Kloosterman Sums and Shifted Character Sums with Multiplicative Coefficients

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1. Introduction on Kloosterman sum

For the real number t, write $e(t) = e^{2\pi it}$. Then the sum

$$S(a,\, b;\, q) = \sum_{\substack{x=1 \\ (x,\, q)=1}}^{q} e\Big(\frac{ax + b\bar{x}}{q}\Big)$$

is called Kloosterman sum, where $\bar{x}$ satisfies $\bar{x}x \equiv 1 \pmod{q}$, which is unique modulo q.

The Kloosterman sum plays an important role in number theory. There is systematic and deep study on this sum.

Early in 1926, Kloosterman introduced in this sum. His purpose is to study the problem on the positive integer solutions of the quadratic diagonal form

$$N = a_1 n_1^2 + a_2 n_2^2 + a_3 n_3^2 + a_4 n_4^2,$$

where a_i are fixed positive integers.

This problem is a generalization of Lagrange four squares theorem, the purpose of which is to determine that for which coefficients (a_1, a_2, a_3, a_4), sufficiently large N can be expressed in this form.

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Now it is difficult to apply Hardy-Littlewood circle method directly. Then Kloosterman made improvement on the circle method. In the proof, he met the sum

$$\sum_{\substack{x=1 \\ (x,q)=1}}^{X} e\Big(\frac{b\bar{x}}{q}\Big),$$

which is called incomplete Kloosterman sum.

If $(b, q) = 1$, $X = q$, the above sum is Ramanujan sum

$$\sum_{\substack{x=1 \\ (x,q)=1}}^{q} e\Big(\frac{b\bar{x}}{q}\Big) = \sum_{\substack{y=1 \\ (y,q)=1}}^{q} e\Big(\frac{y}{q}\Big) = \mu(q).$$

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For the incomplete sum, we can transform it to the complete sum by a standard technique. Therefore

$$\Big| \sum_{\substack{x=1 \\ (x,q)=1}}^{X} e\Big(\frac{b\bar{x}}{q}\Big) \Big| \le (1 + \log q) \max_{1 \le a \le q} |S(a,\, b;\, q)|.$$

Now the problem is changed into the estimate for $S(a,\, b;\, q)$.

We confine that q is a prime number p to see the treatment of Kloosterman. Firstly he considered the mean value

$$\sum_{r=0}^{p-1} \sum_{s=0}^{p-1} |S(r,\, s;\, p)|^4.$$

This mean value has arithmetic meaning which is the number of solutions of the equation system

$$x_1 + x_2 - x_3 - x_4 \equiv 0, \qquad \bar{x}_1 + \bar{x}_2 - \bar{x}_3 - \bar{x}_4 \equiv 0 \pmod p.$$

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Then for the fixed a, b, $p \nmid b$, there are at least $p-1$ terms in the above mean value which are equal to $|S(a,\, b;\, p)|^4$.

Combining all of the above, we get

$$|S(a,\, b;\, p)| < 3^{\frac{1}{4}} p^{\frac{3}{4}}, \qquad p \nmid b.$$

This shows that in the above incomplete Kloosterman sum, if $X > p^{\frac{3}{4}+\varepsilon}$, there is some cancellation.

By this kind of estimate, Kloosterman can solve the problem on the quadratic diagonal form in four variables.

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In 1948, Weil proved a stronger result

$$|S(a, b; p)| \le 2p^{\frac{1}{2}}, \qquad p \nmid b,$$

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1) It is conjectured that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be expressed as

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2,$$

where p_i is prime number. This is a development of Lagrange four squares theorem.

The case for five primes was solved by L. Hua. The case for four primes is very difficult.

2. Applications

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In 1994, Brüdern and Fouvry proved that every sufficiently large integer $N \equiv 4 \pmod{24}$ can be expressed as

$$N = P_1^2 + P_2^2 + P_3^2 + P_4^2,$$

where the number of prime factors of every P_i is at most 34.

In their proof, the improvement on circle method by Kloosterman and the estimate for Kloosterman sum were used.

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2) Let p be prime, $(a, p) = 1$. Solve

$$mn \equiv a \pmod p,$$

where positive integers m and n are small as possible.

Write $M(a)$ as the minimum of $\max(m, n)$. It is obvious that

$$M(p-1) \geq \sqrt{p-1}, \qquad M(a) \leq p-1.$$

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Write

$$A_n = \begin{cases} 1, & \text{if } mn \equiv a \pmod p \text{ has solution } 1 \le m \le M, \\ 0, & \text{others.} \end{cases}$$

By a standard technique, we get

$$\begin{aligned} \Big| \sum_{0<n\le M} A_n - \frac{M^2}{p} \Big| &\le \log p \max_{1\le k<p} \Big| \sum_{n=1}^{p} \sum_{\substack{m=1 \\ mn\equiv a \pmod p}}^{M} e\Big(\frac{kn}{p}\Big) \Big| \\ &= \log p \max_{1\le k<p} \Big| \sum_{m=1}^{M} e\Big(\frac{ka\bar{m}}{p}\Big) \Big| \\ &< 4(\log p)^2 p^{\frac{1}{2}}. \end{aligned}$$

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The sum

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is the number of solutions of the equation

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Hence, if

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then the above equation must has a solution, which means

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3) Let $d(n)$ be divisor function. We consider the behaviour of the sum

$$\sum_{n \le x} d(n)d(n+1).$$

This sum counts the number of integer arrays $(a,\ b,\ r,\ s)$ which satisfy

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We can get the condition

$$ab \equiv -1 \pmod r.$$

For the fixed r, we calculate how many a, b. A similar problem with the above one appears, which is on $M(-1)$.

By the similar discussion, we get an asymptotic formula

$$\sum_{n \le x} d(n)d(n+1) = xQ(\log x) + O(x^{\frac{5}{6}+\varepsilon}),$$

where ε a sufficient small positive constant, Q is some quadratic polynomial.

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For any positive integer a, the sum

$$\sum_{n\le x} d(n)d(n+a)$$

can be dealt with in the same way. These sums are interested since they appear in the Riemann zeta function theory.

We see the integral

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4 dt.$$

Write $|\zeta|^4 = \zeta^2\overline{\zeta^2}$. Expanding this product to produce a double sum, the near diagonal terms contain $d(n)d(n+a)$. Then the problem is changed into that for

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Heath-Brown proved that

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where F is some fourth power polynomial.

Motohashi improved the exponent $\frac{7}{8}$ to $\frac{2}{3}$. He used the mean value theory of Kloosterman sums which was developed by Kuznetsov and Iwaniec.

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4) Iwaniec made further development on the classic fourth power mean value of ζ function. His object is to estimate the sixth power of ζ function. He proved that

$$\int_0^T |\zeta(\frac{1}{2}+it)|^4 \Big|\sum_{n\le N}\frac{a(n)}{n^{\frac{1}{2}+it}}\Big|^2 dt \ll T^{1+\varepsilon},$$

where $a(n)=O(1)$, $N \ll T^{\frac{1}{10}}$. He used the estimate for the Kloosterman sum.

Afterwards, Iwaniec and Deshouillers, Watt improved the exponent $\frac{1}{10}$ into $\frac{1}{5}$ and $\frac{1}{4}$. They used the estimate for the Kloosterman sum.

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3. Further development

Let us see the Kloosterman sum with some special coefficients.

In 1988, D. Hajela, A. Pollington, B. Smith proved that if $(b, q) = 1$, then

$$\sum_{\substack{n \le N \\ (n,q)=1}} \mu(n) e\Big(\frac{b\bar{n}}{q}\Big) \ll N q^{\varepsilon} \Big(\frac{(\log N)^{\frac{5}{2}}}{q^{\frac{1}{2}}} + \frac{q^{\frac{3}{10}} (\log N)^{\frac{11}{5}}}{N^{\frac{1}{5}}} \Big).$$

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P. Deng pointed out that under GRH, one can get

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which is non-trivial for $q \ll N^{1-4\varepsilon}$. Therefore to break through the limitation $q \le N$ is the next direction of development.

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In 1998, Fouvry and Michel proved that if q is a prime number, $P(x)$ and $Q(x)$ are coprime monic polynomials on $\mathbb{Z}[x]$, $g(x) = \frac{P(x)}{Q(x)}$ is a rational function, then for $N \le q$, one has

$$\sum_{\substack{p \le N \\ (Q(p), q)=1}} e\Big(\frac{g(p)}{q}\Big) \ll q^{\frac{3}{16}+\varepsilon} N^{\frac{25}{32}},$$

where p is the prime number. This estimate is non-trivial for $N \le q \ll N^{\frac{7}{6}-7\varepsilon}$.

They also proved that for $N \le q$,

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In 2011, Fouvry and Shparlinski proved that for $(b, q) = 1$, $N^{\frac{3}{4}} \le q \le N^{\frac{4}{3}}$, one has

$$\sum_{\substack{N<p\le 2N\\(p,q)=1}} e\Big(\frac{b\bar{p}}{q}\Big) \ll q^{\varepsilon}(q^{\frac{1}{4}}N^{\frac{2}{3}} + N^{\frac{15}{16}}).$$

This estimate is non-trivial for $N^{\frac{3}{4}} \le q \ll N^{\frac{4}{3}-6\varepsilon}$.

They also proved that

$$\sum_{\substack{N<p\le 2N\\(p,q)=1}} e\Big(\frac{b\bar{p}}{q}\Big) \ll Nq^{\varepsilon}\Big(\frac{(\log N)^2}{q^{\frac{1}{2}}} + \frac{q^{\frac{1}{4}}(\log N)^{\frac{3}{2}}}{N^{\frac{1}{5}}}\Big).$$

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Naturally one would consider more general situation. When Ke Gong visited Montreal University, Professor Granville suggested him to study the non-trivial estimate for

$$\sum_{\substack{n \le N \\ (n,q)=1}} f(n) e\Big(\frac{b\bar{n}}{q}\Big)$$

where $f(n)$ is a multiplicative function satisfying $|f(n)| \le 1$.

In the above results, one can apply Vaughan's method in which properties that

$$\sum_{d|\,n} \Lambda(d) = \log n$$

or

$$\sum_{d|\,n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1 \end{cases}$$

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But in our situation, we know nothing on

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Ke Gong and Chaohua Jia proved that if $f(n)$ is a multiplicative function, $|f(n)| \le 1$, $q \le N^2$, $(b,\, q) = 1$, then

$$\sum_{\substack{n\le N\\(n,q)=1}} f(n)e\Big(\frac{b\bar{n}}{q}\Big) \ll \sqrt{\frac{d(q)}{q}}N(\log\log 6N) + q^{\frac{1}{4}+\frac{\varepsilon}{2}}N^{\frac{1}{2}}(\log 6N)^{\frac{1}{2}} + \frac{N}{\sqrt{\log\log 6N}}.$$

Then for

$$(\log\log 6N)^{3+\varepsilon} \ll q \ll N^{2-5\varepsilon},$$

we have

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If $f(n) = \mu(n)$, we can get a bigger range of non-trivial estimate than before. But for the sum on prime numbers, our method is not available.

M. A. Korolev improved the upper bound to get

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4. Shifted character sum

We have corresponding result on the shifted character sum.

Let q be a prime number, $(a, q) = 1$, χ be a non-principal Dirichlet character modulo q.

Since the 1930s, I. M. Vinogradov had begun the study on character sums over shifted primes

$$\sum_{p \le N} \chi(p + a),$$

and obtained deep results, where p is prime number. His best known result is a nontrivial estimate for the range $N^{\varepsilon} \le q \le N^{\frac{4}{3} - \varepsilon}$, where ε is a sufficiently small positive constant, which lies deeper than the direct consequence of GRH.

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Later, Karatsuba widen the range to $N^{\varepsilon} \leq q \leq N^{2-\varepsilon}$, where Burgess's method was applied.

For the Möbius function $\mu(n)$, one can get same results on sums

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Ke Gong and Chaohua Jia used the finite version of Vinogradov's inequality to prove the following result:

If $f(n)$ is a multiplicative function satisfying $|f(n)| \le 1$, q $(\le N^2)$ is a prime number and $(a, q) = 1$, χ is a non-principal Dirichlet character modulo q, then we have

$$\sum_{n \le N} f(n)\chi(n+a) \ll \frac{N}{q^{\frac{1}{4}}} \log\log(6N) + q^{\frac{1}{4}} N^{\frac{1}{2}} \log(6N) + \frac{N}{\sqrt{\log\log(6N)}}.$$

Then

$$(\log\log(6N))^6 \ll q \ll \frac{N^2}{(\log(6N))^{4+\varepsilon}},$$

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Using Korolev's method, Ke Gong, Chaohua Jia and M. A. Korolev improved the upper bound to get

$$\sum_{n \le N} f(n)\chi(n+a) \ll \frac{N}{\log N} \log\log N.$$