\ 0 & 0 \end{pmatrix} \frac{k!}{z^{k+1}}$$

$$\varphi(z) = \sum_{j=1}^{\infty} \begin{pmatrix} 0 & b_j \\ 0 & 0 \end{pmatrix} \frac{1}{z^j} \quad \Longrightarrow \quad B\varphi(x) = -\sum_{j=0}^{\infty} \begin{pmatrix} 0 & b_{j+1} \\ 0 & 0 \end{pmatrix} \frac{(x - x_0)^j}{j!}$$

Classically, the map $q(x) \mapsto Lq(z) = -\int_0^\infty q(x)e^{-xz}dx$ is a bijection of Ent_1 onto $\{f \in \mathcal{O}(\infty) \,|\, f(\infty) = 0\}$ and B is the inverse map.

2) *Generic solutions.* If $Lq_0 \in \mathrm{Gev}_\alpha$ for some $\alpha < 1/m$, then the solution $q(x,t)$ of the local holomorphic Cauchy problem for (Eq_m) extends meromorphically to the whole of $\mathbb{C}^2$.

3) *The set of admissible initial data decreases as the number of flow increases.* In particular, if the Cauchy problem is soluble even locally for some flow, then it is soluble globally (in t) for all lower flows.

4) *Trivial-monodromy property.* If the Cauchy problem $q(x,t_0) = q_0(x)$ for an equation Eq_m, $m \geq 2$, is soluble locally, then the auxiliary linear system $E_x = (az + q_0(x))E$ (see (5)) has a globally meromorphic (in x) fundamental system of solutions $E(x,z) = \gamma_+(x,z)$ for every $z \in \mathbb{C}$ (see (7)). In case of the KdV equation $u_t = auu_x + bu_{xxx}$, $a, b \in \mathbb{C}\backslash\{0\}$, this property implies that every local holomorphic solution $u(x,t)$ for every fixed $t = t_0$ takes the following form near every its pole x_0:

$$\text{(8)} \quad u(x,t_0) = -\frac{6b}{a}\frac{k(k+1)}{(x-x_0)^2} + \sum_{n=0}^{\infty} c_n (x-x_0)^n, \quad 0 < |x - x_0| < \varepsilon,$$

where $k = k(t_0) \in \mathbb{N}$ and $k(t_0) = 1$ for almost all t_0 while the coefficients $c_j = c_j(t_0) \in \mathbb{C}$, $j = 0, 1, 2, \ldots$ satisfy $c_{2j-1} = 0$ for all $j = 1, \ldots, k$ (characterization of trivial monodromy by Dujstermaat and Grünbaum 1986). In particular, all poles of $u(x,t_0)$ are of second order and of a rather special kind. These properties were known for finite-gap solutions (Gesztesy and Weikard 1996, Veselov 1999), but not for general holomorphic ones.

5) *Tau functions.* The KdV equation

$$(**) \qquad u_t = auu_x + bu_{xxx}, \qquad a, b \in \mathbb{C} \setminus \{0\},$$

has "running wave" solutions

$$\begin{aligned}
u_1(x,t) =& \frac{-12bA^2}{a}\wp(Ax+Bt+C) + \frac{B}{Aa} \quad \mapsto \text{finite-gap} \\
u_2(x,t) =& \frac{12ba^{-1}A^2}{\operatorname{ch}^2(Ax+4bA^3t+C)} \quad \mapsto \text{one-solition} \\
u_3(x,t) =& \frac{-12ba^{-1}}{(x+C)^2} \quad \mapsto \text{Calogero–Moser rational solutions}
\end{aligned}$$

and the Kontsevich–Witten solution $u_4(x,t) = (x+B)/(C-at)$, where $A \in \mathbb{C}\setminus\{0\}$, $B, C \in \mathbb{C}$, $\wp(s)$ is the Weierstrass function (general solution of the equation $\wp''' = 12\wp\wp'$). The functions u_2 and u_3 are limiting cases of u_1 as one or both periods tend to ∞.

THEOREM 2 (D. 2018). *Every local holomorphic solution $u(x,t)$ of the equation $(**)$ may be written in the form*

$$u(x,t) = \frac{12b}{a}\partial_x^2 \log\tau(x,t) = \frac{12b}{a} \cdot \frac{\tau_{xx}\tau - \tau_x^2}{\tau^2},$$

where $\tau(x,t)$ is an entire function of x for every fixed t.

CONJECTURE. *This entire function always has order $\leqslant 3$.*

For example, the solutions u_1, u_2, u_3, u_4 have tau functions $\tau_1, \tau_2, \tau_3, \tau_4$ of orders $2, 1, 0, 3$ respectively.

COROLLARY 1 OF THE CONJECTURE. If a local holomorphic solution $u(x,t)$ of the equation $(**)$ has no poles $x \in \mathbb{C}$ for at least one value of $t \in \mathbb{C}$, then $u(x,t)$ is either a constant of the Kontsevich–Witten solution $u_4(x,t) = (x+B)/(C-at)$ for some $B, C \in \mathbb{C}$. In particular, the KdV equation has no *non-constant entire solutions* (holomorphic in the whole of $\mathbb{C}^2$).

Corollary 2 of the conjecture. Every local holomorphic solution $u(x,t)$ of $(**)$ having only a finite number of poles $x_1,\dots,x_M\in\mathbb{C}$ for some $t\in\mathbb{C}$, is a rational Calogero–Moser solution (a sum of the principal parts of several Laurent series (8))

$$u(x,t)=-\frac{6b}{a}\sum_{j=1}^{M(t)}\frac{k_j(t)(k_j(t)+1)}{(x-c_j(t))^2},\quad c_j(t)\in\mathbb{C},\ M(t),k_j(t)\in\mathbb{N}.$$

Here we have $k_j(t)=1$ and $M(t)=M$ for all j and almost all $t\in\mathbb{C}$ and, for such t, the trajectories of poles are described by the differential equations

$$c_j'(t)=\sum_{l\neq j}\frac{12b}{(c_l(t)-c_j(t))^2},\qquad j=1,\dots,M$$

and necessarily satisfy $\sum_{l\neq j}(c_l(t)-c_j(t))^{-3}=0$, $j=1,\dots,M$. Moreover, the number of poles (counting multiplicities) must be triangular:

$$2M=\sum_{j=1}^{M(t)}k_j(t)(k_j(t)+1)=n(n+1)\quad\text{for some}\quad n\in\mathbb{N}.$$