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QUANTUM CALCULUS AND IDEALS IN THE ALGEBRA OF COMPACT OPERATORS

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I. INTRODUCTION

One of the goals of noncommutative geometry is the translation of basic notions of analysis into the language of Banach algebras. This translation is done using the quantization procedure. The arising operator calculus is called, following Connes, the quantum calculus. In our talk we shall give several assertions from this calculus concerning the interpretation of Schatten ideals of compact operators in a Hilbert space in terms of function theory. The main attention is paid to the case of Hilbert–Schmidt operators.

II. IDEALS IN THE ALGEBRA OF COMPACT OPERATORS

As the first step in the construction of noncommutative version of analysis on Banach algebras we should introduce, as in classical analysis, the notion of ”infinitely small” quantities. Their role in the algebra of bounded linear operators is played by the compact operators, their order of ”smallness” is measured by the order of decreasing of their singular numbers.

Let T be a compact operator in a Hilbert space H and $|T| = \sqrt{T^*T}$. Denote by $\{s_n(T)\}$ the sequence of singular numbers (s-numbers) of operator T given by the eigenvalues of the operator $|T|$ numerated in the decreasing order:

$$s_0(T) \geq s_1(T) \geq \dots$$

so that $s_n(T) \to 0$ for $n \to \infty$.

The singular numbers of the operator T may be computed using the minimax principle, namely:

$$s_n(T) = \inf_E \{\|T|E^\perp\| : \ \dim E = n\}$$

so that $s_n(T)$ coincides with the infimum of the norms of restrictions of T to the orthogonal complements $E^\perp$ of different n-dimensional subspaces $E \subset H$. In fact, this infimum is attained at the subspace E_n generated by the first n eigenvectors of $|T|$ corresponding to the eigenvalues $s_0, \dots, s_{n-1}$.

In another way, we can define $s_n(T)$ as the distance from T to the subspace Fin_n of operators of rank$\leq n$. Namely,

$$s_n(T) = \inf_R \{\|T - R\| : \ R \in \mathrm{Fin}_n\}.$$

Definition. Let T be a compact operator in the Hilbert space H. We shall say that T belongs to the space $\mathfrak{S}^p = \mathfrak{S}^p(H)$, $1 \le p < \infty$, if

$$\sum_{n=0}^{\infty} s_n(T)^p < \infty.$$

The space $\mathfrak{S}^p$ is an ideal in the algebra $\mathcal{K} = \mathcal{K}(H)$ of compact operators and in the algebra $\mathcal{L} = \mathcal{L}(H)$ of bounded linear operators, acting in the Hilbert space H, and is called the Schatten ideal.

We are especially interested in the class of Hilbert–Schmidt operators. A compact operator T in the Hilbert space H is called the Hilbert–Schmidt operator if

$$\sum_{n=0}^{\infty} s_n(T)^2 < \infty.$$

The quantity

$$\|T\|_2 := \left(\sum_{n=0}^{\infty} s_n(T)^2 \right)^{1/2}$$

is called the Hilbert–Schmidt norm of T.

If T is a Hilbert–Schmidt operator then for any orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in H the series

$$\sum_{k=1}^{\infty} \|Te_k\|^2$$

converges. Its sum does not depend on the choice of the orthonormal basis $\{e_k\}$ and coincides with $\|T\|_2^2$.

The space $\mathfrak{S}_2(H)$ of Hilbert–Schmidt operators, acting in the Hilbert space H, is a Hilbert space with the norm $\|\cdot\|_2$ and an ideal in the algebra $\mathcal{K}(H)$ of compact operators closed with respect to the Hilbert–Schmidt norm.

III. QUANTIZATION

Using the quantization procedure, we associate with functions defined on the unit circle the operators in a Hilbert space.

Let A be an algebra of observables, i.e. an associative algebra provided with involution. Suppose that A has the exterior differential $d : A \to \Omega^1(A)$, i.e. a linear map from A to the space $\Omega^1(A)$ of 1-forms on this algebra satisfying Leibniz rule.

The quantization of A is an irreducible linear representation π of observables from A by closed linear operators acting in a complex Hilbert space H called the quantization space.

It is required that the involution in A should transform into Hermitian conjugation, and the action of the exterior derivative operator $d: A \to \Omega^1(A)$ corresponds to the commutator with some symmetry operator S which is a selfadjoint operator on H with square $S^2 = I$. In other words,

$$\pi : df \longmapsto d^q f := [S, \pi(f)], \quad f \in A.$$

Recall that the differentiation of the algebra A is a linear map $D: A \to A$ satisfying the Leibniz rule $D(ab) = (Da)b + a(Db)$. Denote by $\mathrm{Der}(A)$ the Lie algebra of all differentiations of the algebra A. In terms of $\mathrm{Der}(A)$ the quantization is an irreducible representation of the Lie algebra $\mathrm{Der}(A)$ in the Lie algebra of linear operators on H provided with commutator as the Lie bracket.

IV. SOBOLEV SPACE OF HALF-DIFFERENTIABLE FUNCTIONS

We introduce now a Hilbert space which will play the role of the quantization space.

The Sobolev space of half-differentiable functions is the Hilbert space

$$V = H_0^{1/2}(S^1, \mathbb{R})$$

consisting of functions $f \in L_0^2(S^1, \mathbb{R})$ with zero average along the circle having the generalized derivative of order $1/2$ in $L^2(S^1, \mathbb{R})$.

In other words, it consists of functions $f \in L^2(S^1, \mathbb{R})$ having Fourier series of the form

$$f(z) = \sum_{n \neq 0} f_n z^n, \quad \bar{f}_n = f_{-n},\ z = e^{i\theta},$$

with finite Sobolev norm of order 1/2:

$$\|f\|^2_{1/2} = \sum_{n \neq 0} |n||f_n|^2 = 2 \sum_{n=1}^{\infty} n|f_n|^2 < \infty.$$

The inner product on V in terms of Fourier coefficients is given by the formula

$$(\xi, \eta) = \sum_{n \neq 0} |n| \xi_n \bar{\eta}_n = 2 \operatorname{Re} \sum_{n=1}^{\infty} n \xi_n \bar{\eta}_n,$$

for vectors $\xi, \eta \in V$.

The complexification $V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C})$ of the space V is a complex Hilbert space consisting of functions $f \in L^2(S^1, \mathbb{C})$ with Fourier decompositions of the form

$$f(z) = \sum_{n \neq 0} f_n z^n, \qquad z = e^{i\theta},$$

and finite Sobolev norm $\|f\|_{1/2}^2 = \sum_{n\neq 0} |n||f_n|^2 < \infty$. Complexified Sobolev space $V^{\mathbb{C}}$ decomposes into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of subspaces $W_\pm$ consisting of functions

$$f(z) = \sum_{n \neq 0} f_n z^n, \qquad z = e^{i\theta},$$

with Fourier coefficients f_n vanishing for $\mp n > 0$.

The space V admits a realization as the Dirichlet space $\mathcal{D}$ of functions harmonic in the unit disk $\mathbb{D}$. This space consists of harmonic functions $h : \mathbb{D} \to \mathbb{R}$ normalized by the condition $h(0) = 0$ and having finite energy

$$E(h) = \frac{1}{2\pi} \int_{\mathbb{D}} |\operatorname{grad} h(z)|^2 dxdy = \\ = \frac{1}{2\pi} \int_{\mathbb{D}} \left(\left| \frac{\partial h}{\partial x} \right|^2 + \left| \frac{\partial h}{\partial y} \right|^2 \right) dxdy < \infty.$$

Proposition

The Poisson transform

$$Pf(z) = \frac{1}{2\pi}\int_0^{2\pi} P(\zeta, z) f(\zeta) d\theta, \quad \zeta = e^{i\theta},$$

where $P(\zeta, z)$ is the Poisson kernel in the disk $\mathbb{D}$:

$$P(\zeta, z) = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2},$$

establishes an isometric isomorphism

$$P : V \longrightarrow \mathcal{D}$$

between the Sobolev space V and the Dirichlet space $\mathcal{D}$ provided with the norm

$$\|h\|^2_{\mathcal{D}} := E(h).$$

In the case of upper halfplane $\mathbb{H}$ there is one more useful interpretation of the Sobolev space $H_0^{1/2}(S^1, \mathbb{R})$. In this case the Sobolev space V coincides with $H^{1/2}(\mathbb{R})$ and the following Douglas formula gives an expression for the energy of a map Pf with $f \in H^{1/2}(\mathbb{R})$ in terms of the finite-difference derivative of f:

$$E(Pf) = \|f\|^2_{1/2} = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{f(x) - f(y)}{x - y} \right]^2 dxdy.$$

It follows in particular that functions $f \in H^{1/2}(\mathbb{R})$ have L^2-bounded finite-difference derivatives.

V. QUANTUM DIFFERENTIALS

We return now to the quantization problem formulated above and take as the algebra of observables the algebra $A = L^\infty(S^1, \mathbb{C})$ of bounded functions on S^1 provided with the natural involution given by the complex conjugation.

An arbitrary function $f \in A$ determines the bounded multiplication operator M_f in the Hilbert space $H = L_0^2(S^1, \mathbb{C})$ acting by the formula:

$$M_f : h \in H \longmapsto fh \in H.$$

The assignment $f \mapsto M_f$ defines an irreducible linear representation of the algebra A in H.

The differential of a general observable $f \in A$ is not defined in the classical sense but its quantum analogue may be correctly defined. Introduce for that the symmetry operator S on H given by the Hilbert transform

$$(Sh)(\phi) = \frac{1}{2\pi} \text{P.V.} \int_0^{2\pi} K(\phi,\psi) f(\psi) d\psi, \quad f \in H,$$

where the integral is taken in the principal value sense, i.e.

$$\text{P.V.} \int_0^{2\pi} K(\phi,\psi) f(\psi) d\psi := \lim_{\epsilon \to 0} \left[\int_0^{\phi-\epsilon} + \int_{\phi+\epsilon}^{2\pi} \right] K(\phi,\psi) f(\psi) d\psi.$$

(We identify here the functions $f(z)$ on the circle S^1 with functions $f(\phi) := f(e^{i\phi})$ on the interval $[0, 2\pi]$.)

The Hilbert kernel in the above formula is given by the expression

$$K(\phi, \psi) = 1 + i \cot \frac{\phi - \psi}{2} .$$

Note that for $\phi \to \psi$ it behaves like $1 + \frac{2i}{\phi - \psi}$.

The Hilbert operator S on V may be rewritten in a more convenient form using the orthogonal projectors $P_{\pm} : V \to W_{\pm}$ with the sum equal to $P_+ + P_- = I$ and the product $P_+ P_- = 0$. In their terms $S = P_+ - P_-$.

Define the quantum differential

$$d^q f := [S, M_f].$$

This operator is correctly defined as an operator on H for functions $f \in A$ (and even for functions from the space $\mathrm{BMO}(S^1)$).

It is an integral operator given by the formula

$$(d^q f)(h)(\phi) = \frac{1}{2\pi}\int_0^{\pi} k_f(\phi,\psi)h(\psi)d\psi, \quad h \in H,$$

where

$$k_f(\phi,\psi) = K(\phi,\psi)(f(\phi) - f(\psi)),$$

and $K(\phi,\psi)$ is the Hilbert kernel. For $\phi \to \psi$ the kernel $k_f(\phi,\psi)$ behaves (up to a constant) like

$$\frac{f(\phi) - f(\psi)}{\phi - \psi}.$$

The quasiclassical limit of this operator, obtained by the restriction of this operator to the subspace of smooth functions and taking its trace for $\phi = \psi$, is proportional to the multiplication operator: $h \mapsto f' \cdot h$.

So in the considered example the quantization is essentially reduced to the replacement of the derivative by its finite-difference analogue. Such quantization, given by the correspondence $A \ni f \mapsto d^q f : H \to H$, Connes calls the "quantum calculus" by analogy with the finite-difference calculus.

VI. QUANTUM CALCULUS

Here are some examples of the correspondence between the functions $f \in A$ and operators $d^q f$ on H:

1) the differential $d^q f$ is a finite rank operator if and only if f is a rational function (Kronecker theorem);

2) the differential $d^q f$ is a compact operator if and only if the function f belongs to the class $\mathrm{VMO}(S^1)$;

3) the differential $d^q f$ is a bounded operator if and only if the function f belongs to the class $\mathrm{BMO}(S^1)$.

Recall the definitions of the space BMO of functions with bounded mean oscillation and the space VMO of functions with vanishing mean oscillation. It is more convenient to do that for functions $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ defined on the real line $\mathbb{R}$ rather than on the circle.

Denote by

$$f_I := \frac{1}{|I|} \int_I f(x)dx$$

the average of such function over the interval I of the real line of length $|I|$. If

$$M(f) := \sup_I \frac{1}{|I|} \int_I |f(x) - f_I| dx < \infty$$

then we shall say that the function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ belongs to the space BMO($\mathbb{R}$).

Introduce one more notation

$$M_\delta(f) := \sup_{|I|<\delta} \frac{1}{|I|} \int_I |f(x) - f_I| dx$$

where $\delta > 0$. In terms of this function $f \in \mathrm{BMO}(\mathbb{R})$ if and only if the supremum $\sup_{\delta>0} M_\delta(f)$ is finite. We say that a function $f \in \mathrm{BMO}(\mathbb{R})$ belongs to the space $\mathrm{VMO}(\mathbb{R})$ if $M_\delta(f) \to 0$ for $\delta \to 0$.

VII. INTERPRETATION OF SCHATTEN IDEALS

The Sobolev space $V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C})$ of half-differentiable functions on the circle admits the following interpretation in terms of quantum correspondence.

Proposition.

A function f belongs to the Sobolev space $V^{\mathbb{C}}$ if and only if its quantum differential $d^q f$ is a Hilbert–Schmidt operator in H, hence on $V^{\mathbb{C}}$. Moreover, the Hilbert–Schmidt norm of the operator $d^q f$ coincides with the Sobolev norm of the function f.

Indeed, the commutator $d^q f := [S, M_f]$ is an integral operator on H with the kernel equal to

$$k_f(\phi, \psi) = K(\phi, \psi)(f(\phi) - f(\psi)).$$

This operator is Hilbert–Schmidt if and only if its kernel $k_f(\phi, \psi)$ is square integrable on $S^1 \times S^1$ which is equivalent to the condition

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(\phi) - f(\psi)|^2}{\sin^2\left(\frac{\phi-\psi}{2}\right)} d\phi\, d\psi < \infty.$$

Now the assertion of Proposition follows from the Douglas formula. In order to see that it is sufficient to switch in the above formula from the circle S^1 to the real line $\mathbb{R}$. Then the left hand side of the above inequality will be replaced by the expression

$$\frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{f(x) - f(y)}{x - y}\right]^2 dxdy = \|f\|_{1/2}^2$$

which implies the assertion of Proposition.

Recall now the definition of Hankel operators.

Suppose that a function φ belongs to the space $H = L^2_0(S^1, \mathbb{C})$. Denote by $H^2_\pm$ the Hardy subspace in H consisting of functions h given by the Fourier series of the form

$$h(z) = \sum_{n \neq 0} h_n z^n$$

with Fourier coefficients $h_n = 0$ for $\mp n > 0$. Denote by $P_\pm$ the orthogonal projectors $H \to H^2_\pm$.

The Hankel operator $H_\varphi : H^2_+ \to H^2_-$ is given by the formula

$$H_\varphi h := P_-(\varphi h).$$

It is bounded in H^2_+ if $P_-\varphi \in \mathrm{BMO}(S^1)$. In analogous way we can introduce the Hankel operators $\widetilde{H_\varphi} : H^2_- \to H^2_+$ given by the formula $\widetilde{H_\varphi} h := P_+(\varphi h)$.

The quantum differential $d^q_f h = [S, M_f]h$, where $f \in A = L^\infty(S^1, \mathbb{C})$, $h \in H$, may be rewritten using relations $S = P_+ - P_-$, $P_+ + P_- = I$ and $P_+P_- = P_-P_+ = 0$. Namely,

$$[S, M_f] = SM_f - M_f S =$$
$$= (P_+ - P_-)f(P_+ + P_-) - f(P_+ - P_-)(P_+ + P_-) =$$
$$= P_+fP_+ + P_+fP_- - P_-P_-fP_+ - P_-fP_- - fP_+ + fP_- =$$
$$= -P_-(fP_+) + P_+(fP_-) + P_+fP_- - P_-fP_+ = -2P_-fP_+ + 2P_+fP_-$$

implying that

$$[S, M_f]h = -2P_-fP_+h + 2P_+fP_-h.$$

The last expression coincides with $-2P_-fh$ for $h \in H^2_+$ and with $2P_+fh$ for $h \in H^2_-$. In other words, the operator $d^q f$ with $f \in A$ is the direct orthogonal sum of two Hankel operators.

So the description of quantum differentials $d^q f$, belonging to the Schatten class $\mathfrak{S}^p$, is reduced to the description of the Hankel operators from this class. The latter was obtained by Peller.

In order to formulate Peller's result we should recall the definition of Besov classes B_p^s. Denote by Δ_ζ the difference operator

$$(\Delta_\zeta f)(z) := f(\zeta z) - f(z), \quad \zeta, z \in S^1,$$

and define the nth difference Δ_ζ^n as the nth power of operator Δ_ζ.

Then the Besov space B^s_p, $s > 0$, $1 < p < \infty$, is defined as

$$B^s_p = \left\{ f \in L^p : \quad \int_{S^1} \frac{\|\Delta^n_\zeta f\|^p_p}{|1-\zeta|^{1+sp}} d\vartheta < \infty \right\}, \ \zeta = e^{i\vartheta}$$

where n is an arbitrary integer greater than s.

In particular, for $s = 1/p$ we obtain

$$B^{1/p}_p = \left\{ f \in L^p : \quad \int_{S^1} \frac{\|\Delta_\zeta f\|^p_p}{|1-\zeta|^2} d\vartheta < \infty \right\}, \ \zeta = e^{i\vartheta}.$$

Peller's theorem

Let $f \in A$. Then the Hankel operator H_f belongs to the Schatten class $\mathfrak{S}^p$ with $1 < p < \infty$ if and only if $P_- f \in B_p^{1/p}$.

It implies that the quantum differential $d^q f$ belongs to the Schatten class $\mathfrak{S}^p$ with $1 < p < \infty$ if and only if $P_- f \in B_p^{1/p}$. Note that the analogous result holds for Hankel operators from the classes $\mathfrak{S}^p$ with $0 < p < 1$.